Multiple solutions for a fractional Choquard problem with slightly subcritical exponents on bounded domains

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Abstract

This paper is devoted to study a fractional Choquard problem with slightly subcritical exponents on bounded domains. When the exponent of the convolution type nonlinearity tends to the fractional critical one in the sense of Hardy-Littlewood-Sobolev inequality, we prove the existence of multiple positive solutions via Lusternik-Schnirelmann category and nonlocal global compactness. Moreover, the topology of the domain furnishes a lower bound for the number of positive solutions.

Keywords: Fractional Choquard problem; Slightly subcritical exponent; Category; Nonlocal global compactness.

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1 Introduction and a main result

In the present paper, we are concerned with the following fractional Choquard problem with slightly subcritical exponents on bounded domains:

$$\begin{cases} (-\Delta)^s u + \lambda u = \left(\int_\Omega \frac{u^{p_\varepsilon}(y)}{|x-y|^\sigma} \mathrm{d}y \right) u^{p_\varepsilon - 1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ u > 0 & \text{in } \Omega \end{cases}, \tag{1.1}$$

where $s \in (0,1),\ \lambda \geq 0,\ N > 4s,\ 0 < \sigma < N,\ \varepsilon > 0,\ p_\varepsilon := 2^*_{\sigma,s} - \varepsilon,\ 2^*_{\sigma,s} := \frac{2N-\sigma}{N-2s}$ is the fractional critical exponent in the sense of Hardy-Littlewood-Sobolev inequality, Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and $(-\Delta)^s$ is the fractional Laplacian defined by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\rho \to 0} \int_{\mathbb{R}^N \setminus B_\rho(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y \tag{1.2}$$

with $C_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1 - \cos \zeta_1}{|\zeta|^{N+2s}} \mathrm{d}\zeta \right)^{-1}$ being a dimensional constant and $B_\rho(x)$ being an open ball centered at x with radius ρ. As $\varepsilon \to 0$, namely, $p_\varepsilon \to 2^*_{\sigma,s}$, we prove that the problem (1.1) possesses at

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least $cat_{\bar{\Omega}}(\bar{\Omega})+1$ solutions if Ω is not contractible. Here $cat_{\bar{\Omega}}(\bar{\Omega})$ denotes the Lusternik-Schnirelmann category of Ω.

On this type of problem in history, we can go back to Bahri and Coron [3], Benci and Cerami [5] and Benci, Cerami and Passaseo [6]. In the celebrated work [6], the authors considered

$$\begin{cases} -\Delta u + \lambda u = u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ , \\ u > 0 & \text{in } \Omega \end{cases} \tag{1.3}$$

where $N \geq 3$, $\lambda \geq 0$, $p \in (2, 2^* := \frac{2N}{N-2})$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. They showed to us how the number of solutions for (1.3) is affected by the topology of Ω, and the nonlinearity acts strongly on the problem (1.3) if the domain Ω is perturbed by cutting off or adding pieces, small in some sense. To be precise, they applied variational methods to prove that there exists a function $\bar{p} : [0, +\infty) \to (2, 2^*)$ such that for every $p \in [\bar{p}(\lambda), 2^*)$, the problem (1.3) has at least $cat_{\bar{\Omega}}(\bar{\Omega})$ distinct solutions. Moreover, they also showed that the number of solutions is greater than $cat_{\bar{\Omega}}(\bar{\Omega})+1$ when the domain is not contractible. More than a decade later, Benci, Bonanno and Micheletti [4] extended this kind of result to a nonlinear elliptic problem on a Riemannian manifold and proved that the number of solutions depends on the topological properties of the manifold. In the same spirit, Siciliano [23] investigated the existence of multiple positive solutions to a Schrödinger-Poisson-Slater system

$$\begin{cases} -\Delta u + u + \lambda \phi u = |u|^{p-2}u & \text{in } \Omega \\ -\Delta \phi = u^2 & \text{in } \Omega \quad , \\ u = \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

and showed that the number of positive solutions to (1.4) is bounded from below by $cat_{\bar{\Omega}}(\bar{\Omega}) + 1$ under appropriate assumptions. Afterwards, Alves, Figueiredo and Furtado [1] also used this kind of thought to establish an existence result of multiple solutions for a complex equation with magnetic field when the parameter λ has large values.

Recently, based on the work of Gao and Yang [14], Ghimenti and Pagliardini [15] used the concentration properties of the Talenti-Aubin functions to balance the nonlocal effect of the nonlinerity and studied a Choquard equation in a bounded domain

$$\begin{cases} -\Delta u + \lambda u = \left(\int_\Omega \frac{u^{q_\varepsilon}(y)}{|x-y|^\sigma} \mathrm{d}y \right) u^{q_\varepsilon - 1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ , \\ u > 0 & \text{in } \Omega \end{cases} \tag{1.5}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$, $N > 3$, $\sigma \in (0, N)$, $\lambda \geq 0$, $\varepsilon > 0$ and $q_\varepsilon = 2^*_\sigma - \varepsilon$ with $2^*_\sigma := \frac{2N-\sigma}{N-2}$. They also achieved a similar result, which is the existence of at least $cat_{\bar{\Omega}}(\bar{\Omega}) + 1$ solutions for (1.5) when the exponent q_ε is close to 2^*_σ and the domain is not contractible.

The aim of this paper is to investigate the fractional counterpart of the above problem (1.5). Indeed, according to the large amount of literatures on fractional Laplacian appearing in the last years, it is very natural to ask whether a similar result holds for the corresponding problem to (1.5) with the fractional Laplacian. Therefore, we would like to study the slightly critical problem (1.1) with two nonlocal framework caused by the fractional Laplacian and the convolution term, and finally implement a multiplicity result.

It is known that if Ω is a bounded starshaped domain and the nonlinearity is of critical growth, namely, p_ε is replaced by $2^*_{\sigma,s}$, then the problem (1.1) has no solution according to the Pohozaev identity (refer to [21])

$$-s\lambda \int_\Omega u^2 \mathrm{d}x = \frac{\Gamma(s+1)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x\cdot\nu)\mathrm{d}S, \quad \text{where } \delta = \mathrm{dist}(x, \partial\Omega).$$

However, if the critical exponent is perturbed slightly, namely, $p_\varepsilon = 2^*_{\sigma,s} - \varepsilon$, then the existence of solution can be shown and even multiple solutions exist. That is the theme of this paper. We also want to mention [20] in which equation (1.1) with critical nonlinearity (that is $\varepsilon = 0$) is studied on a bounded domain and the analogous of Brézis-Nirenberg problem is proved. The main difference between the two problems is that we use the perturbation of the exponent to obtain the multiplicity result, while in the paper of Mukherjee and Sreenadh is the parameter λ in front of the linear term which plays a key role.

In order to complete the study to (1.1), we shall apply variational methods, nonlocal global compactness, Lusternik-Schnirelmann category theory and the technique introduced by [5,6]. We hope to make a comparison between the category of some sublevel set of the functional and the category of the domain Ω. Hence we need to consider two limit problems and make a careful study on the behavior of the associated functionals and their related minimal levels, where the concentration property of the Talenti-Aubin functions is applied for more than once. About the nonlocal global compactness, we would like to mention a recent work by He and R*ă*dulescu [16], in which they obtained a very useful property (see Lemma 4.6) for the (PS)-sequence to the limit problem (4.1). By means of this property, we establish a crucial splitting lemma (Lemma 4.7) to the second limit problem (4.8). Together with the barycenter map (5.1), we can find the sublevels of the functional with barycenter not away from Ω and category greater than $cat_{\bar{\Omega}}(\bar{\Omega})$, and then get the existence of at least $cat_{\bar{\Omega}}(\bar{\Omega})$ solutions as $\varepsilon \to 0$ according to the Lusternik-Schnirelmann category theory. However, the barycenter map here is different from that in Ghimenti and Pagliardini [15], since the fractional Laplacian makes nonlocal effect. In order to give the existence of another solution when Ω is not contractible, we construct a compact and contractible set T_ε containing only positive functions and being a subset of the Nehari manifold.

Apart from the case of $p \to 2^*$, the works in [5,6] also treat the situation of $\lambda \to +\infty$. Although in this paper we don't explore the second situation in detail, our result still tells that the domain topology gives a lower bound on the number of solutions to (1.1) when the parameter λ is very large.

Now we state our main result, which gives an affirmative answer on the possibility of extending the result in Ghimenti and Pagliardini [15] to the fractional case.

Theorem 1.1. *Assume that $s \in (0,1)$, $\lambda \geq 0$, $N > 4s$ and $\sigma \in (0, N)$. Then there exists $\hat{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \hat{\varepsilon}]$, the problem (1.1) has at least $cat_{\bar{\Omega}}(\bar{\Omega})$ low energy solutions. Moreover, if Ω is not contractible, there is another solution with higher energy.*

Remark 1.2. *The assumption $N > 4s$ is needed to estimate some integrals related to the bubbles of the limit problem, such as (4.4), (4.5) and (4.6). Indeed, it is (4.6) that requires $N > 4s$, while for (4.4) and (4.5), $N > 2s$ is enough.*

Remark 1.3. *If $cat_{\bar{\Omega}}(\bar{\Omega}) = 1$, then it's easy to achieve the existence of one solution. Indeed, it can be obtained in a simplier way by the famous Mountain Pass Theorem. Additionally, if Ω is not contractible, we finally obtain $cat_{\bar{\Omega}}(\bar{\Omega})$ low energy solutions and another high energy solution. Therefore, we assume $cat_{\bar{\Omega}}(\bar{\Omega}) > 1$ in what follows.*

Our paper in the following is organized like this: Section 2 contains the working spaces, some essential definitions, several classical results and useful relations. Section 3 gives the variational structure of this problem, the functional setting and some preliminary results. Section 4 serves two limit problems and especially a nonlocal splitting lemma, which play a key role in the arguments of the main theorem. Section 5 contributes to the proof of our main result.

