CUP PRODUCT IN BOUNDED COHOMOLOGY OF NEGATIVELY CURVED MANIFOLDS

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ABSTRACT. Let M be a negatively curved compact Riemannian manifold with (possibly empty) convex boundary. Every closed differential 2-form $\xi \in \Omega^2(M)$ defines a bounded cocycle $c_\xi \in \mathrm{C}_b^2(M)$ by integrating ξ over straightened 2-simplices. In particular Barge and Ghys proved that, when M is a closed hyperbolic surface, $\Omega^2(M)$ injects this way in $\mathrm{H}_b^2(M)$ as an infinite dimensional subspace. We show that any class of the form $[c_\xi]$, where ξ is an exact differential 2-form, belongs to the radical of the cup product on the graded algebra $\mathrm{H}_b^\bullet(M)$.

1. INTRODUCTION

Bounded cohomology is a rich research field with various applications, but direct computation of bounded cohomology modules is a hard task. An important case is the free non-abelian group with $n \geqslant 2$ generators $F_n = F$. The bounded cohomology modules with real coefficients $\mathrm{H}_b^k(F)$ are infinite dimensional when $k = 2$ or $k = 3$, while it is still not known whether $\mathrm{H}_b^k(F) \neq 0$ when $k \geqslant 4$. All the classes in $\mathrm{H}_b^2(F)$ can notoriously be represented as coboundaries of *quasi-morphisms*. There are various recent results investigating whether it is possible to construct a non-trivial bounded cocycle of degree $k \geqslant 4$ as the cup product of non-trivial quasi-morphisms; see [BM18], [Heu20], [FF20] and [AB21]. All these results seem to suggest that $\cup\colon \mathrm{H}_b^2(F) \times \mathrm{H}_b^k(F) \to \mathrm{H}_b^{k+2}(F)$ could be trivial. In particular, in [AB21] the authors prove the following:

Theorem 1. Let φ be a Δ-decomposable quasi-morphism and $\alpha \in \mathrm{H}_b^k(F)$, then

$$[\delta^1\varphi] \cup \alpha = 0 \in \mathrm{H}_b^{k+2}(F).$$

The main result of this paper has a similar flavour, but in a different context. Let M be a negatively curved compact Riemannian manifold with (possibly empty) convex boundary. Every differential k-form $\psi \in \Omega^k(M)$ defines a singular k-cochain $c_\psi \in \mathrm{C}^k(M)$ by integrating ψ over straightened simplices. As we will see in Section 2.2, c_ψ is bounded when $k \geqslant 2$. Moreover, for every $\varphi \in \Omega^1(M)$ we have $\delta^1 c_\varphi = c_{d\varphi}$, hence c_φ is a *quasi-cocycle* c_φ, i.e. a cochain with bounded differential. Degree one quasi-cocycles play in singular cohomology the very same role of quasi-morphisms in group cohomology. If we denote by $\mathrm{E}\Omega^2(M) \subset \Omega^2(M)$ the space of exact forms we will show the following:

Main Theorem. Let $\xi \in \mathrm{E}\Omega^2(M)$ and $\alpha \in \mathrm{H}_b^k(M)$, then

$$[c_\xi] \cup \alpha = 0 \in \mathrm{H}_b^{k+2}(M).$$

This is particularly interesting when $M = \Sigma$, a closed hyperbolic surface. In this case all the quasi-cocycles defined by non-trivial exact forms are non-trivial and thus $\mathrm{E}\Omega^2(\Sigma)$ is an infinite dimensional subspace of $\mathrm{H}_b^2(\Sigma)$. This is true thanks to Theorem 3.2 of [BG]:

Theorem 2. The map $\Omega^2(\Sigma) \to \mathrm{H}_b^2(\Sigma)$ that sends ψ to $[c_\psi]$ is injective.

