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A Non-Standard Kripke Semantics for the Minimal Deontic Logic

Abstract. In this paper we study a new operator of strong modality $\boxplus$, related to the non-contingency operator Δ. We then provide soundness and completeness theorems for the minimal logic of the $\boxplus$-operator.

Keywords: modal logic; non-normal modal logic; seriality; non-contingency

Introduction

The goal of this paper is to study a new non-normal operator of strong, concrete, necessity that we represent as $\boxplus$ and that we will call the *box-plus*. Semantically, we interpret $\boxplus\varphi$ as saying that "φ is *both* necessary and possible".

This papers is also a contribution to the study of non-normal modal logic and develops a line of work started by the study of RI-logics (Gilbert and Venturi, 2016) and their metaphysical interpretation (Gilbert and Venturi, 2018). In order to explain this connection we recall that in (Gilbert and Venturi, 2017) an interesting connection was shown between the $\circ$-operator of RI-logics and another non-normal operator $\boxdot$, the boxdot operator — first defined in the context of provability logic by Boolos (1993). Indeed, at the core of the characterization results for RI-logics is found the so-called $\boxdot$-translation, which uniformly replaces the box with the boxdot: $(\Box\varphi)^{\boxdot} = \boxdot\varphi^{\boxdot}$, where $\boxdot\varphi \models\!\!\!\!\!\dashv \Box\varphi \land \varphi$, and therefore forces a form of reflexivity in the modal operator.

In this paper we study an analogous form of forcing semantic properties in the basic modal operator. This time we deal with the $\boxplus$-

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translation, which uniformly replaces the box with the boxplus: $(\Box\varphi)^{\boxplus} = \boxplus\varphi^{\boxplus}$, where $\boxplus\varphi \mathrel{\models\!\!\!\dashv} \Box\varphi \wedge \Diamond\varphi$.

Besides the intrinsic interest of giving characterization results for the $\boxplus$-operator we ask whether the $\boxplus$-translation might be useful for the study of other non-normal operators, as the $\boxdot$-translation showed for the $\circ$-operator.

To this end it is important to remember that $\circ\varphi := \varphi \to \Box\varphi$, which represents the converse of the characteristic axiom of **KT**. Then the connection between $\circ$ and $\boxdot$ can be explained by noting that both operators express a form of reflexivity by way of the syntax. Specifically, these operators capture a similar semantic property, which is expressed in the very definition of the basic modality. While $\circ\varphi$ cannot detect the difference between reflexive and non-reflexive frames, and therefore its minimal logic is (also) determined by validity in the class of reflexive frames, on the other hand $\boxdot\varphi$ expresses necessity in a reflexive context, since $\Box\varphi$ and $\Box\varphi \wedge \varphi$ are equivalent in reflexive frames. It is as if these operators were rescaling the order of the modal systems by setting the origin at the level of reflexive frames. While the $\circ$-operator achieves this effect by collapsing validity between **K**-frames and **KT**-frames (Gilbert and Venturi, 2016), on the other hand the $\boxdot$-operator expresses necessity as in **KT**, therefore forcing reflexivity as the basic modal property of the operator.

Coming back to the $\boxplus$-operator, we have that $\boxplus$ represents necessity as in **KD**, since in serial frames we have that $\Box\varphi$ is equivalent to $\Box\varphi\wedge\Diamond\varphi$. Consequently, we could enquire about the connections between $\boxplus$ and another non-normal operator expressing the converse of the characteristic axiom of **KD**. Such an operator is well-known in the literature (Cresswell, 1988; Fan et al., 2015; Humberstone, 1995; Montgomery and Routley, 1966) and is meant to formalize the notion of non-contingency as $\Delta\varphi := \Box\neg\varphi \vee \Box\varphi$. Parallel to the case of reflexivity, the Δ-operator collapses validity between **K**-frames and **KD**-frames (Venturi and Yago, 2020), while the $\boxplus$-operator expresses necessity as in **KD**, therefore forcing seriality as the basic modal property of the operator.

However, there are important differences between the pair of operators which encode a form of reflexivity and those which encode seriality. While $\circ$ and $\boxdot$ are inter-definable, as is easily shown by the following semantic equivalences: $\circ\varphi \wedge \varphi \mathrel{\models\!\!\!\dashv} \boxdot\varphi$ and $\boxdot\varphi \vee \neg\varphi \mathrel{\models\!\!\!\dashv} \circ\varphi$, this is not the case for the pair Δ and $\boxplus$. This seems to be a fundamental impediment to using the characterization results for logics in the $\boxplus$-language in order to study logics in the Δ-language.

