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A road map to the blow-up for a Kirchhoff equation with external force

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Abstract

It is well-known that the classical hyperbolic Kirchhoff equation admits infinitely many simple modes, namely time-periodic solutions with only one Fourier component in the space variables. In this paper we assume that, for a suitable choice of the nonlinearity, there exists a heteroclinic connection between two simple modes with different frequencies. Under this assumption, we cook up a forced Kirchhoff equation that admits a solution that blows-up in finite time, despite the regularity and boundedness of the forcing term. The forcing term can be chosen with the maximal regularity that prevents the application of the classical global existence results in analytic and quasi-analytic classes.

Keywords: hyperbolic Kirchhoff equation, simple modes, heteroclinic connection, blow up, quasi-analytic functions

Mathematics Subject Classification numbers: 35B44, 37J46, 35L90, 35L72

1. Introduction

Let H be a real Hilbert space, and let A be a positive self-adjoint operator on H with dense domain $D(A)$. Let $m : [0,+\infty) \to [0,+\infty)$ and $f : [0,+\infty) \to H$ be two continuous functions. In this paper we consider the forced evolution equation

$$u''(t) + m\left(|A^{1/2}u(t)|^2\right) Au(t) = f(t) \tag{1.1}$$

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with initial data

$$u(0) = u_0, \qquad\qquad u'(0) = u_1. \tag{1.2}$$

Equation (1.1) is an abstract version of the hyperbolic partial differential equation introduced by Kirchhoff in the celebrated monograph [20, section 29.7] as a model for the small transversal vibrations of elastic strings or membranes.

1.1. Local and global existence results

Existence of solutions to problem (1.1) and (1.2) has been extensively investigated in the literature. For the sake of shortness, unless otherwise stated, here we limit ourselves to recall the main results for the case in which the nonlinearity is locally Lipschitz continuous and satisfies the strict hyperbolicity assumption

$$m(\sigma) \geqslant \mu_1 > 0 \qquad \forall \sigma \geqslant 0. \tag{1.3}$$

Under these assumptions, problem (1.1) and (1.2) admits a local-in-time strong solution

$$u \in C^0\left([0,T], D\left(A^{3/4}\right)\right) \cap C^1\left([0,T], D\left(A^{1/4}\right)\right) \tag{1.4}$$

provided that

$$(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4}) \qquad \text{and} \qquad f \in C^0\left([0,+\infty), D(A^{1/4})\right),$$

and this solution is unique in the class of strong solutions, namely solutions with the regularity (1.4). This result was substantially established by Bernstein in the pioneering paper [3], and then refined by many authors (see [1] for a modern version).

Global-in-time strong solutions are known to exist in many different special cases, which we briefly describe below.

1. (Analytic case). Problem (1.1) and (1.2) admits a global solution if both the initial data and the forcing term are analytic with respect to the space variables. Actually in this case it is enough to assume that the nonlinearity m is just continuous and nonnegative. We refer to [2, 3, 5, 6] for more details (see also [15, 16]).
2. (Quasi-analytic case). Problem (1.1) and (1.2) admits a global solution if both the initial data and the forcing term are quasi-analytic with respect to the space variables. This is not just a refinement of the analytic case, because here the known proofs require in an essential way the Lipschitz continuity and the strict hyperbolicity of the nonlinearity (see [14, 23]).
3. (Special nonlinearities). In the case where $m(\sigma) = (a + b\sigma)^{-2}$ for some positive real numbers a and b, problem (1.1) and (1.2) admits a global solution provided that

   $$(u_0, u_1) \in D(A) \times D(A^{1/2}) \qquad \text{and} \qquad f \in C^0([0,+\infty), D(A)).$$

   The technical reason is that in this case (and in some sense only in this case) the equation admits a higher order quantity whose growth can be controlled for all positive times. We refer to [24] for the details.

4. (Dispersive equations). Global existence results have been obtained in the concrete case where A is the usual Laplace operator in the whole space $\mathbb{R}^d$ or in an external domain. The prototype of these results is global existence provided that the initial data and the forcing term have Sobolev regularity in the space variables, and satisfy suitable smallness assumptions and decay conditions at infinity. We refer to [7, 17, 22, 25] for precise statements.
5. (Spectral-gap data and operators). Global existence results are known in cases where both the initial data and the forcing term are 'lacunary', in the sense that their spectrum contains a sequence of large 'holes'. The same is true whenever the eigenvalues of the operator A are a sequence that grows fast enough. We refer to [13, 14, 18, 19, 21] for precise statements. For the sake of completeness, we point out that the spectral gap theory has been recently extended in order to show the existence of global weak solutions in the energy space $D(A^{1/2})\times H$ (see [10]).

The main open problem for Kirchhoff equations is the existence of global solutions for initial data and forcing terms below the analytic or quasi-analytic regularity, for example in Gevrey spaces or in the Sobolev spaces $D(A^{\alpha})$.

1.2. Simple modes

Let us consider the unforced equation

$$u''(t)+m\left(|A^{1/2}u(t)|^2\right)Au(t)=0. \tag{1.5}$$

If e_k is an eigenvector of A with eigenvalue $\lambda_k^2>0$, and both u_0 and u_1 are multiples of e_k, for example $u_0=\alpha e_k$ and $u_1=\beta e_k$, then the solution to (1.5)–(1.2) remains a multiple of e_k for all times, and more precisely $u(t)=z(t)e_k$, where $z(t)$ is the solution to the ordinary differential equation

$$z''(t)+\lambda_k^2 m\left(\lambda_k^2 z(t)^2\right)z(t)=0$$

with initial data $z(0)=\alpha$ and $z'(0)=\beta$.

These special solutions are called *simple modes*, and it is well-known that they are time-periodic. Their stability has been studied extensively in the literature (see [4, 8, 9, 11, 12]). In particular, there are many examples of unstable simple modes, and when this is the case there exist non-periodic trajectories that are asymptotic to them as $t\to+\infty$ or as $t\to-\infty$. What is not known yet is whether the stable manifold of a simple mode can intersect the unstable manifold of a different simple mode. This intersection would deliver a trajectory that is asymptotic, as $t\to-\infty$ and as $t\to+\infty$, to two simple modes corresponding to different frequencies. Such a trajectory, which we call *heteroclinic connection*, would realise a transfer of the energy from a low frequency to a higher frequency (due to reversibility we can always change the verse of time).

1.3. Our result

In this paper we *assume* that, for some choice of the nonlinearity m, the unforced equation (1.5), in the special case where $H=\mathbb{R}^2$ and A is an operator with two eigenvalues equal to 1 and $\lambda^2>1$, admits a heteroclinic connection between the simple modes corresponding to the two eigenvalues (see definition 2.1).