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2. Preliminaries

2.1. **Bounded cohomology and differential forms.** Let X be a topological space, we denote by $(\mathrm{C}^\bullet(X), \delta^\bullet)$ its singular cochain complex with real coefficients and by $\mathrm{H}^\bullet(X)$ its singular cohomology with real coefficients. Let $\mathrm{S}_k(X) = \{s\colon \Delta^k \to X\}$ be the set of k-singular simplices of X and define an ℓ^∞ norm on $\mathrm{C}^k(X)$ by setting, for every $\omega \in \mathrm{C}^k(X)$,

$$\|\omega\|_\infty = \sup\left\{|\omega(s)| \;\middle|\; s \in \mathrm{S}_k(X)\right\}.$$

The subspaces of bounded k-cochains

$$\mathrm{C}_b^k(X) = \left\{\omega \in \mathrm{C}^k(X) \;\middle|\; \|\omega\|_\infty < \infty\right\}$$

form a subcomplex $\mathrm{C}_b^\bullet(X) \subset \mathrm{C}^\bullet(X)$, whose homology will be denoted by $\mathrm{H}_b^\bullet(X)$. The ℓ^∞ norm descends to a seminorm on $\mathrm{H}^\bullet(X)$ and $\mathrm{H}_b^\bullet(X)$ by defining the seminorm of a class as the infimum of the norms of its representatives. The inclusion $\mathrm{C}_b^\bullet(X) \hookrightarrow \mathrm{C}^\bullet(X)$ induces a map

$$c^\bullet\colon\ \mathrm{H}_b^\bullet(X) \to \mathrm{H}^\bullet(X)$$

called the *comparison map*. The kernel of c^k is denoted by $\mathrm{EH}_b^k(X)$ and called the *exact bounded cohomology* of X.

Now let X be a Riemannian manifold, we denote by $\Omega^k(X)$ the space of smooth k-forms on X and by $d\colon \Omega^k(X) \to \Omega^{k+1}(X)$ the usual differential. The subspaces of closed and exact k-forms will be denoted by $\mathrm{C}\Omega^k(X)$ and $\mathrm{E}\Omega^k(X)$, respectively. We denote the De Rham cohomology of X by

$$\mathrm{H}_{dR}^\bullet(X) = \frac{\mathrm{C}\Omega^\bullet(X)}{\mathrm{E}\Omega^\bullet(X)}.$$

For every $\psi \in \Omega^k(X)$ and $x \in X$ set

$$\|\psi_x\|_\infty = \sup\left\{|\psi_x(\underline{v})| \;\middle|\; \underline{v} \in T_xX \text{ is a } k\text{-orthonormal frame}\right\}$$

so that we can define an ℓ^∞ norm on $\Omega^k(X)$ as follows:

$$\|\psi\|_\infty = \sup_{x\in X}\{\|\psi_x\|_\infty\} \in [0, +\infty].$$

Of course, if X is compact, then $\|\psi\|_\infty < \infty$ for every $\psi \in \Omega^\bullet(X)$. Observe that for any k-dimensional immersed submanifold $D \hookrightarrow X$ we have that

$$\left|\int_D \psi\right| \leqslant \int_D \|\psi\|_\infty d\mathrm{Vol} = \mathrm{Vol}_X(D) \cdot \|\psi\|_\infty.$$

2.2. **Negatively curved manifolds and 2-forms.** Throughout the whole paper, let M be a negatively curved orientable compact Riemannian manifold with (possibly empty) convex boundary. The universal covering $\widetilde{M}$ is continuously uniquely geodesic and thus for every $(x_0, \dots, x_k) \in \widetilde{M}^{k+1}$, by repeatedly coning on the x_i one can define the straight k-simplex $[x_0, \dots, x_k] \in \mathrm{S}_k(\widetilde{M})$ as constructed in Section 8.4 of [Fri]. The fundamental group $\pi_1(M) = \Gamma$ acts on the universal covering $\widetilde{M}$ via deck transformations and this defines in turn an action of Γ on $C_b^\bullet(\widetilde{M})$. We denote by $\mathrm{C}_b^\bullet(\widetilde{M})^\Gamma$ the subcomplex of Γ-invariant cochains. The covering map $p\colon \widetilde{M} \to M$ induces an isometric isomorphism of normed complexes $\mathrm{C}_b^\bullet(M) \xrightarrow{\cong} \mathrm{C}_b^\bullet(\widetilde{M})^\Gamma$. Similarly, Γ acts on $\Omega^k(\widetilde{M})$, we denote by $\Omega^k(\widetilde{M})^\Gamma$ the space of Γ-invariant k-forms of $\widetilde{M}$. By pulling-back via the covering projection we get the identification $\Omega^k(M) \xrightarrow{\cong} \Omega^k(\widetilde{M})^\Gamma$.

