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Shape Sensitivity Analysis of a 2D Fluid–Structure Interaction Problem

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Abstract

We study the shape differentiability of a general functional depending on the solution of a bidimensional stationary Stokes–elasticity system with small loads, with respect to the reference domain of the elastic structure immersed in a viscous fluid. The differentiability with respect to reference elastic domain variations is considered under shape perturbations with diffeomorphisms. The shape derivative is then calculated with the use of the velocity method. This derivative involves the material derivatives of the solution of this fluid–structure interaction problem. The adjoint method is then used to obtain a simplified expression for the shape derivative.

Keywords Fluid–structure system · Stokes and elasticity equations · Shape optimisation · Shape sensitivity

1 Introduction

Fluid–structure interaction (FSI) problems model physical systems in which a solid body (rigid or deformable) interacts with a fluid (internal or external to the body). In

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this work, we consider an elastic body in plain strain, clamped to a rigid support in its interior and immersed in a viscous incompressible fluid. The system is infinite in the anti-plane dimension. From the mathematical point of view, we consider a system of bidimensional stationary PDEs which involves on the one hand the Stokes equations for the fluid flow and, on the other hand, incompressible linearised elasticity equations for the deformation of the structure. These two sub-systems are coupled through a boundary condition on the interface between the solid and the fluid, by imposing the force continuity across the interface.

In this paper, we are interested in a shape optimisation issue for this fluid–structure interaction problem. We aim to study the shape sensibility with respect to the reference domain Ω_0 of the elastic body, also called the reference configuration (i.e. the domain at rest, before deformation) of a given shape functional. This functional depends on the elastic reference domain Ω_0 as well as on the corresponding solution of the full PDE system. We point out that in this context, we do not directly control the shape of the deformed elastic body which actually interacts with the fluid.

The goal of this paper is to show the differentiability of a broad family of shape functionals (e.g. energy functional, drag functional) in which the shape is the reference configuration Ω_0 of the elastic body, and also to calculate the associated shape derivatives. The differentiability is tackled with respect to the reference configuration Ω_0 by considering a class of perturbations of Ω_0, obtained by diffeomorphism. We also provide formulas for the associated shape derivatives.

These derivatives would be useful in a numerical shape optimisation procedure (as, for example, steepest descent methods) to determine an optimal elastic reference domain that minimises a given shape functional (see, e.g. [2, 25, 34, 45]).

Dealing with an FSI problem, the first mathematical issue is proving *existence* of solutions. Early important contributions can be found in [4, 16, 17] in which the authors study stationary flows in nonlinear elastic shells and also nonlinear elastic tubes and shells under external flow for which the velocity is prescribed. In the early 2000, mathematicians started to investigate more intensively the interaction of a viscous liquid with elastic bodies in steady and unsteady regimes. For steady-state problems, one can cite [6, 24, 26, 38, 41] and for the unsteady case, we refer for example to [8, 9, 14, 19, 27, 35]. One of the difficulties in the study of this kind of FSI problems is that the fluid, described in Eulerian coordinates, turns out to be defined on a domain depending on the structure displacement, which is instead described in Lagrangian coordinates. For the FSI problem under consideration in this paper, we will first establish the existence and uniqueness of the solution.

The second issue in FSI problems is to find *optimal structures* which optimise a suitable desired efficiency in fluid dynamics, possibly under constraint. Great interest has been shown in the minimisation of the drag in fluid mechanics optimisation (see, e.g. [5, 25, 34]), in the shape minimisation of the dissipated energy in a pipe (see, e.g. [7, 30]) or in the optimisation of fluid flow with or without body forces (see, e.g. [18]). In all these mentioned works, the shape or the geometry in which PDEs lie, are fixed and known. Shape optimisation applied to FSI problems, where the geometry is one of the unknowns, is more recent. One can cite [3, 31, 32, 44, 45] where level-set methods are used to characterise the fluid and the structure domains, and also [33, 36, 37] in which the FSI problem is relaxed by a density design variable. The work presented in this paper is an extension of what is done in [39], where the shape differentiability of a simplified free-boundary one-dimensional problem is studied, and for which it is proved that the shape optimisation problem is well-posed. In the recent papers [21, 22], the shape and topological optimisation of a multiphysics thermal–fluid–structure interaction problem is studied with a velocity and adjoint method, for which the structure domain is assumed to be fixed. In [43], differentiability results are shown for the solutions of a stationary fluid–structure interaction problem in an ALE framework. The differentiability is considered with respect to variations of the given data (volume forces and boundary values) but not with respect to the reference domain of the elastic structure, as it is done in this present paper. Finally, we mention the work of Haubner et al. [28] where the method of mappings is used for proving differentiability results with respect to domain variations, for unsteady fluid–structure interaction problems that couple the Navier–Stokes equations and the Lamé system.

The paper is organised as follows: we start, in Sect. 2, with a presentation of the FSI problem under study. In Sect. 3, we prove an existence and uniqueness result for the FSI problem for small data, first by analysing separately the fluid equations and the structure problem, and finally by coupling the two sub-systems through a fixed-point procedure. Then, in Sect. 4, after an introduction to the calculus of shape derivatives by the *velocity method*, we apply this approach to our FSI problem: the sensitivity analysis allows us to show that the solutions of the FSI system are shape-differentiable. Section 5 is devoted to the calculation of the shape derivative of an abstract shape functional. Using the *adjoint method*, we also give a simplified expression of the shape derivative, not depending on the *material derivatives* of the solutions of the FSI problem but involving the solutions of adjoint problems. Our results, together with possible future research lines, are discussed in Sect. 6.

2 A Two-Dimensional FSI Model with a Shape Optimisation Problem

In this section, we first present the FSI model under study and then the related shape optimisation problem that will be addressed in this paper. The FSI model couples the Stokes equations with the elasticity equation and follows essentially [26] and [39]. The difference with respect to the literature is the assumption of linear incompressible elasticity for the structure, which results in a divergence-free condition for the structure's displacement.

2.1 Notations

In this preliminary paragraph, we fix the notations that will be used throughout the paper. Let $\{\mathrm{e}_1, \mathrm{e}_2\}$ be the canonical orthogonal basis of $\mathbb{R}^2$. Let u and v be two vectors of $\mathbb{R}^2$, A and B be two second-order tensors of $\mathbb{R}^2$. Using the Einstein summation convention, we set:

$$AB = A_{ik}B_{kj}\,\mathrm{e}_i \otimes \mathrm{e}_j, \qquad Au = A_{ij}u_j\,\mathrm{e}_i,$$
$$A : B = A_{ij}B_{ij}, \qquad u \cdot v = u_i v_i,$$

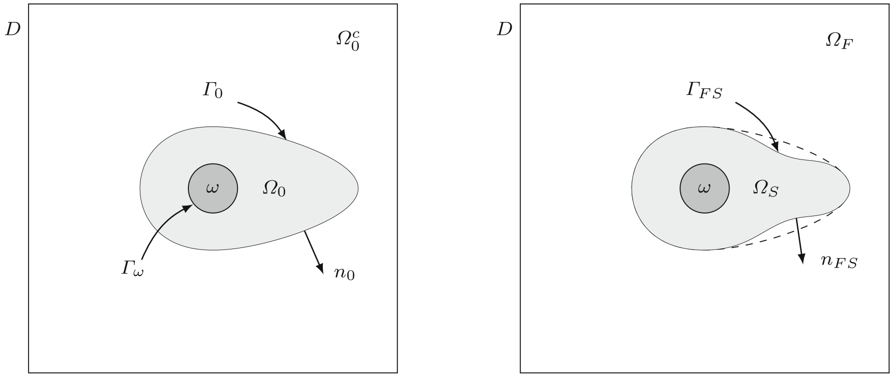

Fig. 1 Geometry of the FSI system, before (left) and after (right) deformation induced by the interaction between the fluid and the structure

where $\{\mathrm{e}_i \otimes \mathrm{e}_j\}_{1\le i,j\le 2}$ forms the canonical basis of the second-order tensors on $\mathbb{R}^2$. Denoting by I the identity matrix, we, respectively, define the trace $\mathrm{tr}(A)$ of a matrix A, its symmetric part A^s, and its norm $|A|$ by:

$$\mathrm{tr}(A) = \mathrm{I} : A, \qquad A^s := \frac{1}{2}\left(A + A^\top\right), \qquad |A| = (A : A)^{1/2}. \tag{1}$$

Moreover, if A is an invertible matrix, we define the cofactor matrix of A by:

$$\mathrm{cof}(A) = \det(A)A^{-\top}.$$

Let Ω be an open subset of $\mathbb{R}^2$. The functions involved in the equations we study in this paper belong to Sobolev spaces $W^{m,p}(\Omega)$, for $m \ge 0$ a positive integer, and $1 \le p \le +\infty$. With this convention, $W^{0,p}(\Omega)$ stands for the Lebesgue space $L^p(\Omega)$. The norm in $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{m,p,\Omega}$, or, when no ambiguity may arise, simply by $\|\cdot\|_{m,p}$. Finally, the space $W^{m,2}(\Omega)$ will simply be denoted by $H^m(\Omega)$.

2.2 The Fluid–Structure Interaction Model

We consider a two-dimensional elastic body (the structure) immersed in an incompressible viscous fluid and clamped from a part of its boundary, while applying volume forces to both fluid and elastic parts. This results in the deformation of the free boundary of the elastic body, which is the interface where the interaction between the elastic body and the fluid takes place (see Fig. 1).

In order to describe this setting, we fix three simply connected bounded open sets $\omega, D_0, D \subset \mathbb{R}^2$, such that $\omega \subset\subset D_0 \subset\subset D$. We denote by Γ_0 and Γ_ω the boundaries of D_0 and ω, respectively. The annular domain:

$$\Omega_0 := D_0 \setminus \overline{\omega}, \tag{2}$$

represents the region occupied by the elastic body, that we assume to be clamped on the boundary part Γ_ω. The complementary set in the box D, namely the annular domain:

$$\Omega_0^c := D \setminus \overline{D_0}, \tag{3}$$

is the region occupied by the fluid, that we take incompressible. The elastic body and the fluid interact through the interface Γ_0, which is deformable.

The fluid and the structure are subject to volume forces which result in a deformation of the elastic part. In our analysis, we assume that the system is at equilibrium, in particular, the time variable will not appear in the model.

The deformed elastic body, denoted by Ω_S, is described in Lagrangian coordinates, that is, through a function defined in the reference configuration:

$$\Omega_S := T(\mathrm{w})(\Omega_0),$$

with:

$$T(\mathrm{w}) : \Omega_0 \to D \setminus \overline{\omega}, \quad T(\mathrm{w}) = \mathrm{id}_{\mathbb{R}^2} + \mathrm{w}, \tag{4}$$

where $\mathrm{id}_{\mathbb{R}^2}$ is the identity in $\mathbb{R}^2$ and w is the *elastic displacement field* in Ω_0. Accordingly, the deformed fluid–structure interface is:

$$\Gamma_{FS} := T(\mathrm{w})(\Gamma_0) = (\mathrm{id}_{\mathbb{R}^2} + \mathrm{w})(\Gamma_0). \tag{5}$$

On the other hand, the fluid is described in Eulerian coordinates, namely through functions defined in the region surrounding the deformed elastic body:

$$\Omega_F := D \setminus \overline{\Omega_S \cup \omega}.$$

The functions describing the fluid are the *velocity field* $\mathrm{u} : \Omega_F \to \mathbb{R}^2$ and the *pressure field* $\mathrm{p} : \Omega_F \to \mathbb{R}$.

In the following paragraphs, we will specify the PDEs governing the two phases of the system, and their interaction.

2.2.1 Fluid Equations

In the framework of incompressibility, the velocity field u and the pressure field p are governed by Stokes equations:

$$\begin{aligned} -\mathrm{div}\varsigma(\mathrm{u}, \mathrm{p}) &= f && \text{in } \Omega_F, \\ \mathrm{d}ivu &= 0 && \text{in } \Omega_F, \\ \mathrm{u} &= 0 && \text{on } \partial\Omega_F. \end{aligned}$$

In the system, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the applied force, defined in the whole space, whereas ς is the *Cauchy stress tensor*, defined by:

$$\varsigma(\mathrm{u}, \mathrm{p}) := 2\nu\nabla^s \mathrm{u} - \mathrm{pI},$$

with $\nu > 0$ the viscosity of the fluid. We recall that the superscript s stands for the symmetrization operator (see (1)).

2.2.2 Structure Equations

We suppose that the elastic body is attached to the rigid support ω via its boundary Γ_ω. This assumption results in a Dirichlet boundary condition for the elastic displacement w:

$$\mathrm{w} = 0 \quad \text{on } \Gamma_\omega.$$

A given volume force g is applied to the structure in Ω_0 and the elastic displacement w satisfies the elasticity equation:

$$-\operatorname{div}\sigma(\mathrm{w}) = g \ \text{ in } \Omega_0, \tag{6}$$

where σ is the *linearised stress tensor* (also called the second Piola–Kirchoff stress tensor) or simply *stress tensor*:

$$\sigma(\mathrm{w}) := 2\mu\nabla^s \mathrm{w} + \lambda(\operatorname{div} \mathrm{w})\mathrm{I}.$$

Here, λ and μ are the so-called Lamé coefficients (see, e.g. [15]). Furthermore, we impose the equilibrium of the surface forces on the free boundary Γ_0 which reads as:

$$\int_{\Gamma_0} \sigma(\mathrm{w})n_0 \cdot (v \circ (\mathrm{id}_{\mathbb{R}^2} + \mathrm{w}))\mathrm{d}\Gamma_0 = \int_{\Gamma_{FS}} \varsigma(\mathrm{u}, \mathrm{p})n_{FS} \cdot v \, \mathrm{d}\Gamma_{FS}, \tag{7}$$

for all functions v defined on Ω_F. In the above relation, Γ_{FS} is defined in (5) and denotes the boundary between the fluid domain Ω_F and the deformed elastic body Ω_S, whereas $\mathrm{d}\Gamma_0$ and $\mathrm{d}\Gamma_{FS}$ are the length elements of the boundaries Γ_0 and Γ_{FS}, respectively, and finally n_0 and n_{FS} are the outer unit normal vectors to Γ_0 and Γ_{FS}, respectively. Recalling that Γ_{FS} is the image of Γ_0 via $T(\mathrm{w}) = \mathrm{id}_{\mathbb{R}^2} + \mathrm{w}$, cf. (4)–(5), we infer (see, e.g. [15]) that:

$$n_{FS}\mathrm{d}\Gamma_{FS} = \left[\det\left(\nabla(T(\mathrm{w}))\right)\nabla(T(\mathrm{w}))^{-\mathsf{T}}n_0\right]\mathrm{d}\Gamma_0. \tag{8}$$

Thus, using $T(\mathrm{w})$ for a change of variables in (7) together with (8), we get the following boundary condition:

$$\sigma(\mathrm{w})n_0 = \left(\varsigma(\mathrm{u}, \mathrm{p}) \circ T\right)\operatorname{cof}(\nabla T)n_0 \quad \text{on } \Gamma_0, \tag{9}$$

where:

$$\operatorname{cof}(\nabla T) = \det(\nabla T)(\nabla T)^{-T},$$

is the cofactor matrix of the Jacobian matrix of $T := T(\mathrm{w})$.

In this paper, we consider the special case of linear incompressible elasticity for the structure, by imposing the following equation for the displacement:

$$\operatorname{div} \mathrm{w} = 0. \tag{10}$$

We introduce a Lagrange multiplier function s associated with the incompressibility constraint (10). Then, the structure equation (6) together with the continuity condition of forces (9), for (w, s), becomes:

$$\begin{aligned} -\operatorname{div}\sigma(\mathrm{w}) + \nabla \mathrm{s} &= g && \text{in } \Omega_0, \\ (\sigma(\mathrm{w}) - \mathrm{sI})\, n_0 &= \big(\varsigma(\mathrm{u}, \mathrm{p}) \circ T\big)\operatorname{cof}(\nabla T) n_0 && \text{on } \Gamma_0. \end{aligned}$$

2.2.3 Full FSI Coupled System

Using the fact that both the velocity u and the displacement w are divergence free, the FSI system for (u, p) and (w,s) that we consider in this paper is the following:

$$\begin{aligned} -\nu\operatorname{div}(\nabla \mathrm{u}) + \nabla \mathrm{p} &= f && \text{in } \Omega_F, \\ \operatorname{div}\mathrm{u} &= 0 && \text{in } \Omega_F, \\ \mathrm{u} &= 0 && \text{on } \partial\Omega_F, \\ -\mu\operatorname{div}(\nabla \mathrm{w}) + \nabla \mathrm{s} &= g && \text{in } \Omega_0, \\ \operatorname{div} \mathrm{w} &= 0 && \text{in } \Omega_0, \\ \mathrm{w} &= 0 && \text{on } \Gamma_\omega, \\ (\mu\nabla \mathrm{w} - \mathrm{sI})\, n_0 &= ((\nu\nabla \mathrm{u} - \mathrm{p}\,\mathrm{I}) \circ T)\operatorname{cof}(\nabla T) n_0 && \text{on } \Gamma_0. \end{aligned} \tag{11}$$

Remark 1 In view of the incompressibility of the fluid, we consider the case where $|\Omega_F|$ is conserved, $|\Omega_F|$ denoting the Lebesgue measure of Ω_F. For this, the following constraint can be considered:

$$|\Omega_S| = \int_{\Omega_0} \det(\mathrm{I} + \nabla \mathrm{w})\, dx = |\Omega_0|. \tag{12}$$

In our case, we have that:

$$\det(\mathrm{I} + \nabla \mathrm{w}) = 1 + \operatorname{div}(\mathrm{w}) + \det(\nabla \mathrm{w}) = 1 + \operatorname{div}(\mathrm{w}) + O\left(\|\nabla \mathrm{w}\|_\infty^2\right). \tag{13}$$

So, under the condition that div w $= 0$ and neglecting the second-order terms in (13), we obtain that the area constraint (12) is satisfied. This implies that the volume of the fluid is conserved, since, by definition, $|\Omega_F| := |D| - |\Omega_S| - |\omega|$.

We can observe that the coupling of the FSI problem (11) is twofold:

- the structure displacement w affects and defines the domain Ω_F on which the fluid equations are posed and where the velocity u and the pressure p are defined,
- the velocity u and the pressure p of the fluid give rise to a surface force which influences the calculation of the displacement w.

One of the main difficulties lies in the fact that there are two kinds of variables under consideration. On the one hand, the FSI problem involves Eulerian variables with the fluid velocity u and pressure p, and on the other hand, the elastic displacement w and the multiplier s are Lagrangian variables.

Moreover, the domain Ω_F on which the fluid equations are written is unknown. To overcome these difficulties, we need to transport the fluid equations into a reference domain matching with the elastic reference domain Ω_0. This domain transformation technique is also known as the ALE method (Arbitrary Eulerian Lagrangian). It is commonly used for computing the numerical solution (with a finite element method) of coupled Eulerian–Lagrangian systems or for free-boundary problems involving a time derivative (see [20]).

2.3 Fixed Domain Formulation of the FSI Problem

In order to tackle the FSI problem (11), we transpose the fluid equations posed on the fluid domain Ω_F onto the fixed domain Ω_0^c defined by:

$$\Omega_0^c := D \setminus \overline{\Omega_0 \cup \omega}. \tag{14}$$

Thus, we need a C^1-diffeomorphism which maps Ω_0^c to Ω_F. To this aim, we consider an extension of the map T, initially defined on Ω_0 in (4), to the whole box D. With a slight abuse of notation, we use the same letter T and we set:

$$T(\mathrm{w}) = \mathrm{id}_{\mathbb{R}^2} + P(\mathrm{w}), \tag{15}$$

where w is a displacement field defined in the initial elastic body domain Ω_0, and P is an extension operator from Ω_0 to D, such that $P(\mathrm{w})$ is defined in D and $T(\mathrm{w})$ is one to one in D. This allows us to consider the fluid domain Ω_F defined as:

$$\Omega_F = T(\mathrm{w})(\Omega_0^c),$$

where Ω_0^c is defined in (14) (see also Fig. 1). We will go through this extension procedure in details later on, to give a rigorous definition of T.