Under this assumption, in theorem 2.3 we show that in every infinite dimensional Hilbert space H there exists an operator A, and actually a rather general class of operators, for which there exist an external force $f(t)$, smooth for every $t \geqslant 0$, and a solution to the forced equation (1.1) that blows-up in a finite time T_∞. By 'blow-up' we mean that the pair $(u(t), u'(t))$ does not admit a limit as $t \to T_\infty^-$ in the energy space $D(A^{1/2}) \times H$ (where the solution is necessarily bounded), while all higher order norms of the form

$$|A^\alpha u'(t)|^2 + |A^{\alpha+1/2} u(t)|^2 \qquad (\text{with } \alpha > 0)$$

tend to $+\infty$ as $t \to T_\infty^-$.

The initial datum of this solution has only one Fourier component, and in particular it is analytic, and hence necessarily the forcing term does not lie in any analytic or quasi-analytic class, because otherwise the solution would be global. Nevertheless, the forcing term can be chosen to lie in any class that is less than quasi-analytic, and in particular in all Gevrey spaces $\mathcal{G}_s$ with $s > 1$. In other words, the existence of a heteroclinic connection would imply that the classical global existence result in quasi-analytic classes is optimal.

1.4. Overview of the technique

Our proof involves three main steps.

- In the first step (proposition 3.2) we consider the heteroclinic connection that we assumed to exist, and we show that for every interval $[a, b]$ we can modify it in order to obtain a new trajectory that *coincides* with the first limiting simple mode for every $t \leqslant a$, and *coincides* with the second limiting simple mode for every $t \geqslant b$. We think of this new trajectory as a sort of bridge that connects the two simple modes in the interval $[a, b]$. In general this bridge is not a solution of an unforced Kirchhoff equation, but rather of a Kirchhoff equation with a forcing term whose size depends on the length of the interval (the longer is the length of the bridge, the smaller is the external force required).
- In the second step (proposition 3.3) we exploit a natural rescaling property of simple modes, and from the bridge between the frequencies 1 and λ^2 we obtain a bridge between the frequencies λ^2 and λ^4, and more generally between the frequencies λ^{2k} and λ^{2k+2}.
- In the third step we consider an operator that admits the sequence λ^{2k} among its eigenvalues, and we consider the solution obtained by glueing all the bridges. This solution moves the energy toward higher and higher frequencies. More precisely, there exists an increasing sequence $\{T_k\}$ of times with the property that at time T_k the solution coincides with the simple mode corresponding to the frequency λ^{2k}. The key point is that we can arrange things in such a way that T_k converges to some finite time T_∞, and the size of the corresponding forcing terms vanishes as $t \to T_\infty^-$.

In summary, the existence of a heteroclinic connection between the frequencies 1 and λ^2 for the unforced equation (1.5) implies the existence of a heteroclinic connection between any two 'consecutive' frequencies λ^{2k} and λ^{2k+2}, again for the unforced equation. Thanks to a suitable external force, we can switch from one connection to the next one, and in this way we obtain a trajectory that visits all frequencies. We can keep the external force small, and actually vanishing as $t \to T_\infty^-$, because the bulk of the work is done by the nonlinearity, and the only role of the external force is to put the solution on the right track from time to time.

It could be interesting to realise a similar path without the aid of the external force, but relying only on a suitable choice of initial data. This remains a challenging open problem.

1.5. Consequences

Our result in some sense proves nothing, because we do not know yet whether a heteroclinic connection exists or not.

If one believes that Kirchhoff equations are not well-posed in Sobolev or Gevrey spaces, we have reduced the search of a counterexample to the existence of a special trajectory for a Hamiltonian system in dimension two. We do hope that the community working on dynamical systems could contribute in this direction.

On the contrary, if one believes that Kirchhoff equations do admit global solutions for all data in Sobolev or Gevrey spaces, our result shows that the proof has to exclude the existence of heteroclinic connections, and therefore it is very likely that it has to involve some 'global' property of the nonlinearity.

1.6. Structure of the paper

This paper is organised as follows. In section 2 we present formally the heteroclinic connection assumption and we state our main result. In section 3 we prove the main result.

2. Statements

Let us state formally the main assumption of this paper.

Definition 2.1 (heteroclinic connection assumption). Let $m:[0,+\infty)\to(0,+\infty)$ be a function of class C^1 satisfying the strict hyperbolicity assumption (1.3), and let $\lambda>1$ be a real number. We say that the pair (m,λ) satisfies the *heteroclinic connection assumption* if there exist two functions $v:\mathbb{R}\to\mathbb{R}$ and $w:\mathbb{R}\to\mathbb{R}$ of class C^2 with the following properties.

- They satisfy the non-triviality condition

$$v'(0)^2+w'(0)^2+v(0)^2+w(0)^2>0. \tag{2.1}$$

- They solve the system of ordinary differential equations

$$\begin{cases} v''(t)+m\left(v(t)^2+\lambda^2 w(t)^2\right)v(t)=0 \\ w''(t)+\lambda^2 m\left(v(t)^2+\lambda^2 w(t)^2\right)w(t)=0 \end{cases} \qquad \forall t\in\mathbb{R}. \tag{2.2}$$

- There exist two positive real numbers A_0 and B_0 such that

$$|v'(t)|^2+|v(t)|^2\leqslant B_0\exp(-A_0 t) \qquad \forall t\geqslant 0, \tag{2.3}$$

$$|w'(t)|^2+\lambda^2|w(t)|^2\leqslant B_0\exp(-A_0|t|) \qquad \forall t\leqslant 0. \tag{2.4}$$

Before moving to our main result, we need to recall the usual notion of Gevrey spaces with respect to an operator.

Definition 2.2 (Gevrey spaces). Let H be a real Hilbert space, let $\{e_k\}_{k\geqslant 0}\subseteq H$ be a sequence of orthonormal vectors (not necessarily a basis), let $\{\lambda_k\}_{k\geqslant 0}$ be a sequence of positive real numbers, and let A be a linear operator on H such that

$$Ae_k=\lambda_k^2 e_k \qquad \forall k\geqslant 0.$$

Given two positive real numbers r and s, and a vector $u \in H$, we say that u is a Gevrey vector with exponent s and radius r, and we write $u \in \mathcal{G}_{r,s}(A)$, if

$$u = \sum_{k=0}^{\infty} \langle u, e_k \rangle e_k \qquad \text{and} \qquad \|u\|^2_{\mathcal{G}_{r,s}(A)} := \sum_{k=0}^{\infty} \langle u, e_k \rangle^2 \exp\left(r\lambda_k^{1/s}\right) < +\infty.$$

We recall that $\|u\|_{\mathcal{G}_{r,s}(A)}$ induces a structure of Hilbert space on the set $\mathcal{G}_{r,s}(A)$, and that the case $s = 1$ corresponds to analytic vectors. More general spaces can be introduced by considering different functions of λ_k in the exponential.

We are now ready to state our main result.

