An introduction to TWG24: Representations in mathematics teaching and learning

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Introduction

CERME13 was the fourth full conference to feature TWG24, *Representations in mathematics teaching and learning*. This year we accepted 18 papers and 1 poster. In terms of participants, Italy and Germany were strongly represented, with the group also including researchers from Australia, Denmark, Israel, Mexico, Netherlands, Switzerland, Turkey, United Arab Emirates, and United Kingdom. This year over half the group were early career researchers. All papers were shared with the participants in advance of the conference, then presented by the author(s), followed by group discussions. This year we introduced a semi-formal 'response' procedure, where each participant was assigned two papers for which they were asked to prepare a question or comment to start the group discussion. This ensured the voices of every member of the group were heard in discussion, including those who may be less confident in such settings, and/or who require more time to prepare a response. For this year's final 'workshop' session in breakout groups, we asked each group to identify, discuss and present an emerging theme of the week that they had found particularly engaging. These themes chosen by the TWG participants are used to structure this report.

Representation and digital environments

Several contributions presented studies of representations of mathematical objects within digital environments, where the affordances go beyond those of 'static' media. There was agreement that such digital environments make the mathematical content more accessible to some learners because of the ways they afford access to the representations through different senses. They allow the user to manipulate mathematical objects in ways that can support the formation of mental representations, that can then be manipulated mentally. The mathematical content included arithmetic word problems concerning addition of integers (Erbilgin & Macur), fractions represented on orthogonal axes (Kadan-Tabaja & Yesushalmy), part-whole relations (Kortenkamp et al.), equations (Bonadiman; Robotti et al.), inequalities (Méndez Franco & Acuña Soto), and geometry (Etzold & Larkin).

These contributions outline some common features and emerging trends, the first of which entails theoretical frameworks that promote the central role played by artifacts: e.g., Artifact Centric Activity Theory (Ladel & Kortenkamp, 2016) and the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008). Artifacts are conceived in a broader sense as the whole digital environment or software (Kortenkamp et al.), but also as a composition of different environments constantly in communication (Bonadiman; Etzold & Larkin; Robotti et al.). Another common feature is the possibility for learners to interact with dynamic manipulable virtual objects and receive immediate

visual feedback resulting from their actions. The feedback can have a variety of effects on students' practice and ways of thinking; indeed, it can support the evolution of signs (Robotti et al.), elicit spatial reasoning and computational thinking (Etzold & Larkin), change children's strategies (Kortenkamp et al.), and support mathematical sense-making (Bonadiman). Another emerging trend in the research showcased is the wider use of mobile devices with touch screens (Bonadiman; Etzold & Larkin; Kortenkamp et al.; Robotti et al.), where interaction can become more haptic and sensorial.

Accessibility and inclusion through representations

This theme has always had a natural place within TWG24: many of our researchers have observed wide diversity in their participants' representational strategies, including certain groups who are structurally disadvantaged compared to their peers by the representational environment of their classrooms and curricula. Providing appropriately inclusive mathematics education requires varied and often nonstandard representational forms and formats, and we have a tradition in particular of case studies of learners with a history of struggle with mathematics. Note that while 'diverse learners' sometimes means those with Special Educational Needs and/or Disabilities, educational disadvantages are much broader than this, and better understood as multifaceted and intersectional. Finesilver's paper provides an overview of CERME research from the last 10 years at the intersection of representation and inclusion, and there are substantial further additions on the theme in this volume.

Several papers continue the tradition of fine-grained case studies of the representational activity of students who face particular challenges in mathematics education. Three focus on students who were either formally diagnosed with mathematical learning disability or considered to be low-attaining: Robotti et al. focus on one student's verbal signs during a digital artifact-mediated algebraic activities, Bonadiman looks at the relations between artifact signs and mathematical signs in one student's artifact-integrated activity, and Macchioni et al. take a commognitive perspective on two students' attempts at sense-making. Wille & Ott's research is also set within an individual learning support context, but provides a visual analysis of the intertwining interaction patterns between diagrammatic activity and communication about the activity.

As with representations more generally, there is overlap between research on inclusive mathematics and embodied mathematics. Miragliotta et al.'s case study is of an adult mathematician who 'sees' with her hands rather than her eyes, while Angeloni et al. continue their work on mathematics as communicated through Austrian Sign Language. O'Brien's study of the mathematics within tapestry weaving, and van Wijk's promotion of mathematical paper folding in secondary education both fit into the broader definition of inclusion, invoking ancient cultural mathematical practices that are outside current classroom norms, and which can inclusively engage learners with powerful concepts.