We leave for another occasion a more detailed study of the relationships between Δ and $\boxplus$. For now, it is important to stress that $\boxplus$ represents a strong form of necessity, able to express the semantic property of seriality. We will show that although the boxplus is a non-normal operator, normality will be restored from serial frames upwards. This phenomenon suggests that the concept of normality is a relative one and that it might depend, among other things, on the class of frames for which a modality expresses necessity. In other words, as $\Box$ expresses the necessity of the class of all frames, $\boxdot$ expresses that of reflexive ones, while $\boxplus$ that of serial ones. We think that this perspective suggests a more general phenomenon that is partially confirmed by the results of this paper.

Another interesting aspect worth discussing is the relation between non-normality and expressivity. Indeed, the non-normality of $\boxplus$ does not prevent it to be as expressive as the normal $\Box$. This observation then seems to confirm the intuition that non-normality cannot be reduced only to the failure of axioms and rules of the normal minimal logic **K**. In other words, there seem to be also semantic constraints which constitute non-normality.

This paper is organized as follows. In Section 1, we introduce a syntax and semantics. In Section 2 we introduce an axiom system for the minimal $\boxplus$-logic, called $\mathbf{K}^{\boxplus}$ and we present soundness and completeness results for this logic. We end by discussing possible extensions of the results presented here.

1. Language and semantics

We will work with two modal languages: the language $\mathcal{L}^{\Box}$ of *normal modal logics*, and the language $\mathcal{L}^{\boxplus}$ of $\boxplus$-*logics*. They all share a countable set Var of variables and, respectively, the sets $\mathrm{Form}_{\Box}$ and $\mathrm{Form}_{\boxplus}$ of formulas which are recursively defined as usual:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi,$$
$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \boxplus\varphi.$$

The other boolean connectives $\vee$, $\rightarrow$, $\bot$ and $\leftrightarrow$ are also defined as usual. Moreover, in the language $\mathcal{L}^{\Box}$ as standard we put $\Diamond := \neg\Box\neg$.

1.1. Modal semantics

We will use a standard Kripke semantics. The novelty will be represented by the semantic interpretation of the different modalities.

A *Kripke frame* is any ordered pair $\mathcal{F} = \langle W, R\rangle$, where W is the nonempty set of *possible worlds* and $R \subseteq W \times W$ is a binary relation, called the *accessibility relation*. Let $\mathbb{C}$ be the class of all frames. A *model* $\mathcal{M}$ based on $\mathcal{F} = \langle W, R\rangle$ is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $V \colon \mathrm{Var} \to \wp(W)$ is a valuation. The interpretation of the basic Boolean connectives is as usual, i.e., for all $p \in \mathrm{Var}$, $\varphi, \psi \in \mathrm{Form}_{\Box}$ (resp. $\varphi, \psi \in \mathrm{Form}_{\boxplus}$) and $w \in W$ we have:

$\mathcal{M}, w \models p$ iff $w \in V(p)$,
$\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$,
$\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$.

On the other hand for the modalities we have the following interpretations. Firstly, for any $\varphi \in \mathrm{Form}_{\Box}$ we have:

$$\mathcal{M}, w \models \Box\varphi \text{ iff for any } z \in W \text{ such that } wRz,\ \mathcal{M}, z \models \varphi.$$

Obviously, for $\Diamond$ $(:= \neg\Box\neg)$ we have:

$$\mathcal{M}, w \models \Diamond\varphi \text{ iff there is a } z \in W \text{ such that } wRz \text{ and } \mathcal{M}, z \models \varphi.$$

Secondly, for any $\varphi \in \mathrm{Form}_{\boxplus}$ we have:

$$\mathcal{M}, w \models \boxplus\varphi \text{ iff both for any } z \in W \text{ such that } wRz,\ \mathcal{M}, z \models \varphi, \text{ and there is an } x \in W \text{ such that } wRx \text{ and } \mathcal{M}, x \models \varphi.$$

We say that a formula is true in a model $\mathcal{M}$ when it is true at every world of the frame on which $\mathcal{M}$ is based. While we say a formula is valid at a frame $\mathcal{F}$ when it is true in every model based on $\mathcal{F}$.