Theorem 2.3 (from a heteroclinic connection to a blow-up for a Kirchhoff equation). *Let $m : [0, +\infty) \to (0, +\infty)$ be a function of class C^1 satisfying the strict hyperbolicity assumption (1.3), and let $\lambda > 1$ be a real number. Let us assume that the pair (m, λ) satisfies the heteroclinic connection assumption of definition 2.1.*

Let H be a Hilbert space, and let A be a self-adjoint linear operator on H for which there exists a sequence $\{e_k\}_{k \geqslant 0} \subseteq H$ of orthonormal vectors (not necessarily a basis) such that

$$Ae_k = \lambda^{2k} e_k \qquad \forall k \geqslant 0.$$

Then there exist a function $f : [0, +\infty) \to H$ such that

$$f \in C^0([0, +\infty), \mathcal{G}_{s,r}(A)) \qquad \forall s > 1, \quad \forall r > 0, \tag{2.5}$$

a real number T_∞, and a solution

$$u \in C^2([0, T_\infty), \mathcal{G}_{s,r}(A)) \qquad \forall s > 0, \quad \forall r > 0, \tag{2.6}$$

to equation (1.1) such that

$$\limsup_{t \to T_\infty^-} |A^\alpha u'(t)|^2 + |A^{\alpha+1/2} u(t)|^2 = +\infty \qquad \forall \alpha > 0, \tag{2.7}$$

and

$$\lim_{t \to T_\infty^-} u'(t) \quad \textit{does not exist}. \tag{2.8}$$

We conclude with some comments on theorem 2.3 above.

Remark 2.4 (time regularity). The regularity of u and f with respect to time depends only on the regularity of the nonlinearity m. If m is of class C^1, then any pair of solutions v and w to (2.2) is automatically of class C^3. At this point, a careful inspection of our construction reveals that actually we can improve (2.5) and (2.6), respectively, to C^1 and C^3 regularity. We spare the reader from the details, which would require only longer but standard estimates, without introducing new ideas.

In the same way, if the nonlinearity m is of class C^r, then we obtain C^r regularity in (2.5), and C^{r+2} regularity in (2.6).

Remark 2.5 (space regularity). We observe that in (2.6) we allow also values $s < 1$, which means that the solution u is more than analytic. More precisely, for every $T \in (0, T_\infty)$ there exists a finitely dimensional subspace H_T of H such that $u(t) \in H_T$ for every $t \in [0, T]$.

Concerning the forcing term, the function f that we construct in the proof satisfies

$$\sum_{k=0}^{\infty} \langle f(t), e_k \rangle^2 \exp\left(\frac{c\lambda^k}{(k+1)^2}\right) < +\infty \qquad \forall t \geqslant 0$$

for a suitable $c > 0$. This implies the Gevrey regularity (2.5). More generally, for every increasing function $\varphi : [1, +\infty) \to [0, +\infty)$ such that

$$\int_1^{+\infty} \frac{\varphi(\sigma)}{\sigma^2} \, \mathrm{d}\sigma < +\infty \qquad \text{and} \qquad \frac{\varphi(\sigma)}{\sigma^2} \text{ is nonincreasing,} \tag{2.9}$$

we can modify our construction (it is enough to modify the choice of S_k in the last step) in such a way that

$$\sum_{k=0}^{\infty} \langle f(t), e_k \rangle^2 \exp\left(\varphi\left(\lambda^k\right)\right) < +\infty \qquad \forall t \geqslant 0. \tag{2.10}$$

We recall that, if $f(t)$ satisfies (2.10) for some increasing function φ for which the integral in (2.9) is divergent, then $f(t)$ is quasi-analytic.

Remark 2.6 (size of the external force). A careful inspection of the proof reveals that we can choose the norm of the external force (in any fixed non quasi-analytic class) to be smaller than any given positive constant. To this end, again it is enough to modify the definition of S_k in the last step. The other side of the coin is that the blow-up time T_∞ depends on the norm of the external force, and tends to infinity when the norm of the external force vanishes.

Remark 2.7 (more general heteroclinic connections). For the sake of simplicity the heteroclinic connection assumption, as stated in definition 2.1, involves a system that corresponds to an unforced Kirchhoff equation with two components. We can generalise the assumption by considering systems that originate from unforced Kirchhoff equations with a finite number of components. The request becomes that all components, with the exception of one, decay exponentially as $t \to -\infty$, and all components, with the exception of another one, corresponding to a different eigenvalue, decay exponentially as $t \to +\infty$.

Under this more general assumption we can again reproduce the phenomenon of theorem 2.3, more or less with the same proof. We just need to be more careful in the choice of the eigenvalues of the operator, in order to reproduce the bridge at different scales.

Remark 2.8 (more general operators). For the sake of simplicity we decided to construct the counterexample for an operator that admits the sequence $\{\lambda^{2k}\}$ among its eigenvalues. With some extra (but rather standard) work, it is possible to extend the construction to more general multiplication operators, provided that the sequence $\{\lambda^{2k}\}$ is contained in the support of the spectrum. We point out that this is always true, for example, in the concrete case where A is the Laplacian in the whole space $\mathbb{R}^d$, in which case the support of the spectrum is the half-line $[0, +\infty)$.

Remark 2.9 (relation with known global existence results). We observe that our construction dodges carefully all the key assumptions of the known global existence results quoted in the introduction. Indeed, we already observed that in our construction we can not take the forcing term $f(t)$ to be analytic or quasi-analytic. Moreover, a heteroclinic connection does not exist when m is the nonlinearity considered in [24] (this can be proved by using Pohozaev's invariant), or when the energy is small enough, because simple modes with small energy are known to be stable (see [8, 11]), and therefore the existence of a heteroclinic connection does not prevent global existence for small initial data. Finally, the operator and the forcing term that we consider do not fall in the spectral gap regime, because the sequence λ^{2k} does not grow fast enough as required by those results.

3. Proof of the main result

3.1. Boundedness and Lipschitz continuity of the nonlinearity

To begin with, we observe that the solution $(v(t), w(t))$ to system (2.2) satisfies the classical energy equality

$$v'(t)^2 + w'(t)^2 + M\left(v(t)^2 + \lambda^2 w(t)^2\right) = H_0^2 \qquad \forall t \in \mathbb{R}, \tag{3.1}$$

for some real number $H_0 > 0$, where

$$M(\sigma) := \int_0^\sigma m(s)\,\mathrm{d}s \qquad \forall \sigma \geqslant 0, \tag{3.2}$$

and the positivity of H_0 follows from (2.1). Due to the strict hyperbolicity assumption (1.3), the function (3.2) satisfies $M(\sigma) \geqslant \mu_1\sigma$ for every $\sigma \geqslant 0$. Thus from (3.1) it follows that

$$v(t)^2 + \lambda^2 w(t)^2 \leqslant \frac{H_0^2}{\mu_1} \qquad \text{and} \qquad v'(t)^2 + w'(t)^2 \leqslant H_0^2$$

for every $t \in \mathbb{R}$, and in particular

$$\sup_{t\in\mathbb{R}} \left\{|v'(t)|, |v(t)|, |w'(t)|, \lambda|w(t)|\right\} \leqslant \max\left\{1, \frac{1}{\sqrt{\mu_1}}\right\} H_0 =: H_1. \tag{3.3}$$

As a consequence, in the sequel we can assume that

$$0 < \mu_1 \leqslant m(\sigma) \leqslant \mu_2 \qquad \forall \sigma \geqslant 0, \tag{3.4}$$

and that m is Lipschitz continuous with some Lipschitz constant L.