Building and interrogating theory

In our TWG this year, several papers endeavoured to go beyond using theory to understand or interpret results, and sought to explore, expand and even interrogate the very theoretical framings they invoked. Miragliotta et al. took up Fischbein's (1993) theory of figural concepts to ask about how this implicitly visual account of geometrical reasoning might be reinterpreted in light of the work of a "blind solver". Through close examination of the speech, gestures and tool-use of one such solver, Miragliotta et al. concluded that Fischbein's theory was, in fact, capacious enough to support the

interpretation of this special case. However, this paper clearly evidenced that the spatial aspects of geometrical reasoning need not be visually grounded. Efforts to interrogate how we theorize the visuospatial elements of mathematical representations were also in the work of Etzold & Larkin, whose paper pointed to an under-theorization of the connections between computational thinking and spatial reasoning. Although in a preliminary stage, the presentation and discussion of this research at the conference hinted at ambitions to develop new ways of thinking about spatial reasoning (connected to other skills and experiences), as well as new directions for artifact-centred theories of representation (which might find innovative means for taking up the multiple, interrelated artifacts simultaneously at play in apps like Cubeling).

Importantly, some of our working group members are working in fields where key concepts and theories are still in the process of being fleshed out. This is particularly the case for Angeloni et al., whose continuing efforts to characterize mathematical discourse of users of Austrian Sign Language (ÖGS) has led them to emphasize the important role of *classifiers* and *simultaneity* in algebraic dialogs conducted in ÖGS. Similarly, Wille & Ott draw on the work of Peirce and Sfard to forge a novel theoretical approach for the understudied domain of individual learning support.

At the opposite end of the spectrum, other contributors to our TWG are working in heavily-trod and sometimes contentious spaces, where theoretical debates about the epistemological status of certain concepts can be, well, fractious. Unexpectedly, one of the hot debates this year centred on whether or not "one" can be considered a fraction. Arising in verbal discussion without the representation of a symbol, number-line or bar-model (etc.), the question of which authorities a person might consult, and what pedagogical problems might be at stake, led to a spirited discussion. This highlights the importance of research like that of Lemrich et al., whose paper explored associations and intuitions about fractions among pre-service teachers, arguing that even our earliest mathematical ideas can inadvertently influence our thinking deep into adulthood.

Representations in data collection, analysis, and communication

A variety of approaches were used in our TWG used to collect and organise research data. Some used video freeze frames to focus on gesture (O'Brien), tables to align speech and action (Miragliotta et al.), or graphs to generate quantitative data related to indicative actions or speech (Kortenkamp et al., Robotti et al.; Thomaneck et al.). Others sought to characterise the larger affective landscape of video data (Eich-Høy). An unusual source of data (for our TWG) stemmed from the presence this year of two literature reviews (Finesilver; van Wijk et al.). As discussed above, Finesilver's work sought to identify connections and themes within the CERME research community with an intersecting focus on representation and inclusion. Van Wijk et al. canvased teacher-oriented journals in France and Germany to make wider observations about learning cultures and teacher practice. Additionally, Vangsøe Færch & Ladegaard Pedersen's textual analysis of the representation of fractions in Danish textbooks demonstrated how different mathematical representations populate (or vanish within) textbook-driven curricula. This text-based research, generating meta-analysis of how mathematical representations appear in research literature and beyond, will help to shape future debates in our field.

During the group discussions we noticed a new phenomenon: many of the data representations were not chosen by the researcher, but output by default (or on request) by software producing specific

visual layouts that are minimally modifiable. Some features of such layouts are more standard, like tables or figures, but others are innovative and not well-known by our community – especially those providing visual configurations representing large amounts of data. For example, Robotti et al. describe how they interpret visual layouts produced by MAXQDA to identify emergent phenomena upon which finer analyses are then conducted. Another example is provided by the studies using eye tracking (Thomaneck et al.), where different visual gaze plots are used to report on and analyse recurring patterns in students' navigation of contextual graphs, highlighting different phases and strategies. In the case studies, the range of possibilities evidences the mercurial nature of video input. Participants discussed whether and how such software-generated layouts might influence the analytical tools and theoretical frameworks behind the studies in which they were used. Building on Peirce's perspective on diagrams, Wille & Ott also used a graphical tool to analyse their data, but in their case such a tool is 'home-made' and its representational output developed with purpose. This is a contrasting situation to the one described above, in that the underlying theory influences the graphical representation of data to be analysed. Overall, discussions shed light onto the lively interplay between data-driven and theory-driven analyses, and how visual layouts of data (digital or analogue) affect and are affected by the theoretical framework that the researchers have chosen.

Looking forward

TWG24 is proud of the diversity we contain in terms of research foci, theoretical perspectives, methodologies, and ways of communicating our discoveries. There are certain underpinning themes that we expect to continue, e.g., an interest in the representational affordances of new technologies, and a commitment to inclusive principles – not just widening participation in mathematics, but optimising *meaningful* participation for all. Two emerging interests that we would highlight are the roles of representation *within* the research process, and the way that research into mathematical representations can advance our thinking about embodied learning and more-than-human pedagogies, such as the enfolding of virtual experiences and physical experiences. These foci offer interesting new terrain for our theories of learning to tackle, and will be addressed in our next call for papers.

References

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