Velocity-vorticity geometric constraints for the energy conservation of 3D ideal incompressible fluids

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Abstract

In this paper we consider the 3D Euler equations and we first prove a criterion for energy conservation for weak solutions with velocity satisfying additional assumptions in fractional Sobolev spaces with respect to the space variables, balanced by proper integrability with respect to time. Next, we apply the criterion to study the energy conservation of solution of the Beltrami type, carefully applying properties of products in (fractional and possibly negative) Sobolev spaces and employing a suitable bootstrap argument.

Keywords: Euler equations, energy conservation, Onsager conjecture, Beltrami solutions.
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1 Introduction

We consider the homogeneous incompressible 3D Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)\, u + \nabla p = 0 & (t,x) \in (0,T) \times \mathbb{T}^3, \\ \operatorname{div} u = 0 & (t,x) \in (0,T) \times \mathbb{T}^3, \\ u(0,x) = u_0(x) & x \in \mathbb{T}^3, \end{cases} \tag{1.1}$$

where $\mathbb{T}^3 := \mathbb{R}^3 \backslash \mathbb{Z}^3$, and $u : (0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ and $p : (0,T) \times \mathbb{T}^3 \to \mathbb{R}$ represent respectively the velocity vector field and the kinematic pressure of an ideal fluid. It is well known that for smooth solution to (1.1) (which are known to exists only locally in time) the kinetic energy

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^3} |u(t,x)|^2 \, dx = \frac{1}{2} \|u(t)\|_2^2,$$

is constant. Let u and p smooth enough to perform the following calculations: we rewrite the convective term of $(1.1)_1$ as follows

$$(u \cdot \nabla)\, u = \operatorname{div}(u \otimes u).$$

Multiplying $(1.1)_1$ by the solution itself, and integrating over the domain, we get, that $\|u(t)\|_2^2 = \|u_0\|_2^2$ since

$$\int_{\mathbb{T}^3} \operatorname{div}(u \otimes u) : u\, dx = -\int_{\mathbb{T}^3} (u \otimes u) : \nabla u\, dx = -\int_{\mathbb{T}^3} u \cdot \nabla \frac{|u|^2}{2}\, dx = 0.$$

We report this very basic calculation since we will use it several times and also since we will show how it changes with a curl-formulation of the convective term.

Since physical experiments show that taking the limit as the viscosity vanishes, the energy dissipation seems not to vanish (cf. Frisch [18]), it has been a subject of many studies to understand if this conservation remains valid supposing certain (limited) regularity on the solution of the Euler equation. In 1954, Lars Onsager [23] conjectured that if u is sufficiently regular in space, say $u \in L^\infty(0,T;C^\theta)$, with $\theta > \frac{1}{3}$, then the kinetic energy is preserved; on the other hand for $\theta < \frac{1}{3}$ a dissipation phenomenon could be possible, even in absence of viscosity. The positive part of this conjecture was solved 40 years later by Constantin, E, Titi (see [12]), where they proved a slightly more general result in Besov spaces (which implies the Hölder case). See also Eyink [17]. To be more precise, in [12] it is proved the conservation of energy if $u \in L^3(0,T;B^\theta_{3,\infty})$, for $\theta > \frac{1}{3}$. Sharpest results were proved later on by Duchon and Robert [15] and Cheskidov *et al.* [10]. Results in scales of classical Hölder functions are proved in [7], while the boundary value problem is analyzed in Bardos and Titi [2]. In the last fifteen years –starting from the celebrated result by De Lellis and Székelyhidi [13]– also the negative part of the Onsager conjecture has been addressed, with an endpoint in the work by Isett [20] and Buckmaster *et al.* [9]. Nevertheless there is still a strong activity to determine the minimal space-time assumptions which are sufficient for the energy conservation, and some recent results are those in [8, 26].

Taking inspiration also from the work by De Rosa [14] and Liu, Wang, and Ye [21], we consider here criteria in scales of fractional Sobolev spaces, instead of Besov or Hölder spaces. This will allow us also to obtain sharp results which reach the critical exponents, see the discussion in Lemma 2.7.

The first result we prove concerns the conservation of energy in the fractional Sobolev setting. We restrict to the Hilbertian case $W^{s,2}(\mathbb{T}^3) = H^s(\mathbb{T}^3)$, but similar results in scales of fractional Sobolev spaces $W^{s,p}(\mathbb{T}^3)$ can be obtained along the same lines.

Theorem 1.1. *Let* $u \in L^{\frac{5}{2s}}(0,T,H^s(\mathbb{T}^3))$, *with* $\frac{5}{6} \leqslant s < \frac{5}{2}$, *be a weak solution to the Euler equation* (1.1). *Then, the kinetic energy is conserved,* *that is*

$$\|u(t)\|_{L^2(\mathbb{T}^3)} = \|u_0\|_{L^2(\mathbb{T}^3)} \qquad \textit{for a.e. } t \in [0,T].$$

We observe that the condition $L^3(0,T;H^{5/6}(\mathbb{T}^3))$ is exactly the same condition proved in Cheskidov, Friedlander, and Shvydkoy [11] for the Navier-Stokes equations (even if a more technical setting of the problem with boundaries), see also Beirao and Yiang [5, Prop. 4.5], again in the viscous case. In this paper we identify the same as a sufficient condition also for the problem without viscosity. In this respect note also that the extension to the Euler equations of results known for the Navier-Stokes equations is one of the results proved in [8]. Note also that, similar to other observations in Nguyen, Nguyen, and Tang [22], Wang *et al.* [26], the criteria involve the "critical spaces" and not slightly smaller spaces, as when considering Besov or Hölder spaces, cf. [7, 12], where the smooth functions are not dense with respect to the norm of the space itself.