3.2. Simple modes

Let $z_1 : \mathbb{R} \to \mathbb{R}$ denote the standard simple mode with energy H_0, namely the solution to the ordinary differential equation

$$z_1''(t) + m\left(z_1(t)^2\right) z_1(t) = 0$$

with initial data

$$z_1(0) = 0, \qquad z_1'(0) = H_0.$$

It is well-known that $z_1(t)$ is a periodic function of class C^3, and we call π_1 its minimal period. One can check that, for every real number $\lambda > 0$, the function defined by

$$z_\lambda(t) := \frac{1}{\lambda} z_1(\lambda t) \qquad \forall t \in \mathbb{R},$$

whose minimal period is of course $\pi_\lambda := \pi_1/\lambda$, is a solution to equation

$$z_\lambda''(t) + \lambda^2 m\left(\lambda^2 z_\lambda(t)^2\right) z_\lambda(t) = 0 \tag{3.5}$$

with the same initial data

$$z_\lambda(0) = 0, \qquad z_\lambda'(0) = H_0. \tag{3.6}$$

We call z_λ the simple mode of energy H_0 corresponding to the frequency λ^2. We observe that all simple modes satisfy the energy equality

$$z_\lambda'(t)^2 + M\left(\lambda^2 z_\lambda(t)^2\right) = H_0^2 \qquad \forall t \in \mathbb{R},$$

and therefore

$$\sup_{t\in\mathbb{R}} \left\{ |z_1'(t)|, |z_1(t)|, |z_\lambda'(t)|, \lambda |z_\lambda(t)| \right\} \leqslant H_1, \tag{3.7}$$

where H_1 is the same constant as in (3.3).

The key point is that the heteroclinic connection $(v(t), w(t))$ is exponentially asymptotic to the simple mode z_1 as $t \to -\infty$, and to the simple mode z_λ as $t \to +\infty$. The formal statement is the following (for the convenience of the reader, we include a proof in the appendix at the end of the paper).

Lemma 3.1 (limiting periodic orbits). *Let $m : [0, +\infty) \to (0, +\infty)$ be a function of class C^1 satisfying the strict hyperbolicity assumption (1.3), and let $\lambda > 1$ be a real number. Let us assume that the pair (m, λ) satisfies the heteroclinic connection assumption.*

Then there exist two real numbers $\tau_0 \in [0, \pi_1]$ and $\tau_1 \in [0, \pi_\lambda]$, and two positive real constants A_1 and B_1, such that

$$|v'(t) - z_1'(t - \tau_0)|^2 + |v(t) - z_1(t - \tau_0)|^2 \leqslant B_1 \exp(-A_1 |t|) \qquad \forall t \leqslant 0, \tag{3.8}$$

and

$$|w'(t) - z_\lambda'(t - \tau_1)|^2 + \lambda^2 |w(t) - z_\lambda(t - \tau_1)|^2 \leqslant B_1 \exp(-A_1 t) \qquad \forall t \geqslant 0. \tag{3.9}$$

3.3. The basic bridge between simple modes

Thanks to lemma 3.1, we can think of the heteroclinic connection $(v(t), w(t))$ as a trajectory that connects the two simple modes z_1 and z_λ in an infinite time, without the aid of any external force. Now we show that, if we allow an external force, then we can find a trajectory $(v_S(t), w_S(t))$ that connects the same two simple modes in a finite time interval $[-2S, 2S]$. The size of the required external force decays exponentially when S grows.

Proposition 3.2 (transition between two simple modes in a finite time interval). *Let $m : [0, +\infty) \to (0, +\infty)$ be a function of class C^1 satisfying the strict hyperbolicity assumption (1.3), and let $\lambda > 1$ be a real number. Let us assume that the pair (m, λ) satisfies the heteroclinic connection assumption.*

Then for every $S > 0$ there exist two functions $v_S : \mathbb{R} \to \mathbb{R}$ and $w_S : \mathbb{R} \to \mathbb{R}$ of class C^2, and two continuous functions $\varphi_S : \mathbb{R} \to \mathbb{R}$ and $\psi_S : \mathbb{R} \to \mathbb{R}$, with the following properties.

(1) (Kirchhoff equation with two modes and forcing term). *The functions v_S and w_S are solutions to the system*

$$\begin{cases} v_S''(t) + m\left(v_S(t)^2 + \lambda^2 w_S(t)^2\right) v_S(t) = \varphi_S(t) \\ w_S''(t) + \lambda^2 m\left(v_S(t)^2 + \lambda^2 w_S(t)^2\right) w_S(t) = \psi_S(t) \end{cases} \qquad \forall t \in \mathbb{R}. \quad (3.10)$$

(2) (Conditions at infinity). *The functions v_S and w_S satisfy*

$$v_S(t) = z_1(t - \tau_0) \quad \textit{and} \quad w_S(t) = 0 \qquad \forall t \leqslant -2S, \tag{3.11}$$

and

$$v_S(t) = 0 \quad \textit{and} \quad w_S(t) = z_\lambda(t - \tau_1) \qquad \forall t \geqslant 2S. \tag{3.12}$$

In particular, there exist $S_1 \in [2S, 2S + \pi_1]$ and $S_2 \in [2S, 2S + \pi_\lambda]$ such that

$$w_S(-S_1) = w_S'(-S_1) = v_S(-S_1) = 0, \qquad v_S'(-S_1) = H_0,$$

and

$$v_S(S_2) = v_S'(S_2) = w_S(S_2) = 0, \qquad w_S'(S_2) = H_0.$$

(3) (Bound on the forcing term). *The functions φ_S and ψ_S are such that*

$$\varphi_S(t) = \psi_S(t) = 0 \qquad \forall t \in (-\infty, -2S] \cup [-S, S] \cup [2S, +\infty), \tag{3.13}$$

and satisfy the estimate

$$|\varphi_S(t)|^2 + |\psi_S(t)|^2 \leqslant \left(\frac{1}{S^2} + 1\right)^2 B_2 \exp(-A_2 S) \qquad \forall t \in \mathbb{R}, \tag{3.14}$$

where A_2 and B_2 are two positive real numbers, both independent of S and t.

Proof. Let $\theta \in C^\infty(\mathbb{R})$ be a cutoff function such that

- $\theta(x) = 1$ for every $x \leqslant 1$,
- $\theta(x) = 0$ for every $x \geqslant 2$,
- $0 \leqslant \theta(x) \leqslant 1$ for every $x \in [1, 2]$,

and let Γ be a constant such that

$$|\theta'(x)| + |\theta''(x)| \leqslant \Gamma \qquad \forall x \in \mathbb{R}. \tag{3.15}$$

The idea is to use the function θ in order to define $v_S(t)$ and $w_S(t)$ as a convex combination of trajectories that coincides

- with the heteroclinic connection $(v(t), w(t))$ for $t \in [-S, S]$,
- with the limiting periodic trajectory $(z_1(t - \tau_0), 0)$ for $t \leqslant -2S$,
- with the limiting periodic trajectory $(0, z_\lambda(t - \tau_1))$ for $t \geqslant 2S$.