Notice that the connective $\boxplus$ can be defined in the language $\mathcal{L}^{\Box}$ by the following:

$$\boxplus\varphi := \Box\varphi \wedge \Diamond\varphi.$$

The reverse definition is also possible, using the following formula:[1]

$$\Box\varphi := \boxplus\top \to \boxplus\varphi.$$

Moreover, in the class of serial frames, $\Box\varphi$ and $\boxplus\varphi$ are simply interchangeable. We define the $\boxplus$-*translation* from $\mathrm{Form}_{\Box}$ to $\mathrm{Form}_{\boxplus}$ that

[1] We thank Lloyd Humberstone for suggesting this definition in a private communication.

acts by uniformly substituting all occurrences of $\Box$ with $\boxplus$ (then serial frames cannot detect the difference between the two modalities):

$$
\begin{aligned}
(p)^{\boxplus} &= p,\\
(\neg\varphi)^{\boxplus} &= \neg\varphi^{\boxplus},\\
(\varphi\wedge\psi)^{\boxplus} &= \varphi^{\boxplus}\wedge\psi^{\boxplus},\\
(\varphi\vee\psi)^{\boxplus} &= \varphi^{\boxplus}\vee\psi^{\boxplus},\\
(\varphi\to\psi)^{\boxplus} &= \varphi^{\boxplus}\to\psi^{\boxplus},\\
(\Box\varphi)^{\boxplus} &= \boxplus\varphi^{\boxplus}.
\end{aligned}
$$

By induction, we get the following:

Proposition 1.1. *Given a serial frame $\mathcal{F} = \langle W, R\rangle$ and a model $\mathcal{M}$ based on $\mathcal{F}$, for all $\varphi \in \mathrm{Form}_{\Box}$ and $w \in W$ we have*:

$$\mathcal{M}, w \models \varphi \ \text{ iff } \ \mathcal{M}, w \models \varphi^{\boxplus}.$$

We remark that in the formulation of the above proposition we are slightly abusing notation, since the symbol $\models$ stands for two different consequence relations: one is the normal modal one, while the other should more appropriately be presented as $\models_{\boxplus}$ to mark that we are evaluating formulas by the semantic clauses characteristic to the use of the $\boxplus$-operator or, equivalently, that we are evaluating formulas from the language $\mathcal{L}^{\boxplus}$. We warn the reader that we will continue in this way throughout the text, since we believe it will be clear from the context which clauses should be used for the evaluation of a given formula.

A last word on the non-normality of the operator $\boxplus$, which stems from the invalidity of the necessitation rule. Indeed, to validate a formula like $\boxplus\varphi$ a frame needs to be serial, which is not always the case in the class of all frames.

2. The minimal logic $\mathbf{K}^{\boxplus}$

In the language $\mathcal{L}^{\boxplus}$, we propose the following axiom system for the minimal $\boxplus$-logic $\mathbf{K}^{\boxplus}$:

$$\text{all instances of classical tautologies} \qquad (\texttt{Taut})$$

$$\boxplus(\varphi\to\psi)\to(\boxplus\varphi\to\boxplus\psi) \qquad (\texttt{K}_{\boxplus})$$

$$\boxplus\varphi\to\neg\boxplus\neg\varphi \qquad (\texttt{D}_{\boxplus})$$

$$\text{from } \varphi \text{ and } \varphi \to \psi \text{ we infer } \psi \tag{MP}$$

$$\text{from } \varphi \to \psi \text{ we infer } \boxplus\varphi \to \boxplus\psi \tag{RK$_\boxplus$}$$

By definition, we see that the logic $\mathbf{K}^{\boxplus}$ is a regular logic in the language $\mathcal{L}^{\boxplus}$ and standardly for regular logics, from (Taut), ($\mathtt{K}_{\boxplus}$), (MP) and ($\mathtt{RK}_{\boxplus}$) we obtain the following thesis for any $n > 0$:

$$\boxplus(\varphi_1 \wedge \cdots \wedge \varphi_n) \leftrightarrow (\boxplus\varphi_1 \wedge \cdots \wedge \boxplus\varphi_n) \tag{R$_\boxplus$}$$

It it easy to see that $\mathbf{K}^{\boxplus}$ is sound with respect to the class $\mathbb{C}$ of all frames. Therefore, the following formula:

$$\boxplus\varphi \to \varphi \tag{T$_\boxplus$}$$

is not a thesis of $\mathbf{K}^{\boxplus}$. Hence we have $\mathbf{K}^{\boxplus} \subsetneq \mathbf{E2}_{\boxplus}$, where $\mathbf{E2}_{\boxplus}$ is the smallest regular logic in $\mathcal{L}^{\boxplus}$ containing ($\mathtt{T}_{\boxplus}$).