For any $\psi \in \Omega^k(\widetilde{M})^\Gamma$, we define a cochain $c_\psi \in \mathrm{C}^k(\widetilde{M})^\Gamma$ by setting for every $s \in \mathrm{S}_k(\widetilde{M})$,

$$c_\psi(s) = \int_{[s(e_0),\dots,s(e_k)]} \psi.$$

where $e_0, \dots, e_k$ are the vertices of the standard simplex Δ^k.

Applying Stoke's Theorem we see that for every $s \in \mathrm{S}_{k+1}(\widetilde{M})$,

$$(\delta^k c_\psi)(s) = c_\psi(\partial_{k+1} s) = \int_{\partial_{k+1}[s(e_0),\dots,s(e_{k+1})]} \psi = \int_{[s(e_0),\dots,s(e_{k+1})]} d\psi = c_{d\psi}(s)$$

and thus mapping ψ to c_ψ defines a morphism of cochain complexes $I^\bullet \colon \Omega^\bullet(\widetilde{M})^\Gamma \to \mathrm{C}^\bullet(\widetilde{M})^\Gamma$. Furthermore, the fact that the *straightening operator* $s \mapsto [s(e_0), \dots, s(e_k)]$ is Γ-equivariantly homotopic to the identity of $\mathrm{C}^k(\widetilde{M})$ (see e.g. [Fri] Proposition 8.11) implies that the map induced by $I^\bullet$ on cohomology corresponds to the *De Rham isomorphism* $\mathrm{H}^\bullet_{dR}(M) \xrightarrow{\cong} \mathrm{H}^\bullet(M)$ defined e.g. in Chapter 18 of [Lee18].

Since the action of Γ is cocompact we have $\|\psi\|_\infty < \infty$, for every $\psi \in \Omega^k(\widetilde{M})^\Gamma$. Furthermore, as shown in the second section of [IY82], when $k \geqslant 2$ the volume of $[x_0, \dots, x_k]$ is bounded by a constant V_k that depends only on k and an upper bound of the curvature of M. This means that for every $s \in \mathrm{S}_k(\widetilde{M})$,

$$|c_\psi(s)| = \left| \int_{[s(e_0),\dots,s(e_k)]} \psi \right| < V_k \cdot \|\psi\|_\infty$$

and thus $c_\psi \in \mathrm{C}^k_b(\widetilde{M})^\Gamma$ is a bounded cochain.

We have a well defined map for $k \geqslant 2$:

$$\begin{aligned} I^k_b \colon \mathrm{C}\Omega^k(M) &\to \mathrm{H}^k_b(M) \\ \psi &\mapsto [c_\psi]. \end{aligned}$$

Interestingly, since $I^\bullet$ induces the De Rham isomorphism we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{C}\Omega^k(M) \arrow[r, two heads] \arrow[d, "I^k_b"] & \mathrm{H}^k_{dR}(M) \arrow[d, "\cong"] \\ \mathrm{H}^k_b(M) \arrow[r, "c^k"] & \mathrm{H}^k(M) \end{tikzcd}$$

showing that the comparison map c^k is surjective for $k \geqslant 2$ (this is true in the much more general context of aspherical manifolds with Gromov hyperbolic fundamental group, see [Min01]).

Furthermore, when $k > 2$, for any $d\varphi \in \mathrm{E}\Omega^k(M)$,

$$I^k_b(d\varphi) = [c_{d\phi}] = [\delta^{k-1} c_\varphi] = 0 \in \mathrm{H}^k_b(M),$$

meaning that the restriction $I^k_b \colon \mathrm{E}\Omega^k(M) \to \mathrm{H}^k_b(M)$ is the zero map. This implies that I^k_b descends on the quotient $\mathrm{C}\Omega^k(M)/\mathrm{E}\Omega^k(M) = \mathrm{H}^k_{dR}(M)$ to a map $\hat{I}^k_b \colon \mathrm{H}^k_{dR}(M) \to \mathrm{H}^k_b(M)$. We now have the following commutative diagram:

$$\begin{tikzcd} & \mathrm{H}^k_{dR}(M) \arrow[dl, "\hat{I}^k_b"'] \arrow[d, "\cong"] \\ \mathrm{H}^k_b(M) \arrow[r, "c^k"] & \mathrm{H}^k(M). \end{tikzcd}$$

Therefore, up to the identification $\mathrm{H}^k_{dR}(M) \cong \mathrm{H}^k(M)$, for $k > 2$ the map $\hat{I}^k_b$ provides a right inverse of the comparison map. On the one hand, this raises the interesting question of understanding the possible geometric properties of the elements in the image of $\hat{I}^k_b$; on the other hand, for $k > 2$ differential forms produce only a finite dimensional subsbace of $\mathrm{H}^k_b(M)$.