2 Notations and preliminaries

For $s \in (0,1)$, we denote $2_s^* := \frac{2N}{N-2s}$ as the fractional critical Sobolev exponent, and define two Hilbert spaces:

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

with the inner product

$$(u,v)_{H^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} uv\mathrm{d}x$$

and the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} u^2 \mathrm{d}x \right)^{\frac{1}{2}},$$

and

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2_s^*}(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

with the inner product

$$(u,v)_{D^{s,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y$$

and the norm

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}.$$

Here we have used the fractional Sobolev inequality (see Theorem 1.1 in Cotsiolis et al [8] and Proposition 3.6 in Di Nezza et al [10])

$$\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leq S_{N,s} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y, \quad \forall u \in C_0^\infty(\mathbb{R}^N), \tag{2.1}$$

where $S_{N,s} := \frac{1}{2}(4\pi)^{-s} \frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})} \left[\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right]^{\frac{2s}{N}} C_{N,s}$ is achieved if and only if

$$u(x) = A(\alpha^2 + (x-x_0)^2)^{-\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N$$

for some fixed constant $A \in \mathbb{R}$ and parameters $\alpha \in \mathbb{R} \setminus \{0\},\ x_0 \in \mathbb{R}^N$. Meanwhile, (2.1) means that $D^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^N)$.

Let

$$H_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}$$

and

$$D_0^{s,2}(\Omega) = \left\{u \in D^{s,2}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}.$$

Since Ω is a bounded domain with Lipschitz boundary, it follows from Corollary 5.5 in Di Nezza [10] that $H_0^s(\Omega)$ can be regarded as the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^N)$ and $D_0^{s,2}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $D^{s,2}(\mathbb{R}^N)$. Moreover, $H_0^s(\Omega) = D_0^{s,2}(\Omega)$. According to Theorems 6.5 and 7.2 in Di Nezza et al [10], $H_0^s(\Omega)$ is continuously embedded in $L^q(\Omega)$ for $q \in [1, 2_s^*]$, and is compactly embedded in $L^q(\Omega)$ for $q \in [1, 2_s^*)$. Notice that $\lambda \geq 0$, we can choose

$$\|u\|_\lambda := \left(\int_\Upsilon \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega u^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

as an equivalent norm in $H_0^s(\Omega)$, where $\Upsilon := \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$ with $\Omega^c := \mathbb{R}^N \setminus \Omega$.

Recall the well-known Lion's Lemma, which will be useful for showing the achievement of the least energy on Nehari manifold.

Lemma 2.1. *(Felmer et al [12]) Let $s \in (0, 1)$ and $r > 0$. If $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N} \int_{B_r(y)} |u_n(x)|^2 dx \to 0 \quad \text{as } n \to \infty,$$

then $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$.

Now we recall some information on Lusternik-Schnirelmann category.

Definition 2.2. *Let A be a closed subset of a topological space X. The category of A in X, denoted by $cat_X(A)$, is the least integer k such that $A \subseteq A_1 \cup A_2 \cup \cdots \cup A_k$ with A_i closed and contractible in X for each $i = 1, 2, \cdots, k$.*

We set $cat_X(\varnothing) = 0$, and $cat_X(A) = +\infty$ if there is no integer with the above property. Without confusion, we write $cat(X)$ for $cat_X(X)$.

Remark 2.3. *(Benci et al [5]) Let X and Y be two topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continuous operators such that $g \circ f$ is homotopic to the identity on X, then $cat(X) \leq cat(Y)$.*

Proposition 2.4. *(Benci et al [6]) Let J be a C^1 real functional on a complete $C^{1,1}$ Banach manifold M. If J is bounded from below and satisfies Palais-Smale condition on M, then J has at least $cat(J^d)$ critical points in J^d, where $J^d := \{u \in M : J(u) \leq d\}$. Moreover, if M is contractible and $cat(J^d) > 1$, then there is at least one critical point $u \notin J^d$.*

We also remind the famous Hardy-Littlewood-Sobolev inequality and some results linked to it.

Lemma 2.5. *(Lieb et al [17]) Let $r, t > 1$ and $0 < \sigma < N$ with $\frac{1}{r} + \frac{\sigma}{N} + \frac{1}{t} = 2$, $f \in L^r(\mathbb{R}^N)$ and $g \in L^t(\mathbb{R}^N)$. Then there exists a constant $C_{N,\sigma,r}$ independent of f and g such that*

$$\left|\int_{\mathbb{R}^{2N}} \frac{f(x)g(y)}{|x - y|^\sigma} dxdy\right| \leq C_{N,\sigma,r} \|f\|_{L^r(\mathbb{R}^N)} \|g\|_{L^t(\mathbb{R}^N)}.$$

Moreover, this equality holds if and only if $r = t = \frac{2N}{2N-\sigma}$, $g = Cf$ (C is a constant) and

$$f(x) = B(\alpha^2 + |x - x_0|^2)^{-\frac{2N-\sigma}{2}}, \quad x \in \mathbb{R}^N$$

for some $B \in \mathbb{C}$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $x_0 \in \mathbb{R}^N$.

Due to Lemma 2.5, embedding result and the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_\lambda$, we have the following result.

Remark 2.6. *For any $u \in H_0^s(\Omega)$, there exists a constant $C > 0$ such that*

$$\int_{\Omega\times\Omega} \frac{|u(x)|^{p_\varepsilon}|u(y)|^{p_\varepsilon}}{|x-y|^\sigma} dxdy \le C_{N,\sigma,s,\varepsilon}\Big\|u\Big\|^{2p_\varepsilon}_{L^{\frac{2Np_\varepsilon}{2N-\sigma}}(\Omega)} \le C\|u\|_\lambda^{2p_\varepsilon}.$$

Using again the Hardy-Littlewood-Sobolev inequality and embedding result, we also get that for all $u \in D^{s,2}(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^*_{\sigma,s}}|u(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\sigma,s}}} \le C_{N,\sigma,s}\|u\|^2_{L^{2^*_s}(\mathbb{R}^N)}. \tag{2.2}$$

Define

$$S^s_{H,L} := \inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)|^{2^*_{\sigma,s}}|u(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\sigma,s}}}} \tag{2.3}$$

and it follows from Lemma 2.5, (2.1) and (2.2) that $S^s_{H,L}$ is achieved if and only if

$$u(x) = U_{R,a}(x) := A\left(\frac{R}{1+R^2|x-a|^2}\right)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N, \tag{2.4}$$

where $A > 0$ is some fixed constant, $R > 0$ and $a \in \mathbb{R}^N$ are parameters. Absolutely, $S^s_{H,L} = (S_{N,s}C_{N,\sigma,s})^{-1}$. We can also define

$$S^{s,\Omega}_{H,L} := \inf_{u\in D_0^{s,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\Upsilon} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y}{\left(\displaystyle\int_{\Omega\times\Omega} \frac{|u(x)|^{2^*_{\sigma,s}}|u(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\sigma,s}}}}. \tag{2.5}$$

By means of Lemma 2.2 in Mukherjee et al [20], $S^{s,\Omega}_{H,L} = S^s_{H,L}$ and $S^{s,\Omega}_{H,L}$ is never achieved unless $\Omega = \mathbb{R}^N$.

In the following arguments, we denote

$$\|u\|_{s,2} := \|u\|_{D^{s,2}(\mathbb{R}^N)}, \quad |u|_q := \|u\|_{L^q(\mathbb{R}^N)},$$

$$u^+ := \max\{0,u\}, \quad u^- := \min\{0,u\}, \quad \mathbb{R}^+ := (0,+\infty).$$

For simplicity and without destruction, we drop the constant $C_{N,s}$ in the definition of $(-\Delta)^s$, and we shall use C to represent various positive constants, which may be different in different places.

3 Variational setting

To study the problem (1.1), we consider the associated functional $J_\varepsilon : H_0^s(\Omega) \to \mathbb{R}$ given by

$$J_\varepsilon(u) := \frac{1}{2}\|u\|^2_\lambda - \frac{1}{2p_\varepsilon}\int_{\Omega\times\Omega} \frac{(u^+(x))^{p_\varepsilon}(u^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y,$$

and the associated Nehari manifold

$$\mathcal{N}_\varepsilon := \left\{u \in H_0^s(\Omega) \setminus \{0\} : \left\langle J_\varepsilon'(u), u\right\rangle = 0\right\}.$$

We set the least energy by

$$\vartheta_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

It is easy to verify that $J_\varepsilon \in C^1(H_0^s(\Omega), \mathbb{R})$, since for any $v \in H_0^s(\Omega)$,

$$\begin{aligned}\left\langle J_\varepsilon'(u), v\right\rangle = &\int_\Upsilon \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y + \lambda\int_\Omega uv\mathrm{d}x \\ &- \int_{\Omega\times\Omega} \frac{(u^+(y))^{p_\varepsilon}(u^+(x))^{p_\varepsilon-1}v(x)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Now we establish some preliminary results.

Proposition 3.1. *The nonzero critical points of J_ε coincide with the solutions of the problem* (1.1).

Proof. It is clear that the nonzero critical points of J_ε are solutions of

$$\begin{cases} (-\Delta)^s u + \lambda u = \left(\displaystyle\int_\Omega \frac{(u^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}y\right)(u^+)^{p_\varepsilon-1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ u \neq 0 & \text{in } \Omega \end{cases}. \tag{3.1}$$

We claim that (3.1) is equivalent to (1.1).

On one hand, if $u \in H_0^s(\Omega)$ is a solution of (1.1), then $u = u^+$, and thus u is a solution of (3.1).