As an application of the criteria in Theorem 1.1 (which are obviously valid also for periodic Leray-Hopf weak solutions to the Navier-Stokes equations), we analyze the energy conservation of a family of solutions with a particular geometric meaning: the Beltrami (also known as Trkal) flows. Beltrami solutions are well known in fluid dynamics as they provide a family of stationary solutions to the Euler equations (1.1). These are such that the curl of the velocity field, denoted by $\omega := \nabla \times u$, is proportional to the field itself, i.e.

$$\omega(x,t) = \lambda(x,t)u(x,t), \tag{1.2}$$

where $\lambda(\cdot,\cdot)$ is a suitable scalar function of the space and/or time variables.

Note that these flows, despite being in some cases very simple and smooth (note that for instance potential flows are Beltrami flows with $\lambda \equiv 0$) they are genuinely 3D flows, since in 2D the (scalar) vorticity is orthogonal to the plane of motion.

In addition, we observe that, by using the so-called *Lamb vector* $\omega \times u$, it is possible to write the alternative *rotational formulation* of the convective term

$$(u \cdot \nabla)\, u = \omega \times u + \frac{1}{2}\nabla |u|^2.$$

This implies that in the case of Beltrami flows the convective term is equal to a gradient (Bernoulli pressure) which can be included in the pressure: Beltrami flows (if smooth) satisfy linear (non-local) evolution equations, since the quadratic term becomes simply a gradient and the nonlinearity disappears. This means that the flow is laminar, but there are two caveat: a) the numerical simulation of the pressure and especially that of the Bernoulli pressure is particularly critical: if not using *pressure-robust* numerical methods, the result at very high Reynolds numbers could be affected by large oscillations (see Gauger, Linke, and Schroeder [19]); b) more important from the theoretical point of view is the fact that $(\omega \times u) \cdot u$ formally vanishes; for non-smooth functions the fact $(\omega \times u) \cdot u = 0$ (at most almost everywhere) does not directly imply that

$$\int_0^t \int_{\mathbb{T}^3} (\omega \times u + \frac{1}{2}\nabla|u|^2) \cdot u \, dx d\tau = 0,$$

since the integration could be not justified. One sufficient condition could be that of showing that the above integral exists: then it will vanish, but unfortunately this is not the case for weak solutions. At least with respect to the space variables, $u \in L^2(\mathbb{T}^3)$ and the vorticity field is a distribution in $H^{-1}(\mathbb{T}^3)$ and so the term $(\omega \times u) \cdot u$ could be not defined.

We start making some observations on the regularity which follows from the geometric constraint (1.2). If $\lambda(x, t) \equiv \lambda \in \mathbb{R}$ (such a condition corresponds to the circularly polarized plane waves used in electromagnetism), then u is smooth and the conservation of energy is an obvious consequence. This follows by a standard bootstrap argument using the Biot-Savart formula: in fact using that $-\Delta u = \operatorname{curl} \omega$, from $u \in L^\infty(0, T; L^2(\mathbb{T}^3))$ we can infer by elliptic regularity in the space variables that then $\omega = \lambda u \in L^\infty(0, T; L^2(\mathbb{T}^3))$, which implies $u \in L^\infty(0, T; H^1(\mathbb{T}^3))$. Iterating we get $u \in L^\infty(0, T; H^3)$, which is a class of classical solutions. This implies that a continuation argument for smooth solutions is valid, provided that the initial datum is smooth.

The second observation comes from a simple computation in the case in which $\lambda(x, t) = \lambda(t) \in L^p(0, T)$, for some $p \geqslant 1$. Observe that in this case $\nabla\cdot\omega = \lambda(t)\nabla\cdot u = 0$, so the divergence-free constraint is satisfied without any further assumption on $\lambda(t)$. Then (1.2) implies that $\omega \in L^p(0, T; L^2(\mathbb{T}^3))$, and consequently we have more regularity on u, indeed $u \in L^p(0, T; H^1(\mathbb{T}^3))$. Iterating this procedure we get that if $\lambda(t) \in L^p(0, T)$, for some $p \geqslant 1$, and $u \in L^\infty(0, T; L^2(\mathbb{T}^3))$ then

$$\omega \in L^{\frac{p}{3}}(0, T; H^2(\mathbb{T}^3)) \hookrightarrow L^{\frac{p}{3}}(0, T; L^\infty(\mathbb{T}^3)).$$

Hence, if $p \geqslant 3$, this is the Beale-Kato-Majda [3] criterion for continuation of smooth solutions (which conserve the energy). Hence, a first elementary results is the following.