In the same way as in [26], we can define the transported velocity and pressure fields:

$$\mathrm{v} := \mathrm{u} \circ T(\mathrm{w}), \quad \text{and} \quad \mathrm{q} := \mathrm{p} \circ T(\mathrm{w}).$$

With these new variables, we can write the fluid equations transported onto the reference domain Ω_0^c (e.g. by using the variational formulation as in [12, Sect. 3.2.2]), and the complete FSI problem reads as:

$$
\begin{aligned}
-\nu \operatorname{div}((\nabla \mathrm{v}) F(\mathrm{w})) + G(\mathrm{w}) \nabla \mathrm{q} &= (f \circ T(\mathrm{w})) J(\mathrm{w}) && \text{in } \Omega_0^c, \\
\operatorname{div}(G(\mathrm{w})^\top \mathrm{v}) &= 0 && \text{in } \Omega_0^c, \\
\mathrm{v} &= 0 && \text{on } \partial \Omega_0^c, \\
-\mu \operatorname{div}(\nabla \mathrm{w}) + \nabla \mathrm{s} &= g && \text{in } \Omega_0, \\
\operatorname{div} \mathrm{w} &= 0 && \text{in } \Omega_0, \\
\mathrm{w} &= 0 && \text{on } \Gamma_\omega, \\
(\mu \nabla \mathrm{w} - \mathrm{s}\, \mathrm{I}) n_0 &= \nu (\nabla \mathrm{v}) F(\mathrm{w}) n_0 \\
&\quad - \mathrm{q} G(\mathrm{w}) n_0 && \text{on } \Gamma_0,
\end{aligned}
\tag{16}
$$

where we have set:

$$
J(\mathrm{w}) := \det(\nabla T(\mathrm{w})), \quad G(\mathrm{w}) := \operatorname{cof}(\nabla T(\mathrm{w})), \quad F(\mathrm{w}) := (\nabla T(\mathrm{w}))^{-1} \operatorname{cof}\,(\nabla T(\mathrm{w}))\,.
$$

The boundary condition on Γ_0 appearing in (16) comes from the computation of the surface force applied on the structure, given in (9) by $(\varsigma(\,\mathrm{u}, \mathrm{p}) \circ T) \operatorname{cof}(\nabla T) n_0$, in terms of the new variables v and q:

$$
(\varsigma(\mathrm{u}, \mathrm{p}) \circ T) \operatorname{cof}(\nabla T) n_0 = (\nu (\nabla \mathrm{v}) F(\mathrm{w}) - \mathrm{q} G(\mathrm{w})) n_0.
$$

We point out that the FSI problem (16) is a sort of hybrid model compared to [26], coming from the linearisation of the equilibrium equation of the structure (that is to say the Piola–Kirchhoff stress tensor) and the area constraint (12). This has been done in order to simplify the shape optimisation analysis performed in this paper. Moreover, we do not have linearised the terms arising from the fluid equations change of variables, i.e. $J(\mathrm{w})$, $G(\mathrm{w})$, and $F(\mathrm{w})$, because we want to compute shape derivatives by keeping as much information as possible, for possible further applications and calculation purposes for a general system.

2.4 Optimisation of the FSI System

The shape sensitivity analysis of the FSI model (16) carried out in this article is motivated by a shape optimisation problem. This problem consists in seeking for an optimal shape of the elastic reference domain Ω_0 that minimises a functional depending on the solution of the FSI system associated to Ω_0. The shape optimisation problem we consider is of the following form:

$$
\min_{\Omega_0 \in \mathcal{U}_{\mathrm{ad}}} \mathcal{J}(\Omega_0), \tag{17}
$$

where $\mathcal{J}(\Omega_0)$ is a quite general shape functional depending on the initial elastic domain, defined by:

$$\mathcal{J}(\Omega_0) = \int_{\Omega_0} j_S(Y, \mathrm{w}(Y), \nabla \mathrm{w}(Y))\, \mathrm{d}Y + \int_{\Omega_F} j_F(x, \mathrm{u}(x), \nabla \mathrm{u}(x))\, \mathrm{d}x,$$

where j_F and j_S are smooth functions depending, respectively, on $\mathrm{u} = \mathrm{v} \circ T(\mathrm{w})^{-1}$ and w. The fields v and w are the velocity and the displacement solutions of the FSI problem (16) posed on $\Omega_0 \cup \Omega_0^c$. The domain $\Omega_0 \in \mathcal{U}_{\mathrm{ad}}$ belongs to a class $\mathcal{U}_{\mathrm{ad}}$ of smooth domains admissible for the FSI problem. For example, we can consider:

$$\mathcal{U}_{\mathrm{ad}} := \{A \subset \mathbb{R}^2,\ A = B \setminus \overline{\omega} \text{ with } B \text{ smooth, simply connected, } \omega \subset B \subset D \text{ and } |A| = |\Omega_0|\}.$$

In this paper, we do not go as far as to solve the complete optimisation problem (17). We will restrict our study to the shape sensitivity analysis of the FSI model (16).

3 Existence and Uniqueness Result for the FSI Problem

In this section, we establish an existence and uniqueness result written in Theorem 1 for the FSI problem (16). In [26], an existence result is obtained for the Navier–Stokes equations coupled with a St. Venant–Kirchhoff material in 3D with a volume constraint, for small enough volume forces. Existence and uniqueness for small data are achieved in [43] for a 3D Stokes and linear elasticity system, without volume constraint, and with small data not affected by the ALE change of variable. For our purpose, the existence and the uniqueness of the solution are required to address the associated optimisation problem and its shape sensitivity analysis. Moreover, since the body force $f \circ T(\mathrm{w})$ applied to the fluid is affected by the change of variable, we need higher regularity of the data. Indeed, from [29, Lemma 5.3.9], we need $f \in H^2$ in order to have that the map $W^{1,\infty} \ni \theta \mapsto f \circ (\mathrm{id}_{\mathbb{R}^2} + \theta) \in H^1$, is of class C^1 in the vicinity of 0 (see (48)). The existence and uniqueness result for our semi-linearised model is obtained by adapting what is done in [26].

Theorem 1 *Let D, Ω_0, Ω_0^c and ω be domains of the form (2)–(3) with boundary components ∂D and Γ_ω of class C^3 and Γ_0 of class $C^{3,1}$. Let $f \in \left(H^2(\mathbb{R}^2)\right)^2$ and $g \in \left(H^1(\Omega_0)\right)^2$. There exists a positive constant C such that if $\|f\|_{2,2} \leq C$ and $\|g\|_{1,2} \leq C$, then there exists a unique solution:*

$$(\mathrm{v}, \mathrm{q}, \mathrm{w}, \mathrm{s}) \in (H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2 \times (L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c)) \times (H_{0,\Gamma_\omega}^1(\Omega_0) \cap H^3(\Omega_0))^2 \times H^2(\Omega_0)$$

to the FSI problem (16). Furthermore, there exists a positive constant C_{FS} such that:

$$\|\mathrm{v}\|_{3,2,\Omega_0^c} + \|\mathrm{q}\|_{2,2,\Omega_0^c} + \|\mathrm{w}\|_{3,2,\Omega_0} + \|\mathrm{s}\|_{2,2,\Omega_0} \leq C_{FS}(\|f\|_{2,2,\mathbb{R}^2} + \|g\|_{1,2,\Omega_0}). \quad (18)$$

Before going through the proof of Theorem 1, let us introduce some preliminary elements that allow to well define the bijective map T introduced in (15).

3.1 Preliminaries

Let $\mathbf{b}$ be a vector field belonging to $(H^3(\Omega_0))^2$. We define the following transformation map:

$$\begin{aligned} T : (H^3(\Omega_0))^2 &\longrightarrow \quad (H^3(\Omega_0^c))^2 \\ \mathbf{b} \quad &\longmapsto \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}(\gamma(\mathbf{b})), \end{aligned} \tag{19}$$

where γ is the trace operator on Γ_0 and $\mathcal{R}$ is a lifting operator from Γ_0 to Ω_0^c:

$$\gamma : H^3(\Omega_0) \to H^{3-1/2}(\Gamma_0), \quad \text{and} \quad \mathcal{R} : H^{3-1/2}(\Gamma_0) \to H^3(\Omega_0^c). \tag{20}$$

We note that γ and $\mathcal{R}$ are continuous linear operators. The extension operator $P = \mathcal{R} \circ \gamma$ can then be used to define the transformation map $T(\mathrm{w})$ introduced in (15). This map has to be a C^1-diffeomorphism, which requires some regularity property of the displacement field w. The following lemma ensures that for a function $\mathbf{b}$ regular enough, the map $T(\mathbf{b})$ defined in (19) can be used as a change of variable in the Stokes equations. A proof of this result can be found in [26].

Lemma 1 *There exists a positive constant $\mathcal{M}$ such that if $\mathbf{b} \in (H^3(\Omega_0))^2$ satisfies:*

$$\|\mathbf{b}\|_{H^3(\Omega_0)} \leq \mathcal{M},$$

then the following properties hold true:

(i) $\nabla(\mathrm{id}_{\mathbb{R}^2} + \mathcal{R}(\gamma(\mathbf{b}))) = \mathrm{I} + \nabla\mathcal{R}(\gamma(\mathbf{b}))$ *is an invertible matrix in* $(H^2(\Omega_0^c))^{2\times 2}$,
(ii) $T(\mathbf{b}) = \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}(\gamma(\mathbf{b}))$ *is one to one on* $\overline{\Omega_0^c}$,
(iii) $T(\mathbf{b})$ *is a* C^1*-diffeomorphism from* Ω_0^c *onto* $T(\mathbf{b})(\Omega_0^c)$.

Note that the change of variables in the Stokes equations shows up some terms such as $(\nabla \mathrm{v})F(\mathrm{w})$ or $G(\mathrm{w})\nabla \mathrm{q}$, see (16). If we want them to be well defined, we still need higher regularity for w, and we need an algebra structure allowing products of functions. This is done with the following result offering an algebra structure for Sobolev spaces (see [1, Theorem 4.39, p. 106]).

Lemma 2 *Let Ω be a bounded domain of $\mathbb{R}^2$ of class C^1. There exists a positive constant C_a such that for all $u,\ v \in H^2(\Omega)$, we have $uv \in H^2(\Omega)$ and:*

$$\|uv\|_{2,2,\Omega} \leq C_\mathrm{a}\, \|u\|_{2,2,\Omega}\, \|v\|_{2,2,\Omega}\, . \tag{21}$$

Furthermore, for all $w \in H^1(\Omega)$ and $u \in H^2(\Omega)$, we have $uw \in H^1(\Omega)$ and:

$$\|uw\|_{1,2,\Omega} \leq C_\mathrm{a}\, \|u\|_{2,2,\Omega}\, \|w\|_{1,2,\Omega}\, . \tag{22}$$

Now, we define the set:

$$B_{\mathcal{M}} := \{\mathbf{b} \in (H^3(\Omega_0))^2 \mid \|\mathbf{b}\|_{3,2} \leq \mathcal{M}\}. \tag{23}$$

Then, from the two preceding lemmas, the following maps $J : (H^3(\Omega_0))^2 \to H^2(\Omega_0^c)$ and $G, F : B_{\mathcal{M}} \to (H^2(\Omega_0^c))^{2\times 2}$ are well defined by:

$$J(\mathbf{b}) = \det(\nabla T(\mathbf{b})), \quad G(\mathbf{b}) = \mathrm{cof}(\nabla T(\mathbf{b})), \quad F(\mathbf{b}) = (\nabla T(\mathbf{b}))^{-1}\mathrm{cof}(\nabla T(\mathbf{b})). \tag{24}$$

Now we give a result concerning the regularity of J, G, F (see [26]).

Lemma 3 *The maps J, G and F are of class C^∞ in $B_{\mathcal{M}}$ defined in (23).*

We conclude the paragraph with some remarks which will turn out useful in Sects. 3.3 and 4.4. From Lemmas 2 and 3, we have that J defined from $B_{\mathcal{M}}$ into $H^2(\Omega_0^c)$ and G and F defined from $B_{\mathcal{M}}$ into $(H^2(\Omega_0^c))^{2\times 2}$ are of class C^∞, and the norms of their derivatives are bounded on $B_{\mathcal{M}}$. We set:

$$\begin{aligned}
\|DJ\|_{\mathcal{M}} &:= \sup_{\mathbf{b}\in B_{\mathcal{M}}} \|DJ(\mathbf{b})\|_{\mathcal{L}(H^3(\Omega_0), H^2(\Omega_0^c))},\\
\|DG\|_{\mathcal{M}} &:= \sup_{\mathbf{b}\in B_{\mathcal{M}}} \|DG(\mathbf{b})\|_{\mathcal{L}(H^3(\Omega_0), (H^2(\Omega_0^c))^{2\times 2})},\\
\|DF\|_{\mathcal{M}} &:= \sup_{\mathbf{b}\in B_{\mathcal{M}}} \|DF(\mathbf{b})\|_{\mathcal{L}(H^3(\Omega_0), (H^2(\Omega_0^c))^{2\times 2})}.
\end{aligned} \tag{25}$$

noting that $J(0) \equiv 1$, $\nabla T(0) \equiv \mathrm{I}$, and that from Sobolev injection theorem, $H^2(\Omega_0^c)$ is continuously embedded into $L^\infty(\Omega_0^c)$, we can choose $\mathcal{M}$ small enough in (23), so that there exist two positive constants $0 < C_1 < C_2$, such that for all $\mathbf{b} \in B_{\mathcal{M}}$ we have:

$$C_1 \leq \|J(\mathbf{b})\|_{2,2},\ \|J(\mathbf{b})^{-1}\|_{2,2},\ \|\nabla T(\mathbf{b})\|_{2,2},\ \|\nabla T(\mathbf{b})^{-1}\|_{2,2} \leq C_2, \tag{26}$$

and:

$$C_1 \leq \|J(\mathbf{b})\|_{0,\infty},\ |J(\mathbf{b})^{-1}\|_{0,\infty},\ \|\nabla T(\mathbf{b})\|_{0,\infty},\ \|\nabla T(\mathbf{b})^{-1}\|_{0,\infty} \leq C_2. \tag{27}$$

Finally, let $\eta \in H^1(\mathbb{R}^2)$. In view of Lemma 1, $T(\mathbf{b})$ is a C^1-diffeomorphism. Thus, we have $\eta \circ T(\mathbf{b}) \in H^1(\Omega_0^c)$ and $\nabla(\eta \circ T(\mathbf{b})) = \big((\nabla\eta)\circ T(\mathbf{b})\big)\nabla T(\mathbf{b})$, where $\nabla T(\mathbf{b})$ is bounded in $H^2(\Omega_0^c)$ and then in $L^\infty(\Omega_0^c)$. It follows that for all $\mathbf{b} \in B_{\mathcal{M}}$:

$$\|\eta \circ T(\mathbf{b})\|_{1,2,\Omega_0^c} \leq C\|\eta\|_{1,2,\mathbb{R}^2}, \tag{28}$$

for all $\eta \in H^1(\mathbb{R}^2)$, where C is a positive constant depending on Ω_0, C_1, and C_2.

Furthermore, we recall a useful calculus property called *Piola's identity* (see, e.g. [15]). For $1 \leq n < p$, and $\Psi \in (W^{2,p})^n$, we have:

$$\mathrm{div}\,(\mathrm{cof}\nabla\Psi) = 0. \tag{29}$$

3.2 Fixed-Point Procedure

The proof of Theorem 1 for the existence and uniqueness of the solution of the FSI problem (16) relies on a fixed-point argument that we present in this subsection, by first considering the two following problems.

1. Let $f \in (H^2(\mathbb{R}^2))^2$ and let $(\mathrm{v}(\mathbf{b}), \mathrm{q}(\mathbf{b}))$ be the solution of the system:

$$\begin{aligned} -\nu \operatorname{div}((\nabla \mathrm{v}(\mathbf{b}))F(\mathbf{b})) + G(\mathbf{b})\nabla \mathrm{q}(\mathbf{b}) &= J(\mathbf{b})(f \circ T(\mathbf{b})) && \text{in } \Omega_0^c, \\ \operatorname{div}(G(\mathbf{b})^T \mathrm{v}(\mathbf{b})) &= 0 && \text{in } \Omega_0^c, \\ \mathrm{v}(\mathbf{b}) &= 0 && \text{on } \partial\Omega_0^c, \end{aligned} \tag{30}$$

   where the maps J, G and F are defined by (24).
2. Let $g \in (H^1(\Omega_0))^2$ and let $(\mathrm{w}(\mathbf{b}), \mathrm{s}(\mathbf{b}))$ be the solution of the system:

$$\begin{aligned} -\mu \operatorname{div}(\nabla \mathrm{w}(\mathbf{b})) + \nabla \mathrm{s}(\mathbf{b}) &= g && \text{in } \Omega_0, \\ \operatorname{div} \mathrm{w}(\mathbf{b}) &= 0 && \text{in } \Omega_0, \\ \mathrm{w}(\mathbf{b}) &= 0 && \text{on } \Gamma_\omega, \\ (\mu \nabla \mathrm{w}(\mathbf{b}) - \mathrm{s}(\mathbf{b})\mathrm{I})n_0 &= (\nu \nabla \mathrm{v}(\mathbf{b})F(\mathbf{b}) - \mathrm{q}(\mathbf{b})G(\mathbf{b}))n_0 && \text{on } \Gamma_0. \end{aligned} \tag{31}$$

For a fixed $\mathbf{b}$ small enough, we will show that the problem (30) admits a unique solution $(\mathrm{v}(\mathbf{b}), \mathrm{q}(\mathbf{b}))$, and then that the problem (31) depending on $(\mathrm{v}(\mathbf{b}), \mathrm{q}(\mathbf{b}))$ admits also a unique solution denoted by $(\mathrm{w}(\mathbf{b}), \mathrm{s}(\mathbf{b}))$. In particular, we will see that $\mathrm{w}(\mathbf{b})$ belongs to $H^3(\Omega_0)$. Thus, we will be able to define a map:

$$\begin{aligned} \boldsymbol{S} : B_{\mathcal{M}} &\longrightarrow (H^3(\Omega_0))^2 \\ \mathbf{b} &\longmapsto \mathrm{w}(\mathbf{b}), \end{aligned} \tag{32}$$

and we will show in Sect. 3.3 that this map is actually a contraction, so that we can apply the Banach fixed-point theorem, and deduce that the solution we search for the FSI problem is unique and is given by the fixed point of $\boldsymbol{S}$.

Problem (30) is a slightly perturbed incompressible Stokes problem with non-slip boundary condition, having a solution for which the pressure field is defined up to a constant. For this, we introduce the null mean-value pressure space:

$$L_0^2(\Omega_0^c) := \left\{ q \in L^2(\Omega_0^c) \;\middle|\; \int_{\Omega_0^c} q \, \mathrm{d}x = 0 \right\}.$$

In the case of the structure problem (31) with mixed boundary conditions, the velocity together with the pressure are completely determined, and no zero mean value has to be imposed for the pressure. Recalling that Ω_0 is defined by (2) with boundary components Γ_0 and Γ_ω (see Fig. 1), we set:

$$H^1_{0,\Gamma_\omega}(\Omega_0) := \{u \in H^1(\Omega_0) \mid u = 0 \text{ on } \Gamma_\omega\}.$$

We give existence, uniqueness, and regularity results for the solutions to the fluid and the structure problems (30) and (31), compiled in the following theorem.