On the other hand, if $u \in H_0^s(\Omega)$ is a solution of (3.1), then u is a critical point of J_ε. It follows from

$$\begin{aligned}\int_\Upsilon \frac{|u^-(x)-u^-(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = &\int_{\{u>0\}\times\{u<0\}} \frac{|-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\{u<0\}\times\{u>0\}} \frac{|u(x)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\{u<0\}\times\{u<0\}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ \leq &\int_{\{u>0\}\times\{u<0\}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\{u<0\}\times\{u>0\}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ &+ \int_{\{u<0\}\times\{u<0\}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\ \leq &\int_\Upsilon \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\end{aligned}$$

that $u^- \in H_0^s(\Omega)$. Now we use u^- as a test function and obtain that

$$\begin{aligned}
0 = \left\langle J_\varepsilon'(u), u^- \right\rangle = & \int_\Upsilon \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega (u^-)^2 \mathrm{d}x \\
& + \int_{\Omega\times\Omega} \frac{(u^+(y))^{p_\varepsilon}(u^+(x))^{p_\varepsilon-1}u^-(x)}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y \\
= & \int_{\{u<0\}\times\{u<0\}} \frac{(u^-(x)-u^-(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
& + \int_{\{u>0\}\times\{u<0\}} \frac{(u^+(x)-u^-(y))(-u^-(y))}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\
& + \int_{\{u<0\}\times\{u>0\}} \frac{(u^-(x)-u^+(y))u^-(x)}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega (u^-)^2 \mathrm{d}x \\
\geq & \int_\Upsilon \frac{(u^-(x)-u^-(y))^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega (u^-)^2 \mathrm{d}x \\
= & \|u^-\|_\lambda^2,
\end{aligned}$$

which implies $u^- = 0$. Hence $u = u^+ \geq 0$. By Theorem 3.2 in d'Avenia et al [9], we get $u \in C^{0,\mu}(\mathbb{R}^N)$ with $\mu \in (0,1)$. Thus $u \in L^\infty(\mathbb{R}^N)$. Due to Theorem 1 and its proof in Du Plessis [11], we have $|x|^{-\sigma} * u^{p_\varepsilon} \in C^{0,\mu+\sigma}(\mathbb{R}^N)$ if $\mu+\sigma \in (0,1)$, and $|x|^{-\sigma} * u^{p_\varepsilon} \in C^{0,1}(\mathbb{R}^N)$ if $\mu+\sigma \in [1, N+1)$. As Theorem 1.4 in Felmer et al [12], we obtain that $u \in C^{0,2s+\mu}(\mathbb{R}^N)$ if $2s+\mu \leq 1$, and $u \in C^{1,2s+\mu-1}(\mathbb{R}^N)$ if $2s+\mu > 1$. This regularity makes sure (1.2) hold for u in the pointwise sense. Suppose $u(x_0) = 0$ for some $x_0 \in \Omega$, then the equation in (3.1) deduces $((-\Delta)^s u)(x_0) = 0$. By using (1.2) again, we get $u \equiv 0$, which is impossible. Hence $u > 0$ in Ω. Concequently, u is also a solution of (1.1). □

By Remark 2.6, we can get the common property of Nehari manifold, that is, there exists a constant $C > 0$ such that $\|u\|_\lambda > C$ and $J_\varepsilon(u) > C$ for all $u \in \mathcal{N}_\varepsilon$. In addition, we also see that for any $0 \neq u \in H_0^s(\Omega)$, there is an unique constant $t_\varepsilon(u) > 0$ such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$. Based on these information and a standard proof, we have the following result.

Lemma 3.2. *There holds*

$$0 < \vartheta_\varepsilon = \inf_{u\in H_0^s(\Omega)\setminus\{0\}} \max_{t\geq 0} J_\varepsilon(tu) = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)),$$

where

$$\Gamma_\varepsilon := \left\{\gamma \in C\left([0,1], H_0^s(\Omega)\right) : \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\right\}.$$

From Lemma 2.1, Lemma 3.2, and a similar proof to that of Theorem 1 in Moroz et al [19] or Lemma 3.5 in Liu et al [18], the following result is true.

Lemma 3.3. *ϑ_ε is achieved by a function $u_\varepsilon \in \mathcal{N}_\varepsilon$, that is $\vartheta_\varepsilon = J_\varepsilon(u_\varepsilon)$.*

Finally, we review Palais-Smale sequence simplified by (PS)-sequence.

Definition 3.4. *A sequence $\{u_n\}_{n\in\mathbb{N}}$ is called a (PS)-sequence of J_ε, if*

$$\{J_\varepsilon(u_n)\}_{n\in\mathbb{N}} \text{ is bounded in } \mathbb{R} \text{ and } J_\varepsilon'(u_n) \to 0 \text{ in } H_0^{-s}(\Omega),$$

where $H_0^{-s}(\Omega)$ signifies the dual space of $H_0^s(\Omega)$. If every (PS)-sequence of J_ε has a convergent subsequence, then we say J_ε satisfies the (PS)-condition on $H_0^s(\Omega)$.

In fact, J_ε does satisfy the (PS)-condition on $H_0^s(\Omega)$ globally.

Lemma 3.5. *If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ is a (PS)-sequence of the constrained functional $J_\varepsilon|_{\mathcal{N}_\varepsilon}(u) = \frac{p_\varepsilon-1}{2p_\varepsilon}\|u\|_\lambda^2$, then it is a (PS)-sequence of the free functional J_ε on $H_0^s(\Omega)$.*

The above lemma means that the Nehari manifold is a natural constraint for J_ε, whose proof is standard and is omitted (see [2] for details). For simplicity, we write J_ε instead of $J_\varepsilon|_{\mathcal{N}_\varepsilon}$.

4 Two Limit problems

In this section, we track $\lim\limits_{\varepsilon\to 0}\vartheta_\varepsilon$, for which we consider two limit problems.

The first limit problem is

$$\begin{cases} (-\Delta)^s u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{u^{2^*_{\sigma,s}}(y)}{|x-y|^\sigma}\mathrm{d}y\right) u^{2^*_{\sigma,s}-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \end{cases}. \tag{4.1}$$

For (4.1), we define the corresponding functional $J_* : D^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_*(u) := \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y - \frac{1}{2\cdot 2^*_{\sigma,s}}\int_{\mathbb{R}^{2N}} \frac{(u^+(x))^{2^*_{\sigma,s}}(u^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y,$$

the Nehari manifold associated to J_* by

$$\mathcal{N}_* := \left\{u \in D^{s,2}(\mathbb{R}^N)\setminus\{0\} : \left\langle J_*'(u), u\right\rangle = 0\right\},$$

and the least energy by

$$\vartheta_* := \inf_{u\in\mathcal{N}_*} J_*(u).$$

In the following result, we compute the relation between ϑ_* and $S^s_{H,L}$ given in (2.3).

Lemma 4.1. *There holds*

$$\vartheta_* = \left(\frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}}\right)(S^s_{H,L})^{\frac{2^*_{\sigma,s}}{2^*_{\sigma,s}-1}},$$

and ϑ_* is achieved only by functions $(S^s_{H,L})^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}}U_{R,a}$ with $U_{R,a}$ defined in (2.4).

Proof. For $u \in D^{s,2}(\mathbb{R}^N)\setminus\{0\}$, we have

$$\max_{t>0} J_*(tu) = \left(\frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}}\right)\left(\frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{(u^+(x))^{2^*_{\sigma,s}}(u^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_{\sigma,s}}}}\right)^{\frac{2^*_{\sigma,s}}{2^*_{\sigma,s}-1}}.$$

Therefore, $\vartheta_* = \inf_{u\in D^{s,2}(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} J_*(tu) \geq \left(\frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}}\right)(S^s_{H,L})^{\frac{2^*_{\sigma,s}}{2^*_{\sigma,s}-1}}$.

Additionally, notice that $\tilde{u} := (S^s_{H,L})^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}}U_{R,a}$ satisfies (4.1), we get that $\tilde{u} \in \mathcal{N}_*$ and

$$\vartheta_* \leq J_*(\tilde{u}) = \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right)\int_{\mathbb{R}^{2N}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \left(\frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}}\right)(S^s_{H,L})^{\frac{2^*_{\sigma,s}}{2^*_{\sigma,s}-1}}.$$

□

We want to use the minimizers of ϑ_* to construct the approximating sequences for ϑ_ε. Since Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, we choose $r > 0$ small enough such that

$$\Omega_r^+ := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq r\}$$

and

$$\Omega_r^- := \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq r\}$$

are homotopically equivalent to Ω. For $R > 1$ and $x_0 \in \Omega_r^-$, define

$$u_{R,x_0}(\cdot) := R^{\frac{N-2s}{2}} U_{1,0}(R(\cdot - x_0))\phi_{r,x_0}(\cdot) = U_{R,x_0}(\cdot)\phi_{r,x_0}(\cdot), \tag{4.2}$$

where $U_{1,0}$ is the standard bubble function defined in (2.4) with $R = 1$ and $a = 0$, and ϕ_{r,x_0} is a cut-off function defined by

$$\phi_{r,x_0}(x) := \begin{cases} 1 & \text{if } |x - x_0| < \frac{r}{2} \\ (0,1) & \text{if } \frac{r}{2} \leq |x - x_0| \leq r \\ 0 & \text{if } |x - x_0| > r \end{cases}. \tag{4.3}$$

It follows from $x_0 \in \Omega_r^-$ that $u_{R,x_0} \in H_0^s(\Omega)$.

Noticing $N > 4s$ and referring to Proposition 21 in Servadei et al [22], we can verify that

$$\int_\Upsilon \frac{|u_{R,x_0}(x) - u_{R,x_0}(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \leq \int_{\mathbb{R}^{2N}} \frac{|U_{1,0}(x) - U_{1,0}(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + o_R(1), \tag{4.4}$$

$$\int_{\Omega\times\Omega} \frac{|u_{R,x_0}(x)|^{2^*_{\sigma,s}} |u_{R,x_0}(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{2N}} \frac{|U_{1,0}(x)|^{2^*_{\sigma,s}} |U_{1,0}(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y + o_R(1), \tag{4.5}$$

$$\int_\Omega |u_{R,x_0}(x)|^2 \mathrm{d}x = \frac{1}{R^{2s}} \int_{\mathbb{R}^N} |U_{1,0}(x)|^2 \mathrm{d}x + o_R(1) = o_R(1), \tag{4.6}$$

where $o_R(1)$ denotes the quantities that tend to 0 as $R \to +\infty$.