On the contrary, in degree 2, for every $\xi = d\varphi \in \mathrm{E}\Omega^2(M)$, the primitive $c_\varphi \in \mathrm{C}^1(M)$ of c_ξ is not necessarily bounded since the length of geodesic segments in M is arbitrarily big and thus $[c_\xi] \in \mathrm{EH}^2_b(M)$ may be non-trivial. In particular, when $M = \Sigma$, a closed hyperbolic surface, thanks to Theorem 2 $[c_\xi]$ is never trivial if $\xi \neq 0$, and the space of exact forms $\mathrm{E}\Omega^2(\Sigma)$ defines a infinite dimensional subspace of $\mathrm{EH}^2_b(\Sigma)$:

$$\begin{tikzcd}
\mathrm{E}\Omega^2(\Sigma) \arrow[r, hook] \arrow[d, hook, "I^2_b"] & \Omega^2(\Sigma) \arrow[r, two heads] \arrow[d, hook, "I^2_b"] & \mathrm{H}^2_{dR}(\Sigma) \arrow[d, "\cong"] \\
\mathrm{EH}^2_b(\Sigma) \arrow[r, hook] & \mathrm{H}^2_b(\Sigma) \arrow[r, two heads] & \mathrm{H}^2(\Sigma).
\end{tikzcd}$$

2.3. **Smooth cohomology.** In this section we show that every class $\alpha \in \mathrm{H}^k_b(M)$ admits a representative that smoothly depends on the vertices of simplices. Moreover, in Lemma 3 we show an additional property of this representative that we will use in the next section.

Let X be a topological space, we endow the set of singular k-simplices $\mathrm{S}_k(X)$ with the compact-open topology to define the subcomplex of the continuous cochains of X

$$\mathrm{C}^k_c(X) = \{\omega \in \mathrm{C}^k(X) \mid \omega_{|\mathrm{S}_k(X)} \text{ is continuous}\}.$$

Moreover, we set $\mathrm{C}^k_{c,b}(X) = \mathrm{C}^k_c(X) \cap \mathrm{C}^k_b(X)$ and denote the homology of these complexes by $\mathrm{H}^\bullet_c(X)$ and $\mathrm{H}^\bullet_{c,b}(X)$, respectively.

Theorem 1.4 of [Fri11] states that if X is path connected, paracompact and with contractible universal covering $\widetilde{X}$, then the inclusion of bounded continuous cochains in classical cochains

$$i^\bullet_b \colon \mathrm{C}^\bullet_{c,b}(X) \to \mathrm{C}^\bullet_b(X)$$

induces isometric isomorphisms on cohomology

$$i^\bullet_b \colon \mathrm{H}^\bullet_{c,b}(X) \to \mathrm{H}^\bullet_b(X).$$

Furthermore, there is an explicit formula for the inverse of these isomorphisms

$$\theta^\bullet_b = (i^\bullet_b)^{-1} \colon \mathrm{H}^\bullet_b(X) \to \mathrm{H}^\bullet_{c,b}(X).$$

In what follows we will give the explicit formula of $\theta^\bullet_b$ in the case $X = M$. It is shown in Lemma 6.1 of [Fri11] that the isometric isomorphism $\mathrm{C}^\bullet_b(M) \cong \mathrm{C}^\bullet_b(\widetilde{M})^\Gamma$ induced by $p \colon \widetilde{M} \to M$ can be restricted to

$$p^\bullet_{c,b} \colon \mathrm{C}^\bullet_{c,b}(M) \to \mathrm{C}^\bullet_{c,b}(\widetilde{M})^\Gamma.$$