Proposition 1.2. *Let u be a weak solution to the Euler equations, which is a Beltrami solution with $\lambda \in L^p(0, T)$, with $p \geqslant 3$. Let $u_0 \in H^3(\mathbb{T}^3)$, with $\nabla \cdot u_0 = 0$, then u is the unique a classical solution of* (1.1) *in* $[0, T]$ *and conserves the energy.*

If λ depends also on the space variables a compatibility condition to preserve the divergence condition is that $\nabla\lambda\cdot u = 0$. This has some consequences on the effective velocity fields to be considered, especially if the solutions are classical, see Beltrami [6] and Trkal [25]. For recent results on possible existence of non-trivial Beltrami fields, see Enciso and Peralta [16] and Abe [1]. In our case we suppose to have a weak solution, which is also a Beltrami field and work directly on it.

The second Theorem we prove is about the conservation of energy when u is a Beltrami field.

Theorem 1.3. *Let be a weak solution to the Euler equation* (1.1)*, such that it is a Beltrami field, i.e.* (1.2) *is satisfied. If* $\lambda \in L^{\beta}(0, T, H^{\tau}(\mathbb{T}^3))$*, with* $\beta > \frac{5}{2\tau-1}$*, for* $\frac{1}{2} < \tau \leqslant \frac{3}{2}$*, or if* $\lambda \in L^{5/2}(0, T; H^{\tau}(\mathbb{T}^3))$*, with* $\tau > \frac{3}{2}$*, then the kinetic energy is conserved.*

This result derives from Theorem 1.1 after a proper (even iterated) use of some precise results about the continuity of the multiplication operator in (negative) Sobolev spaces, see the results summarized in the next section.

Plan of the paper: In Section 2 we give some basic definition and introduce functional spaces we will work with, recalling different results that will be used later on in the paper. In Section 3 we will give the proofs of the two main results, namely Theorems 1.1-1.3.

2 Preliminaries

2.1 Functional Spaces and weak solutions

In this paper we will use the classical periodic Lebesgue spaces $(L^p(\mathbb{T}^3), \|\cdot\|_p)$ and Sobolev spaces $(W^{k,p}(\mathbb{T}^3), \|\cdot\|_{W^{k,p}})$ of natural order $k \in \mathbb{N}$, all considered with zero mean value. When $p = 2$ we also use the notation $H^k = W^{k,2}(\mathbb{T}^3)$. Here we will denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the scalar product and the norm in $L^2(\mathbb{T}^3)$ respectively. In addition we do not distinguish norm of scalar or vector valued functions.

A central role in this paper will be played by the fractional Sobolev spaces that we define in the following (see [4]).

Definition 2.1 (Fractional Sobolev spaces)**.** *Let* $s \in \mathbb{R}$ *and* $1 \leqslant p \leqslant \infty$*. We define the Sobolev-Slobodeckij spaces as follows*

- *Let* $s \in (0, 1)$*, then we will say that* $u \in W^{s,p}(\mathbb{T}^3)$ *if*

$$\|u\|_{W^{s,p}} := \|u\|_p + [u]_{W^{s,p}} < \infty,$$

where

$$[u]_{W^{s,p}} = \begin{cases} \left(\displaystyle\int_{\mathbb{T}^3} \int_{\mathbb{T}^3} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dxdy \right)^{\frac{1}{p}} & p \in [1, +\infty) \\ \\ \operatorname*{ess\,sup}_{\substack{x,y \in \mathbb{T}^3 \\ x \neq y}} \dfrac{|u(x) - u(y)|}{|x - y|^s} & p = \infty; \end{cases}$$

- *Let $s = \theta + k$, with $\theta \in (0,1)$ and $k \in \mathbb{N}_0$. Then, we will say that $u \in W^{s,p}(\mathbb{T}^3)$ if*

$$\|u\|_{W^{s,p}} := \|u\|_{W^{k,p}} + \sum_{|r|=k} [\partial_r u]_{W^{\theta,p}} < \infty;$$

- *If $s < 0$, and p' is the Sobolev conjugate exponent of p, then*

$$W^{s,p}(\mathbb{T}^3) = (W^{-s,p'}(\mathbb{T}^3))^*,$$

where $*$ denotes the topological dual space.

Remark 2.2. Note that for $p = 2$ one can use also an equivalent semi-norm $[u]_{W^{\alpha,2}} = \|(-\Delta)^{\alpha/2}u\|_{L^2}$.

In the case of the whole space, but also for bounded domains with a proper definition of the restriction fractional spaces can be defined even by means of Bessel potentials, and in this case the space is denoted in literature by $H^{s,p}(\Omega)$. For our purposes, we do not give this definition since for $s \in \mathbb{R}$ and $p = 2$, the two definitions coincide, see Triebel [24] . That is why we will denote $W^{s,2}(\Omega)$ as $H^s(\Omega)$.

Here we state some propositions that will be useful when considering the product of two fractional Sobolev functions $u \in H^{s_1}(\mathbb{T}^3)$, $\lambda \in H^{s_2}(\mathbb{T}^3)$ (see [4, Thm 6.1, 7.3, 8.1, 8.2] for the whole space case and the results in the periodic setting follow along the same lines. Here results are rephrased in the simpler case $p_i = p = 2$).

The first proposition regards the case of non-negative exponents.

Proposition 2.3. *Let $s, s_i \in \mathbb{R}$ be parameters such that for $i = 1, 2$*

1. $s \geqslant 0$;
2. $s_i \geqslant s$;
3. $s_1 + s_2 - s > \frac{3}{2}$.