Lemma 4.2. $\limsup\limits_{\varepsilon\to 0} \vartheta_\varepsilon \leq \vartheta_*$.

Proof. For any $\varepsilon > 0$, there is an unique $t_\varepsilon(u_{R,x_0}) > 0$ satisfying $t_\varepsilon(u_{R,x_0})u_{R,x_0} \in \mathcal{N}_\varepsilon$. Thus

$$\|t_\varepsilon(u_{R,x_0})u_{R,x_0}\|_\lambda^2 = \int_{\Omega\times\Omega} \frac{|t_\varepsilon(u_{R,x_0})u_{R,x_0}(x)|^{p_\varepsilon} |t_\varepsilon(u_{R,x_0})u_{R,x_0}(y)|^{p_\varepsilon}}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y,$$

which implies

$$t_\varepsilon^{2p_\varepsilon - 2}(u_{R,x_0}) = \frac{\|u_{R,x_0}\|_\lambda^2}{\displaystyle\int_{\Omega\times\Omega} \frac{|u_{R,x_0}(x)|^{p_\varepsilon} |u_{R,x_0}(y)|^{p_\varepsilon}}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y}.$$

In virtue of (4.4), (4.5) and (4.6), we get

$$
\begin{aligned}
\lim_{\varepsilon\to 0} t_\varepsilon(u_{R,x_0}) &= \left(\frac{\|u_{R,x_0}\|_\lambda^2}{\displaystyle\int_{\Omega\times\Omega} \frac{|u_{R,x_0}(x)|^{2^*_{\sigma,s}}|u_{R,x_0}(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y} \right)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} \\
&\le \left(\frac{\|U_{1,0}\|_{s,2}^2 + o_R(1)}{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|U_{1,0}(x)|^{2^*_{\sigma,s}}|U_{1,0}(y)|^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y + o_R(1)} \right)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} \\
&= (S^s_{H,L})^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} + o_R(1),
\end{aligned}
\tag{4.7}
$$

and then

$$
J_\varepsilon(t_\varepsilon(u_{R,x_0})u_{R,x_0}) = \frac{p_\varepsilon - 1}{2p_\varepsilon}\|t_\varepsilon(u_{R,x_0})u_{R,x_0}\|_\lambda^2 = \frac{p_\varepsilon-1}{2p_\varepsilon}t_\varepsilon^2(u_{R,x_0})\|U_{1,0}\|_{s,2}^2 + o_R(1).
$$

Therefore,

$$
\lim_{\varepsilon\to 0} J_\varepsilon(t_\varepsilon(u_{R,x_0})u_{R,x_0}) \le \frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}}(S^s_{H,L})^{\frac{1}{2^*_{\sigma,s}-1}}\|U_{1,0}\|_{s,2}^2 + o_R(1).
$$

For any $\delta > 0$, we can choose R large enough such that $o_R(1) < \delta$. By Lemma 4.1, we obtain that

$$
\limsup_{\varepsilon\to 0} \vartheta_\varepsilon \le \lim_{\varepsilon\to 0} J_\varepsilon(t_\varepsilon(u_{R,x_0})u_{R,x_0}) < \frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}}(S^s_{H,L})^{\frac{1}{2^*_{\sigma,s}-1}}\|U_{1,0}\|_{s,2}^2 + \delta = \vartheta_* + \delta.
$$

This completes the proof by letting $\delta \to 0$. □

Remark 4.3. *The groundstates u_ε are also bounded uniformly in ε. Indeed, by Lemma 3.3,*

$$
\|u_\varepsilon\|_\lambda^2 = \frac{2p_\varepsilon}{p_\varepsilon - 1}J_\varepsilon(u_\varepsilon) = \frac{2p_\varepsilon}{p_\varepsilon-1}\vartheta_\varepsilon.
$$

Now we introduce the second limit problem which acts as the mediator between the problem (1.1) and the first limit problem (4.1). In particular, it will play an important role in computing $\lim\limits_{\varepsilon\to 0}\vartheta_\varepsilon$. Consider

$$
\begin{cases}
(-\Delta)^s u + \lambda u = \left(\displaystyle\int_\Omega \frac{u^{2^*_{\sigma,s}}(y)}{|x-y|^\sigma}\mathrm{d}y\right)u^{2^*_{\sigma,s}-1} & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\
u > 0 & \text{in } \Omega
\end{cases}.
\tag{4.8}
$$

The existence, nonexistence and regularity results of weak solutions to (4.8) have been studied in Mukherjee et al [20]. As usual, we define the energy functional $J_*^\Omega : H_0^s(\Omega) \to \mathbb{R}$ for (4.8) by

$$
J_*^\Omega(u) := \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2\cdot 2^*_{\sigma,s}}\int_{\Omega\times\Omega}\frac{(u^+(x))^{2^*_{\sigma,s}}(u^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y,
$$

the associated Nehari manifold by

$$
\mathcal{N}_*^\Omega := \left\{u \in H_0^s(\Omega)\setminus\{0\} : \left\langle (J_*^\Omega)'(u), u\right\rangle = 0\right\},
$$

and the least energy by

$$
\vartheta_*^\Omega := \inf_{u\in\mathcal{N}_*^\Omega} J_*^\Omega(u).
$$

Lemma 4.4. *$\vartheta_*^\Omega = \vartheta_*$ and ϑ_*^Ω is not achieved.*

Proof. We first show $\vartheta_*^\Omega = \vartheta_*$. For one thing, for any $u \in \mathcal{N}_*^\Omega$, we extend u to zero outside Ω, and then there is an unique $t_*(u) \in (0,1)$ such that $t_*(u)u \in \mathcal{N}_*$. Hence

$$\vartheta_* \leq J_*(t_*(u)u) = \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right)\|t_*(u)u\|_{s,2}^2 < \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right)\|u\|_\lambda^2,$$

which implies $\vartheta_* \leq \vartheta_*^\Omega$. For another, for all $x_0 \in \Omega_r^-$ and $R > 1$, we take u_{R,x_0} defined in (4.2) and an unique $t_*^\Omega(u_{R,x_0}) > 0$ satisfying $t_*^\Omega(u_{R,x_0})u_{R,x_0} \in \mathcal{N}_*^\Omega$. Proceeding as the proof of Lemma 4.2, we obtain that for any $\delta > 0$, there exists $R > 1$ such that

$$\vartheta_*^\Omega \leq J_*^\Omega(t_*^\Omega(u_{R,x_0})u_{R,x_0}) < \vartheta_* + \delta.$$

We get $\vartheta_*^\Omega \leq \vartheta_*$ by means of the arbitrariness of δ. Hence $\vartheta_*^\Omega = \vartheta_*$.

Next we show that ϑ_*^Ω can not be achieved. Indeed, suppose by contradiction that $v \in \mathcal{N}_*^\Omega$ satisfies $J_*^\Omega(v) = \vartheta_*^\Omega$. We extend v to zero outside Ω. There is an unique $t_*(v) > 0$ satisfying $t_*(v)v \in \mathcal{N}_*$. Thus

$$\begin{aligned} t_*^2(v)\|v\|_{s,2}^2 =& t_*^{2\cdot 2^*\sigma,s}(v)\int_{\mathbb{R}^{2N}} \frac{(v^+(x))^{2^*_{\sigma,s}}(v^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \\ =& t_*^{2\cdot 2^*\sigma,s}(v)\int_{\Omega\times\Omega} \frac{(v^+(x))^{2^*_{\sigma,s}}(v^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \\ =& t_*^{2\cdot 2^*\sigma,s}(v)\|v\|_\lambda^2, \end{aligned}$$

which together with $\lambda > 0$ implies that

$$t_*(v) = \left(\frac{\|v\|_{s,2}^2}{\|v\|_\lambda^2}\right)^{\frac{1}{2\cdot 2^*\sigma,s-2}} < 1.$$

Hence

$$\begin{aligned} \vartheta_* \leq J_*(t_*(v)v) &= \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right) t_*^{2\cdot 2^*\sigma,s}(v)\int_{\mathbb{R}^{2N}} \frac{(v^+(x))^{2^*_{\sigma,s}}(v^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \\ &= \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right) t_*^{2\cdot 2^*\sigma,s}(v)\int_{\Omega\times\Omega} \frac{(v^+(x))^{2^*_{\sigma,s}}(v^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \\ &< \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right)\int_{\Omega\times\Omega} \frac{(v^+(x))^{2^*_{\sigma,s}}(v^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y = \vartheta_*^\Omega = \vartheta_*, \end{aligned}$$

which is a contradiction.

If $\lambda = 0$, then it follows from $v \in \mathcal{N}_*^\Omega$ that $t_*(v) = 1$ and then $v \in \mathcal{N}_*$. Moreover,

$$J_*(v) = J_*^\Omega(v) = \vartheta_*^\Omega = \vartheta_* = \inf_{u\in\mathcal{N}_*} J_*(u),$$

which implies that $v = (S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R,a} > 0$. But it is impossible by the construction of v. □

Now we prove the main result in this section.