3.3.1. Definition when $t \leqslant 0$. In the case $t \leqslant 0$ we set $\theta_S(t) := \theta(-t/S)$ and we consider the functions

$$v_S(t) := \theta_S(t)\, v(t) + (1-\theta_S(t))\, z_1(t-\tau_0) \qquad \text{and} \qquad w_S(t) := \theta_S(t)\, w(t).$$

Since

$$\theta_S(t) = 0 \quad \forall t \leqslant -2S \qquad \text{and} \qquad \theta_S(t) = 1 \quad \forall t \in [-S, 0], \tag{3.16}$$

we deduce that (3.11) holds true, and in addition

$$(v_S(t), w_S(t)) = (v(t), w(t)) \qquad \forall t \in [-S, 0].$$

Computing the second order time derivatives of v_S and w_S, we discover that for $t \leqslant 0$ these functions are solutions to system (3.10) provided that we set

$$\begin{aligned}
\varphi_S(t) := {} & \theta_S''(t)\{v(t) - z_1(t-\tau_0)\} + 2\theta_S'(t)\{v'(t) - z_1'(t-\tau_0)\} \\
& + \theta_S(t)\, v(t) \left\{ m\left(v_S(t)^2 + \lambda^2 w_S(t)^2\right) - m\left(v(t)^2 + \lambda^2 w(t)^2\right) \right\} \\
& + (1-\theta_S(t))\, z_1(t-\tau_0) \left\{ m\left(v_S(t)^2 + \lambda^2 w_S(t)^2\right) - m\left(z_1(t-\tau_0)^2\right) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\psi_S(t) := {} & \theta_S''(t)\, w(t) + 2\theta_S'(t)\, w'(t) + \lambda^2 \theta_S(t)\, w(t) \left\{ m\left(v_S(t)^2 + \lambda^2 w_S(t)^2\right) \right. \\
& \left. - m\left(v(t)^2 + \lambda^2 w(t)^2\right) \right\}.
\end{aligned}$$

Using again (3.16) we obtain that

$$\varphi_S(t) = \psi_S(t) = 0 \qquad \forall t \in (-\infty, -2S] \cup [-S, 0],$$

which proves (3.13) for $t \leqslant 0$.

It remains to prove (3.14) for $t \leqslant 0$. Let $L_1(t)$, $L_2(t)$, $L_3(t)$ denote the three lines in the definition of $\varphi_S(t)$. From (3.15) we deduce that

$$|\theta_S'(t)| \leqslant \frac{\Gamma}{S} \qquad \text{and} \qquad |\theta_S''(t)| \leqslant \frac{\Gamma}{S^2} \qquad \forall t \leqslant 0, \tag{3.17}$$

and therefore

$$|L_1(t)| \leqslant \frac{\Gamma}{S^2} |v(t) - z_1(t-\tau_0)| + \frac{2\Gamma}{S} |v'(t) - z_1'(t-\tau_0)|.$$

Moreover, from (3.3) and (3.7) we deduce that

$$|v_S(t)| \leqslant H_1 \qquad \text{and} \qquad \lambda |w_S(t)| \leqslant H_1 \qquad \forall t \leqslant 0,$$

and therefore from the Lipschitz continuity of m we obtain that

$$\begin{aligned}
&\left|m\left(v_S(t)^2+\lambda^2 w_S(t)^2\right)-m\left(v(t)^2+\lambda^2 w(t)^2\right)\right| \\
&\quad\leqslant L\left(\left|v_S(t)^2-v(t)^2\right|+\lambda^2\left|w_S(t)^2-w(t)^2\right|\right) \\
&\quad\leqslant L\left(|v_S(t)+v(t)|\cdot|v_S(t)-v(t)|+|\lambda w_S(t)+\lambda w(t)|\cdot\lambda|w_S(t)-w(t)|\right) \\
&\quad\leqslant 2H_1 L\left(|v_S(t)-v(t)|+\lambda|w_S(t)-w(t)|\right).
\end{aligned}$$

Finally we observe that

$$|v_S(t)-v(t)|=(1-\theta_S(t))\,|v(t)-z_1(t-\tau_0)|$$

and

$$|w_S(t)-w(t)|=(1-\theta_S(t))\,|w(t)|,$$

so that in conclusion

$$\left|m\left(v_S(t)^2+\lambda^2 w_S(t)^2\right)-m\left(v(t)^2+\lambda^2 w(t)^2\right)\right|\leqslant 2H_1 L\left(|v(t)-z_1(t-\tau_0)|+\lambda|w(t)|\right), \tag{3.18}$$

and therefore

$$|L_2(t)|\leqslant 2H_1^2 L\left\{|v(t)-z_1(t-\tau_0)|+\lambda|w(t)|\right\}.$$

In an analogous way we obtain that

$$\begin{aligned}
&\left|m\left(v_S(t)^2+\lambda^2 w_S(t)^2\right)-m\left(z_1(t-\tau_0)^2\right)\right| \\
&\quad\leqslant L\left(\left|v_S(t)^2-z_1(t-\tau_0)^2\right|+\lambda^2 w_S(t)^2\right) \\
&\quad\leqslant L\left(|v_S(t)+z_1(t-\tau_0)|\cdot|v_S(t)-z_1(t-\tau_0)|+\lambda|w_S(t)|\cdot\lambda|w_S(t)|\right) \\
&\quad\leqslant 2H_1 L\left(|v_S(t)-z_1(t-\tau_0)|+\lambda|w_S(t)|\right) \\
&\quad\leqslant 2H_1 L\left(|v(t)-z_1(t-\tau_0)|+\lambda|w(t)|\right),
\end{aligned}$$

and therefore

$$|L_3(t)|\leqslant 2H_1^2 L\left\{|v(t)-z_1(t-\tau_0)|+\lambda|w(t)|\right\}.$$

From all these estimate we deduce that

$$|\varphi_S(t)|\leqslant\left(\frac{\Gamma}{S^2}+4H_1^2 L\right)|v(t)-z_1(t-\tau_0)|+\frac{2\Gamma}{S}|v'(t)-z_1'(t-\tau_0)|+4H_1^2 L\lambda|w(t)|.$$

Similarly, from (3.17) and (3.18) we obtain that

$$\begin{aligned}
|\psi_S(t)| &\leqslant \frac{\Gamma}{S^2}|w(t)|+\frac{2\Gamma}{S}|w'(t)|+2H_1^2 L\lambda\left\{|v(t)-z_1(t-t_0)|+\lambda|w(t)|\right\} \\
&\leqslant\left(\frac{\Gamma}{S^2\lambda}+2H_1^2 L\lambda\right)\lambda|w(t)|+\frac{2\Gamma}{S}|w'(t)|+2H_1^2 L\lambda|v(t)-z_1(t-t_0)|.
\end{aligned}$$