With the identifications $\mathrm{C}^\bullet_b(M) \cong \mathrm{C}^\bullet_b(\widetilde{M})^\Gamma$ and $\mathrm{C}^\bullet_{c,b}(M) \cong \mathrm{C}^\bullet_{c,b}(\widetilde{M})^\Gamma$ in mind, we will write out the explicit formula for the map

$$\widetilde{\theta}^k_b \colon \mathrm{C}^k_b(\widetilde{M})^\Gamma \to \mathrm{C}^k_{b,c}(\widetilde{M})^\Gamma$$

which induces the map θ^k_b on cohomology. Since M is compact, we can slightly modify the construction in Lemma 5.1 of [Fri11], by using a smooth partition of unity subordinate to a *finite* open cover of M and get a smooth map $h_{\widetilde{M}} \colon \widetilde{M} \to [0,1]$ with the following properties:

(i) There is an $N \in \mathbb{N}$, such that for every $x \in \widetilde{M}$ there is a neighbourhood W_x of x such that the set $\{\gamma \in \Gamma \mid \gamma(W_x) \cap \mathrm{supp}(h_{\widetilde{M}})\}$ has at most N elements.

(ii) For every $x \in \widetilde{M}$, we have $\sum_{\gamma \in \Gamma} h_{\widetilde{M}}(\gamma x) = 1$.
(iii) $\operatorname{supp} h_{\widetilde{M}}$ is compact.

Let $\omega \in \mathrm{C}_b^k(\widetilde{M})^\Gamma$ and pick a basepoint $z \in \widetilde{M}$. We define the function $f_\omega \colon \widetilde{M}^{k+1} \to \mathbb{R}$ as follows:

$$f_\omega(x_0, \dots, x_k) = \sum_{(\gamma_0, \dots, \gamma_k) \in (\Gamma)^{k+1}} h_{\widetilde{M}}(\gamma_0^{-1} x_0) \cdot \dots \cdot h_{\widetilde{M}}(\gamma_k^{-1} x_k) \cdot \omega([\gamma_0 z, \dots, \gamma_k z]).$$

Notice that the sum above is finite because of property (i). Finally we can define

$$\widetilde{\theta}_b^k(\omega)(s) = f_\omega(s(e_0), \dots, s(e_k)).$$

Observe that $\widetilde{\theta}_b^k(\omega)$ is a Γ-invariant cocycle because f_ω is a Γ-invariant function, where Γ acts on $\widetilde{M}^{k+1}$ diagonally.

In order to prove the Main Theorem we will need the following:

Lemma 3. Let $\omega \in \mathrm{C}_b^k(\widetilde{M})^\Gamma$ and let $(x_1, \dots, x_k) \in (\widetilde{M})^k$. Then the function

$$f_\omega(-, x_1, \dots, x_k) \colon \widetilde{M} \to \mathbb{R}$$

is smooth and the norm of its differential $df_\omega(-, x_1, \dots, x_k) \in \Omega^1(\widetilde{M})$ is bounded by a constant that does not depend on $(x_1, \dots, x_k)$.

Proof. It is clear by construction that $f_\omega(-, x_1, \dots, x_k)$ is smooth. Moreover, expanding its differential

$$df_\omega(-, x_1, \dots, x_k) = \sum_{(\gamma_0, \dots, \gamma_k) \in (\Gamma)^{k+1}} dh_{\widetilde{M}}(\gamma_0^{-1} -) \cdot \dots \cdot h_{\widetilde{M}}(\gamma_k^{-1} x_k) \cdot \omega([\gamma_0 z, \dots, \gamma_k z])$$

we see that, by property (i) of $h_{\widetilde{M}}$, there are at most N^{k+1} non-zero summands and thus

$$\|df_\omega(-, x_1, \dots, x_k)\| \leqslant N^{k+1} \cdot \|dh_{\widetilde{M}}\|_\infty \cdot \|\omega\|_\infty.$$

We can conclude since $\|\omega\|_\infty < \infty$ by assumption and $\|dh_{\widetilde{M}}\|_\infty < \infty$ because $h_{\widetilde{M}}$ has compact support. □

3. Proof of the Main Theorem

Let $\varphi \in \Omega^1(\widetilde{M})^\Gamma$ and $[\omega] \in \mathrm{H}_b^k(M)$, we look for a bounded primitive of $c_{d\varphi} \cup \omega \in \mathrm{C}_b^{k+2}(\widetilde{M})^\Gamma$. Observe that $c_\varphi \cup \omega \in \mathrm{C}_b^{k+1}(\widetilde{M})^\Gamma$ is a (not necessarily bounded) primitive, in fact

$$\delta^{k+1}(c_\varphi \cup \omega) = \delta^1(c_\varphi) \cup \omega = c_{d\varphi} \cup \omega.$$

Of course, it is sufficient to find an $\eta \in \mathrm{C}^k(\widetilde{M})^\Gamma$ such that $c_\varphi \cup \omega + \delta^k \eta \in \mathrm{C}_b^{k+1}(\widetilde{M})^\Gamma$ is bounded.