If $u \in H^{s_1}(\mathbb{T}^3)$ and $\lambda \in H^{s_2}(\mathbb{T}^3)$, then $\lambda u \in H^s(\mathbb{T}^3)$ and the map of pointwise multiplication

$$H^{s_1}(\mathbb{T}^3) \times H^{s_2}(\mathbb{T}^3) \to H^s(\mathbb{T}^3),$$

is continuous and bilinear. Moreover if $s \in \mathbb{N}_0$, the strictness of inequalities (2) *and* (3) *can be interchanged.*

In the case of non-negative exponents we have the following proposition

Proposition 2.4. *Let $s, s_i \in \mathbb{R}$ be parameters such that for $i = 1, 2$*

1. $s_i \geqslant s$;

2. $\min\{s_1, s_2\} < 0$*;*

3. $s_1 + s_2 \geqslant 0$*;*

4. $s_1 + s_2 - s > \frac{3}{2}$*.*

If $u \in H^{s_1}(\mathbb{T}^3)$ *and* $\lambda \in H^{s_2}(\mathbb{T}^3)$*, then* $\lambda u \in H^s(\mathbb{T}^3)$ *and the map of pointwise multiplication*

$$H^{s_1}(\mathbb{T}^3) \times H^{s_2}(\mathbb{T}^3) \to H^s(\mathbb{T}^3),$$

is continuous and bilinear. Moreover, if we substitute the condition (2) *by the condition*

5. $\min\{s_1, s_2\} \geqslant 0$ *and* $s < 0$*,*

and the inequality (3) *is strict, the same result holds.*

Additionally, if $s \notin \mathbb{N}_0$, proposition 2.3 is valid even in open and bounded set with Lipschitz boundary.

To complete these definitions we will say that the Bochner measurable function $u \in L^q(0, T, W^{s,p}(\Omega))$ if the following norm

$$\|u\|_{L^q(W^{s,p})} := \begin{cases} \left(\int_0^T \|u(t)\|_{W^{s,p}}^q \, dt \right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_{W^{s,p}} & \text{if } q = \infty, \end{cases}$$

is finite.
We want to introduce now the notion of a weak solution to the Euler equations, it is necessary to introduce the spaces H and V, which are respectively the closure in $L^2(\mathbb{T}^3)$ and $W^{1,2}(\mathbb{T}^3)$ of the smooth, periodic, divergence-free and with zero mean-value vector fields. The space of test functions will be

$$\mathcal{D}_T = \{\varphi \in C_0^\infty([0, T[; C^\infty(\mathbb{T}^3)) : \nabla \cdot \varphi = 0\}.$$

Now we are able to define the notion of weak solutions for the Euler equations which we will consider.

Definition 2.5. *Let* $v_0 \in H$*. A measurable function* $v : (0, T) \times \mathbb{T}^3 \to \mathbb{T}^3$ *is called a weak solution to the Euler equation if* $v \in L^\infty(0, T, H)$ *is such that*

$$\int_0^\infty [(v, \partial_t \varphi) + ((v \otimes v), \nabla \varphi)] \, dt = -(v(0), \varphi(0)) \qquad \forall \, \varphi \in \mathcal{D}_T.$$

2.2 Mollification

A fundamental tool in the sequel will be that of mollification and we recall the most relevant properties, stated for periodic functions. Let us consider a centrally symmetric function $\rho \in C_0^\infty(\mathbb{R}^3)$, such that $\rho \geqslant 0$, $\operatorname{supp}\rho \subset B_1(0)$ and $\|\rho\|_{L^1(\mathbb{R}^3)} = 1$. Let $\varepsilon \in (0,1]$, we define the family of Friederichs mollifiers as follows $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(\varepsilon^{-1}x)$. Then for every $f \in L^1_{\mathrm{loc}}(\mathbb{T}^3)$ we can define the well-posed mollification of f, that is

$$f_\varepsilon(x) = \int_{\mathbb{T}^3} \rho_\varepsilon(x-y) f(y)\, dy = \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x-y)\, dy,$$

which is nothing else the convolution of ρ_ε and f.
Since for small $\varepsilon > 0$, $\operatorname{supp}\rho_\varepsilon \subset B_\varepsilon(0) \subset]-\pi,\pi[^3$, then we can say that

$$f_\varepsilon(x) = \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x-y)\, dy.$$

We note that if $f \in L^1(\mathbb{T}^3)$, then $f \in L^1_{\mathrm{loc}}(\mathbb{T}^3)$ and f_ε is 2π-periodic along the x_i-axis, for $i = 1,2,3$.

Apart classical result on mollification in Lebesgue, Hölder and, Sobolev spaces, most of the results can be extended to fractional Sobolev spaces. An useful Lemma, contained in [14], can be summarized as follows.