The logic $\mathbf{K}^{\boxplus}$ is just Lemmon's **D2** from (Lemmon, 1957), which is the weakest system in the family of regular deontic logics. According to Lemmon, axiom ($\mathtt{T}_{\boxplus}$) and the necessitation rule are undesirable for a deontic interpretation of modality, since "we may question whether a *logical* system interpreted deontically should contain any thesis stating that something is a *moral* (or a *legal*) obligation" (Lemmon, 1957, p. 185).

We now prove the completeness of $\mathbf{K}^{\boxplus}$ by means of a canonical model construction. The proof is inspired by a similar construction from (Steinsvold, 2011).

We say that a set Φ of formulas is $\mathbf{K}^{\boxplus}$-*consistent* iff there is no $\gamma_1, \ldots, \gamma_n \in \Phi$ such that $\neg(\gamma_1 \wedge \cdots \wedge \gamma_n)$ is a thesis of $\mathbf{K}^{\boxplus}$. In virtue of Lindenbaum's Lemma, for any $\mathbf{K}^{\boxplus}$-consistent set Φ there is a maximal $\mathbf{K}^{\boxplus}$-consistent set Γ such that $\Phi \subseteq \Gamma$.

The canonical model $\mathcal{M}^{\boxplus} = \langle W^{\boxplus}, R^{\boxplus}, V^{\boxplus}\rangle$ for the logic $\mathbf{K}^{\boxplus}$ is defined as follows:

1. $W^{\boxplus}$ is the set of maximal $\mathbf{K}^{\boxplus}$-consistent subsets of $\mathrm{Form}_{\boxplus}$ such that $W^{\boxplus} = W^s \cup W^{\neg s}$ where:
   - $w \in W^s$ iff for some $\varphi \in \mathrm{Form}_{\boxplus}$ we have $\boxplus\varphi \in w$;
   - $w \in W^{\neg s}$ iff there is no $\varphi \in \mathrm{Form}_{\boxplus}$ such that $\boxplus\varphi \in w$.
2. The relation $R^{\boxplus} \subseteq W^{\boxplus} \times W^{\boxplus}$ is defined as follows: for all $w, y \in W^{\boxplus}$
   - if $w \in W^s$, then $wR^{\boxplus}y$ iff $\lambda(w) \subseteq y$, where $\lambda(w) = \{\varphi \mid \boxplus\varphi \in w\}$;
   - if $w \in W^{\neg s}$, then there is no $y \in W^{\boxplus}$ such that $wR^{\boxplus}y$.
3. The function $V^{\boxplus} \colon \mathrm{Var} \to \wp(W^{\boxplus})$ is defined as follows:
$$V^{\boxplus}(p) := \{w \in W^{\boxplus} \mid p \in w\}.$$

LEMMA 2.1. *Let w be a maximal $\mathbf{K}^{\boxplus}$-consistent set of formulas such that $\lambda(w) \neq \emptyset$. If $\lambda(w) \cup \{\neg\varphi\}$ is not $\mathbf{K}^{\boxplus}$-consistent, then $\boxplus\varphi \in w$.*

PROOF. Assume that w is a maximal $\mathbf{K}^{\boxplus}$-consistent set, $\lambda(w) \neq \emptyset$ and $\lambda(w) \cup \{\neg\varphi\}$ is not $\mathbf{K}^{\boxplus}$-consistent. Then there are $\gamma_1, \ldots, \gamma_n \in \lambda(w)$ such that $\neg(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi)$ is a thesis of $\mathbf{K}^{\boxplus}$; and so $(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi$ is a thesis of $\mathbf{K}^{\boxplus}$. Hence, by ($\mathtt{RK}_{\boxplus}$), also $\boxplus(\gamma_1 \wedge \cdots \wedge \gamma_n) \to \boxplus\varphi$ is a thesis of $\mathbf{K}^{\boxplus}$. Hence, by ($\mathtt{R}_{\boxplus}$), we obtain the thesis $(\boxplus\gamma_1 \wedge \cdots \wedge \boxplus\gamma_n) \to \boxplus\varphi$. Since w is maximal set, $\boxplus\gamma_1 \wedge \cdots \wedge \boxplus\gamma_n \in w$; and so $\boxplus\varphi \in w$. □