Theorem 2 *Let D, Ω_0, Ω_0^c and ω be domains of the form (2)–(3) with boundary components ∂D and Γ_ω of class C^3 and Γ_0 of class $C^{3,1}$. Let $(f_F, h_F) \in (H^1(\Omega_0^c))^2 \times H^2(\Omega_0^c)$ and $(g, h_S, f_b) \in (H^1(\Omega_0))^2 \times H^2(\Omega_0) \times (H^{3/2}(\Gamma))^2$ be such that:*

$$\int_{\Omega_0^c} h_F \mathrm{d}x = 0. \tag{33}$$

Let $\mathbf{A}, \mathbf{B} \in (H^2(\Omega_0^c))^{2\times 2}$ and $\mathbf{C}$, $\mathbf{D} \in (H^2(\Omega_0))^{2\times 2}$ be matrix fields such that there exist $\psi_F \in (H^3(\Omega_0^c))^2$ and $\psi_S \in (H^3(\Omega_0))^2$ satisfying:

$$\mathbf{B} = \mathrm{cof}(\nabla \psi_F) \quad \textit{and} \quad \mathbf{D} = \mathrm{cof}(\nabla \psi_S).$$

There exists a positive constant C_{pert} such that, if:

$$\|\mathbf{I} - \mathbf{A}\|_{(H^2(\Omega_0))^{2\times 2}} \leq C_{\mathrm{pert}}, \; \|\mathbf{I} - \mathbf{B}\|_{(H^2(\Omega_0))^{2\times 2}} \leq C_{\mathrm{pert}}, \tag{34}$$

$$\|_{(H^2(\Omega_0))^{2\times 2}} \leq C_{\mathrm{pert}}, \; \|\mathbf{I} - \mathbf{D}\|_{(H^2(\Omega_0))^{2\times 2}} \leq C_{\mathrm{pert}}, \tag{35}$$

then there exists a unique solution $(v, p) \in (H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2 \times (L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c))$ of the perturbed Stokes system:

$$\begin{aligned} -\nu \mathrm{div}((\nabla v)\mathbf{A}) + \mathbf{B}\nabla p &= f_F \quad && \textit{in } \Omega_0^c, \\ \mathrm{div}(\mathbf{B}^\top v) &= h_F \quad && \textit{in } \Omega_0^c, \\ v &= 0 \quad && \textit{on } \partial\Omega_0^c, \end{aligned} \tag{36}$$

and there exists a unique solution $(w, s) \in (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^3(\Omega_0))^2 \times H^2(\Omega_0)$ of the problem:

$$\begin{aligned} -\mu \mathrm{div}((\nabla w)\mathbf{C}) + \mathbf{D}\nabla s &= g \quad && \textit{in } \Omega_0, \\ \mathrm{div}(\mathbf{D}^\top w) &= h_S \quad && \textit{in } \Omega_0, \\ w &= 0 \quad && \textit{on } \Gamma_\omega, \\ (\mu(\nabla w)\mathbf{C} - s\mathbf{D})n &= f_b \quad && \textit{on } \Gamma, \end{aligned} \tag{37}$$

where n is the outward normal vector to Γ. Furthermore, there exists a positive constant C_{fs} such that:

$$\|v\|_{3,2,\Omega_0^c} + \|p\|_{2,2,\Omega_0^c} \leq C_{\mathrm{fs}}(\|f_F\|_{1,2,\Omega_0^c} + \|h_F\|_{2,2,\Omega_0^c})$$

and

$$\|w\|_{3,2,\Omega_0} + \|s\|_{2,2,\Omega_0} \leq C_{\mathrm{fs}}(\|g\|_{1,2,\Omega_0} + \|h_S\|_{2,2,\Omega_0} + \|f_b\|_{H^{3/2}(\Gamma)}).$$

We refer the reader to [26], where the proof of this result for Problem (36) is entirely given. The demonstration relies on a fixed-point argument leading to conditions (34), and on the classical regularity result of Stokes problem (see, e.g. [10]). For a complete proof of well-posedness and regularity for Stokes problem, we may refer to [13] for the three-dimensional case, and to [42, Proposition 2.3 p. 35] for the two-dimensional case. A complete development on these questions is carried out in [23].

Problem (37) involves non-standard boundary conditions of different types. In the case where $\mathbf{C} = \mathbf{D} = \mathrm{I}$, a proof of the existence and uniqueness of a weak solution is given in [12, Sect. 3.3.3], and the regularity result can be obtained following the approach presented in [10, Sect. IV.7] in the case where the stress boundary condition lies on the whole boundary $\partial\Omega_0$. From there, the result dealing with Problem (37) can be proved in the very same way as for Problem (36), with a fixed-point argument.

3.3 Proof of Theorem 1

Now, we turn to the proof of Theorem 1 for the existence and uniqueness of the solution of the FSI problem (16) by means of the fixed-point procedure introduced in Sect. 3.2. From now on, we will denote by C any generic positive constant depending only on Ω_0 and on the constants C_1 and C_2 appearing in inequalities (26) and (27). The proof is divided into 3 steps.

• *Step 1: continuity of the fluid problem.* We start by proving that Problem (30) possesses a unique solution. We have that $G(0) = F(0) = \mathrm{I}$. For $\mathbf{b} \in B_{\mathcal{M}}$ (see (23)), we deduce from Lemma 3 that if $\mathcal{M}$ is small enough, then $\|\mathrm{I} - F(\mathbf{b})\|_{(H^2(\Omega))^{2\times 2}} \leq C_{\mathrm{pert}}$ and $\|\mathrm{I} - G(\mathbf{b})\|_{(H^2(\Omega))^{2\times 2}} \leq C_{\mathrm{pert}}$, where $C_{\mathrm{pert}} > 0$ is the positive constant from inequalities (34) of Theorem 2. Moreover, from Lemma 1 we know that $T(\mathbf{b})$ is a C^1-diffeomorphism and consequently $f \circ T(\mathbf{b}) \in \left(H^1(\Omega_0^c)\right)^2$. Since $J(\mathbf{b}) \in H^2(\Omega_0^c)$, we deduce from (21) in Lemma 2 that $J(\mathbf{b})(f \circ T(\mathbf{b})) \in \left(H^1(\Omega_0^c)\right)^2$. Thus, we can apply Theorem 2 with $f_F = J(\mathbf{b}) f \circ T(\mathbf{b})$ and $h_F \equiv 0$ for Problem (30): for all $\mathbf{b} \in B_{\mathcal{M}}$ with $\mathcal{M}$ small enough, Problem (30) admits a unique solution $(\mathrm{v}(\mathbf{b}), \mathrm{q}(\mathbf{b})) \in (H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2 \times (L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c))$, satisfying the following estimate:

$$\|\mathrm{v}(\mathbf{b})\|_{3,2,\Omega} + \|\mathrm{q}(\mathbf{b})\|_{2,2,\Omega} \leq C_{\mathrm{fs}} \|J(\mathbf{b})(f \circ T(\mathbf{b}))\|_{1,2,\Omega}. \tag{38}$$

Now, we prove a continuity property for the solutions of Problem (30). Let then $(\mathrm{v}(\mathbf{b}_1), \mathrm{q}(\mathbf{b}_1))$ and $(\mathrm{v}(\mathbf{b}_2), \mathrm{q}(\mathbf{b}_2))$ be the solutions of Problem (30) for, respectively, $\mathbf{b}_1$ and $\mathbf{b}_2$ in $B_{\mathcal{M}}$. We set $\boldsymbol{\delta}\mathrm{v} := \mathrm{v}(\mathbf{b}_1) - \mathrm{v}(\mathbf{b}_2)$ and $\boldsymbol{\delta}\mathrm{q} := \mathrm{q}(\mathbf{b}_1) - \mathrm{q}(\mathbf{b}_2)$. We want to estimate $\|\boldsymbol{\delta}\mathrm{v}\|_{3,2,\Omega_0^c}$ and $\|\boldsymbol{\delta}\mathrm{q}\|_{2,2,\Omega_0^c}$ with respect to the difference $\|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0}$. In view of (30), by difference, we infer that the pair $(\boldsymbol{\delta}\mathrm{v}, \boldsymbol{\delta}\mathrm{q})$ solves:

$$\begin{aligned} -\nu \mathrm{div}(\nabla(\boldsymbol{\delta}\mathrm{v}) F(\mathbf{b}_1)) + G(\mathbf{b}_1)\nabla\boldsymbol{\delta}\mathrm{q} &= f_F && \text{in } \Omega_0^c, \\ \mathrm{div}(G(\mathbf{b}_1)^\top \boldsymbol{\delta}\mathrm{v}) &= h_F && \text{in } \Omega_0^c, \\ \boldsymbol{\delta}\mathrm{v} &= 0 && \text{on } \partial\Omega_0^c, \end{aligned} \tag{39}$$

where now f_F and h_F are defined by:

$$\begin{aligned} f_F &:= J(\mathbf{b}_1) f \circ T(\mathbf{b}_1) - J(\mathbf{b}_2) f \circ T(\mathbf{b}_2) + \nu \mathrm{div}(\nabla(\mathrm{v}(\mathbf{b}_2))(F(\mathbf{b}_1) - F(\mathbf{b}_2))) \\ &\quad - (G(\mathbf{b}_1) - G(\mathbf{b}_2))\nabla \mathrm{q}(\mathbf{b}_2), \\ h_F &:= -\mathrm{div}((G(\mathbf{b}_1) - G(\mathbf{b}_2))^\top \mathrm{v}(\mathbf{b}_2)). \end{aligned} \tag{40}$$

The compatibility condition (33) for h_F is valid because of the homogeneous Dirichlet condition satisfied by $\mathrm{v}(\mathbf{b}_2)$. In view of the regularity of $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathrm{v}(\mathbf{b}_2)$ and $\mathrm{q}(\mathbf{b}_2)$, we can apply Theorem 2 to Problem (39). Indeed, from Piola's identity (29), we have that $h_F = -\mathrm{div}((G(\mathbf{b}_1) - G(\mathbf{b}_2))^\top \mathrm{v}(\mathbf{b}_2)) = -(G(\mathbf{b}_1) - G(\mathbf{b}_2)) \cdot \nabla \mathrm{v}(\mathbf{b}_2)$, which belongs to $H^2(\Omega_0^c)$ thanks to Lemma 2. Still from Lemma 2, we directly have that $\mathrm{div}(\nabla(\mathrm{v}(\mathbf{b}_2))(F(\mathbf{b}_1) - F(\mathbf{b}_2)))$ is in $H^1(\Omega_0^c)$. From the second part (22) of Lemma 2, $(G(\mathbf{b}_1) - G(\mathbf{b}_2))\nabla \mathrm{q}(\mathbf{b}_2)$ belongs to $H^1(\Omega_0^c)$. As a result from (40), we deduce that $f_F \in (H^1(\Omega_0^c))^2$ and we can apply Theorem 2 to Problem (39). Thus, for all $\mathbf{b}_1$, $\mathbf{b}_2$ in $B_{\mathcal{M}}$, the solution $(\boldsymbol{\delta}\mathrm{v}, \boldsymbol{\delta}\mathrm{q})$ of Problem (39) belongs to $(H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2 \times (L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c))$ and satisfies:

$$\|\boldsymbol{\delta}\mathrm{v}\|_{3,2,\Omega_0^c} + \|\boldsymbol{\delta}\mathrm{q}\|_{2,2,\Omega_0^c} \leq C_{\mathrm{fs}} \left(\|f_F\|_{1,2,\Omega_0^c} + \|h_F\|_{2,2,\Omega_0^c} \right). \tag{41}$$

Let us first estimate the term f_F, starting by considering the terms depending on $\mathrm{v}(\mathbf{b}_2)$ and $\mathrm{q}(\mathbf{b}_2)$ in (40). From Theorem 2 applied to Problem (30) written for $\mathbf{b}_2$, we have the estimate:

$$\|\nabla \mathrm{v}(\mathbf{b}_2)\|_{2,2,\Omega_0^c,} + \|\nabla \mathrm{q}(\mathbf{b}_2)\|_{1,2,\Omega_0^c} \leq C_{\mathrm{fs}} \|J(\mathbf{b}_2)(f \circ T(\mathbf{b}_2))\|_{1,2,\Omega_0^c}. \tag{42}$$

In view of Lemma 2, and inequalities (26) and (28), we have, up to taking a smaller $\mathcal{M}$:

$$\|J(\mathbf{b}_2)(f \circ T(\mathbf{b}_2))\|_{1,2,\Omega_0^c} \leq C C_{\mathrm{a}} \|f\|_{1,2,\mathbb{R}^2}. \tag{43}$$

From Lemma 2 with (42), (43) and (25), we deduce:

$$\begin{aligned} \|\nabla \mathrm{v}(\mathbf{b}_2)(F(\mathbf{b}_1) - F(\mathbf{b}_2))\|_{2,2,\Omega_0^c} &\leq C_{\mathrm{a}} \|\nabla \mathrm{v}(\mathbf{b}_2)\|_{2,2,\Omega_0^c} \|F(\mathbf{b}_1) - F(\mathbf{b}_2)\|_{2,2,\Omega_0^c} \\ &\leq C C_{\mathrm{a}}^2 C_{\mathrm{fs}} \|f\|_{1,2,\mathbb{R}^2} \|DF\|_{\mathcal{M}} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0}, \end{aligned} \tag{44}$$

and similarly we find:

$$\|(G(\mathbf{b}_1) - G(\mathbf{b}_2))\nabla \mathrm{q}(\mathbf{b}_2)\|_{1,2,\Omega_0^c} \leq C C_{\mathrm{a}}^2 C_{\mathrm{fs}} \|f\|_{1,2,\mathbb{R}^2} \|DG\|_{\mathcal{M}} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0}. \tag{45}$$

In order to obtain a bound for f_F, we also need to treat the first two terms in the right-hand side of (40), which we rewrite as follows:

$$\|J(\mathbf{b}_1)f\circ T(\mathbf{b}_1)-J(\mathbf{b}_2)f\circ T(\mathbf{b}_2)\|_{1,2,\Omega_0^c}\le\|(J(\mathbf{b}_1)-J(\mathbf{b}_2))f\circ T(\mathbf{b}_1)\|_{1,2,\Omega_0^c}$$
$$+\|J(\mathbf{b}_2)(f\circ T(\mathbf{b}_1)-f\circ T(\mathbf{b}_2))\|_{1,2,\Omega_0^c}. \quad (46)$$

For the first term of the right-hand side of (46), we have from Lemma 2 and (28):

$$\|(J(\mathbf{b}_1)-J(\mathbf{b}_2))f\circ T(\mathbf{b}_1)\|_{1,2,\Omega_0^c}\le CC_{\mathrm{a}}\|f\|_{1,2,\mathbb{R}^2}\|DJ\|_{\mathcal{M}}\|\mathbf{b}_1-\mathbf{b}_2\|_{3,2,\Omega_0}. \quad (47)$$

For the second term of the right-hand side of (46), we rely on [29, Lemma 5.3.9]. Let us remark that it is at this stage, i.e. for the application of this Lemma, that we need more regularity for f when normally H^1-regularity would have been enough to solve the fluid problem. Indeed, this lemma states that if $f\in H^2(\mathbb{R}^2)$, then the map:

$$(W^{1,\infty}(\mathbb{R}^2))^2\ni\theta\mapsto f\circ(\mathrm{id}_{\mathbb{R}^2}+\theta)\in H^1(\mathbb{R}^2) \quad (48)$$

is of class C^1 in the vicinity of 0, and the differential is given by $D(f\circ(\mathrm{id}_{\mathbb{R}^2}+\theta))\xi=(\nabla f)\circ(\mathrm{id}_{\mathbb{R}^2}+\theta)\cdot\xi$ for all ξ in $(W^{1,\infty}(\mathbb{R}^2))^2$. Yet we have that $T(\mathbf{b})$ defined in (19) can in fact be defined as $T(\mathbf{b})=\mathrm{id}_{\mathbb{R}^2}+\mathcal{R}(\gamma(\mathbf{b}))$ with $B_{\mathcal{M}}\ni\mathbf{b}\mapsto\mathcal{R}(\gamma(\mathbf{b}))\in H^3(\mathbb{R}^2)$. From Sobolev embedding, we have that $(H^3(\mathbb{R}^2))^2$ is continuously embedded into $(W^{1,\infty}(\mathbb{R}^2))^2$, and we denote by C_∞ the embedding constant. We also note that $\mathbf{b}\mapsto T(\mathbf{b})$ is continuously affine and then smooth. As a consequence we have that the map:

$$B_{\mathcal{M}}\ni\mathbf{b}\mapsto f\circ T(\mathbf{b})\in(H^1(\mathbb{R}^2))^2$$

is well defined and is of class C^1 in the vicinity of 0. Its differential is given by $D_{\mathbf{b}}(f\circ T(\mathbf{b}))\xi=(\nabla f)\circ T(\mathbf{b})\cdot\mathcal{R}(\gamma(\xi))$ for all ξ in $(H^3(\Omega_0))^2$. In view of Lemma 2 with $f\in\left(H^2(\mathbb{R}^2)\right)^2$, $D_{\mathbf{b}}(f\circ T(\mathbf{b}))\xi$ is indeed in $(H^1(\mathbb{R}^2))^2$ and:

$$\|D_{\mathbf{b}}(f\circ T(\mathbf{b}))\|_{\mathcal{L}((H^3(\Omega_0))^2,(H^1(\mathbb{R}^2))^{2\times2})}\le CC_{\mathcal{R}\gamma}C_\infty\|(\nabla f)\circ T(\mathbf{b})\|_{1,2,\mathbb{R}^2},$$

where $C_{\mathcal{R}\gamma}$ stands for the continuity constant of the operator $\mathcal{R}\circ\gamma$. Thus, for $f\in(H^2(\mathbb{R}^2))^2$, we have:

$$\|f\circ T(\mathbf{b}_1)-f\circ T(\mathbf{b}_2)\|_{1,2,\Omega_0^c}$$
$$\le CC_{\mathcal{R}\gamma}C_\infty\sup_{\mathbf{b}\in B_{\mathcal{M}}}\left\{\|(\nabla f)\circ T(\mathbf{b})\|_{1,2,\mathbb{R}^2}\right\}\|\mathbf{b}_1-\mathbf{b}_2\|_{3,2,\Omega_0}.$$

In the light of (28), we have similarly for all $\mathbf{b}\in B_{\mathcal{M}}$:

$$\|(\nabla f)\circ T(\mathbf{b})\|_{1,2,\mathbb{R}^2}\le C\|f\|_{2,2,\mathbb{R}^2},$$

and then by arguing in the same way as for (43) we have:

$$\|J(\mathbf{b}_2)(f\circ T(\mathbf{b}_1)-f\circ T(\mathbf{b}_2))\|_{1,2,\Omega_0^c}\le CC_{\mathrm{a}}C_{\mathcal{R}\gamma}C_\infty\|f\|_{2,2,\mathbb{R}^2}\|\mathbf{b}_1-\mathbf{b}_2\|_{3,2,\Omega_0}. \quad (49)$$