Lemma 2.6. *Let $f, g : \mathbb{T}^3 \to \mathbb{T}^3$ are such that $f \in W^{\alpha,p}(\mathbb{T}^3)$ and $g \in W^{\beta,q}(\mathbb{T}^3)$, for some $0 < \alpha,\beta < 1$ and $p,q \geqslant 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{m}$. Then. for every $1 \leqslant m < +\infty$, there exists a constant $C = C(m)$, such that*

$$\|\nabla f_\varepsilon\|_p \leqslant C\varepsilon^{\alpha-1}[f]_{W^{\alpha,p}}, \tag{2.1}$$

$$\|(f \otimes g)_\varepsilon - f_\varepsilon \otimes g_\varepsilon\|_m \leqslant C\varepsilon^{\alpha+\beta}[f]_{W^{\alpha,p}}[g]_{W^{\beta,q}}. \tag{2.2}$$

Moreover, in the case $\alpha = \beta = 0$ it also follows that

$$\limsup_{\varepsilon\to 0} \|(f \otimes g)_\varepsilon - f_\varepsilon \otimes g_\varepsilon\|_m = 0. \tag{2.3}$$

We end this section with a Lemma whose proof comes from the estimate (2.1) and the fact that $C^\infty(\mathbb{T}^3)$ is dense in $W^{\alpha,p}(\mathbb{T}^3)$, as it is done in [22, Lemma 2.1] for the case $\alpha = 0$

Lemma 2.7. *Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a function in $L^q(0,T;W^{\alpha,p}(\mathbb{T}^3))$, for some $\alpha \in]0,1[$. Then, for every $1 \leqslant p,q < \infty$, we have*

$$\limsup_{\varepsilon\to 0^+} \varepsilon^{1-\alpha}\|\nabla f_\varepsilon\|_{L^q(L^p)} = 0.$$

Moreover if $f \in L^q(0,T;L^p(\mathbb{T}^3))$, then

$$\limsup_{\varepsilon\to 0^+} \|f_\varepsilon - f\|_{L^q(L^p)} = 0.$$

Proof. The proof of this result is based on the observation that, if $f \in W^{\alpha,p}(\mathbb{T}^3)$, then f_ε is infinitely smooth but higher norms are not uniform bounded in $\varepsilon > 0$. In particular, from (2.1) one can deduce immediately the boundedness

$$\sup_{\varepsilon>0} \varepsilon^{1-\alpha}\|\nabla f_\varepsilon\|_{L^p} \leqslant C[f]_{W^{\alpha,p}} < +\infty.$$

To prove the Lemma it is enough to observe that for each $\lambda > 0$ we can find $g \in C^\infty(\mathbb{T}^3)$ such that $\|f - g\|_{W^{\alpha,p}} < \lambda$. Then, applying again the estimate (2.1) to $f - g$ one gets

$$\varepsilon^{1-\alpha}\|\nabla f_\varepsilon\|_p \leqslant \varepsilon^{1-\alpha}\|\nabla g_\varepsilon\|_p + C[f-g]_{W^{\alpha,p}} \leqslant \varepsilon^{1-\alpha}\|\nabla g_\varepsilon\|_p + C\lambda.$$

Since λ can be chosen arbitrarily small and since $\lim_{\varepsilon\to 0} \varepsilon^{1-\alpha}\|\nabla g_\varepsilon\|_p = 0$ (being g smooth and fixed) we get the proof. The need for the density of smooth function excludes the case $p = \infty$ and, more generally excludes from this type of results Nikol'skiĭ and Hölder spaces. Then, the extension to f in the Bochner space $L^q(0,T; W^{\alpha,p}(\mathbb{T}^3))$ is simply obtained by raising to the q-th power the above estimate and integrating over $(0,T)$. □

3 Main results

In this section we give the proof of the main results of the paper. We start with the proof of a criterion about conservation of energy for velocities in the fractional Sobolev spaces.

Proof of Theorem 1.1. By following a very standard procedure to deal with non-smooth functions we use as test function in the definition of weak solutions $\rho_\varepsilon * (\rho_\varepsilon * u)$. To be precise the argument will also need another smoothing in time which is nevertheless standard to justify, see [2]. By using the identity

$$\int_0^t \int_{\mathbb{T}^3} (u_\varepsilon \otimes u_\varepsilon) : \nabla u_\varepsilon \, dxd\tau = 0,$$

being u_ε smooth and divergence-free, we get the equality

$$\frac{1}{2}\|u_\varepsilon(t)\|^2 - \frac{1}{2}\|u_{0,\varepsilon}\|^2 = \int_0^t \int_{\mathbb{T}^3} ((u\otimes u)_\varepsilon - u_\varepsilon \otimes u_\varepsilon) : \nabla u_\varepsilon \, dxd\tau.$$

If we manage to estimate the integrand of the previous inequality in such a way the right-hand side goes to zero, we have finished since the L^2-convergence of $u_\varepsilon(t)$ to $u(t)$ as $\varepsilon \to 0$ holds almost everywhere in $t \in (0,T)$, by the properties of smoothing. The proof will be slightly different depending on the values of the exponent "$s \in \mathbb{R}$" in the extra-assumption in the fractional space $H^s(\mathbb{T}^3)$. For this reason we split the proof in two parts but in all cases the main step is a proper estimate of the integral

$$\mathcal{I}_\varepsilon := \int_0^t \int_{\mathbb{T}^3} |u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon||\nabla u_\varepsilon| \, dx d\tau.$$

The case $\frac{5}{6} \leqslant s < 1$. Applying Hölder inequality and the convex interpolation inequality in Lebesgue spaces we get

$$\begin{aligned}
\mathcal{I}_\varepsilon &\leqslant \int_0^t \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_2 \, \|\nabla u_\varepsilon\|_2 \, d\tau \\
&\leqslant \int_0^t \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_1^{1-\theta} \, \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_p^{\theta} \, \|\nabla u_\varepsilon\|_2 \, d\tau,
\end{aligned}$$

where

$$\theta = \frac{p}{2(p-1)}.$$

and clearly $p \geqslant 2$.