THEOREM 2.2. *For the canonical model $\mathcal{M}^{\boxplus}$ for $\mathbf{K}^{\boxplus}$, any $w \in W^{\boxplus}$ and $\varphi \in \mathrm{Form}_{\boxplus}$, we have:*

$$\mathcal{M}^{\boxplus}, w \models \varphi \textit{ iff } \varphi \in w.$$

PROOF. The atomic case and the cases of Boolean connectives are standard. So we assume that $\boxplus\psi \in w$. Then $\lambda(w) \neq \emptyset$ and, by ($\mathtt{D}_{\boxplus}$), also $\neg \boxplus \neg\psi \in w$; and so $\boxplus\neg\psi \notin w$. Hence, by Lemma 2.1, $\lambda(w) \cup \{\neg\neg\psi\}$, thus, as well as $\lambda(w) \cup \{\psi\}$, is $\mathbf{K}^{\boxplus}$-consistent and so, by Lindenbaum's Lemma, it can be extended to a maximal set z. Thus, $\lambda(w) \subseteq z$ and $\psi \in z$. Therefore $\mathcal{M}^{\boxplus}, z \models \psi$, by the induction hypothesis. Furthermore, $w \in W^s$, hence $wR^{\boxplus}z$. Besides, for any $y \in W^{\boxplus}$:

$$\begin{aligned} wR^{\boxplus}y &\text{ iff } \lambda(w) \subseteq y \\ &\text{ iff } \{\varphi \mid \boxplus\varphi \in w\} \subseteq y \\ &\text{ iff for any } \varphi, \text{ if } \boxplus\varphi \in w \text{ then } \varphi \in y. \end{aligned}$$

Hence we obtain that for any $y \in W^{\boxplus}$ such that $wR^{\boxplus}y$ we have $\psi \in y$; and so $\mathcal{M}^{\boxplus}, y \models \psi$, by the induction hypothesis. Thus, $\mathcal{M}^{\boxplus}, w \models \boxplus\psi$.

Conversely, suppose that $\boxplus\psi \notin w$. We consider two cases: $w \in W^s$; $w \in W^{\neg s}$.

Firstly, if $w \in W^s$, then $\lambda(w) \neq \emptyset$ and, by Lemma 2.1, $\lambda(w) \cup \{\neg\psi\}$ is $\mathbf{K}^{\boxplus}$-consistent. So, applying Lindenbaum's Lemma, we can extend $\lambda(w) \cup \{\neg\psi\}$ to a maximal $\mathbf{K}^{\boxplus}$-consistent set y. Then $\lambda(w) \cup \{\neg\psi\} \subseteq y$. So $\lambda(w) \subseteq y$ and $\neg\psi \in y$. Hence $wR^{\boxplus}y$ and $\psi \notin y$. So, by induction hypothesis, we have $\mathcal{M}, y \not\models \psi$. Therefore, $\mathcal{M}, w \not\models \boxplus\psi$.

Secondly, if $w \in W^{\neg s}$, then there is no $y \in W^{\boxplus}$ such that $wR^{\boxplus}y$. Therefore, $\mathcal{M}, w \not\models \boxplus\psi$. □

Now, recall that the non-normality of the $\boxplus$-operator is shown by the failure of the necessitation rule, since serial frames are needed to validate its application. For this reason, when we restrict our attention to serial frames we regain normality by the possibility of safely using the necessitation rule. So, for $\mathbf{D}^{\boxplus}$ — the logic **KD** in the language $\mathcal{L}^{\boxplus}$, which is standardly axiomatised as $\mathbf{K}^{\boxplus}$, but instead of ($\mathtt{RK}_{\boxplus}$) one takes the necessitation rule (from φ we infer $\boxplus\varphi$), it is a trivial exercise to prove that $\mathbf{D}^{\boxplus}$ is is sound and complete with respect to class of serial frames, where sematic conditions for $\boxplus$ stay as they are given on page 128.