Proposition 4.5. *There holds*

$$\lim_{\varepsilon\to 0}\vartheta_\varepsilon=\vartheta_*.$$

Proof. In virtue of Lemma 4.2, it is sufficient to show that

$$\liminf_{\varepsilon\to 0}\vartheta_\varepsilon\geq\vartheta_*. \tag{4.9}$$

By Lemma 3.3, we choose $u_\varepsilon\in\mathcal{N}_\varepsilon$ satisfying $J_\varepsilon(u_\varepsilon)=\vartheta_\varepsilon$. There is an unique $t_*^\Omega(u_\varepsilon)>0$ such that $t_*^\Omega(u_\varepsilon)u_\varepsilon\in\mathcal{N}_*^\Omega$, namely,

$$\left(t_*^\Omega(u_\varepsilon)\right)^2\|u_\varepsilon\|_\lambda^2=\left(t_*^\Omega(u_\varepsilon)\right)^{2\cdot 2^*_{\sigma,s}}\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y.$$

Noticing $u_\varepsilon\in\mathcal{N}_\varepsilon$, we get that

$$\begin{aligned}\left(t_*^\Omega(u_\varepsilon)\right)^{2\cdot 2^*_{\sigma,s}-2}=&\frac{\|u_\varepsilon\|_\lambda^2}{\displaystyle\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y}\\=&\frac{\displaystyle\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{p_\varepsilon}(u_\varepsilon^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y}.\end{aligned}\tag{4.10}$$

We claim that

$$\limsup_{\varepsilon\to 0}t_*^\Omega(u_\varepsilon)\leq 1. \tag{4.11}$$

It follows from

$$\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{p_\varepsilon}(u_\varepsilon^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y=\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{p_\varepsilon}(u_\varepsilon^+(y))^{p_\varepsilon}}{|x-y|^{\sigma\cdot\frac{p_\varepsilon}{2^*_{\sigma,s}}}}\cdot\frac{1}{|x-y|^{\sigma\cdot\frac{\varepsilon}{2^*_{\sigma,s}}}}\mathrm{d}x\mathrm{d}y$$

and the Hölder inequality that

$$\begin{aligned}&\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{p_\varepsilon}(u_\varepsilon^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\\\leq&\left(\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{p_\varepsilon}{2^*_{\sigma,s}}}\cdot\left(\int_{\Omega\times\Omega}\frac{1}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{\varepsilon}{2^*_{\sigma,s}}}.\end{aligned}$$

We take a change of variables $\xi=x-y$, $\eta=x+y$. Then for $\rho=\rho(\Omega)>0$ large enough, we have

$$\int_{\Omega\times\Omega}\frac{1}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\leq\frac{1}{2}\int_{B_\rho(0)\times B_\rho(0)}\frac{1}{|\xi|^\sigma}\mathrm{d}\xi\mathrm{d}\eta\leq C_\rho\int_{B_\rho(0)}\frac{1}{|\xi|^\sigma}\mathrm{d}\xi=C_\rho,$$

where we have used the assumption $\sigma\in(0,N)$. Since ρ depends only on Ω, we obtain that

$$\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{p_\varepsilon}(u_\varepsilon^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\leq\left(\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{p_\varepsilon}{2^*_{\sigma,s}}}\cdot C_\Omega^{\frac{\varepsilon}{2^*_{\sigma,s}}}. \tag{4.12}$$

By inserting (4.12) into (4.10), we get that

$$\begin{aligned}\left(t_*^{\Omega}(u_\varepsilon)\right)^{2\cdot 2^*_{\sigma,s}-2} &\leq \left(\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\right)^{\frac{-\varepsilon}{2^*_{\sigma,s}}}\cdot C_\Omega^{\frac{\varepsilon}{2^*_{\sigma,s}}}\\ &=\left(\frac{C_\Omega}{\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y}\right)^{\frac{\varepsilon}{2^*_{\sigma,s}}}.\end{aligned}\tag{4.13}$$

Due to Remark 4.3 and (2.5), we see that $\int_{\Omega\times\Omega}\frac{(u_\varepsilon^+(x))^{2^*_{\sigma,s}}(u_\varepsilon^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y$ is bounded uniformly in ε. Hence we deduce (4.11) from (4.13).

Concequently,

$$\begin{aligned}\vartheta_*=\vartheta_*^\Omega\leq J_*^\Omega(t_*^\Omega(u_\varepsilon)u_\varepsilon)&=\left(\frac{1}{2}-\frac{1}{2\cdot 2^*_{\sigma,s}}\right)\left(t_*^\Omega(u_\varepsilon)\right)^2\|u_\varepsilon\|_\lambda^2\\ &=\left(t_*^\Omega(u_\varepsilon)\right)^2\frac{\frac{1}{2}-\frac{1}{2\cdot 2^*_{\sigma,s}}}{\frac{1}{2}-\frac{1}{2p_\varepsilon}}\left(\frac{1}{2}-\frac{1}{2p_\varepsilon}\right)\|u_\varepsilon\|_\lambda^2\\ &\leq(1+o_\varepsilon(1))\vartheta_\varepsilon,\end{aligned}$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$ and ϑ_ε is bounded by Lemma 4.2. Thus (4.9) is showed. □

In the end of this section, we give a technical lemma from He and Rădulescu [16], and state a nonlocal splitting lemma which gives a complete description for the functional J_*^Ω. This nonlocal splitting lemma is a variant of the classical one contained in Struwe [24].

Lemma 4.6. *(Lemma 3.1 in [16]) Let $\{v_n\}_{n\in\mathbb{N}}$ be a (PS)$_c$-sequence for the functional J_* with $v_n\rightharpoonup 0$ and $v_n\nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n\to\infty$. Then there exists a sequence $\{R_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^+$, a point sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ and a nontrival solution $v_0\in D^{s,2}(\mathbb{R}^N)$ of* (4.1) *such that, up to a subsequence of $\{v_n\}_{n\in\mathbb{N}}$, we have that*

$$\tilde{v}_n(x)=v_n(x)-R_n^{\frac{N-2s}{2}}v_0(R_n(x-x_n))+o_n(1)$$

is a (PS)$_{c-J_*(v_0)}$-sequence for J_*, where $o_n(1)\to 0$ as $n\to\infty$.

The above property on (PS)-sequence to the first limit problem (4.1) is very important in proving the following nonlocal splitting lemma for the fractional critical Choquard problem (4.8).

Lemma 4.7. *Let $\{v_n\}_{n\in\mathbb{N}}$ be a (PS)-sequence of J_*^Ω in $H_0^s(\Omega)$. Then there exist $k\in\mathbb{N}$, a point sequence $\{x_n^j\}_{n\in\mathbb{N}}\subset\Omega$, a radius sequence $\{R_n^j\}_{n\in\mathbb{N}}\subset\mathbb{R}^+$, a solution $v\in H_0^s(\Omega)$ of* (4.8)*, and nontrivial solutions $v^j\in D^{s,2}(\mathbb{R}^N)$ to* (4.1)*, where $j=1,2,\cdots,k$, such that a subsequence of $\{v_n\}_{n\in\mathbb{N}}$, denoted also by $\{v_n\}_{n\in\mathbb{N}}$, satisfies*

$$\Big\|v_n-v-\sum_{j=1}^k v^j_{R_n^j,x_n^j}\Big\|_{s,2}\to 0\quad \text{as } n\to\infty,$$

where

$$v^j_{R_n^j,x_n^j}(x):=(R_n^j)^{\frac{N-2s}{2}}v^j(R_n^j(x-x_n^j)),\quad j=1,2,\cdots,k.$$

Moreover,

$$J_*^\Omega(v_n)\to J_*^\Omega(v)+\sum_{j=1}^k J_*(v^j)\quad \text{as } n\to\infty.\tag{4.14}$$

Proof. **Step 1.** The (PS)-sequence $\{v_n\}_{n\in\mathbb{N}}$ of J_*^Ω is bounded in $H_0^s(\Omega)$. Thus up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H_0^s(\Omega)$ as $n\to\infty$. Moreover, v solves (4.8). Set $v_n^1 := v_n - v$. Then by Brézis-Lieb Lemma (see Lemma 1.32 and Remark 1.33 in Willem [25]), we get

$$\|v_n\|_{s,2}^2 - \|v_n^1\|_{s,2}^2 \to \|v\|_{s,2}^2 \quad \text{as } n\to\infty,$$
$$|v_n|_2^2 - |v_n^1|_2^2 \to |v|_2^2 \quad \text{as } n\to\infty,$$
$$(v_n^+)^{2^*\sigma,s} - ((v_n^1)^+)^{2^*\sigma,s} \to (v^+)^{2^*\sigma,s} \quad \text{in } L^{\frac{2N}{2N-\sigma}}(\Omega) \ \text{ as } n\to\infty.$$

By Lemma 2.1 in Liu et al [18], we have

$$|x|^{-\sigma} * (v_n^+)^{2^*\sigma,s} - |x|^{-\sigma} * ((v_n^1)^+)^{2^*\sigma,s} \to |x|^{-\sigma} * (v^+)^{2^*\sigma,s} \quad \text{in } L^{\frac{2N}{\sigma}}(\Omega) \ \text{ as } n\to\infty.$$

Proceeding as the arguments of Lemma 2.5 in Liu et al [18] with slight amendment, we obtain

$$\begin{aligned}&\int_{\Omega\times\Omega}\frac{(v_n^+(x))^{2^*\sigma,s}(v_n^+(y))^{2^*\sigma,s}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y - \int_{\Omega\times\Omega}\frac{((v_n^1)^+(x))^{2^*\sigma,s}((v_n^1)^+(y))^{2^*\sigma,s}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\\ \to&\int_{\Omega\times\Omega}\frac{(v^+(x))^{2^*\sigma,s}(v^+(y))^{2^*\sigma,s}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \quad \text{as } n\to\infty,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\int_{\Omega\times\Omega}\frac{(v_n^+(y))^{2^*\sigma,s}(v_n^+(x))^{2^*\sigma,s-1}\psi(x)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\\ &-\int_{\Omega\times\Omega}\frac{((v_n^1)^+(y))^{2^*\sigma,s}((v_n^1)^+(x))^{2^*\sigma,s-1}\psi(x)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y\\ \to&\int_{\Omega\times\Omega}\frac{(v^+(y))^{2^*\sigma,s}(v^+(x))^{2^*\sigma,s-1}\psi(x)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \ \text{ as } n\to\infty \text{ uniformly in } \psi\in H_0^s(\Omega).\end{aligned} \tag{4.16}$$