We first replace ω with $\widetilde{\theta}_b^k(\omega)$, this can be done without loss of generality because as shown in the previous section the map $\widetilde{\theta}_b^\bullet$ induces an isomorphism on bounded cohomology. Under this assumption we have that $\omega(s) = f_\omega(s(e_0), \dots, s(e_k))$ for every $s \in \mathrm{S}_k(\widetilde{M})$. Thus $(c_\varphi \cup \omega)(s)$ only depends on the vertices of $s \in \mathrm{S}_{k+1}(\widetilde{M})$, in fact

$$\begin{aligned}(c_\varphi \cup \omega)(s) &= c_\varphi([s(e_0), s(e_1)]) \cdot \omega([s(e_1), \dots, s(e_{k+1})]) \\ &= \int_{[s(e_0), s(e_1)]} \varphi \cdot f_\omega(s(e_1), \dots, s(e_{k+1})).\end{aligned}$$

Next, we define the function $\zeta \colon \widetilde{M}^{k+2} \to \mathbb{R}$ as follows

$$\zeta(x_0, \dots, x_{k+1}) = \int_{[x_0, x_1]} \varphi \cdot f_\omega(-, x_2, \dots, x_{k+1})$$

where we see $f_\omega(-, x_2, \dots, x_{k+1})$ as a 0-form (i.e. a smooth function). We observe that ζ is a Γ-invariant function (again using the diagonal action of Γ on $\widetilde{M}^{k+2}$), in fact φ and f_ω are Γ-invariant and for any $\gamma \in \Gamma$ we have that $[\gamma x_0, \gamma x_1] = \gamma[x_0, x_1]$.

Lemma 4. For every $(x_0, \dots, x_{k+1}) \in \widetilde{M}^{k+2}$ we have that

$$(c_\varphi \cup \omega)([x_1, \dots, x_{k+1}]) = \zeta(x_0, \dots, x_{k+1}) - \sum_{i=2}^{k+1}(-1)^i \zeta(x_0, x_1, x_1, x_2, \dots, \hat{x}_i, \dots, x_{k+1}).$$

Proof. Since ω is a cocycle we have that for any $z \in \widetilde{M}$,

$$\begin{aligned}
0 &= \delta^k \omega([z, x_1, \dots, x_{k+1}]) \\
&= \omega([x_1, \dots, x_{k+1}]) + \sum_{i=1}^{k+1}(-1)^i \omega([z, x_1, \dots, \hat{x}_i, \dots, x_{k+1}]) \\
&= f_\omega(x_1, \dots, x_{k+1}) + \sum_{i=1}^{k+1}(-1)^i f_\omega(z, x_1, \dots, \hat{x}_i, \dots, x_{k+1})
\end{aligned}$$

and thus

$$f_\omega(x_1, \dots, x_{k+1}) = -\sum_{i=1}^{k+1}(-1)^i f_\omega(-, x_1, \dots, \hat{x}_i, \dots, x_{k+1}).$$

We use this relation to conclude that

$$\begin{aligned}
(c_\varphi \cup \omega)([x_1, \dots, x_{k+1}]) =& c_\varphi([x_0, x_1]) \cdot \omega([x_1, \dots, x_{k+1}]) \\
=& \int_{[x_0, x_1]} \varphi \cdot f_\omega(x_1, \dots, x_{k+1}) \\
=& \int_{[x_0, x_1]} \varphi \cdot \left(-\sum_{i=1}^{k+1}(-1)^i f_\omega(-, x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \right) \\
=& -\sum_{i=1}^{k+1}(-1)^i \int_{[x_0, x_1]} \varphi \cdot f_\omega(-, x_1, \dots, \hat{x}_i, \dots, x_{k+1}) \\
=& \zeta(x_0, \dots, x_{k+1}) - \sum_{i=2}^{k+1}(-1)^i \zeta(x_0, x_1, x_1, x_2, \dots, \hat{x}_i, \dots, x_{k+1}).
\end{aligned}$$