Note that both the L^1-norm of $u_\varepsilon(\tau) \otimes u_\varepsilon(\tau)$ and $(u(\tau) \otimes u(\tau))_\varepsilon$ can be easily estimated by using the properties of mollifiers as follows

$$\|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_1 \leqslant \|u_\varepsilon\|_2^2 + \|(u \otimes u)_\varepsilon\|_1 \leqslant c_1 \|u_\varepsilon\|_2^2 \leqslant c_2 \|u\|_2^2 \leqslant C,$$

hence proving an uniform bounded since u is a weak solution to (1.1).

Next, we fix $p = \frac{5-2s}{5-4s}$, hence $\theta = \frac{5-2s}{4s}$, and we get

$$\mathcal{I}_\varepsilon \leqslant C \int_0^t \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_{\frac{5-2s}{5-4s}}^{\frac{5-2s}{4s}} \|\nabla u_\varepsilon\|_2 \, d\tau,$$

It only remains to estimate the term involving the $L^{\frac{5-2s}{5-4s}}$-norm. Using (2.2), and the assumption $u \in H^s(\mathbb{T}^3)$ we have

$$\|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_p \leqslant C\varepsilon^{2\alpha} [u]^2_{W^{\alpha, 2\frac{5-2s}{5-4s}}} \leqslant C\varepsilon^{2\alpha} \|u\|^2_{H^s},$$

for $\alpha = \frac{2(s-1)s}{2s-5}$, fixed in such a way that

$$\frac{1}{2} - \frac{s}{3} = \frac{1}{2p} - \frac{\alpha}{3} \implies \alpha = \frac{3 - 3p + 2ps}{2p},$$

since this is the value for which the (fractional) Sobolev embedding $H^s(\mathbb{T}^3) = W^{s,2}(\mathbb{T}^3) \hookrightarrow W^{\alpha, 2p}(\mathbb{T}^3)$, holds true. Putting all together, since $2\alpha\theta = 1 - s$ we arrive to the following estimate

$$\mathcal{I}_\varepsilon \leqslant C \int_0^t \|u\|_{H^s}^{\frac{5-2s}{2s}} \varepsilon^{1-s} \|\nabla u_\varepsilon\|_2 \, d\tau.$$

Hence by Hölder inequality with exponents $x = \frac{5}{5-2s}$ and $x' = \frac{5}{2s}$ we get

$$\mathcal{I}_\varepsilon \leqslant C\|u\|_{L^{5/2s}(H^s)}^{\frac{5-2s}{2s}} \varepsilon^{1-s}\|\nabla u_\varepsilon\|_{L^{5/2s}(L^2)}.$$

Finally by using the assumptions of Theorem 1.1 and Lemma 2.7 we get

$$\limsup_{\varepsilon\to 0^+} \mathcal{I}_\varepsilon \leqslant C \limsup_{\varepsilon\to 0^+} \varepsilon^{1-s}\|\nabla u_\varepsilon\|_{L^{5/2s}(L^2)} = 0.$$

This is enough to end the proof since $u_\varepsilon(t) \to u(t)$ for almost all $t \in (0, T)$.

The case $1 \leqslant s < 5/2$. In the case $s \geqslant 1$ the proof is a little different since now we can estimate directly the term ∇u_ε. Observe also that for $s > \frac{5}{2}$, then $H^s(\mathbb{T}^3) \hookrightarrow W^{1,\infty}(\mathbb{T}^3)$ and so one recovers the Beale-Kato-Majda criterion for regularity if $u \in L^1(0, T; H^s(\mathbb{T}^3))$, $s > \frac{5}{2}$.

We first recall that, for $1 \leqslant s < \frac{3}{2}$ have the following embedding $H^s(\mathbb{T}^3) \hookrightarrow W^{1,p}(\mathbb{T}^3)$, where p is such that $\frac{1}{2} + \frac{1-s}{3} = \frac{1}{p}$, and so

$$p = \frac{6}{5-2s} \qquad \text{and} \qquad p' = \frac{p}{p-1} = \frac{6}{1+2s}. \tag{3.1}$$

We distinguish two further sub-cases.

The sub-case $1 \leqslant s < \frac{3}{2}$. With this position we have $2 \leqslant p < 3$. Moreover the bound $s < \frac{3}{2}$, gives even the following embedding $H^s(\mathbb{T}^3) \hookrightarrow L^{p^*}(\mathbb{T}^3)$, where

$$p^* = \frac{6}{3-2s}.$$

Applying to $\mathcal{I}_\varepsilon$ Hölder inequality with conjugate exponents p and p', and an interpolation inequality with suitable exponents (always possible since $p^*/2 > p'$) we get

$$\mathcal{I}_\varepsilon \leqslant \int_0^t \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_1^{1-\theta} \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_{\frac{p^*}{2}}^{\theta} \|\nabla u_\varepsilon\|_p \, d\tau,$$

with θ satisfying the following equality $\frac{1+2s}{6} = 1 - \theta + \frac{\theta}{\frac{p^*}{2}}$, hence

$$\theta = \frac{5-2s}{4s}.$$

As in the previous case we have $\|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_1 \leqslant C$.