Indeed, the main differences with the proof of Lemma 2.5 in Liu et al [18] are

$$\begin{aligned}&\int_\Omega\left(|x|^{-\sigma} * (v^+)^{2^*\sigma,s}\right)\left((v_n^1)^+(x)\right)^{2^*\sigma,s-1}|\psi(x)|\mathrm{d}x\\ \le&\left(\int_\Omega\left(|x|^{-\sigma} * |v|^{2^*\sigma,s}\right)^{\frac{2N}{N+2s}}|v_n^1(x)|^{(2^*\sigma,s-1)\frac{2N}{N+2s}}\mathrm{d}x\right)^{\frac{N+2s}{2N}}|\psi|_{\frac{2N}{N-2s}}\\ \le& o_n(1)\|\psi\|_\lambda,\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega\left(|x|^{-\sigma} * \left((v_n^1)^+\right)^{2^*\sigma,s}\right)\left(v^+(x)\right)^{2^*\sigma,s-1}|\psi(x)|\mathrm{d}x\\ \le&\left(\int_\Omega\left(|x|^{-\sigma} * |v_n^1|^{2^*\sigma,s}\right)^{\frac{2N}{N+2s}}|v(x)|^{(2^*\sigma,s-1)\frac{2N}{N+2s}}\mathrm{d}x\right)^{\frac{N+2s}{2N}}|\psi|_{\frac{2N}{N-2s}}\\ \le& o_n(1)\|\psi\|_\lambda,\end{aligned}$$

where the former is ensured by $(|x|^{-\sigma} * |v|^{2^*\sigma,s})^{\frac{2N}{N+2s}} \in L^{\frac{N+2s}{\sigma}}(\Omega)$ and $|v_n^1|^{(2^*\sigma,s-1)\frac{2N}{N+2s}} \rightharpoonup 0$ in $L^{\frac{N+2s}{N+2s-\sigma}}(\Omega)$ as $n\to\infty$, and the latter is due to $|v|^{(2^*\sigma,s-1)\frac{2N}{N+2s}} \in L^{\frac{N+2s}{N+2s-\sigma}}(\Omega)$ and $|x|^{-\sigma} *$

$|v_n^1|^{2^*\sigma,s})^{\frac{2N}{N+2s}} \rightharpoonup 0$ in $L^{\frac{N+2s}{\sigma}}(\Omega)$ as $n \to \infty$. Therefore, as $n \to \infty$,

$$\begin{aligned} J_*^\Omega(v_n) - J_*^\Omega(v_n^1) &\to J_*^\Omega(v) \\ (J_*^\Omega)'(v_n) - (J_*^\Omega)'(v_n^1) &\to (J_*^\Omega)'(v) \quad \text{in } H_0^{-s}(\Omega). \end{aligned}$$

Since $(J_*^\Omega)'(v_n) \to 0$ in $H_0^{-s}(\Omega)$ as $n \to \infty$ and $(J_*^\Omega)'(v) = 0$, we get

$$(J_*^\Omega)'(v_n^1) \to 0 \quad \text{in } H_0^{-s}(\Omega) \text{ as } n \to \infty.$$

Moreover, it follows from $v_n^1 \to 0$ in $L^2(\Omega)$ as $n \to \infty$ that

$$\begin{aligned} J_*(v_n^1) &= J_*^\Omega(v_n^1) + o_n(1) = J_*^\Omega(v_n) - J_*^\Omega(v) + o_n(1) \\ J_*'(v_n^1) &= (J_*^\Omega)'(v_n^1) + o_n(1) = o_n(1). \end{aligned} \tag{4.17}$$

Thus $\{v_n^1\}_{n\in\mathbb{N}}$ is a (PS)-sequence of J_*.

Step 2. If $v_n^1 \to 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$, then the proof is completed with $k = 0$. If $v_n^1 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$, then it follows from Lemma 4.6 that there exist $\{R_n^1\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, $\{x_n^1\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and a nontrival solution $v^1 \in D^{s,2}(\mathbb{R}^N)$ of (4.1) such that

$$v_n^2(x) := v_n^1(x) - (R_n^1)^{\frac{N-2s}{2}} v^1(R_n^1(x - x_n^1)) + o_n(1)$$

is a (PS)-sequence for J_*.

Define

$$\begin{aligned} \tilde{v}_n^1(x) &:= (R_n^1)^{\frac{2s-N}{2}} v_n^1\Big(\frac{x}{R_n^1} + x_n^1\Big), \\ \tilde{v}_n^2(x) &:= (R_n^1)^{\frac{2s-N}{2}} v_n^2\Big(\frac{x}{R_n^1} + x_n^1\Big). \end{aligned}$$

Obviously, $\tilde{v}_n^2(x) = \tilde{v}_n^1(x) - v^1(x) + o_n(1)$. Meanwhile, $\tilde{v}_n^2 \rightharpoonup 0$ in $D^{s,2}(\mathbb{R}^N)$ due to the proof of Lemma 3.1 in [16]. Similarly as before,

$$\|v_n^2\|_{s,2}^2 = \|\tilde{v}_n^2\|_{s,2}^2 = \|\tilde{v}_n^1\|_{s,2}^2 - \|v^1\|_{s,2}^2 + o_n(1) = \|v_n\|_{s,2}^2 - \|v\|_{s,2}^2 - \|v^1\|_{s,2}^2 + o_n(1).$$

Since $J_*'(v^1) = 0$ and $J_*'(\tilde{v}_n^1) = o_n(1)$, we have $\|J_*'(v_n^2)\| = \|J_*'(\tilde{v}_n^2)\| = o_n(1)$. In addition, by (4.17), we get

$$J_*(v_n^2) = J_*(\tilde{v}_n^2) = J_*(\tilde{v}_n^1) - J_*(v^1) + o_n(1) = J_*^\Omega(v_n) - J_*^\Omega(v) - J_*(v^1) + o_n(1).$$

If $\tilde{v}_n^2 \to 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$, then $v_n^2 \to 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$, and the proof is completed with $k = 1$. If $\tilde{v}_n^2 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$, then $v_n^2 \nrightarrow 0$ in $D^{s,2}(\mathbb{R}^N)$ as $n \to \infty$, and it follows from Lemma 4.6 that there exist $\{R_n^2\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, $\{x_n^2\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and a nontrival solution $v^2 \in D^{s,2}(\mathbb{R}^N)$ of (4.1) such that

$$v_n^3(x) := v_n^2(x) - (R_n^2)^{\frac{N-2s}{2}} v^2(R_n^2(x - x_n^2)) + o_n(1)$$

is a (PS)-sequence for J_*.

Iterating the above procedure, we construct sequences v^j, x_n^j and R_n^j such that

$$\begin{aligned}
&v_n^k := v_n - v - \sum_{j=1}^{k-1} (R_n^j)^{\frac{N-2s}{2}} v^j \left(R_n^j(\cdot - x_n^j)\right) + o_n(1),\\
&\|v_n^k\|_{s,2}^2 = \|v_n\|_{s,2}^2 - \|v\|_{s,2}^2 - \sum_{j=1}^{k-1} \|v^j\|_{s,2}^2 + o_n(1),\\
&J_*(v_n^k) = J_*^\Omega(v_n) - J_*^\Omega(v) - \sum_{j=1}^{k-1} J_*(v^j) + o_n(1),\\
&J_*'(v_n^k) = o_n(1), \quad J_*'(v^j) = 0, \quad j = 1, 2, \cdots, k-1.
\end{aligned}$$

Step 3. By means of (2.1) and (2.2), we obtain that any nontrival critical point u of J_* satisfies

$$|u|_{2_s^*}^2 \le S_{N,s}\|u\|_{s,2}^2 = S_{N,s} \int_{\mathbb{R}^{2N}} \frac{(u^+(x))^{2^*_{\sigma,s}} (u^+(y))^{2^*_{\sigma,s}}}{|x-y|^\sigma} \mathrm{d}x\mathrm{d}y \le S_{N,s} C_{N,\sigma,s} |u|_{2_s^*}^{2\cdot 2^*_{\sigma,s}},$$

which implies that

$$J_*(u) = \left(\frac{1}{2} - \frac{1}{2\cdot 2^*_{\sigma,s}}\right)\|u\|_{s,2}^2 \ge \frac{2^*_{\sigma,s}-1}{2\cdot 2^*_{\sigma,s}} \frac{1}{S_{N,s}} (S_{H,L}^s)^{\frac{1}{2^*_{\sigma,s}-1}} > 0. \tag{4.18}$$

Theorefore, the above iteration must terminate at some finite index k by (4.18). Hence $\|v_n^k\|_{s,2} = \|\tilde{v}_n^k\|_{s,2} \to 0$ as $n \to \infty$. The proof of Lemma 4.7 is finished. □

In virtue of the above nonlocal splitting lemma and Lemmas 4.1, 4.4, we have the following immediate result.

Remark 4.8. *If there exists a (PS)-sequence for J_*^Ω at level ϑ_*^Ω, then*

$$v = 0, \quad k = 1, \quad v^1 = (S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R,a}$$

and $v_n - (S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,a_n} \to 0$ in $D^{s,2}(\mathbb{R}^n)$ as $n \to \infty$.