Finally, taking (2.4) and (3.8) into account, we conclude that

$$|\varphi_S(t)|^2 \leqslant 2\left(\frac{\Gamma}{S^2}+\frac{2\Gamma}{S}+4H_1^2L\right)^2 B_1\exp\left(-A_1|t|\right)+32H_1^4L^2B_0\exp\left(-A_0|t|\right),$$

and analogously

$$|\psi_S(t)|^2 \leqslant 2\left(\frac{\Gamma}{S^2\lambda}+\frac{2\Gamma}{S}+2H_1^2L\lambda\right)^2 B_0\exp\left(-A_0|t|\right)+8H_1^4L^2\lambda^2B_1\exp\left(-A_1|t|\right).$$

Recalling that $\varphi_S(t)=\psi_S(t)=0$ for $t\in[-S,0]$, the last two inequalities imply (3.14) for $t\leqslant 0$.

3.3.2. Definition when $t\geqslant 0$. In the case $t\geqslant 0$ we set $\theta_S(t):=\theta(t/S)$ and we consider the functions

$$v_S(t):=\theta_S(t)\,v(t) \qquad \text{and} \qquad w_S(t):=\theta_S(t)\,w(t)+(1-\theta_S(t))\,z_\lambda(t-\tau_1).$$

As in the previous case we find that

$$(v_S(t),w_S(t))=(v(t),w(t)) \qquad \forall t\in[0,S],$$

and that for $t\geqslant 0$ these functions are solutions to system (3.10) provided that we set

$$\begin{aligned}\varphi_S(t) :=\;& \theta_S''(t)\,v(t)+2\theta_S'(t)\,v'(t)\\ &+\theta_S(t)\,v(t)\left\{m\left(v_S(t)^2+\lambda^2w_S(t)^2\right)-m\left(v(t)^2+\lambda^2w(t)^2\right)\right\}.\end{aligned}$$

and

$$\begin{aligned}\psi_S(t) :=\;& \theta_S''(t)\left\{w(t)-z_\lambda(t-\tau_1)\right\}+2\theta_S'(t)\left\{w'(t)-z_\lambda'(t-\tau_1)\right\}\\ &+\lambda^2\theta_S(t)\,w(t)\left\{m\left(v_S(t)^2+\lambda^2w_S(t)^2\right)-m\left(v(t)^2+\lambda^2w(t)^2\right)\right\}\\ &+\lambda^2(1-\theta_S(t))\,z_\lambda(t-\tau_1)\left\{m\left(v_S(t)^2+\lambda^2w_S(t)^2\right)-m\left(\lambda^2z_\lambda(t-\tau_1)^2\right)\right\}.\end{aligned}$$

As in the previous case we obtain that

$$\left|m\left(v_S(t)^2+\lambda^2w_S(t)^2\right)-m\left(v(t)^2+\lambda^2w(t)^2\right)\right|\leqslant 2H_1L\left(|v(t)|+\lambda|w(t)-z_\lambda(t-\tau_1)|\right),$$

and

$$\left|m\left(v_S(t)^2+\lambda^2w_S(t)^2\right)-m\left(\lambda^2z_\lambda(t-\tau_1)^2\right)\right|\leqslant 2H_1L\left(|v(t)|+\lambda|w(t)-z_\lambda(t-\tau_1)|\right),$$

from which we deduce that

$$|\varphi_S(t)|\leqslant\left(\frac{\Gamma}{S^2}+2H_1^2L\right)|v(t)|+\frac{2\Gamma}{S}|v'(t)|+2H_1^2L\lambda|w(t)-z_\lambda(t-\tau_1)|$$

and

$$|\psi_S(t)|\leqslant\left(\frac{\Gamma}{S^2\lambda}+4H_1^2L\lambda\right)\lambda|w(t)-z_\lambda(t-\tau_1)|+\frac{2\Gamma}{S}|w'(t)-z_\lambda'(t-\tau_1)|+4H_1^2L\lambda|v(t)|.$$

Finally, taking (2.3) and (3.9) into account, we conclude that

$$|\varphi_S(t)|^2 \leqslant 2\left(\frac{\Gamma}{S^2}+\frac{2\Gamma}{S}+2H_1^2L\right)^2 B_0\exp(-A_0t)+8H_1^4L^2B_1\exp(-A_1t),$$

and

$$|\psi_S(t)|^2 \leqslant 2\left(\frac{\Gamma}{S^2\lambda}+\frac{2\Gamma}{S}+4H_1^2L\lambda\right)^2 B_1\exp(-A_1t)+32H_1^4L^2\lambda^2B_0\exp(-A_0t).$$

Recalling that $\varphi_S(t)=\psi_S(t)=0$ for $t\in[0,S]$, the last two inequalities imply (3.14) for $t\geqslant 0$. □

3.4. A sequence of bridges between consecutive simple modes

In the next result we rescale the construction of proposition 3.2, and we obtain a bridge between the simple modes corresponding to the frequencies λ^{2k} and λ^{2k+2}.

Proposition 3.3 (rescaling). *Let $m:[0,+\infty)\to(0,+\infty)$ be a function of class C^1 satisfying the strict hyperbolicity assumption (1.3), and let $\lambda>1$ be a real number. Let us assume that the pair (m,λ) satisfies the heteroclinic connection assumption.*

Let H be a Hilbert space, and let A be an operator as in theorem 2.3. Let k be a positive integer, and let S_k be a positive real number.

Then there exist two functions $u_k:\mathbb{R}\to H$ and $f_k:\mathbb{R}\to H$ with the following properties.

(1) (Regularity). *The functions u_k and f_k satisfy*

$$u_k\in C^2(\mathbb{R},\mathrm{Span}(e_k,e_{k+1})) \qquad \textit{and} \qquad f_k\in C^0(\mathbb{R},\mathrm{Span}(e_k,e_{k+1})). \tag{3.19}$$

(2) (Kirchhoff equation with two modes and forcing term). *The functions u_k is a solutions to the forced Kirchhoff equation*

$$u_k''(t)+m\left(|A^{1/2}u_k(t)|^2\right)Au_k(t)=f_k(t) \qquad \forall t\in\mathbb{R}. \tag{3.20}$$

(3) (Conditions at infinity). *The functions u_k satisfies*

$$u_k(t)=z_{\lambda^k}\left(t-\frac{\tau_0}{\lambda^k}\right)e_k \qquad \forall t\leqslant -2S_k, \tag{3.21}$$

and

$$u_k(t)=z_{\lambda^{k+1}}\left(t-\frac{\tau_1}{\lambda^k}\right)e_{k+1} \qquad \forall t\geqslant 2S_k. \tag{3.22}$$