Moreover, by $H^s(\mathbb{T}^3) \hookrightarrow W^{1,p}(\mathbb{T}^3)$ and Hölder inequality with exponents $5/(5-2s)$ and $5/2s$ we have

$$\mathcal{I}_\varepsilon \leqslant C\|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_{L^{\frac{5}{4s}}(0,T;L^{\frac{p^*}{2}})}^{\frac{5-2s}{4s}} \|u\|_{L^{\frac{5}{2s}}(0,T;H^s)}.$$

Next, we observe that

$$u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon = u_\varepsilon \otimes (u_\varepsilon - u) + (u_\varepsilon - u) \otimes u + u \otimes u - (u \otimes u)_\varepsilon,$$

and, since $u \in L^{\frac{5}{2s}}(0,T;H^s(\mathbb{T}^3))$ implies $u \otimes u \in L^{\frac{5}{4s}}(0,T;L^{\frac{p^*}{2}}(\mathbb{T}^3))$, then estimate (2.3) from Lemma 2.7 implies again that $\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon = 0$, ending the proof.

The sub-case $\frac{3}{2} \leqslant s < \frac{5}{2}$. Again we apply Hölder inequality with conjugate exponents p and p' defined in (3.1) and interpolating the $L^{p'}$-norm between 1 and $q/2 = 3$ we get

$$\mathcal{I}_\varepsilon \leqslant \int_0^t \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_1^{\frac{2s-1}{4}} \|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_3^{\frac{5-2s}{4}} \|\nabla u_\varepsilon\|_p \, d\tau.$$

We use the same control as before for the $L^1(\mathbb{T}^3)$ norm and considering the interpolation of $H^1(\mathbb{T}^3)$ between $L^2(\mathbb{T}^3)$ and $H^s(\mathbb{T}^3)$ and the uniform L^2-bound, we have

$$\|u\|_{H^1} \leqslant \|u\|_{L^2}^{1-1/s} \|u\|_{H^s}^{1/s} \leqslant C\|u\|_{H^s}^{1/s}.$$

This implies also that from $u \in L^{\frac{5}{2s}}(0,T;H^s(\mathbb{T}^3))$ it follows –by interpolation with $L^\infty(0,T;L^2(\mathbb{T}^3))$– that $u \in L^{\frac{5}{2s}}(0,T;H^1(\mathbb{T}^3)) \hookrightarrow L^{\frac{5}{2s}}(0,T;L^6(\mathbb{T}^3))$, hence $u \otimes u \in L^{\frac{5}{4s}}(0,T;L^3(\mathbb{T}^3))$. Hence by using the Hölder inequality we get

$$\mathcal{I}_\varepsilon \leqslant C\|u_\varepsilon \otimes u_\varepsilon - (u \otimes u)_\varepsilon\|_{L^{\frac{5}{4s}}(L^3)}^{\frac{5-2s}{4s}} \|u\|_{L^{\frac{5}{2s}}(H^s)},$$

and Lemma 2.7 implies again that $\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon = 0$. This ends the proof of the conservation of energy. □

After having finished the proof of the criterion for energy conservation in fractional spaces, we can pass to prove to a criterion for energy conservation who employs vorticity/velocity in a sort of "geometric" special situation. This should be compared with the results in [14] where an "analytic" combination of the two quantities is considered.

Proof of Theorem 1.3. Let $u \in L^\infty(0,T;L^2(\mathbb{T}^3))$ be a weak solution to the Euler equation (1.1) and let us consider $\lambda \in L^\beta(0,T;H^\tau(\mathbb{T}^3))$, for some $\beta \geqslant 1$ and $\tau \in \mathbb{R}$. Moreover, we are assuming that u is a Beltrami field, i.e. its curl can be written as the product of λ and itself as in (1.2). We want to apply Proposition 2.3-2.4 in order to infer sharp regularity for the vorticity ω, which –in turn– would give additional regularity for the velocity u. Doing so, possibly iterating, we try to show that u belongs to some of the spaces as those in the hypotheses of Theorem 1.1 to have conservation of energy.

Following the notation of Propositions 2.3 and 2.4, we have $s_1 = 0$, $s_2 = \tau$ and, in both statements, it is required $s_i \geqslant s$, $i = 1, 2$, which gives

$$s \leqslant 0.$$

Moreover, a further requirement is that

$$s < \tau - \frac{3}{2}.$$

We have to distinguish different cases.

The case $0 \leqslant \tau \leqslant \frac{3}{2}$. In this case, we have $s < 0$ and we fall in the hypotheses of Proposition 2.4. Consequently we get

$$\omega \in L^\beta(0, T; H^{\tau-\frac{3}{2}-\varepsilon}(\mathbb{T}^3)), \quad \text{for any arbitrarily small } \varepsilon > 0,$$

where the integrability in time remains unchanged since u is essentially bounded in time.