5 Proof of the main result

In order to apply Proposition 2.4 to prove Theorem 1.1, we need to construct a map from Ω_r^- to $\mathcal{N}_\varepsilon$ and a function from $\mathcal{N}_\varepsilon$ to Ω_r^+. We denote with the same symbol u its trivial extension out of the support of u. Referring to Figueiredo et al [13], we introduce the barycenter of a function $u \in D^{s,2}(\mathbb{R}^N)$ with compact support as $\beta(u) := (\beta^1(u), \beta^2(u), \cdots, \beta^N(u))$, where

$$\beta^i(u) := \frac{\displaystyle\int_{\mathbb{R}^N} x^i |u|^{2_s^*} \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u|^{2_s^*} \mathrm{d}x}, \quad i = 1, 2, \cdots N. \tag{5.1}$$

This barycenter map allows us to compare the topology of Ω with the topology of some suitable sublevels of J_ε. Exectly, we can show the following result according to Remark 4.8.

Proposition 5.1. *There exist* $\delta_0 > 0$ *and* $\varepsilon_0 = \varepsilon_0(\delta_0) > 0$ *such that for any* $\delta \in (0, \delta_0]$ *and for any* $\varepsilon \in (0, \varepsilon_0]$, *it holds*

$$u \in \mathcal{N}_\varepsilon \ \text{ and } \ J_\varepsilon(u) < \vartheta_\varepsilon + \delta \Longrightarrow \beta(u) \in \Omega_r^+.$$

Proof. Suppose on the contrary that there exist sequences $\delta_n \to 0$, $\varepsilon_n \to 0$ and $u_n \in \mathcal{N}_{\varepsilon_n}$ such that

$$J_{\varepsilon_n}(u_n) \le \vartheta_{\varepsilon_n} + \delta_n \quad \text{and} \quad \beta(u_n) \notin \Omega_r^+. \tag{5.2}$$

It follows from (5.2) and Proposition 4.5 that

$$J_{\varepsilon_n}(u_n) \to \vartheta_* \quad \text{as } n \to \infty, \tag{5.3}$$

and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H_0^s(\Omega)$. There is an unique $t_{*,n}^\Omega(u_n) > 0$ such that $t_{*,n}^\Omega(u_n)u_n \in \mathcal{N}_*^\Omega$. Set $p_n := 2_{\sigma,s}^* - \varepsilon_n$, we next evaluate

$$\begin{aligned}
&J_{\varepsilon_n}(u_n) - J_*^\Omega\left(t_{*,n}^\Omega(u_n)u_n\right)\\
&= \left(\frac{1}{2} - \frac{1}{2p_n}\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2} - \frac{1}{2\cdot 2_{\sigma,s}^*}\right)\left(t_{*,n}^\Omega(u_n)\right)^2\|u_n\|_\lambda^2\\
&= \left(\frac{1}{2} - \frac{1}{2p_n}\right)\left(1 - \left(t_{*,n}^\Omega(u_n)\right)^2\right)\|u_n\|_\lambda^2 - \left(\frac{1}{2p_n} - \frac{1}{2\cdot 2_{\sigma,s}^*}\right)\left(t_{*,n}^\Omega(u_n)\right)^2\|u_n\|_\lambda^2.
\end{aligned}$$

Similar to (4.11) in the proof of Proposition 4.5, we have $t_{*,n}^\Omega(u_n) \le 1 + o_n(1)$. Hence

$$\left(\frac{1}{2} - \frac{1}{2p_n}\right)\left(1 - \left(t_{*,n}^\Omega(u_n)\right)^2\right)\|u_n\|_\lambda^2 \ge o_n(1),$$

and by $p_n \to 2_{\sigma,s}^*$ as $n \to \infty$, we have

$$\left(\frac{1}{2p_n} - \frac{1}{2\cdot 2_{\sigma,s}^*}\right)\left(t_{*,n}^\Omega(u_n)\right)^2\|u_n\|_\lambda^2 = o_n(1).$$

Therefore,

$$J_{\varepsilon_n}(u_n) - J_*^\Omega\left(t_{*,n}^\Omega(u_n)u_n\right) \ge o_n(1).$$

Due to (5.3), we get

$$\vartheta_*^\Omega \le \lim_{n\to\infty} J_*^\Omega\left(t_{*,n}^\Omega(u_n)u_n\right) \le \lim_{n\to\infty}\left(J_{\varepsilon_n}(u_n) + o_n(1)\right) = \vartheta_* = \vartheta_*^\Omega,$$

which implies

$$\lim_{n\to\infty} J_*^\Omega\left(t_{*,n}^\Omega(u_n)u_n\right) = \vartheta_*^\Omega.$$

In terms of Ekeland variational principle (see Theorem 8.5 in Willem [25]), there exist sequences $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_*^\Omega$ and $\{\nu_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that as $n \to \infty$,

$$\|v_n - t_{*,n}^\Omega(u_n)u_n\|_\lambda \to 0, \tag{5.4}$$

$$J_*^\Omega(v_n) = \left(\frac{1}{2} - \frac{1}{2\cdot 2_{\sigma,s}^*}\right)\|v_n\|_\lambda^2 \to \vartheta_*^\Omega,$$

$$(J_*^\Omega)'(v_n) - \nu_n(G_*^\Omega)'(v_n) \to 0 \quad \text{in } H_0^{-s}(\Omega),$$

where $G_*^{\Omega}(v_n) := \left\langle (J_*^{\Omega})'(v_n), v_n \right\rangle$. By Lemma 3.5, we get that $\{v_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence for the free functional J_*^{Ω} at level ϑ_*^{Ω}. Then Remark 4.8 implies that

$$v_n - (S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n} \to 0 \quad \text{in } D^{s,2}(\mathbb{R}^N) \ \text{ as } n \to \infty,$$

where $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$ and $R_n \to +\infty$ as $n \to \infty$. Write

$$v_n = (S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n} + w_n,$$

where $\|w_n\|_{s,2} \to 0$ as $n \to \infty$, which implies $|w_n|_{2_s^*} \to 0$ as $n \to \infty$. Unless to relabel w_n, we get from (5.4) that

$$t_{*,n}^{\Omega}(u_n)u_n = (S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n} + w_n.$$

According to (5.1), we get that for $x = (x^1, x^2, \cdots, x^N) \in \mathbb{R}^N$,

$$\begin{aligned}
&\beta^i \left(t_{*,n}^{\Omega}(u_n)u_n\right) \left|t_{*,n}^{\Omega}(u_n)u_n\right|_{2_s^*}^{2_s^*} = \int_{\mathbb{R}^N} x^i \left|t_{*,n}^{\Omega}(u_n)u_n\right|^{2_s^*} \mathrm{d}x \\
=&\int_{\mathbb{R}^N} x^i \Big(\left|(S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n} + w_n\right|^{2_s^*} - \left|(S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n}\right|^{2_s^*} \Big) \mathrm{d}x \\
&+ (S_{H,L}^s)^{\frac{2_s^*}{2\cdot 2^*_{\sigma,s}-2}} \int_{\mathbb{R}^N} x^i |U_{R_n,x_n}|^{2_s^*} \mathrm{d}x \\
=:& I_1 + I_2.
\end{aligned} \tag{5.5}$$

Since $U_{R_n,x_n}(x) = R_n^{\frac{N-2s}{2}} U_{1,0}(R_n(x-x_n))$, we take the change of variables $\tilde{x} = R_n(x - x_n)$ and use the property of integral for odd functions in symmetric domain to obtain that

$$\begin{aligned}
I_2 &= (S_{H,L}^s)^{\frac{2_s^*}{2\cdot 2^*_{\sigma,s}-2}} \Big(x_n^i |U_{1,0}|_{2_s^*}^{2_s^*} + \frac{1}{R_n} \int_{\mathbb{R}^N} \tilde{x}^i |U_{1,0}(\tilde{x})|^{2_s^*} \mathrm{d}\tilde{x} \Big) \\
&= (S_{H,L}^s)^{\frac{2_s^*}{2\cdot 2^*_{\sigma,s}-2}} x_n^i |U_{1,0}|_{2_s^*}^{2_s^*}.
\end{aligned} \tag{5.6}$$

Notice that $\{v_n\}_{n\in\mathbb{R}^N}$ is supported in Ω, we have

$$w_n = -(S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n} \quad \text{in } \Omega^c.$$

By the mean value theorem, the Hölder inequality and the change of variables $\tilde{x} = R_n(x - x_n)$, we get that

$$\begin{aligned}
|I_1| \le& 2_s^* \int_{\mathbb{R}^N} |x^i| \left|(S_{H,L}^s)^{\frac{1}{2\cdot 2^*_{\sigma,s}-2}} U_{R_n,x_n} + \theta w_n\right|^{2_s^*-1} |w_n| \mathrm{d}x \ (0 < \theta < 1) \\
\le& C \int_{\Omega} \left(|U_{R_n,x_n}|^{2_s^*-1}|w_n| + |w_n|^{2_s^*}\right) \mathrm{d}x + C \int_{\Omega^c} |x^i| |U_{R_n,x_n}|^{2_s^*} \mathrm{d}x \\
\le& C\left(|U_{1,0}|_{2_s^*}^{2_s^*-1} |w_n|_{2_s^*} + |w_n|_{2_s^*}^{2_s^*}\right) + C \int_{R_n(\Omega^c - x_n)} \frac{\frac{1}{R_n}|\tilde{x}^i| + |x_n^i|}{(1+|\tilde{x}|^2)^N} \mathrm{d}\tilde{x} \\
=& o_n(1).
\end{aligned} \tag{5.7}$$

Similarly, we also have

$$\left|t_{*,n}^{\Omega}(u_n)u_n\right|_{2_s^*}^{2_s^*} = (S_{H,L}^s)^{\frac{2_s^*}{2\cdot 2^*_{\sigma,s}-2}} |U_{1,0}|_{2_s^*}^{2_s^*} + o_n(1). \tag{5.8}$$

Inserting (5.6), (5.7) and (5.8) into (5.5), we obtain that

$$\beta^i(u_n) = \beta^i\left(t^{\Omega}_{*,n}(u_n)u_n\right) = \frac{x_n^i (S^s_{H,L})^{\frac{N}{N-\sigma+2s}} \left|U_{1,0}\right|^{2^*_s}_{2^*_s} + o_n(1)}{(S^s_{H,L})^{\frac{N}{N-\sigma+2s}} \left|U_{1,0}\right|^{2^*_s}_{2^*_s} + o_n(1)}. \tag{5.9}$$

Noticing $\{x_n\}_{n\in\mathbb{N}} \subset \Omega$, we get from (5.9) that $\beta(u_n) \in \Omega^+_r$ for n sufficiently large, which contradicts with (5.2). This completes the proof. □

Now we are ready to prove the main theorem in this paper.