In particular, there exist $S_{1,k}\in[2S_k,2S_k+\pi_{\lambda^k}]$ and $S_{2,k}\in[2S_k,2S_k+\pi_{\lambda^{k+1}}]$ such that

$$u_k(-S_{1,k})=0, \qquad u_k'(-S_{1,k})=H_0e_k, \tag{3.23}$$

and

$$u_k(S_{2,k})=0, \qquad u_k'(S_{2,k})=H_0e_{k+1}. \tag{3.24}$$

(4) (Properties of the forcing term). *The function f_k is such that*

$$f_k(t) = 0 \qquad \forall t \in (-\infty, -2S_k] \cup [-S_k, S_k] \cup [2S_k, +\infty), \tag{3.25}$$

and for every positive value of r and s its norm in the Gevrey space $\mathcal{G}_{r,s}(A)$ satisfies

$$\|f_k(t)\|^2_{\mathcal{G}_{r,s}(A)} \leqslant \left(\frac{1}{\lambda^k S_k^2} + \lambda^k\right)^2 B_2 \exp\left(r\lambda^{(k+1)/s} - A_2\lambda^k S_k\right) \qquad \forall t \in \mathbb{R}, \tag{3.26}$$

where A_2 and B_2 are the constants that appear in (3.14).

Proof. Let us apply proposition 3.2 with

$$S := \lambda^k S_k,$$

and let v_S, w_S, φ_S and ψ_S be the functions that we obtain. At this point let us set

$$u_k(t) := \frac{1}{\lambda^k} v_S\left(\lambda^k t\right) e_k + \frac{1}{\lambda^k} w_S\left(\lambda^k t\right) e_{k+1}$$

and

$$f_k(t) := \lambda^k \varphi_S\left(\lambda^k t\right) e_k + \lambda^k \psi_S\left(\lambda^k t\right) e_{k+1}. \tag{3.27}$$

Both the regularity (3.19) and equation (3.20) follow from these definitions and from the corresponding properties of v_S, w_S, φ_S and ψ_S.

Moreover, for every $t \leqslant -2S_k$ it turns out that $\lambda^k t \leqslant -2S$, and therefore from (3.11) we obtain that

$$u_k(t) = \frac{1}{\lambda^k} z_1\left(\lambda^k t - \tau_0\right) e_k = \frac{1}{\lambda^k} z_1\left(\lambda^k\left(t - \frac{\tau_0}{\lambda_k}\right)\right) e_k = z_{\lambda^k}\left(t - \frac{\tau_0}{\lambda^k}\right) e_k,$$

which proves (3.21). Similarly, for every $t \geqslant 2S_k$ it turns out that $\lambda^k t \geqslant 2S$, and therefore from (3.12) we obtain that

$$u_k(t) = \frac{1}{\lambda^k} z_\lambda\left(\lambda^k t - \tau_1\right) e_{k+1} = \frac{1}{\lambda^{k+1}} z_1\left(\lambda^{k+1} t - \lambda\tau_1\right) e_{k+1} = z_{\lambda^{k+1}}\left(t - \frac{\tau_1}{\lambda^k}\right) e_{k+1},$$

which proves (3.22).

Finally, (3.25) follows from (3.13), while from (3.27) and (3.14) we obtain that

$$\begin{aligned}\|f_k(t)\|^2_{\mathcal{G}_{s,r}(A)} &\leqslant \lambda^{2k}|\varphi_S\left(\lambda^k t\right)|^2 \exp\left(r\lambda^{k/s}\right) + \lambda^{2k}|\psi_S\left(\lambda^k t\right)|^2 \exp\left(r\lambda^{(k+1)/s}\right)\\ &\leqslant \left(\frac{1}{\lambda^k S_k^2} + \lambda^k\right)^2 B_2 \exp\left(-A_2\lambda^k S_k\right) \exp\left(r\lambda^{(k+1)/s}\right)\end{aligned}$$

for every $t \in \mathbb{R}$, which proves (3.26).

□

3.5. Conclusion of the proof of theorem 2.3

3.5.1. Definitions. For every integer $k \geqslant 0$ we apply proposition 3.3 with

$$S_k := \frac{1}{(k+1)^2}.$$

We consider the functions u_k and f_k, and the times $S_{1,k}$ and $S_{2,k}$, provided by that result. We consider the sequence $\{T_k\} \subseteq [0,+\infty)$ defined by $T_0 := 0$ and

$$T_{k+1} := T_k + S_{1,k} + S_{2,k} \qquad \forall k \geqslant 0.$$

We observe that

$$S_{1,k} \in [2S_k, 2S_k + \pi_{\lambda^k}] \qquad \text{and} \qquad S_{2,k} \in [2S_k, 2S_k + \pi_{\lambda^{k+1}}],$$

and therefore

$$T_{k+1} - T_k = S_{1,k} + S_{2,k} \leqslant \frac{4}{(k+1)^2} + \pi_{\lambda^k} + \pi_{\lambda^{k+1}} = \frac{4}{(k+1)^2} + \pi_1 \left(\frac{1}{\lambda^k} + \frac{1}{\lambda^{k+1}} \right).$$

As a consequence, the sequence $\{T_k\} \subseteq [0,+\infty)$ is increasing and

$$T_\infty := \lim_{k\to+\infty} T_k < +\infty.$$

Let us define $u : [0, T_\infty) \to H$ by

$$u(t) := u_k (t - T_k - S_{1,k}) \qquad \text{if } t \in [T_k, T_{k+1}] \text{ for some } k \geqslant 0,$$

and let us define $f : [0,+\infty) \to H$ by

$$f(t) := \begin{cases} f_k (t - T_k - S_{1,k}) & \text{if } t \in [T_k, T_{k+1}] \text{ for some } k \geqslant 0, \\ 0 & \text{if } t \geqslant T_\infty. \end{cases}$$

We claim that the conclusions of theorem 2.3 hold true with these choices.

3.5.2. Regularity of f. To begin with, we observe that the definition is well-posed because from (3.25) we know that

$$f_k (T_{k+1} - T_k - S_{1,k}) = f_k (S_{2,k}) = 0 = f_{k+1} (-S_{1,k+1}) = f_{k+1} (T_{k+1} - T_{k+1} - S_{1,k+1}).$$

Moreover, from (3.26) we deduce that

$$\|f_k (t)\|^2_{\mathcal{G}_{r,s}(A)} \leqslant \left(\frac{(k+1)^4}{\lambda^k} + \lambda^k \right)^2 B_2 \exp\left(r\lambda^{(k+1)/s} - A_2 \frac{\lambda^k}{(k+1)^2} \right) \qquad \forall t \in [T_k, T_{k+1}]$$

for every positive value of s and r. This implies in particular that

$$\forall s > 1 \quad \forall r > 0 \qquad \lim_{t\to T_\infty^-} f(t) = 0 \quad \text{in } \mathcal{G}_{s,r}(A),$$

which shows that f has the regularity (2.5).