We recall that f_F is given by (40). We have completely estimated $\|f_F\|_{1,2}$ by combining (44), (45), (47), and (49). We obtain:

$$\|f_F\|_{1,2,\Omega_0^c} \leq \|f\|_{1,2,\mathbb{R}^2} \left(C C_{\mathrm{a}}^2 C_{\mathrm{fs}} (\nu \|DF\|_{\mathcal{M}} + \|DG\|_{\mathcal{M}}) + C C_{\mathrm{a}} \|DJ\|_{\mathcal{M}} \right) \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0} + C C_{\mathrm{a}} C_{\mathcal{R}\gamma} C_{\infty} \|f\|_{2,2,\mathbb{R}^2} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0},$$

and finally we have a constant $C_{\mathbf{1}} = C_{\mathbf{1}}(C_{\mathrm{fs}}, C_{\mathrm{a}}, C_{\infty}, C_{\mathcal{R}\gamma}, \mathcal{M})$ such that:

$$\|f_F\|_{1,2,\Omega_0^c} \leq C_{\mathbf{1}} \|f\|_{2,2,\mathbb{R}^2} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0}. \tag{50}$$

Let us now pass to the estimate for $\|h_F\|_{2,2}$. We recall that in view of Piola's identity (29) we can write:

$$h_F = -\mathrm{div}((G(\mathbf{b}_1) - G(\mathbf{b}_2))^{\top} \mathrm{v}(\mathbf{b}_2)) = -(G(\mathbf{b}_1) - G(\mathbf{b}_2)) : \nabla \mathrm{v}(\mathbf{b}_2),$$

so that in a same manner as for (44), we have:

$$\begin{aligned} \|h_F\|_{2,2,\Omega_0^c} &\leq C_{\mathrm{a}} \|G(\mathbf{b}_1) - G(\mathbf{b}_2)\|_{2,2,\Omega_0^c} \|\nabla \mathrm{v}(\mathbf{b}_2)\|_{2,2,\Omega_0^c} \\ &\leq C C_{\mathrm{a}}^2 C_{\mathrm{fs}} \|f\|_{1,2,\mathbb{R}^2} \|DG\|_{\mathcal{M}} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0}. \end{aligned} \tag{51}$$

At this point, we have computed two upper bounds for the norms of f_F and h_F. Thus, by combining (41), (50), and (51), we finally obtain that there exists a constant $\boldsymbol{C}_{\mathbf{F}} = \boldsymbol{C}_{\mathbf{F}}(C_{\mathrm{fs}}, C_{\mathrm{a}}, C_{\infty}, C_{\mathcal{R}\gamma}, \mathcal{M})$ such that for all $\mathbf{b}_1, \mathbf{b}_2$ in $B_{\mathcal{M}}$:

$$\|\boldsymbol{\delta}\mathrm{v}\|_{3,2,\Omega_0^c} + \|\boldsymbol{\delta}\mathrm{q}\|_{2,2,\Omega_0^c} \leq \boldsymbol{C}_{\mathbf{F}} \|f\|_{2,2,\mathbb{R}^2} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2,\Omega_0}. \tag{52}$$

•*Step 2: continuity of the structure problem.* We first prove that Problem (31) has a unique solution. For $\mathbf{b} \in B_{\mathcal{M}}$, Problem (31) involves the source term on Γ_0:

$$f_b = \left[\nu \nabla \mathrm{v}(\mathbf{b}) F(\mathbf{b}) - \mathrm{q}(\mathbf{b}) G(\mathbf{b}) \right] n_0, \tag{53}$$

where $(\mathrm{v}(\mathbf{b}), \mathrm{q}(\mathbf{b}))$ is the unique solution of the fluid equations (30) studied in Step 1. In view of the regularity of the fields involved in the expression (53) and from Lemma 2, we have that:

$$\nu \nabla \mathrm{v}(\mathbf{b}) F(\mathbf{b}) - \mathrm{q}(\mathbf{b}) G(\mathbf{b}) \in H^2(\Omega_0^c). \tag{54}$$

Thus, f_b belongs to $(H^{3/2}(\Gamma_0))^2$ and Theorem 2 for $\mathbf{C} = \mathbf{D} = \mathrm{I}$ can be applied: for all $\mathbf{b} \in B_{\mathcal{M}}$, there exists a unique solution $(\mathrm{w}(\mathbf{b}), \mathrm{s}(\mathbf{b})) \in (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^2(\Omega_0))^2 \times H^2(\Omega_0)$ of Problem (31) and there exists a positive constant C_{fs} such that:

$$\|\mathrm{w}(\mathbf{b})\|_{3,2,\Omega_0} + \|\mathrm{s}(\mathbf{b})\|_{2,2,\Omega_0} \leq C_{\mathrm{fs}} \left(\|g\|_{1,2,\Omega_0} + \|f_b\|_{H^{3/2}(\Gamma_0)} \right). \tag{55}$$

Now, we establish a continuity property for Problem (31). Let $(\mathrm{w}(\mathbf{b}_1), \mathrm{s}(\mathbf{b}_1))$ and $(\mathrm{w}(\mathbf{b}_2), \mathrm{s}(\mathbf{b}_2))$ be the solutions of Problem (31) for, respectively, $\mathbf{b}_1$ and $\mathbf{b}_2$ in $B_{\mathcal{M}}$ (note that in the system $\mathrm{v}(\mathbf{b}_i)$ are given and they solve the fluid equation studied in Step 1). We set $\boldsymbol{\delta}\mathrm{w} := \mathrm{w}(\mathbf{b}_1) - \mathrm{w}(\mathbf{b}_2)$ and $\boldsymbol{\delta}\mathrm{s} := \mathrm{s}(\mathbf{b}_1) - \mathrm{s}(\mathbf{b}_2)$. In view of (31), by difference, we infer that the pair $(\boldsymbol{\delta}\mathrm{w}, \boldsymbol{\delta}\mathrm{s})$ solves:

$$\begin{aligned} -\mu \mathrm{div}(\nabla^s \boldsymbol{\delta}\mathrm{w}) + \nabla \boldsymbol{\delta}\mathrm{s} &= 0 \text{ in } \Omega_0, \\ \mathrm{div}\boldsymbol{\delta}\mathrm{w} &= 0 \text{ in } \Omega_0, \\ \boldsymbol{\delta}\mathrm{w} &= 0 \text{ on } \Gamma_\omega, \\ (\mu \nabla \boldsymbol{\delta}\mathrm{w} - \boldsymbol{\delta}\mathrm{s}\, \mathrm{I}) n_0 &= f_b \text{ on } \Gamma_0, \end{aligned}$$

with f_b the surface force on Γ_0:

$$f_b = [\nu \nabla \mathrm{v}(\mathbf{b}_1) F(\mathbf{b}_1) - \nu \nabla \mathrm{v}(\mathbf{b}_2) F(\mathbf{b}_2) - q(\mathbf{b}_1) G(\mathbf{b}_1) + q(\mathbf{b}_2) G(\mathbf{b}_2)] n_0. \quad (56)$$

In view of (54), $f_b \in (H^{3/2}(\Gamma_0))^2$ and Theorem 2 applies giving the a priori estimate:

$$\|\boldsymbol{\delta}\mathrm{w}\|_{3,2} + \|\boldsymbol{\delta}\mathrm{s}\|_{2,2} \leq C_{\mathrm{fs}} \| f_b \|_{H^{3/2}(\Gamma_0)}. \quad (57)$$

Let us furtherly bound from above the right-hand side, in order to make the norm of the difference $\mathbf{b}_1 - \mathbf{b}_2$ appear. The first two terms of f_b (see expression (56)) satisfy:

$$\begin{aligned} &\|(\nabla \mathrm{v}(\mathbf{b}_1) F(\mathbf{b}_1) - \nabla \mathrm{v}(\mathbf{b}_2) F(\mathbf{b}_2))\, n_0\|_{3/2,2,\Gamma_0} \\ &\quad \leq C \left(\|\nabla \mathrm{v}(\mathbf{b}_2)(F(\mathbf{b}_1) - F(\mathbf{b}_2))\|_{2,2,\Omega_0^c} + \|(\nabla \mathrm{v}(\mathbf{b}_1) - \nabla \mathrm{v}(\mathbf{b}_2)) F(\mathbf{b}_1)\|_{2,2,\Omega_0^c} \right). \end{aligned} \quad (58)$$

We bound the two terms of the right-hand side of (58) by using, respectively, (44) and (52), and noting that H^2 norm of $F(\mathbf{b})$ is bounded in $B_{\mathcal{M}}$ by a positive constant $\boldsymbol{C_2} = \boldsymbol{C_2}(\mathcal{M})$. This gives:

$$\begin{aligned} &\nu \|(\nabla \mathrm{v}(\mathbf{b}_1) F(\mathbf{b}_1) - \nabla \mathrm{v}(\mathbf{b}_2) F(\mathbf{b}_2)) n_0\|_{3/2,2,\Gamma_0} \\ &\qquad \leq \nu \left(C C_{\mathrm{a}}^2 C_{\mathrm{fs}} \|DF\|_{\mathcal{M}} + C_{\mathrm{a}} \boldsymbol{C_F} \boldsymbol{C_2} \right) \|f\|_{2,2,\mathbb{R}^2} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2}. \end{aligned} \quad (59)$$

In a same manner, exploiting (45), (52), and a bound $\boldsymbol{C_3} = \boldsymbol{C_3}(\mathcal{M})$ of the H^2 norm of $G(\mathbf{b})$ for b in $B_{\mathcal{M}}$, we get:

$$\begin{aligned} &\|(q(\mathbf{b}_1) G(\mathbf{b}_1) - q(\mathbf{b}_2) G(\mathbf{b}_2)) n_0\|_{3/2,2,\Gamma_0} \\ &\qquad \leq (C C_{\mathrm{a}}^2 C_{\mathrm{fs}} \|DG\|_{\mathcal{M}} + C_{\mathrm{a}} \boldsymbol{C_F} \boldsymbol{C_3}) \|f\|_{2,2,\mathbb{R}^2} \|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2}. \end{aligned} \quad (60)$$

By combining (57), (59), and (60), we conclude that there exists a positive constant $\boldsymbol{C_{\mathcal{M}}} = \boldsymbol{C_{\mathcal{M}}}(C_{\text{fs}}, C_{\text{a}}, C_{\infty}, C_{\mathcal{R}\gamma}, \mathcal{M})$ such that:

$$\|\text{w}(\mathbf{b}_1) - \text{w}(\mathbf{b}_2)\|_{3,2} + \|\text{s}(\mathbf{b}_1) - \text{s}(\mathbf{b}_2)\|_{2,2} \leq C_{\text{fs}}\boldsymbol{C_{\mathcal{M}}}\|f\|_{2,2,\mathbb{R}^2}\|\mathbf{b}_1 - \mathbf{b}_2\|_{3,2}. \quad (61)$$

• *Step 3: contraction property.* In the sequel, we prove that the map $\boldsymbol{S} : \mathbf{b} \mapsto \text{w}(\mathbf{b})$ defined in (32) is a contraction. From estimate (61), we have the existence of a positive constant C_I with $C_I C_{\text{fs}}\boldsymbol{C_{\mathcal{M}}} < 1$ such that if $\|f\|_{2,2,\mathbb{R}^2} < C_I$, then $\boldsymbol{S}$ is a contraction in $B_{\mathcal{M}}$. From (55), we deduce that there exists a constant C_{II} such that if $\|f\|_{1,2,\mathbb{R}^2} < C_{II}$ and $\|g\|_{1,2,\Omega_0} < C_{II}$, then:

$$\|\text{w}(\mathbf{b})\|_{3,2} + \|\text{s}(\mathbf{b})\|_{2,2} \leq \mathcal{M}. \quad (62)$$

By defining:

$$C_S = \min(C_I, C_{II}),$$

we have that if $\|f\|_{2,2,\mathbb{R}^2} < C_S$ and $\|g\|_{1,2,\Omega_0} < C_S$, then the map $\boldsymbol{S}$ is a contraction which maps $B_{\mathcal{M}}$ onto $B_{\mathcal{M}}$. Thus, the Banach fixed-point theorem ensures that $\boldsymbol{S}$ admits a unique fixed point in $B_{\mathcal{M}}$ denoted by w. It results that the solution (v(w), q(w), w, s(w)) is the unique solution to the fluid–structure interaction problem (16). Finally, combining (38), (53), (55), and (62), we obtain estimate (18). The proof of Theorem 1 is then complete.

4 Velocity Method and Shape Differentiability of the FSI System

After having proved the existence of solutions of the FSI system for a prescribed reference configuration, we now address the so-called *shape sensitivity analysis*: we analyse the behaviour of the solutions with respect to infinitesimal perturbations of the reference configuration. The section is organized as follows: we start, in Sect. 4.1, by introducing the classical velocity method, applied to the FSI problem in Sect. 4.2, and we show its uniform well-posedness in Sect. 4.3. We end with the main result of this section establishing in Sect. 4.4, Theorem 3, the shape differentiability of the solutions of the FSI problem.

4.1 Presentation of the Method

We are interested in the study of the behaviour of a shape functional $\mathcal{J}(\Omega)$ with respect to infinitesimal variations of its argument, the set Ω. This topic, referred to as *shape derivative* or *shape sensitivity analysis*, is now a standard tool in shape optimisation. See, e.g. [40, Chapter 2], [29, Sect. 5.1], or [2, Chapter 6].

Let us present the classical approach: the *velocity method*. Given an admissible domain Ω_0 for $\mathcal{J}$, we consider a 1-parameter family of shapes $(\Omega_{0,t})_t$ of the form:

$$\Omega_{0,t} := \Phi_t(\Omega_0), \quad (63)$$

where $(\Phi_t)_t$ is a family of diffeomorphisms, chosen with the following properties:

- at $t = 0$ there holds $\Phi_0 = \mathrm{id}_{\mathbb{R}^2}$;
- the map $t \mapsto \Phi_t$ is of class C^1;
- each diffeomorphism Φ_t preserves the imposed geometrical constraints on Ω_0, so that every $\Omega_{0,t}$ is admissible for $\mathcal{J}$.

If the function $t \mapsto \mathcal{J}(\Omega_{0,t})$ is differentiable at $t = 0$, then it admits the following development in t:

$$\mathcal{J}(\Omega_{0,t}) = \mathcal{J}(\Omega_0) + t\mathcal{J}'(\Omega_0) + o(t).$$

The coefficient $\mathcal{J}'(\Omega_0)$ of t is the so-called shape derivative of $\mathcal{J}$ at Ω_0 with respect to the deformations $(\Phi_t)_t$. In the literature, it is classical to take diffeomorphisms of the form:

$$\Phi_t = \mathrm{id}_{\mathbb{R}^2} + tV,$$

for a suitable vector field V, representing the velocity (when t is seen as the time) of Φ_t at $t = 0$.

In order to write the expression of $\mathcal{J}'(\Omega_0)$, it is useful to introduce the notion of *material derivative* of a family of functions $(\varphi_t)_t$ defined on the family of transformed domains $(\Omega_{0,t})_{t\geq 0}$ given by (63). By definition, $\varphi_t \circ \Phi_t$ are all defined in the fixed domain Ω_0. If the map $t \mapsto \varphi_t \circ \Phi_t$ is differentiable at $t = 0$, we define the material derivative $\dot{\varphi}$ of φ_t at $t = 0$ as the coefficient of t in the expansion:

$$\varphi_t \circ \Phi_t = \varphi_0 + t\dot{\varphi} + o(t).$$

Note that φ_0 and $\dot{\varphi}$ do not depend on t.

4.2 Shape Transformation of the FSI Problem

In order to apply the velocity method in our framework, let us start by specifying the transformations Φ_t that we choose. We consider $t \geq 0$ small (the threshold will be specified later) and:

$$\Phi_t := \mathrm{id}_{\mathbb{R}^2} + tV. \tag{64}$$

Here V is taken in the space:

$$\Theta := \left\{ V \in H^3(\mathbb{R}^2, \mathbb{R}^2) \,\middle|\, \mathrm{supp} V \subset\subset D \setminus \overline{\omega} \right\}. \tag{65}$$

Let Ω_0 be defined as in Sect. 2, namely the reference configuration of an elastic body contained into D and attached to the rigid support ω. For $t \geq 0$ (small), we set:

$$\Omega_{0,t} := \Phi_t(\Omega_0), \quad \Omega^c_{0,t} := \Phi_t(\Omega^c_0), \quad \text{and} \quad \Gamma_{0,t} = \Phi_t(\Gamma_0). \tag{66}$$

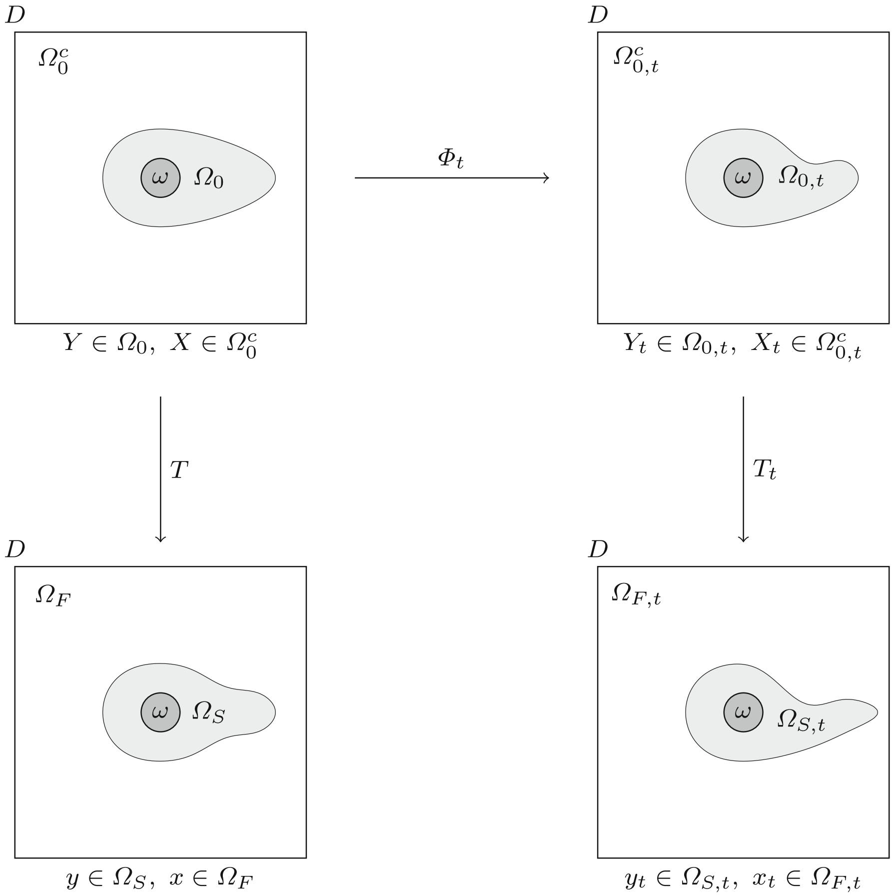

Fig. 2 Geometries of the fluid–elasticity system submitted to the transformation Φ_t and the resolution of the coupled problems, characterised by T_t

We recall that Ω_0^c is the open complementary of Ω_0 in $D \setminus \overline{\omega}$ (see Fig. 2). The assumptions on Θ ensure that every $\Omega_{0,t}$ is contained into D and its boundary is the union of $\Gamma_{0,t}$ and Γ_ω. Let $(\mathrm{u}_t, \mathrm{p}_t, \mathrm{w}_t, \mathrm{s}_t)$ be the solution of the coupled FSI problem (see (11)) posed on the perturbed elastic body $\Omega_{0,t}$ and on the perturbed fluid domain $\Omega_{F,t}$, defined by:

$$\begin{aligned} \Omega_{F,t} &:= D \setminus (\overline{\Omega_{S,t} \cup \omega}), \\ \Omega_{S,t} &:= (\mathrm{id}_{\mathbb{R}^2} + \mathrm{w}_t)(\Omega_{0,t}). \end{aligned}$$

The map $\mathrm{id}_{\mathbb{R}^2} + \mathrm{w}_t$ is one to one from $\Omega_{0,t}$ to $\Omega_{S,t}$ for a w_t small enough (see Lemma 1). Thus, $\Omega_{S,t}$ and $\Omega_{F,t}$ represent, respectively, the shape of the elastic body and the incompressible fluid after resolution of the coupled problem. In the same way as in Sect. 2.3, we can transport the fluid equations on the reference domain $\Omega_{0,t}^c$. In principle, we could repeat the very same steps, by replacing Ω_0 with $\Omega_{0,t}$ and by introducing suitable lifting and trace operators which depend on t. An alternative approach, that we follow here, consists in exploiting the change of variables Φ_t, which allows to use the lifting and trace operators defined in (20) constructed starting from Ω_0, which of course do not depend on t:

$$\mathcal{R} : H^{3-1/2}(\Gamma_0) \longrightarrow H^3(\Omega_0^c) \quad \text{and} \quad \gamma : H^3(\Omega_0) \longrightarrow H^{3-1/2}(\Gamma_0).$$

We set:

$$T_t := \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}(\gamma(\mathrm{w}_t \circ \Phi_t)) \circ \Phi_t^{-1}, \tag{67}$$

where $\mathrm{w}_t \in H^3(\Omega_{0,t})$ is the displacement solving the fluid–structure problem and Φ_t is defined in (64). The transformation T_t maps the domain $\Omega_{0,t}$ onto $\Omega_{S,t}$ and the domain $\Omega_{0,t}^c$ onto $\Omega_{F,t}$ (see Fig. 2).