Proof of Theorem 1.1:

Step 1. We show the existence of $cat(\bar{\Omega})$ low energy solutions for the problem (1.1).
Let's fix $\delta_0 > 0$ and $\varepsilon_0(\delta_0) > 0$ as in Proposition 5.1. Then for all $\varepsilon < \varepsilon_0(\delta_0)$, there holds

$$u \in \mathcal{N}_\varepsilon \text{ and } J_\varepsilon(u) < \vartheta_\varepsilon + \delta_0 \Longrightarrow \beta(u) \in \Omega^+_r. \tag{5.10}$$

By Proposition 4.5, for the above δ_0, there exists $\bar{\varepsilon}(\delta_0)$ such that

$$|\vartheta_\varepsilon - \vartheta_*| \le \frac{\delta_0}{2}, \quad \forall \varepsilon < \bar{\varepsilon}(\delta_0). \tag{5.11}$$

Due to the proof of Lemma 4.2, there exists $\tilde{\varepsilon}(\delta_0) > 0$ such that for all $\varepsilon < \tilde{\varepsilon}(\delta_0)$, there is $R = R(\delta_0, \varepsilon) > 1$ such that

$$J_\varepsilon(t_\varepsilon(u_{R,x_0})u_{R,x_0}(x)) \le \vartheta_* + \frac{\delta_0}{2}, \tag{5.12}$$

where u_{R,x_0} is defined in (4.2) and $t_\varepsilon(u_{R,x_0}) > 0$ is the unique value satisfying $t_\varepsilon(u_{R,x_0})u_{R,x_0} \in \mathcal{N}_\varepsilon$.
After taking $0 < \varepsilon < \min\{\varepsilon_0(\delta_0), \bar{\varepsilon}_0(\delta_0), \tilde{\varepsilon}_0(\delta_0)\}$ and choosing $R = R(\delta_0, \varepsilon) > 1$ sufficiently large, we define

$$\varphi_\varepsilon : \Omega^-_r \to \mathcal{N}_\varepsilon \text{ and } \varphi_\varepsilon(x) := t_\varepsilon(u_{R,x_0})u_{R,x_0}(x). \tag{5.13}$$

It follows from (5.11), (5.12) and (5.13) that

$$\varphi_\varepsilon(\Omega^-_r) \subseteq \mathcal{N}_\varepsilon \cap J_\varepsilon^{\vartheta_*+\delta_0/2} \subseteq \mathcal{N}_\varepsilon \cap J_\varepsilon^{\vartheta_\varepsilon+\delta_0}, \tag{5.14}$$

where $J^c_\varepsilon := \{u \in H^s_0(\Omega) : J_\varepsilon(u) \le c\}$ $(c \in \mathbb{R})$ denotes the level set of J_ε.
By means of (5.10) and (5.14), the following maps are well-defined

$$\Omega^-_r \xrightarrow{\varphi_\varepsilon} \mathcal{N}_\varepsilon \cap J_\varepsilon^{\vartheta_\varepsilon+\delta_0} \xrightarrow{\beta} \Omega^+_r,$$

and $\beta \circ \varphi_\varepsilon$ is homotopic to the identity on Ω^-_r. Due to Remark 2.3, we get that

$$cat_{J_\varepsilon^{\vartheta_\varepsilon+\delta_0}}(\varphi_\varepsilon(\Omega^-_r)) \ge cat(\Omega^-_r) = cat(\bar{\Omega}) > 1.$$

Thus we find a sublevel of J_ε on $\mathcal{N}_\varepsilon$ with category greater than $cat(\bar{\Omega})$. Notice that J_ε satisfies the (PS)-condition on $\mathcal{N}_\varepsilon$, we obtain from Proposition 2.4 that there exist at least $cat(\bar{\Omega})$ critical points of J_ε in $J_\varepsilon^{\vartheta_\varepsilon+\delta_0}$, which correspond to the low energy solutions of the problem (1.1).
Step 2. We prove the existence of another high energy solution for (1.1) when Ω is not contractible.

Given arbitrarily a positive function $v \in D^{s,2}(\mathbb{R}^N)$ and $x_0 \in \Omega_r^-$, we let

$$\bar{v}(x) := v(x)\phi_{r,x_0}(x),$$

where ϕ_{r,x_0} is defined as in (4.3). Then $\bar{v} \in H_0^s(\Omega)$. Set

$$E_\varepsilon := \left\{\theta\bar{v}(x) + (1-\theta)e(x) : \theta \in [0,1], e \in \varphi_\varepsilon(\Omega_r^-)\right\},$$

then $\varphi_\varepsilon(\Omega_r^-) \subset E_\varepsilon \subset H_0^s(\Omega)$. Moreover, E_ε is compact and contractible in $H_0^s(\Omega)$. Notice that the functions in $\varphi_\varepsilon(\Omega_r^-)$ are all positive, we get that E_ε contains only positive functions. For $u \in E_\varepsilon$, there is an unique $t_\varepsilon(u) > 0$ such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$, and then

$$(t_\varepsilon(u))^{2p_\varepsilon-2} = \frac{\|u\|_\lambda^2}{\displaystyle\int_{\Omega\times\Omega}\frac{(u^+(x))^{p_\varepsilon}(u^+(y))^{p_\varepsilon}}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y} = \frac{\|u\|_\lambda^2}{\displaystyle\int_{\Omega\times\Omega}\frac{u^{p_\varepsilon}(x)u^{p_\varepsilon}(y)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y}. \tag{5.15}$$

Define

$$T_\varepsilon := \{t_\varepsilon(u)u : u \in E_\varepsilon\},$$

then $\varphi_\varepsilon(\Omega_r^-) \subset T_\varepsilon \subset \mathcal{N}_\varepsilon$. Additionally, T_ε is compact and contractible in $\mathcal{N}_\varepsilon$, and T_ε contains only positive functions. Finally, we denote

$$m_\varepsilon := \max_{u\in E_\varepsilon} J_\varepsilon(t_\varepsilon(u)u) = \max_{u\in T_\varepsilon} J_\varepsilon(u).$$

Then $T_\varepsilon \subset \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon}$ and $m_\varepsilon \geq \vartheta_\varepsilon$.

Claim: There exists a constant $c > 0$ such that for each $\varepsilon > 0$ small, it holds $m_\varepsilon < c$.

Proof. For $u \in E_\varepsilon$, we have

$$J_\varepsilon(t_\varepsilon(u)u) = \frac{p_\varepsilon - 1}{2p_\varepsilon}(t_\varepsilon(u))^2\|u\|_\lambda^2.$$

It follows from the definition of E_ε and (5.13), (4.7) that

$$\|u\|_{s,2} \leq \|\bar{v}\|_{s,2} + 2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}\|u_{R,x_0}\|_{s,2} \leq C \tag{5.16}$$

and

$$|u|_2 \leq |\bar{v}|_2 + 2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}|u_{R,x_0}|_2 \leq C. \tag{5.17}$$

Since the domain Ω is bounded, we denote with diamΩ as its diameter. For any $x, y \in \Omega$, we have $|x-y| \leq 2\mathrm{diam}\Omega$. Hence for $u \in E_\varepsilon$,

$$\int_{\Omega\times\Omega}\frac{u^{p_\varepsilon}(x)u^{p_\varepsilon}(y)}{|x-y|^\sigma}\mathrm{d}x\mathrm{d}y \geq \frac{1}{(2\mathrm{diam}\Omega)^\sigma}\int_{B_{r/2}(x_0)\times B_{r/2}(x_0)} u^{p_\varepsilon}(x)u^{p_\varepsilon}(y)\mathrm{d}x\mathrm{d}y > C > 0. \tag{5.18}$$

Indeed, we get from (4.7) that for $\varepsilon > 0$ small enough and $x \in B_{r/2}(x_0)$,

$$\begin{aligned}
u(x) \geq& \theta v(x) + (1-\theta)1/2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}U_{R,x_0}(x)\\
\geq& \max\left\{\theta v(x), (1-\theta)1/2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}U_{R,x_0}(x)\right\}\\
\geq& \max\left\{\theta\min\left\{v, 1/2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}U_{R,x_0}\right\}, (1-\theta)\min\left\{v, 1/2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}U_{R,x_0}\right\}\right\}\\
\geq& 1/2\min\left\{v, 1/2(S_{H,L}^s)^{\frac{1}{2\cdot 2_{\sigma,s}^*-2}}U_{R,x_0}\right\}, \quad \forall u \in E_\varepsilon,
\end{aligned}$$

and then $u^{p_\varepsilon}(x) \geq C \min\{v, U_{R,x_0}\}^{2^*_{\sigma,s}}$, which ensures the correctness of (5.18).

According to (5.15), (5.16), (5.17) and (5.18), we see that $t_\varepsilon(u)$ is bounded on E_ε uniformly in ε. Concequently, J_ε is bounded on T_ε uniformly in ε. This finishes the proof of the claim. □

Similarly to Section 6 in Benci et al [4] and as the same argument of Proposition 2.4 in the contractible case, we conclude that there exists another solution $\hat{u}$ to the problem (1.1) such that

$$\vartheta_\varepsilon + \delta_0 < J_\varepsilon(\hat{u}) \leq m_\varepsilon.$$

We complete the proof of Theorem 1.1.

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