In presenting a sound and complete Kripke semantics for the logic $\mathbf{K}^{\boxplus}$, we have thus presented a Kripke semantics for the logics **D2**. This is indeed different from the standard way of using a non-normal semantics for characterising regular logics, as indeed takes place for the case of **D2** and **E2**. Let us recall, for example, that models for **D2** (resp. **E2**) are standardly defined in terms of structures of the form $\mathcal{N} = \langle W, N, R, V\rangle$, where W is a non-empty set (of worlds), $N \subseteq W$ (the elements of N are called *normal worlds*), R is a serial (resp. reflexive) relation on W, and V a is valuation, which interprets Boolean formulas as usual and $\Box$, $\Diamond$ as follows:

$$
\begin{aligned}
\mathcal{N}, x \models \Box\varphi \text{ iff both } & x \in N \text{ and} \\
& \text{for every } y \in W \text{ such that } xRy,\ \mathcal{N}, y \models \varphi, \\
\mathcal{N}, x \models \Diamond\varphi \text{ iff either } & x \notin N \text{ or} \\
& \text{for some } y \in W \text{ such that } xRy,\ \mathcal{N}, y \models \varphi.
\end{aligned}
$$

It is well-known that **D2** (resp. **E2**) is sound and complete with respect to serial (resp. reflexive) models. To provide non-standard semantic conditions applied to normal Kripke frames, for **E2** and other strictly regular modal logics may thus constitute an interesting task, which would highlight interesting peculiarities of the specific canonical model of a given strictly regular modal logic.

3. Concluding remarks

As for the interpretation of the $\boxplus$-operator, however, the matter is a bit more complicated. We saw that the logic $\mathbf{K}^{\boxplus}$ expresses the same modal principles as Lemmon's system **D2**, where $\Box$ is interpreted as "it is obligatory that". Indeed in these systems are expressed some fundamental aspects of deontic logics (see, e.g., Chellas, 1980). As regards the failure of the necessitation rule, even though Lemmon argues that it is a desirable aspect of this logical system, this seems to go against the current state of deontic logics, where it is usually accepted that every tautology should be obligatory.

We leave the problem of the right interpretation of the $\boxplus$-operator open. We just remark that if any can be found, this should be in a context where axiom ($\mathtt{D}_{\boxplus}$) plays a fundamental conceptual role and where the lack of necessitation does not cause harm to a faithful interpretation.

We now go back to the non-normal character of $\boxplus$. Non-normality is usually defined as the failure of some axiom or rule of the modal logic **K**. But the results of the present paper suggests that there might be more to it than that. Indeed, the non-normal modality $\boxplus$ is as expressive as the normal operator $\Box$, since we can offer a definition of the latter in terms of the former in the class of all frames. It seems that also semantic components should play a role in the definition of non-normality. Indeed, it is the semantic definition of $\boxplus$ which is responsible both for the failure of necessitation in the class of all frames and of the impossibility to distinguish between $\Box$ and $\boxplus$ in serial frames, as shown in Proposition 1.1. We leave to another occasion an attempt to offer a definition of non-normality able to take into account both syntactic and semantic aspects.

A second interesting aspect worth commenting on is the relationship between the couples of operators $\{\circ, \boxdot\}$ and $\{\Delta, \boxplus\}$. Although both $\boxdot$ and $\boxplus$ internalize, within their syntax, semantic properties of the standard modal operator (indeed $\Box\varphi \equiv_{\mathbf{KT}} \boxdot\varphi$ and $\Box\varphi \equiv_{\mathbf{KD}} \boxplus\varphi$), they do it in slightly different ways. To see this remember that the $\boxdot$-operator is insensitive to reflexivity. This means that adding or removing reflexive arrows does not change the validity of $\boxdot$-formulas in a given frame. On the contrary, as already noted, the $\boxplus$-operator is very sensitive to the presence or the absence of serial frames. A possible explanation for this phenomenon is the interdefinability of $\circ$ and $\boxdot$, where this phenomenon is absent in the case of Δ and $\boxplus$. Indeed, $\circ$ and Δ are insensitive with respect to reflexivity (Gilbert and Venturi, 2016) and seriality (Humberstone, 1995), respectively. Therefore, one might speculate that the insensitivity property of $\boxdot$ is just a property derived from the $\circ$-operator.

A last word on the possibility of using the $\boxplus$-operator for the study of the Δ-operator. Although these two operators are not in general interdefinable, they are when we restrict them to reflexive frames. Indeed, it is easy to check that $\Delta\varphi \wedge \varphi \equiv_{\mathbf{KT}} \Box\varphi \equiv_{\mathbf{KT}} \boxplus\varphi$, but also $\Delta\varphi \equiv_{\mathbf{KT}} \boxplus\varphi \vee \boxplus\neg\varphi$ (the latter being the case already in **KD**). We think that this observation raises the hope that the trivial characterization results we have for $\boxplus$ above **KT** can be used to shed light on the corresponding results for the Δ-operation. We leave this task for future work.

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