Now we can define the Lagrangian fluid velocity and pressure variables:

$$\mathrm{v}_t := \mathrm{u}_t \circ T_t, \quad \mathrm{q}_t := \mathrm{p}_t \circ T_t, \tag{68}$$

and we find that the transported FSI problem for $(\mathrm{v}_t, \mathrm{q}_t, \mathrm{w}_t, \mathrm{s}_t)$ can be written as follows (see (16)):

$$\begin{aligned}
-\nu \mathrm{div}((\nabla \mathrm{v}_t) F(T_t)) + G(T_t)\nabla \mathrm{q}_t &= (f \circ T_t) J(T_t) && \text{in } \Omega_{0,t}^c,\\
\mathrm{div}(G(T_t)^\top \mathrm{v}_t) &= 0 && \text{in } \Omega_{0,t}^c,\\
\mathrm{v}_t &= 0 && \text{on } \partial\Omega_{0,t}^c,\\
-\mu \mathrm{div}(\nabla \mathrm{w}_t) + \nabla \mathrm{s}_t &= g && \text{in } \Omega_{0,t},\\
\mathrm{div}\, \mathrm{w}_t &= 0 && \text{in } \Omega_{0,t},\\
\mathrm{w}_t &= 0 && \text{on } \Gamma_\omega,\\
(\mu \nabla \mathrm{w}_t - \mathrm{s}_t \mathrm{I}) n_{0,t} &= \nu (\nabla \mathrm{v}_t) F(T_t) n_{0,t} - \mathrm{q}_t G(T_t) n_{0,t} && \text{on } \Gamma_{0,t},
\end{aligned} \tag{69}$$

where we formally define for any vector field φ:

$$J(\varphi) = \det(\nabla\varphi), \quad G(\varphi) = \mathrm{cof}(\nabla\varphi), \quad F(\varphi) = (\nabla\varphi)^{-1}\mathrm{cof}(\nabla\varphi). \tag{70}$$

It has to be noted that these maps, which will be used in the rest of the article, differ from the ones defined in (24) and used in Sect. 3. Nevertheless, we still denote them by F, G, and J for the sake of readability.

Before investigating in Sect. 4.4 the differentiability in the t variable at 0 of the solutions to the FSI problems (69), we transport these problems, which are defined on the t-dependent domains $\Omega_{0,t}$ and $\Omega_{0,t}^c$, onto the fixed reference domains Ω_0 and Ω_0^c.

We briefly explain how to obtain the transported system of equations. For details, we refer to [12, Sect. 3.4.4] (see also [26]). The main idea is to write the variational formulation of Problem (69) with test functions $(\mathfrak{v} \circ \Phi_t^{-1}, \mathfrak{q} \circ \Phi_t^{-1})$ and $(\mathfrak{w} \circ \Phi_t^{-1}, \mathfrak{s} \circ$

Φ_t^{-1}), for any $(\mathfrak{v}, \mathfrak{q}) \in (H_0^1(\Omega_0^c))^2 \times L_0^2(\Omega_0^c)$ and $(\mathfrak{w}, \mathfrak{s}) \in (H^1_{0,\Gamma_\omega}(\Omega_0))^2 \times L^2(\Omega_0)$, recalling that Φ_t is defined in (64). It can be shown that the compositions

$$\mathrm{v}^t := \mathrm{v}_t \circ \Phi_t, \quad \mathrm{q}^t := \mathrm{q}_t \circ \Phi_t, \quad \mathrm{w}^t := \mathrm{w}_t \circ \Phi_t, \quad \mathrm{s}^t := \mathrm{s}_t \circ \Phi_t, \tag{71}$$

of the solutions $(\mathrm{v}_t, \mathrm{q}_t, \mathrm{w}_t, \mathrm{s}_t)$ of the transformed FSI problem (69) with the change of variable $x = \Phi_t(X)$, solve the following problem:

$$\begin{aligned}
-\nu \operatorname{div}((\nabla \mathrm{v}^t) F(T_t \circ \Phi_t)) + G(T_t \circ \Phi_t) \nabla \mathrm{q}^t &= (f \circ T_t \circ \Phi_t) J(T_t \circ \Phi_t) \text{ in } \Omega_0^c, \\
\operatorname{div}(G(T_t \circ \Phi_t)^\top \mathrm{v}^t) &= 0 \text{ in } \Omega_0^c, \\
\mathrm{v}^t &= 0 \text{ on } \partial\Omega_0^c, \\
-\mu \operatorname{div}((\nabla \mathrm{w}^t) F(\Phi_t)) + G(\Phi_t) \nabla \mathrm{s}^t &= (g \circ \Phi_t) J(\Phi_t) \text{ in } \Omega_0, \\
\operatorname{div}(G(\Phi_t)^\top \mathrm{w}^t) &= 0 \text{ in } \Omega_0, \\
\mathrm{w}^t &= 0 \text{ on } \Gamma_\omega, \\
(\mu(\nabla \mathrm{w}^t) F(\Phi_t) - \mathrm{s}_t G(\Phi_t)) n_0 &= \nu(\nabla \mathrm{v}^t) F(T_t \circ \Phi_t) n_0 \\
&\quad - \mathrm{q}^t G(T_t \circ \Phi_t) n_0 \text{ on } \Gamma_0,
\end{aligned} \tag{72}$$

where we recall that T_t is defined above in (67), whereas J, G, and F are given in (70).

4.3 Uniform Well-Posedness for Small *t*

By directly applying Theorem 1 to Problem (69), we can obtain a solution to Problem (69) and consequently to Problem (72). However, the constants C and C_{FS} in Theorem 1, the former controlling the data and the latter appearing in the a priori estimate, should depend on t. In order to make this resolution uniform with respect to t, we have the following result.

Proposition 1 *Let $f \in (H^2(\mathbb{R}^2))^2$ and $g \in (H^1(\mathbb{R}^2))^2$. There exist three positive constants $t_{\mathcal{M}}$, $C_{\mathbf{S}}$ and C_{FS} such that if $\|f\|_{2,2} \leq C_{\mathbf{S}}$ and $\|g\|_{1,2} \leq C_{\mathbf{S}}$ then for all $t \in [0, t_{\mathcal{M}})$, Problem (72) admits a unique solution $(\mathrm{v}^t, \mathrm{q}^t, \mathrm{w}^t, \mathrm{s}^t) \in (H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2 \times (L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c)) \times (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^3(\Omega_0))^2 \times H^2(\Omega_0)$. Furthermore, there exists a positive constant C_{FS} which does not depend on t, such that:*

$$\|\mathrm{v}^t\|_{3,2,\Omega_0^c} + \|\mathrm{q}^t\|_{2,2,\Omega_0^c} + \|\mathrm{w}^t\|_{3,2,\Omega_0} + \|\mathrm{s}^t\|_{2,2,\Omega_0} \leq C_{FS}(\|f\|_{2,2,\mathbb{R}^2} + \|g\|_{1,2,\mathbb{R}^2}).$$

Proof To solve Problem (72), we copy the fixed point procedure built in Sect. 3.2, applied this time to w^t. With the new definition of the transformation T_t in (67), and from (71), we have then that $T_t := \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}(\gamma(\mathrm{w}^t)) \circ \Phi_t^{-1}$. This suggests to consider the adapted transformation:

$$T(\mathbf{b}) = \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}(\gamma(\mathbf{b})) \circ \Phi_t^{-1},$$

for $\mathbf{b} \in H^3(\Omega_0)^2$ and to introduce the following version of Problem (72), for which the change of variable is given by $\mathbf{b}$:

$$\begin{aligned}
-\nu \mathrm{div}((\nabla \mathrm{v}^t(\mathbf{b}))F(T(\mathbf{b}) \circ \Phi_t)) & \\
+G(T(\mathbf{b}) \circ \Phi_t)\nabla \mathrm{q}^t(\mathbf{b}) &= (f \circ T(\mathbf{b}) \circ \Phi_t)J(T(\mathbf{b}) \circ \Phi_t) && \text{in } \Omega_0^c, \\
\mathrm{div}(G(T(\mathbf{b}) \circ \Phi_t)^\top \mathrm{v}^t(\mathbf{b})) &= 0 && \text{in } \Omega_0^c, \\
\mathrm{v}^t(\mathbf{b}) &= 0 && \text{on } \partial\Omega_0^c, \\
-\mu \mathrm{div}((\nabla \mathrm{w}^t(\mathbf{b}))F(\Phi_t)) + G(\Phi_t)\nabla \mathrm{s}^t(\mathbf{b}) &= (g \circ \Phi_t)J(\Phi_t) && \text{in } \Omega_0, \\
\mathrm{div}(G(\Phi_t)^\top \mathrm{w}^t(\mathbf{b})) &= 0 && \text{in } \Omega_0, \\
\mathrm{w}^t &= 0 && \text{on } \Gamma_\omega, \\
(\mu(\nabla \mathrm{w}^t(\mathbf{b}))F(\Phi_t) - \mathrm{s}^t(\mathbf{b})G(\Phi_t))n_0 &= \nu(\nabla \mathrm{v}^t(\mathbf{b}))F(T(\mathbf{b}) \circ \Phi_t)n_0 \\
&\quad - \mathrm{q}^t(\mathbf{b})G(T(\mathbf{b}) \circ \Phi_t)n_0 && \text{on } \Gamma_0.
\end{aligned} \tag{73}$$

We can adapt the proof of Theorem 1 in Sect. 3 to prove that the map:

$$\begin{aligned}
\boldsymbol{S}_t : (H^3(\Omega_0))^2 &\longrightarrow (H^3(\Omega_0))^2 \\
\mathbf{b} &\longmapsto \mathrm{w}^t(\mathbf{b}),
\end{aligned}$$

has a unique fixed point w^t such that $\left(\mathrm{v}^t(\mathrm{w}^t), \mathrm{q}^t(\mathrm{w}^t), \mathrm{w}^t, \mathrm{s}^t(\mathrm{w}^t)\right)$ corresponds to the solution of Problem (72).

We recall that $\Phi_t = \mathrm{id}_{\mathbb{R}^2} + tV$ and we have $T(\mathbf{b}) \circ \Phi_t = \mathrm{id}_{\mathbb{R}^2} + \eta_t(\mathbf{b})$ with:

$$\eta_t(\mathbf{b}) := tV + \mathcal{R}(\gamma(\mathbf{b})).$$

We know that $\|\mathcal{R}(\gamma(\mathbf{b}))\|_{3,2,D} \leq C_{\mathcal{R}\gamma}\|\mathbf{b}\|_{3,2,\Omega_0}$. Then, let $t_{\mathcal{M}} > 0$ be such that $t_{\mathcal{M}}\|V\|_{3,2} \leq C_{\mathcal{R}\gamma}\mathcal{M}/2$. Thus, we have that:

$$\|\eta_t(\mathbf{b})\|_{3,2,D} \leq C_{\mathcal{R}\gamma}\mathcal{M},$$

for all $t \in [0, t_{\mathcal{M}})$ and for all $\mathbf{b} \in B_{\mathcal{M}/2} := \{\mathbf{b} \in (H^3(\Omega_0))^2 \mid \|\mathbf{b}\|_{3,2,\Omega_0} \leq \mathcal{M}/2\}$. Now, we can choose the constant $\mathcal{M} > 0$ independent of t such that for all $u \in H^3(D)$ with $\|u\|_{3,2,D} \leq C_{\mathcal{R}\gamma}\mathcal{M}$, then $(\mathrm{id}_{\mathbb{R}^2} + u)$ satisfies all the properties required in Sect. 3. In particular, we have that, for all $t \in [0, t_{\mathcal{M}})$ and for all $\mathbf{b} \in B_{\mathcal{M}/2}$:

- Lemma 1 and inequalities (26) and (27) are satisfied for both Φ_t and $T(\mathbf{b}) \circ \Phi_t$,
- Conditions (34) are satisfied for $\mathbf{A} = F(T(\mathbf{b}) \circ \Phi_t)$, $\mathbf{B} = G(T(\mathbf{b}) \circ \Phi_t)$, and (35) are satisfied for $\mathbf{C} = F(\Phi_t)$, $\mathbf{D} = G(\Phi_t)$.

As a consequence, we can proceed as in Sect. 3.3 by applying Theorem 2 in order to solve Problem (73). Thereafter, we show that there exists a constant $C_{\boldsymbol{S}}$ which depend only on $\mathcal{M}$ and Ω_0—and not on t—such that if $\|f\|_{2,2} \leq C_{\boldsymbol{S}}$ and $\|g\|_{1,2} \leq C_{\boldsymbol{S}}$, then $\boldsymbol{S}_t$ is a contraction and $\boldsymbol{S}_t(B_{\mathcal{M}/2}) \subset B_{\mathcal{M}/2}$. □

4.4 Differentiability with Respect to the Domain

We want to analyse the shape sensitivity of these solutions, namely their behaviour with respect to small variations of t, which amounts to study the differentiability of $(\mathrm{v}_t, \mathrm{q}_t, \mathrm{w}_t, \mathrm{s}_t) \circ \Phi_t$. For this, we apply the classical method presented in [29, Sects. 5.3.3 and 5.3.4]. The main result of this section is the following.

Theorem 3 *Under the assumptions of Proposition 1, let* $(\mathrm{v}^t, \mathrm{q}^t, \mathrm{w}^t, \mathrm{s}^t)$ *be the unique solution to the FSI problem (72) for all* $t \in [0, t_{\mathcal{M}})$. *In addition, assume that g belongs to* $(H^2(\mathbb{R}^2))^2$. *Then the map:*

$$t \in [0, t_{\mathcal{M}}) \mapsto (\mathrm{v}^t, \mathrm{q}^t, \mathrm{w}^t, \mathrm{s}^t),$$

is differentiable in the vicinity of 0 *in:*

$$(H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2 \times L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c) \times (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^3(\Omega_0))^2 \times H^2(\Omega_0).$$

Proof The key argument is the implicit function theorem, that will be applied to an adequate operator characterising the problem, and which depends on both t and the state variables representing the solution.

Let us set:

$$\begin{aligned}
&\mathrm{H}_1 := (H_0^1(\Omega_0^c) \cap H^3(\Omega_0^c))^2, && \mathrm{H}_2 := L_0^2(\Omega_0^c) \cap H^2(\Omega_0^c),\\
&\mathrm{H}_3 := (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^3(\Omega_0))^2, && \mathrm{H}_4 := H^2(\Omega_0),\\
&\mathrm{K}_1 := (H^1(\Omega_0^c))^2, && \mathrm{K}_3 := (H^1(\Omega_0))^2,\\
&\mathrm{K}_4 := H^1(\Omega_0), && \mathrm{K}_5 := H^{3/2}(\Gamma_0),
\end{aligned}$$

and:

$$\mathrm{K}_2 := \left\{ h \in H^1(\Omega_0^c) \,\middle|\, \int_{\Omega_0^c} h = 0 \right\}.$$

From this, we define the following sets:

$$\mathbf{H} := \mathrm{H}_1 \times \mathrm{H}_2 \times \mathrm{H}_3 \times \mathrm{H}_4,$$
$$\mathbf{K} := \mathrm{K}_1 \times \mathrm{K}_2 \times \mathrm{K}_3 \times \mathrm{K}_4 \times \mathrm{K}_5.$$

Before defining the adequate operator we want to study, we can remark that the map T_t defined in (67) and involved in the FSI problem, depends on the parameter t through the map Φ_t given by (64) and through the field w^t. To make a distinction between these two dependencies, we introduce the following map defined from $\mathbb{R}_+ \times \mathrm{H}_3$ to $(H^3(\Omega_0^c))^2$ by:

$$T_w^t := \Phi_t + \mathcal{R}\gamma(w), \quad \forall t \geq 0, \ \forall w \in \mathrm{H}_3. \tag{74}$$

In this manner, the map T^t_w depends on functions w belonging to the fixed space H_3, and we have furthermore that:

$$T_t \circ \Phi_t = T^t_{\mathrm{w}^t}.$$

Let us denote by $\mathcal{X}^t$ the vector of $\mathbf{H}$ solution of the FSI problem defined for all $t \geq 0$ by:

$$\mathcal{X}^t := (\mathrm{v}^t, \mathrm{q}^t, \mathrm{w}^t, \mathrm{s}^t),$$

while:

$$\mathcal{X} = (v, q, w, s)$$

stands for an arbitrary vector of $\mathbf{H}$. The FSI coupling problem (72) leads us to define the following operator. Let:

$$\boldsymbol{F} : \mathbb{R} \times \mathbf{H} \to \mathbf{K}$$

be the map defined by:

$$\begin{aligned}
\boldsymbol{F}_1(t, \mathcal{X}) &:= -\nu \mathrm{div}((\nabla v) F(T^t_w)) + G(T^t_w)\nabla q - (f \circ T^t_w) J(T^t_w),\\
\boldsymbol{F}_2(t, \mathcal{X}) &:= \mathrm{div}(G(T^t_w)^\top v),\\
\boldsymbol{F}_3(t, \mathcal{X}) &:= -\mu \mathrm{div}((\nabla w) F(\Phi_t)) + G(\Phi_t)\nabla s - J(\Phi_t)(g \circ \Phi_t),\\
\boldsymbol{F}_4(t, \mathcal{X}) &:= \mathrm{div}(G(\Phi_t)^\top w),\\
\boldsymbol{F}_5(t, \mathcal{X}) &:= \left[\mu(\nabla w) F(\Phi_t) - s G(\Phi_t) - \nu(\nabla v) F(T^t_w) + q G(T^t_w)\right] n_0,
\end{aligned} \tag{75}$$

where we recall that $F(T^t_w)$, $G(T^t_w)$, and $J(T^t_w)$ are given by the expressions in (70). As we said, for $t = 0$, the vector $\mathcal{X}^0 = (\mathrm{v}^0, \mathrm{q}^0, \mathrm{w}^0, \mathrm{s}^0)$ is the solution of the coupling FSI problem (72) posed on Ω_0 and Ω_0^c. Thus, by definition (75) of $\boldsymbol{F}$, we have $\boldsymbol{F}(0, \mathcal{X}^0) = 0$. From there, we want to apply the implicit functions theorem to $\boldsymbol{F}$, by showing that:

1. $\boldsymbol{F}$ is of class C^1 in a neighbourhood of $(0, \mathcal{X}^0)$,
2. $D_{\mathcal{X}}\boldsymbol{F}(0, \mathcal{X}^0)$ is a bi-continuous isomorphism.