But again, ω is the curl of u, so that by elliptic regularity

$$u \in L^\beta(0, T; H^{\tau-\frac{1}{2}-\varepsilon}(\mathbb{T}^3)), \quad \text{for any arbitrarily small } \varepsilon > 0.$$

We first note that this will give an improvement in the known regularity for the velocity of a weak solution only if $\tau > 1/2$, hence from now on we consider τ in the restricted range $\tau \in]1/2, 3/2]$. Next, we can directly apply Theorem 1.1 to prove conservation of energy if,

$$\frac{5}{6} < \tau - \frac{1}{2} \leqslant \frac{5}{2},$$

which holds if $\tau \in]4/3, 3/2]$, and if in addition

$$\beta > \frac{5}{2\tau - 1}.$$

Within this range for both τ and β, the weak solution u satisfies the hypotheses of Theorem 1.1.

Let us now see what we can infer for smaller τ, that is $\tau \in]1/2, 4/3]$: We iterate the same process with a bootstrap argument. We start the iteration of the result on product in Sobolev spaces from $u \in L^\beta(0, T; H^{\tau-\frac{1}{2}-\varepsilon}(\mathbb{T}^3))$, $\lambda \in L^\beta(0, T; H^\tau(\mathbb{T}^3))$ and for this reason we define two sequences $\{\beta_n\}$, $\{\sigma_n\}$, as follows

$$\beta_1 = \beta, \qquad \text{and} \qquad \sigma_1 = \tau - \frac{1}{2}.$$

Next, we define by recursion (which follows as a formal application of Proposition 2.3 in the limiting case $\varepsilon = 0$)

$$\beta_{n+1} := \frac{\beta_n \beta}{\beta_n + \beta} \qquad \text{and} \qquad \sigma_{n+1} := \min\left\{\sigma_n, \tau, \sigma_n + \tau - \frac{3}{2}\right\} + 1.$$

Remark 3.1. The real index s for the space regularity of the velocity field after n applications of the product theorem will be any number strictly less than σ_n. While β_n will be the exact Lebesgue index with respect to the time variable.

Note that, since $\tau < \frac{3}{2}$ and if $\sigma_n \leqslant \frac{3}{2}$ then

$$\min\left\{\sigma_n, \tau, \sigma_n, \sigma_n + \tau - \frac{3}{2}\right\} = \sigma_n + \tau - \frac{3}{2}.$$

These relations imply, that

$$\beta_n = \frac{\beta}{n} \quad \text{and} \quad \sigma_n = n(\tau - \frac{1}{2}) \qquad \forall\, n \in \mathbb{N}.$$

By a now rigorous application of Proposition 2.3 this finally proves that

$$u \in L^{\frac{\beta}{n}}(0,T; H^{n(\tau-\frac{1}{2})-\varepsilon}(\mathbb{T}^3)), \quad \text{for any arbitrarily small } \varepsilon > 0,$$

and this argument can be iterated as long as $n(\tau-1/2)-\varepsilon < n(\tau-1/2) \leqslant \frac{3}{2}$.

Since we are considering the range $\tau \in]1/2, 4/3]$ we have now $\tau - 1/2 \leqslant \frac{5}{6}$. We then fix $n_0 \in \mathbb{N}$ such that

$$n_0(\tau - \frac{1}{2}) \leqslant \frac{5}{6} < (n_0+1)(\tau - \frac{1}{2}),$$

and iterate till reaching the regularity

$$u \in L^{\frac{\beta}{n_0+1}}(0,T; H^{(n_0+1)(\tau-\frac{1}{2})-\varepsilon}(\mathbb{T}^3)), \quad \text{for any arbitrarily small } \varepsilon > 0,$$

which is a suitable class for energy conservation. In fact, since $n_0(\tau - 1/2) - \varepsilon < \frac{5}{6}$ the iteration is well-defined and moreover $\tau < 3/2$ implies $(n_0+1)(\tau-1/2) < 5/6+1 < 5/2$. Finally we observe that if $\beta > \frac{5}{2\tau-1}$, then $\beta_{n_0+1} = \frac{\beta}{(n_0+1)} > \frac{5}{(n_0+1)(2\tau-1)}$, showing that the hypotheses of Theorem 1.1 are then satisfied.

The case $\tau > \frac{3}{2}$. In this case note that $H^\tau(\mathbb{T}^3) \subset L^\infty(\mathbb{T}^3)$, hence we get immediately that $\omega \in L^\beta(0,T; H^0(\mathbb{T}^3))$, which implies

$$u \in L^\beta(0,T; H^1(\mathbb{T}^3)),$$

and if $\beta \geqslant \frac{5}{2}$ we are done, since it falls within the assumptions of Theorem 1.1. Note that the result will not be improved with a further iteration. At least in the case $\tau > \frac{5}{2}$ (but the other case $\tau \in [3/2, 5/2[$ is similar) one will get $\omega \in L^{\beta/2}(0,T; H^1(\mathbb{T}^3))$, which gives $u \in L^{\beta/2}(0,T; H^2(\mathbb{T}^3)) \hookrightarrow L^{\beta/2}(0,T; C^{1/2}(\mathbb{T}^3))$, which is an energy conservation class if $\beta \geqslant 4$, see [7].

□

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Conflicts of interest and data availability statement

The authors declare that there is no conflict of interest. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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