□

We define $\eta \in \mathrm{C}^k(\widetilde{M})^\Gamma$ so that for every $s \in \mathrm{S}_k(\widetilde{M})$,

$$\eta(s) = \zeta(s(e_0), s(e_1), s(e_1), s(e_2), \dots, s(e_k)).$$

This cochain is Γ-invariant because the function ζ is. As anticipated we will conclude by showing that $c_\varphi \cup \omega + \delta^k \eta$ is a bounded cochain. Since both $c_\varphi \cup \omega$ and $\delta^k \eta$ only depend on the vertices of simplices it will be enough to show that the function $(x_0, \dots, x_{k+1}) \in$

$\widetilde{M}^{k+2} \mapsto (c_\varphi \cup \omega + \delta^k\eta)([x_0, \dots, x_{k+1}]) \in \mathbb{R}$ is bounded:

$$\begin{aligned}(c_\varphi \cup \omega + \delta^k\eta)([x_0, \dots, x_{k+1}]) &= \zeta(x_0, \dots, x_{k+1}) \\ &- \sum_{i=2}^{k+1} (-1)^i \zeta(x_0, x_1, x_1, x_2, \dots, \hat{x_i}, \dots, x_{k+1}) \\ &+ \sum_{i=0}^{k+1} (-1)^i \eta([x_0, \dots, \hat{x_i}, \dots, x_{k+1}]) \\ &= \zeta(x_0, x_1, x_2, \dots, x_{k+1}) \\ &+ \zeta(x_1, x_2, x_2, \dots, x_{k+1}) \\ &- \zeta(x_0, x_2, x_2, \dots, x_{k+1}).\end{aligned}$$

Next we use Stoke's Theorem:

$$\begin{aligned}&\zeta(x_0, x_1, x_2, \dots, x_{k+1}) \\ &+\zeta(x_1, x_2, x_2, \dots, x_{k+1}) \\ &-\zeta(x_0, x_2, x_2, \dots, x_{k+1}) \\ &= \int_{[x_0,x_1]} \varphi \cdot f_\omega(-, x_2, \dots, x_{k+1}) \\ &+ \int_{[x_1,x_2]} \varphi \cdot f_\omega(-, x_2, \dots, x_{k+1}) \\ &- \int_{[x_0,x_2]} \varphi \cdot f_\omega(-, x_2, \dots, x_{k+1}) \\ &= \int_{[x_0,x_1]\cup[x_1,x_2]\cup[x_2,x_0]} \varphi \cdot f_\omega(-, x_2, \dots, x_{k+1}) \\ &= \int_{\partial[x_0,x_1,x_2]} \varphi \cdot f_\omega(-, x_2, \dots, x_{k+1}) \\ &= \int_{[x_0,x_1,x_2]} d(\varphi \cdot f_\omega(-, x_2, \dots, x_{k+1})).\end{aligned}$$

The integration domain is a 2-simplex with bounded area, this means that we only need to check that the norm of $d(\varphi \cdot f_\omega(-, x_2, \dots, x_{k+1})) \in \Omega^2(\widetilde{M})^\Gamma$ is bounded by a constant that does not depend on $(x_2, \dots, x_{k+1})$. We expand

$$d(\varphi \cdot f_\omega(-, x_2, \dots, x_{k+1})) = d\varphi \cdot f_\omega(-, x_2, \dots, x_{k+1}) + \varphi \wedge df_\omega(-, x_2, \dots, x_{k+1}).$$

Both φ and $d\varphi$ are Γ-invariant and since Γ is cocompact $\|\varphi\|_\infty < \infty$ and $\|d\varphi\|_\infty < \infty$. The function $f_\omega(-, x_2, \dots, x_{k+1})$ is bounded by $\|\omega\|_\infty$. Finally, as we saw in Lemma 3, $\|df_\omega(-, x_2, \dots, x_{k+1})\|_\infty$ is also bounded by a constant that does not depend on $(x_2, \dots, x_{k+1})$. This concludes the proof of our main Theorem.

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