In this case, by uniqueness of the FSI problem, we will have as a result that the map $t \mapsto \mathcal{X}^t$ is of class C^1 in a neighbourhood of $(0, \mathcal{X}^0)$. □

4.4.1 Step (1).

We first show that the map $\boldsymbol{F}$ is of class C^1 in a neighbourhood of $(0, \mathcal{X}^0)$. Obviously, $\boldsymbol{F} = \boldsymbol{F}(t, v, q, w, s)$ is of class C^1 with respect to v, q and s since it is linear in these variables. So we only have to check that $\boldsymbol{F}$ is also of class C^1 in t and w. We have that the map $(t, w) \in \mathbb{R}_+ \times H^3(\Omega_0) \mapsto \nabla(\Phi_t + \mathcal{R}\gamma(w)) \in H^2(\Omega_0^c)$ is of class

C^∞. Indeed, $w \mapsto \mathcal{R}\gamma(w)$ is linear and continuous and $t \mapsto \Phi_t$ is affine since $\Phi_t := \mathrm{id}_{\mathbb{R}^2} + tV$ with $V \in (H^3(\mathbb{R}^2))^2$. We can also show that $A \in (H^2(\Omega_0^c))^{2\times 2} \mapsto A^{-1} \in (H^2(\Omega_0^c))^{2\times 2}$ is of class C^∞ in a neighbourhood of the identity matrix I. Thus, the maps $t \mapsto J(\Phi_t) \in H^2(\Omega_0^c)$ and $t \mapsto (\nabla\Phi_t)^{-1} \in (H^2(\Omega_0^c))^{2\times 2}$ are C^∞. Moreover, from Lemma 3, we have that the three maps $(t, w) \in \mathbb{R}_+ \times H^3(\Omega_0) \mapsto F(T_w^t)$, $G(T_w^t) \in (H^2(\Omega_0^c))^{2\times 2}$, and $J(T_w^t) \in H^2(\Omega_0^c)$ are of class C^∞. Finally, because of the regularity of $f \in (H^2(\mathbb{R}^2))^2$ and $g \in (H^2(\mathbb{R}^2))^2$, we have from [29, Lemma 5.3.9] that $(t, w) \mapsto (f \circ T_w^t)J(T_w^t)$ and $(t, w) \mapsto J(\Phi_t)(g \circ \Phi_t)$ are C^1 in the vicinity of 0.

4.4.2 Step (2).

For a $\mathcal{X} = (v, q, w, s)$ in $\mathbf{H}$, we calculate the following element of $\mathbf{K}$:

$$D_{\mathcal{X}}\boldsymbol{F}(0, \mathcal{X}^0)\mathcal{X} = \begin{pmatrix} D_{\mathcal{X}}\boldsymbol{F}_1(0, \mathcal{X}^0)\mathcal{X} \\ D_{\mathcal{X}}\boldsymbol{F}_2(0, \mathcal{X}^0)\mathcal{X} \\ D_{\mathcal{X}}\boldsymbol{F}_3(0, \mathcal{X}^0)\mathcal{X} \\ D_{\mathcal{X}}\boldsymbol{F}_4(0, \mathcal{X}^0)\mathcal{X} \\ D_{\mathcal{X}}\boldsymbol{F}_5(0, \mathcal{X}^0)\mathcal{X} \end{pmatrix}^\top,$$

whose components are given by:

$$\begin{aligned}
D_{\mathcal{X}}\boldsymbol{F}_1(0, \mathcal{X}^0)\mathcal{X} &= -\nu\mathrm{div}((\nabla v)F(T^0)) - \nu\mathrm{div}((\nabla \mathrm{v}^0)D_w F(T^0)w) \\
&\quad + G(T^0)\nabla q + (D_w G(T^0)w)\nabla \mathrm{q}^0 - D_w(J(T^0)f\circ T^0)w, \\
D_{\mathcal{X}}\boldsymbol{F}_2(0, \mathcal{X}^0)\mathcal{X} &= \mathrm{div}(G(T^0)^\top v) + \mathrm{div}((D_w G(T^0)w)\mathrm{v}^0), \\
D_{\mathcal{X}}\boldsymbol{F}_3(0, \mathcal{X}^0)\mathcal{X} &= -\mu\mathrm{div}(\nabla w) + \nabla s, \\
D_{\mathcal{X}}\boldsymbol{F}_4(0, \mathcal{X}^0)\mathcal{X} &= \mathrm{div}(w), \\
D_{\mathcal{X}}\boldsymbol{F}_5(0, \mathcal{X}^0)\mathcal{X} &= [\mu\nabla w - s\mathrm{I} - \nu(\nabla v)F(T^0) - \nu(\nabla \mathrm{v}^0)(D_w F(T^0)w)]n_0 \\
&\quad - [qG(T^0) + \mathrm{q}^0(D_w G(T^0)w)]n_0.
\end{aligned}$$

Here, $T^0 := \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}\gamma(\mathrm{w}^0)$, whereas the expressions of $(D_w J(T^0)w)$, $(D_w G(T^0)w)$, and $(D_w F(T^0)w)$ are given in the Appendix (cf. (105)–(107)). Moreover, we set:

$$D_w(J(T^0)f\circ T^0)w := (D_w J(T^0)w)(f \circ T^0) + J(T^0)(\nabla f \circ T^0)\nabla T^0.$$

Given $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_4, \mathscr{F}_5) \in \mathbf{K}$, we want to show that there exists a unique $\mathcal{X} = (v, q, w, s) \in \mathbf{H}$ such that:

$$D_{\mathcal{X}}\boldsymbol{F}(0, \mathcal{X}^0)\mathcal{X} = \mathscr{F}. \tag{76}$$

This amounts to solving the following problem: find $(v, q, w, s) \in \mathbf{H}$ such that:

$$\begin{aligned} -\nu \mathrm{div}((\nabla v)F(T^0)) + G(T^0)\nabla q &= \mathbf{f}_1(w) + \mathscr{F}_1 && \text{in } \Omega_0^c, \\ \mathrm{div}(G(T^0)^\top v) &= \mathbf{f}_2(w) + \mathscr{F}_2 && \text{in } \Omega_0^c, \\ v &= 0 && \text{on } \partial\Omega_0^c, \\ -\mu \mathrm{div}(\nabla w) + \nabla s &= \mathbf{f}_3(w) + \mathscr{F}_3 && \text{in } \Omega_0, \\ \mathrm{div} w &= \mathbf{f}_4(w) + \mathscr{F}_4 && \text{in } \Omega_0, \\ w &= 0 && \text{on } \Gamma_\omega, \\ \Big(\mu\nabla w - s\mathrm{I} - \nu(\nabla v)F(T^0) + qG(T^0)\Big) n_0 &= \mathbf{f}_5(w) + \mathscr{F}_5 && \text{on } \Gamma_0, \end{aligned} \tag{77}$$

where the maps $\mathbf{f}_j$ for $j = 1, \cdots, 5$, are, respectively, linear forms from H_3 to K_j, given by $\mathbf{f}_3 \equiv \mathbf{f}_4 \equiv 0$, and:

$$\mathbf{f}_1(w) := \nu \mathrm{div}((\nabla \mathrm{v}^0)D_w F(T^0)w) - (D_w G(T^0)w)\nabla \mathrm{q}^0 + D_w(J(T^0) f \circ T^0)w, \tag{78}$$

$$\mathbf{f}_2(w) := -\mathrm{div}((D_w G(T^0)w)\mathrm{v}^0), \tag{79}$$

$$\mathbf{f}_5(w) := [\nu(\nabla \mathrm{v}^0)(D_w F(T^0)w) - \mathrm{q}^0(D_w G(T^0)w)]n_0. \tag{80}$$

Let $\mathbf{b} \in H^3(\Omega_0)$ be an arbitrary field. In order to prove that Problem (77) admits a unique solution, we introduce the following parametrised problem, with parameter $\mathbf{b}$:

$$\begin{aligned} -\nu \mathrm{div}((\nabla v(\mathbf{b}))F(T^0)) + G(T^0)\nabla q(\mathbf{b}) &= \mathbf{f}_1(\mathbf{b}) + \mathscr{F}_1 && \text{in } \Omega_0^c, \\ \mathrm{div}(G(T^0)^\top v(\mathbf{b})) &= \mathbf{f}_2(\mathbf{b}) + \mathscr{F}_2 && \text{in } \Omega_0^c, \\ v(\mathbf{b}) &= 0 && \text{on } \partial\Omega_0^c, \\ -\mu \mathrm{div}(\nabla w(\mathbf{b})) + \nabla s(\mathbf{b}) &= \mathscr{F}_3 && \text{in } \Omega_0, \\ \mathrm{div} w(\mathbf{b}) &= \mathscr{F}_4 && \text{in } \Omega_0, \\ w(\mathbf{b}) &= 0 && \text{on } \Gamma_\omega, \\ \big(\mu\nabla w(\mathbf{b}) - s(\mathbf{b})\mathrm{I} - \nu(\nabla v(\mathbf{b}))F(T^0) + q(\mathbf{b})G(T^0)\big) n_0 &= \mathbf{f}_5(\mathbf{b}) + \mathscr{F}_5 && \text{on } \Gamma_0. \end{aligned} \tag{81}$$

In the same way as done in Sect. 3.3, we can prove that for any $\mathbf{b} \in (H^3(\Omega_0))^2$, there exists a unique solution $(v(\mathbf{b}), q(\mathbf{b}), w(\mathbf{b}), s(\mathbf{b})) \in \mathbf{H}$ to this problem, allowing us to define the map $b \mapsto \boldsymbol{S}(\mathbf{b}) := w(\mathbf{b})$. Indeed, the fields $\mathbf{f}_1(w)$, $\mathbf{f}_2(w)$, and $\mathbf{f}_5(w)$ have the required regularity to apply Theorem 2, and $\mathbf{f}_2(w)$ together with $\mathscr{F}_2$ satisfy the compatibility condition (33). Moreover, since w^0 is the displacement solution of the coupling FSI problem (72) for $t = 0$, we have that $T^0 := \mathrm{id}_{\mathbb{R}^2} + \mathcal{R}\gamma(\mathrm{w}^0)$ is such that $F(T^0)$ and $G(T^0)$ satisfy the assumption (34) of Theorem 2. Thus, applying Theorem 2, we obtain a unique solution $(v(\mathbf{b}), q(\mathbf{b}), w(\mathbf{b}), s(\mathbf{b})) \in \mathbf{H}$ to Problem (81).

Now we want to show that this map is a contraction for data f and g small enough. Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be in $(H^3(\Omega_0))^2$. We set $\boldsymbol{\delta v} := v(\mathbf{b}_1) - v(\mathbf{b}_2)$, $\boldsymbol{\delta q} := q(\mathbf{b}_1) - q(\mathbf{b}_2)$,

$\boldsymbol{\delta} w := w(\mathbf{b}_1) - w(\mathbf{b}_2)$, and $\boldsymbol{\delta} s := s(\mathbf{b}_1) - s(\mathbf{b}_2)$. By linearity of Problem (81), and applying Theorem 2 for $(\boldsymbol{\delta} v, \boldsymbol{\delta} q)$ and $(\boldsymbol{\delta} w, \boldsymbol{\delta} s)$, we have:

$$\|\boldsymbol{\delta} v\|_{3,2,\Omega_0^c} + \|\boldsymbol{\delta} q\|_{2,2,\Omega_0^c} \leq C_{\mathrm{fs}}(\|\mathbf{f}_1(\mathbf{b}_1 - \mathbf{b}_2)\|_{1,2,\Omega_0^c} + \|\mathbf{f}_2(\mathbf{b}_1 - \mathbf{b}_2)\|_{2,2,\Omega_0^c}),$$

and

$$\|\boldsymbol{\delta} w\|_{3,2} + \|\boldsymbol{\delta} s\|_{2,2} \leq C_{\mathrm{fs}}(\|\mathbf{f}_5(\mathbf{b}_1 - \mathbf{b}_2)\|_{H^{3/2}(\Gamma_0)} + C(\|\boldsymbol{\delta} v\|_{3,2,\Omega_0^c} + \|\boldsymbol{\delta} q\|_{2,2,\Omega_0^c})),$$

where C_{fs} depends only on Ω_0 and C_1, C_2 in (26), (27). We can see in expressions (78), (79), and (80), by using Lemma 2 and the same kind of estimates written in Sect. 3.3, that the norms of the linear maps $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_5$ are bounded by the norms of v^0, q^0, and the volume force f. Yet, from Theorem 1, we have that:

$$\|\mathrm{v}^0\|_{3,2,\Omega_0^c} + \|\mathrm{q}^0\|_{2,2,\Omega_0^c} + \|\mathrm{w}^0\|_{3,2,\Omega_0} + \|\mathrm{s}^0\|_{2,2,\Omega_0} \leq C_{FS}(\|f\|_{2,2,\mathbb{R}^2} + \|g\|_{1,2,\Omega_0}).$$

Then, we can choose the data f and g of our problem small enough so that $\boldsymbol{S}$ is a contraction on $(H^3(\Omega_0))^2$. Therefore, $\boldsymbol{S}$ admits a unique fixed point showing that Problem (76) has a unique solution $\mathcal{X} = (v, q, w, s) \in \mathbf{H}$.

Finally, from Problem (77) we have the following estimates:

$$\begin{aligned} &\|v\|_{3,2,\Omega_0^c} + \|q\|_{2,2,\Omega_0^c} + \\ &\quad \leq C_{\mathrm{fs}}\Big[\sum_{i=1}^{2} \|\mathscr{F}_i\|_{K_i} + (\|\mathbf{f}_1\|_{\mathcal{L}(\mathrm{H}_3,\mathrm{K}_1)} + \|\mathbf{f}_2\|_{\mathcal{L}(\mathrm{H}_3,\mathrm{K}_2)})\|w\|_{3,2,\Omega_0}\Big], \end{aligned} \tag{82}$$

and

$$\begin{aligned} \|w\|_{3,2,\Omega_0} + \|s\|_{2,2,\Omega_0} \leq C_{\mathrm{fs}}\Big[\sum_{i=3}^{5} \|\mathscr{F}_i\|_{K_i} + \|\mathbf{f}_5\|_{\mathcal{L}(\mathrm{H}_3,\mathrm{K}_5)}\|w\|_{3,2,\Omega_0} \\ + C(\|v\|_{3,2,\Omega_0^c} + \|q\|_{2,2,\Omega_0^c})\Big]. \end{aligned} \tag{83}$$

Once again, $\|\mathbf{f}_1\|_{\mathcal{L}(\mathrm{H}_3,\mathrm{K}_1)}$, $\|\mathbf{f}_2\|_{\mathcal{L}(\mathrm{H}_3,\mathrm{K}_2)}$, and $\|\mathbf{f}_5\|_{\mathcal{L}(\mathrm{H}_3,\mathrm{K}_5)}$ can be chosen small enough so that combining (82) and (83), we obtain that the solution $\mathcal{X} = (v, q, ws) \in \mathbf{H}$ of the linear elliptic system (76) (see also (77)), satisfies the following estimate:

$$\|v\|_{3,2,\Omega_0^c} + \|q\|_{2,2,\Omega_0^c} + \|w\|_{3,2,\Omega_0} + \|s\|_{2,2,\Omega_0} \leq C \sum_{i=1}^{5} \|\mathscr{F}_i\|_{K_i}$$

where C is a positive constant depending on the norms of $(\mathrm{v}^0, \mathrm{q}^0, \mathrm{w}^0, \mathrm{s}^0)$, f and g. Then, $D_{\mathcal{X}}\boldsymbol{F}(0, \mathcal{X}^0)$ is a bi-continuous isomorphism.

5 Shape Derivative of $\mathcal{J}(\boldsymbol{\Omega})$

5.1 Direct Calculus

In this paragraph, we compute the shape derivative of functionals depending on the FSI problem.

We consider a functional of the form:

$$\mathcal{J}(\Omega_0) = \mathcal{J}_S(\Omega_0) + \mathcal{J}_F(\Omega_0) = \int_{\Omega_0} j_S(Y, \mathrm{w}(Y), \nabla \mathrm{w}(Y))\, \mathrm{d}Y + \int_{\Omega_F} j_F(x, \mathrm{u}(x), \nabla \mathrm{u}(x))\, \mathrm{d}x, \tag{84}$$

where j_S and j_F are differentiable functions. As we have done in the previous section, we consider a 1-parameter family of shapes $\Omega_{0,t}$ defined in (66).

Computing the shape derivative of $\mathcal{J}$ with respect to the deformation chosen amounts to evaluate the derivative of $t \mapsto \mathcal{J}(\Omega_{0,t})$ at $t = 0$. The shape functional evaluated on the domain $\Omega_{0,t}$ is given by:

$$\mathcal{J}(\Omega_{0,t}) = \mathcal{J}_S(\Omega_{0,t}) + \mathcal{J}_F(\Omega_{0,t}) = \int_{\Omega_{0,t}} j_S(Y, \mathrm{w}_t(Y), \nabla \mathrm{w}_t(Y))\, \mathrm{d}Y + \int_{\Omega_{F,t}} j_F(x, \mathrm{u}_t(x), \nabla \mathrm{u}_t(x))\, \mathrm{d}x.$$

where $(\mathrm{w}_t, \mathrm{u}_t)$ are the solution fields of the FSI problem (69).

Let us first compute the derivative of $\mathcal{J}_S(\Omega_{0,t})$. After transporting the integral from $\Omega_{0,t}$ to Ω_0, we obtain:

$$\mathcal{J}_S(\Omega_{0,t}) = \int_{\Omega_0} j_S\left(\Phi_t(Y), \mathrm{w}_t \circ \Phi_t(Y), (\nabla \mathrm{w}_t) \circ \Phi_t(Y)\right) \det(\nabla \Phi_t)\, \mathrm{d}Y.$$

Thus, the shape derivative of $\mathcal{J}_S$ is given by:

$$\begin{aligned}\mathcal{J}_S'(\Omega_0) = &\int_{\Omega_0} j_S(Y, \mathrm{w}(Y), \nabla \mathrm{w}(Y)) \mathrm{div} V\, \mathrm{d}Y \\ &+ \int_{\Omega_0} D_1 j_S(Y, \mathrm{w}(Y), \nabla \mathrm{w}(Y)) V\, \mathrm{d}Y \\ &+ \int_{\Omega_0} D_2 j_S(Y, \mathrm{w}(Y), \nabla \mathrm{w}(Y)) \dot{\mathrm{w}}\, \mathrm{d}Y \\ &+ \int_{\Omega_0} D_3 j_S(Y, \mathrm{w}(Y), \nabla \mathrm{w}(Y))(\nabla \dot{\mathrm{w}} - \nabla \mathrm{w} \nabla V)\, \mathrm{d}Y,\end{aligned} \tag{85}$$

where $\dot{\mathrm{w}}$ is the material derivative of w_t at $t = 0$, defined by:

$$\dot{\mathrm{w}} := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathrm{w}^t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathrm{w}_t \circ \Phi_t),$$

and D_1, D_2, D_3 stand for the differential on each argument of j_S. In (85), we have used the relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \det(\nabla\Phi_t) = \mathrm{div} V, \tag{86}$$

with the definition (64) of Φ_t (see (101) in the Appendix). The term $(\nabla\dot{\mathrm{w}} - \nabla\mathrm{w}\nabla V)$ comes from the differentiation of $(\nabla\mathrm{w}_t) \circ \Phi_t(Y)$.

Secondly, we consider the shape derivative of $\mathcal{J}_F$ with respect to t. We perform a change of variable $x = T_t \circ \Phi_t(X)$, in order to rewrite the integrals from $\Omega_{F,t}$ to Ω_0^c. This gives

$$\mathcal{J}_F(\Omega_{0,t}) = \int_{\Omega_0^c} \Big(j_F(T_t \circ \Phi_t(X), \mathrm{u}_t \circ T_t \circ \Phi_t(X), (\nabla\mathrm{u}_t) \circ T_t \circ \Phi_t(X)) \\ \det(\nabla(T_t \circ \Phi_t(X)))\Big)\, \mathrm{d}X. \tag{87}$$

We compute the shape derivative of $\mathcal{J}_F$, setting: $\mathrm{v} = \mathrm{u} \circ T$, where $T = T_0 = \mathrm{id}_{\mathbb{R}^2} + R\gamma(\mathrm{w})$. This gives:

$$\begin{aligned}
\mathcal{J}_F'(\Omega_0) = &\int_{\Omega_0^c} j_F(T, \mathrm{v}, \nabla\mathrm{v}(\nabla T)^{-1})\mathrm{tr}(\mathrm{cof}(\nabla T)^\top \nabla\dot{T})\, \mathrm{d}X \\
&+ \int_{\Omega_0^c} D_1 j_F(T, \mathrm{v}, \nabla\mathrm{v}(\nabla T)^{-1})\dot{T} \det(\nabla T)\, \mathrm{d}X \\
&+ \int_{\Omega_0^c} D_2 j_F(T, \mathrm{v}, \nabla\mathrm{v}(\nabla T)^{-1})\dot{\mathrm{v}} \det(\nabla T)\, \mathrm{d}X \\
&+ \int_{\Omega_0^c} D_3 j_F(T, \mathrm{v}, \nabla\mathrm{v}(\nabla T)^{-1}) \left(\nabla\dot{\mathrm{v}} - \nabla\mathrm{v}(\nabla T)^{-1}\nabla\dot{T}\right) \mathrm{cof}(\nabla T)^\top\, \mathrm{d}X,
\end{aligned} \tag{88}$$

where we denote by $\dot{\mathrm{v}}$ and $\dot{T}$ the material derivatives of v and T_t, respectively, defined by:

$$\dot{\mathrm{v}} := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathrm{v}^t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\mathrm{v}_t \circ \Phi_t), \quad \text{and} \quad \dot{T} := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(T_t \circ \Phi_t)\,.$$

From the definitions of T_t in (67) and of $\dot{T}$, we have:

$$\dot{T} = V + \mathcal{R}\gamma(\dot{\mathrm{w}}).$$

The term $\mathrm{tr}(\mathrm{cof}(\nabla T)^\top \nabla \dot{T})$ in (88) comes from the differentiation of $\det(\nabla(T_t \circ \Phi_t(X)))$ in (87). The terms $\dot{T}$ and $\dot{\mathrm{v}}$ in (88) are, respectively, the results of the differentiation through the chain rule of the terms $T_t \circ \Phi_t(X)$ and $\mathrm{u}_t \circ T_t \circ \Phi_t(X)$ in (87). For the last term $(\nabla\dot{\mathrm{v}} - \nabla \mathrm{v}(\nabla T)^{-1}\nabla\dot{T})\mathrm{cof}(\nabla T)^\top$ in (88) deriving from $(\nabla \mathrm{u}_t) \circ T_t \circ \Phi_t(X)$ in (87), we can write:

$$
\begin{aligned}
(\nabla \mathrm{u}_t) \circ T_t \circ \Phi_t(X) &= (\nabla(\mathrm{u}_t \circ T_t \circ \Phi_t))(X)(\nabla(T_t \circ \Phi_t))^{-1}(X),\\
&= (\nabla(\mathrm{v}_t \circ \Phi_t))(X)(\nabla(T_t \circ \Phi_t))^{-1}(X),
\end{aligned}
$$

with $\mathrm{v}_t = \mathrm{u}_t \circ T_t$ (see (68)). From there, we can write in the following proposition the formula of the shape derivative $\mathcal{J}'(\Omega_0)$.

Proposition 2 *Let $\mathcal{J}$ be the shape functional defined by (84), where j_S and j_F are differentiable functions. Let V be a velocity field belonging to the space Θ introduced in (65). Then, the shape derivative of $\mathcal{J}$ in the direction V computed at Ω_0 is given by:*

$$
\begin{aligned}
\mathcal{J}'(\Omega_0) &= \int_{\Omega_0} j_S(Y, \mathrm{w}, \nabla \mathrm{w})\mathrm{div}V \,\mathrm{d}Y + \int_{\Omega_0} D_1 j_S(Y, \mathrm{w}, \nabla \mathrm{w}) V \,\mathrm{d}Y\\
&+ \int_{\Omega_0} D_2 j_S(Y, \mathrm{w}, \nabla \mathrm{w})\dot{\mathrm{w}} \,\mathrm{d}Y + \int_{\Omega_0} D_3 j_S(Y, \mathrm{w}, \nabla \mathrm{w})(\nabla\dot{\mathrm{w}} - \nabla \mathrm{w}\nabla V) \,\mathrm{d}Y\\
&+ \int_{\Omega_0^c} j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1})\mathrm{tr}(\mathrm{cof}(\nabla T)^\top \nabla \dot{T}) \,\mathrm{d}X\\
&+ \int_{\Omega_0^c} D_1 j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1})\dot{T} \det(\nabla T) \,\mathrm{d}X\\
&+ \int_{\Omega_0^c} D_2 j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1})\dot{\mathrm{v}} \det(\nabla T) \,\mathrm{d}X\\
&+ \int_{\Omega_0^c} D_3 j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1}) \left(\nabla\dot{\mathrm{v}} - \nabla \mathrm{v}(\nabla T)^{-1}\nabla\dot{T}\right) \mathrm{cof}(\nabla T)^\top \,\mathrm{d}X.
\end{aligned}
\tag{89}
$$

Notice that the expression (89) of $\mathcal{J}'$ depends on the material derivatives $\dot{\mathrm{v}}$ and $\dot{\mathrm{w}}$ of the velocity and of the displacement. These material derivatives can be computed as solutions of boundary value problems which depend on the direction V (see [12, Sect. 3.4.4]). For a practical use of the shape derivative—within a shape optimisation algorithm for example—it is suitable to find an expression which does not depend on $\dot{\mathrm{v}}$ and $\dot{\mathrm{w}}$. For this, we apply in the next section the classical *adjoint method* allowing for a simplified expression of $\mathcal{J}'$.

5.2 Adjoint Method

The *adjoint method* allows to guess straightforwardly the *adjoint states* we need to introduce in order to simplify the expression of the shape derivative formula (89). In this section, we use a *mixed variational formulation method* as presented in [21, Sect. 3.4.4].

5.2.1 Shape Functional and Its Related Lagrangian

We consider the shape functional defined by (84) written on a perturbed domain $\Omega_{0,t}$. According to (87), we can rewrite it as follows:

$$\mathcal{J}(\Omega_{0,t}) = \int_{\Omega_{0,t}^c} j_F(T_t, \mathrm{v}_t, \nabla \mathrm{v}_t(\nabla T_t)^{-1}) \det(\nabla T_t)\, \mathrm{d}X_t + \int_{\Omega_{0,t}} j_S(Y_t, \mathrm{w}_t, \nabla \mathrm{w}_t)\, \mathrm{d}Y_t, \tag{90}$$

where T_t is defined in (67), whereas $\mathrm{v}_t \in (H_0^1(\Omega_{0,t}^c))^2$ and $\mathrm{w}_t \in (H^1_{0,\Gamma_\omega}(\Omega_{0,t}))^2$ are, respectively, the velocity and the displacement solutions of Problem (69). To find suitable adjoint states, we need to define a Lagrangian related to $\mathcal{J}$ having independent variables lying in the space $(H_0^1(\Omega_0^c))^2 \times L_0^2(\Omega_0^c) \times (H^1_{0,\Gamma_\omega}(\Omega_0) \times H^3(\Omega_0))^2 \times L^2(\Omega_0)$ independent of t. To this aim, we first transport $\mathcal{J}(\Omega_{0,t})$ on the reference configuration, as it is done in the previous section, by means of the change of variable $x = \Phi_t(X)$. Then, we exploit the variational formulation of Problem (72), taking the same test function $\mathfrak{y}$ for both the equilibrium equation of the fluid (72)(i) and the equilibrium equation of the structure (72)(iv). This suggests the following definition of Lagrangian, for $t \geq 0$, for all (v, q, w, s) in $(H_0^1(\Omega_0^c))^2 \times L_0^2(\Omega_0^c) \times (H^1_{0,\Gamma_\omega}(\Omega_0) \times H^3(\Omega_0))^2 \times L^2(\Omega_0)$, and for all $(\mathfrak{y}, \mathfrak{q}, \mathfrak{s})$ in $H_0^1(D\backslash\overline{\omega}) \times L_0^2(\Omega_0^c) \times L^2(\Omega_0)$:

$$\begin{aligned}
\mathcal{L}(t, (v, q, w, s), (\mathfrak{y}, \mathfrak{q}, \mathfrak{s})) &:= \mathcal{J}^t(\Omega_0, v, w) \\
&+ \int_{\Omega_0^c} \Big(\nu(\nabla v)F(T_w^t) : \nabla\mathfrak{y} - q(G(T_w^t) : \nabla\mathfrak{y}) - (f \circ T_w^t \cdot \mathfrak{y})J(T_w^t)\Big) \\
&+ \int_{\Omega_0} \Big(\mu(\nabla w)F(\Phi_t) : (\nabla\mathfrak{y}) - sG(\Phi_t) : \nabla\mathfrak{y} - ((g \circ \Phi_t)\cdot\mathfrak{y})J(\Phi_t)\Big) \\
&- \int_{\Omega_0^c} \mathfrak{q}(G(T_w^t) : \nabla v) - \int_{\Omega_0} \mathfrak{s}G(\Phi_t) : \nabla w,
\end{aligned}$$

recalling that the transformation T_w^t defined in (74) is given by $T_w^t = \Phi_t + \mathcal{R}\gamma(w)$, and setting:

$$\mathcal{J}^t(\Omega_0, v, w) := \int_{\Omega_0^c} j_F(T_w^t, v, \nabla v \nabla (T_w^t)^{-1})J(T_w^t) + \int_{\Omega_0} j_S(\Phi_t, w, \nabla w \nabla \Phi_t^{-1})J(\Phi_t).$$

Recalling that $(\mathrm{v}^t, \mathrm{q}^t, \mathrm{w}^t, \mathrm{s}^t)$ defined in (71) are the transported solutions of the coupling Problem (72), and that $T^t_{\mathrm{w}^t} = T_t \circ \Phi_t$ (see (67)), we have that the following equality holds:

$$\mathcal{J}^t(\Omega_0, \mathrm{v}^t, \mathrm{w}^t) = \mathcal{J}(\Omega_{0,t}),$$

where $\mathcal{J}(\Omega_{0,t})$ is given by (90). Then, in view of Problem (72) multiplied by the corresponding test functions $\mathfrak{y}$, $\mathfrak{q}$, $\mathfrak{s}$, and after integration by parts, we obtain that for all $(\mathfrak{y}, \mathfrak{q}, \mathfrak{s})$ in $H_0^1(D\backslash\overline{\omega}) \times L_0^2(\Omega_0^c) \times L^2(\Omega_0)$:

$$\mathcal{L}(t, (\mathrm{v}^t, \mathrm{q}^t, \mathrm{w}^t, \mathrm{s}^t), (\mathfrak{y}, \mathfrak{q}, \mathfrak{s})) = \mathcal{J}(\Omega_{0,t}). \tag{91}$$

5.2.2 Derivatives of the Lagrangian

In order to obtain the adjoint problems, we need to differentiate the Lagrangian $\mathcal{L}$ with respect to the variables v, q, w, and s. The derivatives of $\mathcal{L}$ are evaluated at $t \geq 0$, $(v, q, w, s) \in (H_0^1(\Omega_0^c))^2 \times L_0^2(\Omega_0^c) \times (H^1_{0,\Gamma_\omega}(\Omega_0) \times H^3(\Omega_0))^2 \times L^2(\Omega_0)$, and $(\mathfrak{y}, \mathfrak{q}, \mathfrak{s}) \in H_0^1(D\backslash\overline{\omega}) \times L_0^2(\Omega_0^c) \times L^2(\Omega_0)$. For the sake of readability, we group the variables of the Lagrangian as follows, by setting:

$$X := (v, q, w, s) \quad \text{and} \quad \mathfrak{Y} := (\mathfrak{y}, \mathfrak{q}, \mathfrak{s}).$$

We first differentiate the Lagrangian with respect to the variables q and s. For $d \in L_0^2(\Omega_0^c)$ and $e \in L^2(\Omega_0)$, we have:

$$\left\langle \frac{\partial \mathcal{L}}{\partial q}(t, X, \mathfrak{Y}), d \right\rangle = -\int_{\Omega_0^c} d(G(T_w^t) : \nabla \mathfrak{y}), \quad \left\langle \frac{\partial \mathcal{L}}{\partial s}(t, X, \mathfrak{Y}), e \right\rangle = -\int_{\Omega_0} eG(\Phi_t) : \nabla \mathfrak{y}. \tag{92}$$

When differentiating the Lagrangian with respect to the variables v and w, we shall simply write $D_\alpha j_F$ and $D_\alpha j_S$ instead of $D_\alpha j_F(T_w^t, v, \nabla v \nabla (T_w^t)^{-1})$ and $D_\alpha j_S(\Phi_t, w, \nabla w (\nabla \Phi_t)^{-1})$, respectively, for $\alpha = 1, 2, 3$. For $h \in (H_0^1(\Omega_0^c))^2$ and $k \in (H^1_{0,\Gamma_\omega}(\Omega_0) \times H^3(\Omega_0))^2$, we have:

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}}{\partial v}(t, X, \mathfrak{Y}), h \right\rangle = &\int_{\Omega_0^c} ((D_2 j_F) h + (D_3 j_F) \nabla h \nabla (T_w^t)^{-1}) J(T_w^t) \\ &+ \int_{\Omega_0^c} \Big(\nu (\nabla h) F(T_w^t) : \nabla \mathfrak{y} - \mathfrak{q} G(T_w^t) : \nabla h \Big),\end{aligned} \tag{93}$$

and:

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}}{\partial w}(t, X, \mathfrak{Y}), k \right\rangle = &\int_{\Omega_0} \Big((D_2 j_S) k J(\Phi_t) + (D_3 j_S) \nabla k \nabla \Phi_t^{-1} J(\Phi_t) \Big) \\ &+ \int_{\Omega_0^c} \Big((j_F) D_w J(T_w^t) k + \big[(D_1 j_F) D_w (T_w^t) k + (D_3 j_F) \nabla v D_w (\nabla (T_w^t)^{-1}) k \big] J(T_w^t) \Big) \\ &+ \int_{\Omega_0^c} \Big([\nu \nabla v D_w F(T_w^t) k - q D_w G(T_w^t) k] : \nabla \mathfrak{y} - (D_w G(T_w^t) k : \nabla v) \mathfrak{q} \Big) \\ &- \int_{\Omega_0^c} \Big((D_w (f \circ T_w^t) k \cdot \mathfrak{y}) J(T_w^t) + (f \circ T_w^t \cdot \mathfrak{y}) D_w J(T_w^t) k \Big)\end{aligned}$$

$$+\int_{\Omega_0} \mu(\nabla k) F(\Phi_t) : \nabla \mathfrak{y} - \int_{\Omega_0} \mathfrak{s} G(\Phi_t) : \nabla k, \tag{94}$$

where the derivatives $D_w(\cdot)$ with respect to the variable w are detailed in the Appendix (cf. (104)–(107)), whereas $D_w(f \circ T_w^t)k = (\nabla f) \circ T_w^t \cdot \mathcal{R}(\gamma(k))$.

5.2.3 Definition of the adjoint states

Let us write the adjoint equations. For this, the partial derivatives of the Lagrangian calculated in the previous section are evaluated at $t = 0$ and at $X^0 := (\mathrm{v}^0, \mathrm{q}^0, \mathrm{w}^0, \mathrm{s}^0)$, solution to Problem (72) written at $t = 0$ (which is in fact Problem (16)). Because of the terms written on Ω_0^c in (94) involving $\mathcal{R}(\gamma(k))$, it is not straightforward to write a strong formulation of the adjoint problem, then we need to use an abstract weak form result for which the test function k lies in H^1. In view of the regularity of X^0 given by Theorem 1, all the terms in (94) are well defined. Then, the adjoint problem associated to the shape functional $\mathcal{J}$ defined in (84) and to the FSI problem (16) is defined as follows:

$$\begin{aligned}
&\text{Find } \mathfrak{Y}^0 := (\mathfrak{y}^0, \mathfrak{q}^0, \mathfrak{s}^0) \in H_0^1(D \setminus \overline{\omega}) \times L_0^2(\Omega_0^c) \times L^2(\Omega_0) \text{ such that:}\\
&\left\langle \frac{\partial \mathcal{L}}{\partial v}(0, X^0, \mathfrak{Y}^0), h \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial q}(0, X^0, \mathfrak{Y}^0), d \right\rangle\\
&\qquad + \left\langle \frac{\partial \mathcal{L}}{\partial w}(0, X^0, \mathfrak{Y}^0), k \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial s}(0, X^0, \mathfrak{Y}^0), e \right\rangle = 0,\\
&\forall (h, d, k, e) \in (H_0^1(\Omega_0^c))^2 \times L_0^2(\Omega_0^c) \times (H_{0,\Gamma_\omega}^1(\Omega_0))^2 \times L^2(\Omega_0),
\end{aligned} \tag{95}$$

where the partial derivatives are given by expressions (92), (93), and (94).

Proposition 3 *Let $X^0 := (\mathrm{v}^0, \mathrm{q}^0, \mathrm{w}^0, \mathrm{s}^0)$ be the solution of Problem (72) with $f \in (H^2(\Omega_0^c))^2$ and $g \in (H^2(\Omega_0))^2$. There exists a constant $C > 0$ such that if $\|f\|_{2,2} \leq C$ and $\|g\|_{2,2} \leq C$, then the adjoint problem (95) admits a unique solution.*

Proof Given $\mathfrak{y} \in H_0^1(D \setminus \overline{\omega})$, we define the restrictions:

$$\mathfrak{v} := \mathfrak{y}_{|\Omega_0^c} \quad \text{and} \quad \mathfrak{w} := \mathfrak{y}_{|\Omega_0}.$$

From expressions (92), (93), and (94), we can rewrite Problem (95) as follows: find $(\mathfrak{y}, \mathfrak{q}, \mathfrak{s}) \in H_0^1(D \setminus \overline{\omega}) \times L_0^2(\Omega_0^c) \times L^2(\Omega_0)$ such that:

$$\begin{aligned}
a_F(\mathfrak{v}, h) + b_F(h, \mathfrak{q}) &= c_F(h), && \forall h \in V_F, && V_F := (H_0^1(\Omega_0^c))^2,\\
b_F(\mathfrak{v}, d) &= 0, && \forall d \in W_F, && W_F := L_0^2(\Omega_0^c),\\
\mathfrak{v} &= \mathfrak{w} && \text{on } \Gamma_0,\\
a_S(\mathfrak{w}, k) + b_S(k, \mathfrak{s}) &= c_S(k) + d_S(\mathfrak{v}, \mathfrak{q})(k), && \forall k \in V_S, && V_S := (H_{0,\Gamma\omega}^1(\Omega_0))^2,\\
b_S(\mathfrak{w}, e) &= 0, && \forall e \in W_S, && W_S. := L_0^2(\Omega_0),
\end{aligned} \tag{96}$$

where a_F, b_F, and c_F are bilinear and linear forms defined in (93), a_S, b_S, c_S are bilinear and linear forms defined in (94), and d_S is a continuous linear map from $(H^1(\Omega_0))^2 \times L^2_0(\Omega_0)$ to $[(H^1(\Omega_0))^2]'$ defined in (94). To solve Problem (96), we apply a fixed-point procedure by fixing $(\mathfrak{v}, \mathfrak{q})$ in the structure system, and homogenising the Dirichlet condition in the fluid system. Given $\mathbf{b} := (\mathfrak{v}_{\mathbf{b}}, \mathfrak{q}_{\mathbf{b}})$, we define the following problem:

$$\begin{aligned} a_F(\tilde{\mathfrak{v}}(\mathbf{b}), h) + b_F(h, \mathfrak{q}(\mathbf{b})) &= c_F(h) - a_F(\mathrm{E}\mathfrak{w}(\mathbf{b}), h), && \forall h \in V_F, \\ b_F(\tilde{\mathfrak{v}}(\mathbf{b}), d) &= 0, && \forall d \in W_F, \\ \tilde{\mathfrak{v}}(\mathbf{b}) &= 0 && \text{on } \partial\Omega_0^c, \\ a_S(\mathfrak{w}(\mathbf{b}), k) + b_S(k, \mathfrak{s}(\mathbf{b})) &= c_S(k) + d_S(\mathfrak{v}_{\mathbf{b}}, \mathfrak{q}_{\mathbf{b}})(k), && \forall k \in V_S, \\ b_S(\mathfrak{w}(\mathbf{b}), e) &= 0, && \forall e \in W_S, \end{aligned} \tag{97}$$

whose solution is $(\tilde{\mathfrak{v}}(\mathbf{b}), \mathfrak{q}(\mathbf{b}), \mathfrak{w}(\mathbf{b}), \mathfrak{s}(\mathbf{b}))$. Here, E denotes an extension operator from $H^1_{0,\Gamma_\omega}(\Omega_0)$ to $H^1_0(D\backslash\overline{\omega})$, which in particular fixes the trace zero on ∂D. We define:

$$\mathfrak{v}(\mathbf{b}) := \tilde{\mathfrak{v}}(\mathbf{b}) + \mathrm{E}\mathfrak{w}(\mathbf{b}). \tag{98}$$

From what is done in [26], we have that a_F, b_F, a_S, and a_S satisfy the conditions required to apply the abstract result from [11]. Namely, there exists $M > 0$ such that for $\alpha = F, S$, and for any $(f_\alpha, g_\alpha) \in V'_\alpha \times W'_\alpha$, there exists a unique $(v, q) \in V_\alpha \times W_\alpha$ satisfying:

$$\begin{aligned} a_\alpha(v, h) + b_\alpha(h, q) &= f_\alpha(h), \quad \forall h \in V_\alpha, \\ b_\alpha(v, d) &= g_\alpha(d), \quad \forall d \in W_\alpha, \end{aligned}$$

and:

$$\|v\|_{V_\alpha} + \|q\|_{W_\alpha} \leq M(\|f\|_{V'_\alpha} + \|g\|_{W'_\alpha}). \tag{99}$$

Thus, we first obtain existence and uniqueness of a solution $(\mathfrak{w}(\mathbf{b}), \mathfrak{s}(\mathbf{b})) \in (H^1_{0,\Gamma\omega}(\Omega_0))^2 \times L^2_0(\Omega_0)$ to Problem (97)(iv)–(v), then we obtain existence and uniqueness of a solution $(\tilde{\mathfrak{v}}(\mathbf{b}), \mathfrak{q}(\mathbf{b}))$ in the space $(H^1_0(\Omega_0^c))^2 \times L^2_0(\Omega_0^c)$ to Problem (97)(i)–(ii). Let map $\mathfrak{S}$ be the map from $(H^1_{0,\partial D}(\Omega_0^c))^2 \times L^2_0(\Omega_0^c)$ into itself defined by:

$$\mathfrak{S}(\mathbf{b}) := (\mathfrak{v}(\mathbf{b}), \mathfrak{q}(\mathbf{b})).$$

It remains us to show that $\mathfrak{S}$ is a contraction. Take $\mathbf{b}_1, \mathbf{b}_2 \in (H^1_{0,\partial D}(\Omega_0^c))^2 \times L^2_0(\Omega_0^c)$. By writing the difference of Problem (97) for $\mathbf{b}_1$ and $\mathbf{b}_2$, thanks to (98) and (99), we infer that:

$$\begin{aligned} &\|\mathfrak{v}(\mathbf{b}_1) - \mathfrak{v}(\mathbf{b}_2)\|_{1,2} + \|\mathfrak{q}(\mathbf{b}_1) - \mathfrak{q}(\mathbf{b}_2)\|_{0,2} \\ &\leq (1 + \|a_F\|M)cM\|d_S\|\Big(\|\mathfrak{v}_{\mathbf{b}_1} - \mathfrak{v}_{\mathbf{b}_2}\|_{1,2} + \|\mathfrak{q}_{\mathbf{b}_1} - \mathfrak{q}_{\mathbf{b}_2}\|_{0,2}\Big), \end{aligned}$$

where c is the norm of the extension operator E depending only on Ω_0 and D. From the definition of d_S in lines 3 and 4 of the right-hand side of (94), we infer that $\|d_S\|$ is controlled by the norms of v^0, q^0, and f, and, in the end, from Theorem 1, it is controlled by the norms of f and g. This concludes the proof. □

5.3 Simplified Formula for the Shape Derivative

We can simplify the formula of the shape derivative $\mathcal{J}'(\Omega_0)$ obtained in (89), Sect. 5.1. Indeed, in view of (91), we have that:

$$\mathcal{J}'(\Omega_0) = \frac{\partial \mathcal{L}}{\partial t}(0, X^0, \mathfrak{Y}) + \left\langle \frac{\partial \mathcal{L}}{\partial X}(0, X^0, \mathfrak{Y}), \dot{X^0} \right\rangle,$$

for any $\mathfrak{Y} \in H_0^1(D\backslash\overline{\omega}) \times L_0^2(\Omega_0^c) \times L^2(\Omega_0)$. Here $X^0 := (\mathrm{v}^0, \mathrm{q}^0, \mathrm{w}^0, \mathrm{s}^0)$ is the solution of the FSI Problem (16) and $\dot{X^0}$ denotes its material derivative. Thus, by definition of the adjoint state $\mathfrak{Y}^0$ solution of Problem (95), we obtain:

$$\mathcal{J}'(\Omega_0) = \frac{\partial \mathcal{L}}{\partial t}(0, X^0, \mathfrak{Y}^0).$$

Referring to the expressions (108)–(110) in the Appendix of the time derivatives $D_t(\cdot)$ of $J(T_w^t)$, $G(T_w^t)$, and $F(T_w^t)$, and using (86), we write the expression of the time derivative of the Lagrangian:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial t}(t, X, \mathfrak{Y}) \\
&\quad = \int_{\Omega_0^c} \Big((j_F) D_t J(T_w^t) + (D_1 j_F) D_t(T_w^t) J(T_w^t) + (D_3 j_F) \nabla v D_t(\nabla (T_w^t)^{-1}) J(T_w^t) \Big) \\
&\quad + \int_{\Omega_0} \Big((j_S) \mathrm{div} V + (D_1 j_S) V J(\Phi_t) + (D_3 j_S) \nabla w D_t \nabla \Phi_t^{-1} J(\Phi_t) \Big) \\
&\quad + \int_{\Omega_0^c} \Big([\nu(\nabla v) D_t(F(T_w^t)) - q D_t(G(T_w^t))] : \nabla \mathfrak{y} - \mathfrak{q} D_t(G(T_w^t)) : \nabla v \Big) \\
&\qquad\qquad - \int_{\Omega_0^c} \Big((f \circ T_w^t \cdot \mathfrak{y}) D_t J(T_w^t) + (D_t(f \circ T_w^t) \cdot \mathfrak{y}) J(T_w^t) \Big) \\
&\quad + \int_{\Omega_0} \Big([\mu(\nabla w) D_t F(\Phi_t) - \mathrm{s} D_t G(\Phi_t)] : \nabla \mathfrak{y} - \mathfrak{s} D_t G(\Phi_t) : \nabla w \Big),
\end{aligned}$$

where we recall that $T_w^t = \Phi_t + \mathcal{R}(\gamma(w))$ (see (74)). This formula can be simplified by noticing that $D_t T_w^t = V_t := D_t \Phi_t$, and $D_t(f \circ T_w^t) = (\nabla f) \circ T_w^t \cdot V_t$. From this expression evaluated at $(t, X, \mathfrak{Y}) = (0, X^0, \mathfrak{Y}^0)$, we have the following result.

Theorem 4 *Let $\mathcal{J}(\Omega_0)$ be the shape functional defined by (84). Let* $(\mathrm{v}, \mathrm{q}, \mathrm{w}, \mathrm{s})$ *be the solution of the FSI problem (16), and* $(\mathfrak{y}, \mathfrak{q}, \mathfrak{s})$ *be the adjoint state solution of the* *adjoint problem (95). Then the shape derivative of $\mathcal{J}(\Omega_0)$ can be written as follows:*

$$
\begin{aligned}
&\mathcal{J}'(\Omega_0)\\
&\quad = \int_{\Omega_0^c} j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1}) DJ(V) + D_1 j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1}) V J(T)\\
&\qquad + \int_{\Omega_0^c} D_3 j_F(T, \mathrm{v}, \nabla \mathrm{v}(\nabla T)^{-1}) \nabla \mathrm{v}(-\nabla T^{-1} \nabla V \nabla T^{-1}) J(T)\\
&\qquad + \int_{\Omega_0} \Big(j_S(Y, \mathrm{w}, \nabla \mathrm{w}) \mathrm{div} V + D_1 j_S(Y, \mathrm{w}, \nabla \mathrm{w}) V + D_3 j_S(Y, \mathrm{w}, \nabla \mathrm{w}) \nabla \mathrm{w}(-\nabla V) \Big)\\
&\qquad + \mathcal{A}'\big((u, q, \mathrm{w}, \mathrm{s}), (\mathfrak{y}, \mathfrak{q}, \mathfrak{y}, \mathfrak{s}), V \big),
\end{aligned}
$$

where $\mathcal{A}'$ is given by:

$$
\begin{aligned}
&\mathcal{A}'\big((u, q, \mathrm{w}, \mathrm{s}), (\mathfrak{y}, \mathfrak{q}, \mathfrak{y}, \mathfrak{s}), V \big)\\
&\quad := \int_{\Omega_0^c} \Big([\nu \nabla \mathrm{v} DF(V) - \mathfrak{q} DG(V)] : \nabla \mathfrak{y} - \mathfrak{q} DG(V) : \nabla \mathrm{v} \Big)\\
&\qquad - \int_{\Omega_0^c} \Big((f \circ T \cdot \mathfrak{y}) DJ(V) + (D_t(f \circ T) \cdot \mathfrak{y}) J(T) \Big)\\
&\qquad + \int_{\Omega_0} \Big([\mu(\nabla \mathrm{w}) DF(V) - \mathrm{s} DG(V)] : \nabla \mathfrak{y} - \mathfrak{s} DG(V) : \nabla \mathrm{w} \Big),
\end{aligned}
$$

and where $T := T_0$ is given by (67), V is the velocity of the transformation Φ_t given by (64), whereas $DJ(V)$, $DG(V)$, and $DF(V)$ denote, respectively, the time derivatives of $J(T_w^t)$, $G(T_w^t)$, and $F(T_w^t)$, computed in (108)–(110) and evaluated at $t = 0$ and $w = \mathrm{w}$, and are given by:

$$
\begin{aligned}
DJ(V) &= \mathrm{tr}(\mathrm{cof}(\nabla T)^\top \nabla V),\\
DG(V) &= \mathrm{cof}(\nabla T) \Big[\mathrm{tr}\Big((\nabla T)^{-1} \nabla V \Big) \mathrm{I} - [(\nabla T)^{-1} \nabla V]^\top \Big],\\
DF(V) &= \mathrm{cof}(\nabla T)^\top \Big[\mathrm{tr}\Big((\nabla T)^{-1} \nabla V \Big) \mathrm{I} - 2[\nabla V (\nabla T)^{-1}]^s \Big] (\nabla T)^{-\top}.
\end{aligned}
$$

6 Conclusions

In this paper, we have addressed a stationary 2D FSI problem. The mathematical model that we propose couples Stokes equations (for the fluid) and incompressible linearised elasticity equations (for the structure), through a boundary condition (for the common interface). This system of PDEs is shown to have a unique solution, when the applied forces are small. Then, the shape differentiability of the solution is established, and the shape derivative of a general functional is computed and simplified by showing the existence of suitable adjoint states.

Among the possible extensions of this work, offering a simplified but rigorous baseline, we can consider enriched models for the fluid (e.g. Navier–Stokes), the structure (e.g. compressible linear or nonlinear elasticity), and the boundary conditions (e.g. slip conditions for the fluid). More general functionals could also be studied, such as boundary integrals.

Finally, the shape derivative computed here allows to engage a numerical investigation, for instance for an energy minimization problem with the functional $\mathcal{J}(\Omega_0) = \int_{\Omega_0} |\nabla \mathrm{w}|^2 + \int_{\Omega_F} |\nabla \mathrm{u}|^2$, taken into account by our study.

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Appendix

Let $\alpha : U \subset \mathbb{V} \mapsto \varphi_\alpha \in (H^3(\Omega))^2$ be a differentiable map, where $\mathbb{V}$ is a normed vector space endowed with the norm $\|\cdot\|_{\mathbb{V}}$, and U is an open subset of $\mathbb{V}$, and Ω is an open subset of $\mathbb{R}^2$. Thus, $\alpha : U \subset \mathbb{V} \mapsto \nabla\varphi_\alpha \in (H^2(\Omega))^{2\times 2}$ is differentiable, and we denote by $D_\alpha(\nabla\varphi_\alpha)$ the differential of $\alpha \mapsto \nabla\varphi_\alpha$ at α. Namely $D_\alpha(\nabla\varphi_\alpha)$ is the continuous linear map from $\mathbb{V}$ to $(H^2(\mathbb{R}^2))^{2\times 2}$ such that for all $d\alpha \in \mathbb{V}$:

$$\nabla\varphi_{\alpha+d\alpha} = \nabla\varphi_\alpha + D_\alpha(\nabla\varphi_\alpha)d\alpha + o(\|d\alpha\|_{\mathbb{V}}).$$

Assuming $\nabla\varphi_\alpha$ being invertible, we define the following maps depending on φ_α:

$$J(\varphi_\alpha) := \det(\nabla\varphi_\alpha), \quad G(\varphi_\alpha) := \operatorname{cof}(\nabla\varphi_\alpha), \quad F(\varphi_\alpha) := (\nabla\varphi_\alpha)^{-1}\operatorname{cof}(\nabla\varphi_\alpha), \tag{100}$$

where $\operatorname{cof}(\nabla\varphi_\alpha)$ is the cofactor matrix of $\nabla\varphi_\alpha$ defined by:

$$\operatorname{cof}(\nabla\varphi_\alpha) = \det(\nabla\varphi_\alpha)\nabla\varphi_\alpha^{-T}.$$

We recall that the determinant $\det(\cdot)$, the inverse $(\cdot)^{-1}$, and the cofactor $\operatorname{cof}(\cdot)$ matrix are differentiable maps defined on the open set of invertible matrices, and their differentials are given by the following expressions. Let $A, B \in \mathbb{R}^{2\times 2}$, A being invertible, and $|B|$ sufficiently small so that $A + B$ is invertible, where $|B|$ is given in (1). We have:

$$\det(A+B) = \det(A) + \operatorname{tr}(\operatorname{cof}(A)^\top B) + o(|B|), \tag{101}$$

$$(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + +o(|B|), \tag{102}$$

$$\operatorname{cof}(A+B) = \operatorname{cof}(A) + \left(\operatorname{tr}(\operatorname{cof}(A)^\top B)\mathrm{I} - \operatorname{cof}(A)B^\top\right)A^{-\top} + o(|B|). \tag{103}$$

As it is shown in Sect. 3.1, the maps J, G, and F are well defined and differentiable because of the Banach algebra structure of $H^2(\Omega)$. From there, applying the chain rule and using expressions (101), (102), and (103), we can compute the differentials $D_\alpha J(\varphi_\alpha)$, $D_\alpha G(\varphi_\alpha)$, and $D_\alpha F(\varphi_\alpha)$. We give their expressions in the case where $\alpha = t$, $\alpha = w$, and $\varphi_\alpha = T_w^t$.

We recall that Φ_t is the map defined in (64) in Sect. 4.2 by:

$$\Phi_t := \mathrm{id}_{\mathbb{R}^n} + tV,$$

and that we have defined in (74) the following $H^3(\Omega_0^c)$-valued map for all $(t, w) \in \mathbb{R}_+ \times (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^3(\Omega_0))^2$ by:

$$T_w^t := \Phi_t + \mathcal{R}\gamma(w).$$

This map is differentiable, and we have the following derivatives with respect to w:

$$D_w(T_w^t)k = \mathcal{R}(\gamma(k)), \quad \text{and} \quad D_w(\nabla T_w^t)k = \nabla\mathcal{R}(\gamma(k)), \tag{104}$$

for all $k \in (H^1_{0,\Gamma_\omega}(\Omega_0) \cap H^3(\Omega_0))^2$. Thus, from the definitions in (100), the expressions (101), (102), (103) and (104) and in view of the chain rule, we can deduce the values of the following differentials:

$$D_w J(T_w^t)k = \operatorname{tr}(\operatorname{cof}(\nabla T_w^t)^\top \nabla\mathcal{R}(\gamma(k))), \tag{105}$$

$$D_w G(T_w^t)k = \left[\operatorname{tr}((\nabla T_w^t)^{-1}\nabla\mathcal{R}(\gamma(k)))\mathrm{I} - (\nabla T_w^t)^{-\top}\nabla\mathcal{R}(\gamma(k))^\top\right]\operatorname{cof}(\nabla T_w^t), \tag{106}$$

$$D_w F(T_w^t)k = \operatorname{cof}(\nabla T_w^t)^\top\left[\operatorname{tr}((\nabla T_w^t)^{-1}\nabla\mathcal{R}(\gamma(k)))\mathrm{I} - 2(\nabla\mathcal{R}(\gamma(k))(\nabla T_w^t)^{-1})^s\right](\nabla T_w^t)^{-\top}. \tag{107}$$

Noting that the derivative of T_w^t with respect to t is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}T_w^t = V_t := \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t,$$

we can also deduce the time derivatives of $J(T_w^t)$, $G(T_w^t)$, and $F(T_w^t)$, given by:

$$D_t J(T_w^t) = \operatorname{tr}(\operatorname{cof}(\nabla T_w^t)^\top \nabla V_t), \tag{108}$$

$$D_t G(T_w^t) = \operatorname{cof}(\nabla T_w^t)\left[\operatorname{tr}\left((\nabla T_w^t)^{-1}\nabla V_t\right)\mathrm{I} - [(\nabla T_w^t)^{-1}\nabla V_t]^\top\right], \tag{109}$$

$$D_t F(T_w^t) = \mathrm{cof}(\nabla T_w^t)^\top \left[\mathrm{tr}\left((\nabla T_w^t)^{-1}\nabla V_t\right)\mathrm{I} - 2[\nabla V_t(\nabla T_w^t)^{-1}]^s\right](\nabla T_w^t)^{-\top}. \tag{110}$$

By setting $T_w^t = T_0$ and $V_t = V$ in these expressions, we retrieve the fields $DJ(V)$, $DG(V)$, and $DF(V)$ involved in Theorem 4.

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