3.5.3. Regularity of u. To begin with, we observe that the definition is well-posed because from (3.24) with k and (3.23) with $k+1$ we know that

$$u_k(T_{k+1}-T_k-S_{1,k}) = u_k(S_{2,k}) = 0 = u_{k+1}(-S_{1,k+1}) = u_{k+1}(T_{k+1}-T_{k+1}-S_{1,k+1}),$$

and

$$u_k'(T_{k+1}-T_k-S_{1,k}) = u_k'(S_{2,k}) = H_0 e_{k+1} = u_{k+1}'(-S_{1,k+1}) = u_{k+1}'(T_{k+1}-T_{k+1}-S_{1,k+1}).$$

In the same way from (3.20) and (3.25) we obtain that

$$u_k''(T_{k+1}-T_k-S_{1,k}) = 0 = u_{k+1}''(T_{k+1}-T_{k+1}-S_{1,k+1}),$$

and therefore the regularity of u follows from the regularity of u_k.

3.5.4. Blow-up of u. To this end, it is enough to observe that

$$\begin{aligned}\limsup_{t\to T_\infty^-}\left(|A^\alpha u'(t)|^2+|A^{\alpha+1/2}u(t)|^2\right) &\geqslant \limsup_{k\to+\infty}|A^\alpha u'(T_k)| \\ &= \limsup_{k\to+\infty}|A^\alpha(H_0 e_k)|^2 \\ &= \limsup_{k\to+\infty} H_0^2\lambda^{4k\alpha} \\ &= +\infty\end{aligned}$$

for every $\alpha > 0$, which proves (2.7).

In the same way the sequence $u'(T_k) = H_0 e_k$ has no limit as $k \to +\infty$, and therefore the limit of $u'(t)$ as $t \to T_\infty^-$ does not exist, which proves (2.8). □

Data availability statement

No new data were created or analysed in this study.

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Appendix

A.1. Proof of lemma 3.1

We limit ourselves to proving the asymptotic behavior as $t \to +\infty$, because the case when $t \to -\infty$ is analogous. As observed at the beginning of section 3, the functions v and w satisfy (3.1) and (3.3), and we can assume that m satisfies the bounds (3.4) and is Lipschitz continuous with some constant L. We observe also that the simple mode z_λ that solves (3.5) and (3.6) is an odd periodic function whose graph, roughly speaking, has the same qualitative behaviour of the graph of the function $t \mapsto \sin t$, and in particular z'_λ has exactly two zeroes in a period (in $\pi_\lambda/4$ and $3\pi_\lambda/4$).

Before entering into details, we describe briefly the general strategy. The constants H_2, H_3 and H_4 that we introduce in the sequel depend only on μ_1, μ_2, L, λ, and on the constants H_0 and H_1 that appear in (3.1) and (3.3).

- In the first step we consider the set

$$\mathcal{S} := \{t \geqslant 0 : w(t) = 0,\ w'(t) > 0\},$$

and we show that it consists of a sequence of real numbers $\{T_n\}_{n\geqslant 0} \subseteq [0,+\infty)$ such that

$$T_0 \leqslant \frac{2\pi}{\lambda\sqrt{\mu_1}} \qquad \text{and} \qquad \frac{2\pi}{\lambda\sqrt{\mu_2}} \leqslant T_{n+1} - T_n \leqslant \frac{2\pi}{\lambda\sqrt{\mu_1}} \qquad \forall n \geqslant 0. \tag{A.1}$$

- In the second step we show that, for every $n \geqslant 0$ and every $t \geqslant T_n$, it turns out that

$$|z'_\lambda(t-T_n) - w'(t)|^2 + \lambda^2 |z_\lambda(t-T_n) - w(t)|^2 \leqslant H_2 \exp\left(H_3(t-T_n) - 2A_0 T_n\right), \tag{A.2}$$

where z_λ is the simple mode that solves (3.5) and (3.6), and A_0 is the constant that appears in (2.3).

- In the third step we show that there exists a real number S_∞ such that

$$|T_n - n\pi_\lambda - S_\infty| \leqslant H_4 \exp\left(-A_0 T_n\right) \qquad \forall n \geqslant 0, \tag{A.3}$$

where we recall that π_λ denotes the period of z_λ.

- In the fourth step we show that (3.9) holds true with $\tau_1 = S_\infty$.

A.1.1. Step one—the set of zeroes with positive derivative. The structure of the set $\mathcal{S}$, and the estimates in (A.1), are consequences of the following lemma, applied with

$$u(t) := w(t), \qquad c(t) := \lambda^2 m\left(v(t)^2 + \lambda^2 w(t)^2\right), \qquad m_1 := \lambda^2\mu_1, \qquad m_2 := \lambda^2\mu_2.$$

Lemma A.1 (oscillation lemma). *Let m_1 and m_2 be two positive real numbers, let $c : [0,+\infty) \to \mathbb{R}$ be a continuous function such that*

$$0 < m_1 \leqslant c(t) \leqslant m_2 \qquad \forall t \geqslant 0, \tag{A.4}$$

and let $u : [0,+\infty) \to \mathbb{R}$ be a function of class C^2 such that $u'(0)^2 + u(0)^2 > 0$, and

$$u''(t) + c(t)u(t) = 0 \qquad \forall t \geqslant 0.$$

Then the following conclusions hold true.

(1) The set

$$\mathcal{S} := \{t \geqslant 0 : u(t) = 0,\ u'(t) > 0\}$$

consists of a sequence of real numbers $\{T_n\}_{n \geqslant 0} \subseteq [0, +\infty)$ *such that*

$$T_0 \leqslant \frac{2\pi}{\sqrt{m_1}} \qquad \textit{and} \qquad \frac{2\pi}{\sqrt{m_2}} \leqslant T_{n+1} - T_n \leqslant \frac{2\pi}{\sqrt{m_1}} \quad \forall n \geqslant 0. \tag{A.5}$$

(2) For every $n \geqslant 0$ *there exists* $S_n \in (T_n, T_{n+1})$ *such that*

$$u(S_n) = 0 \qquad \textit{and} \qquad u'(S_n) < 0. \tag{A.6}$$

Proof. Since u in not identically zero, we know that actually $u'(t)^2 + u(t)^2 > 0$ for every $t \geqslant 0$. Therefore, there exists two functions $\rho_1 : [0, +\infty) \to \mathbb{R}$ and $\theta_1 : [0, +\infty) \to \mathbb{R}$ of class C^1 such that

$$u(t) = \frac{1}{\sqrt{m_1}} \rho_1(t) \cos\theta_1(t) \qquad \text{and} \qquad u'(t) = -\rho_1(t) \sin\theta_1(t) \tag{A.7}$$

for every $t \geqslant 0$. We observe that $t \in \mathcal{S}$ if and only if $\theta_1(t) = -\pi/2 + 2k\pi$ for some $k \in \mathbb{Z}$.

Moreover, with some computations we obtain that ρ_1 and θ_1 are solutions to the system of ordinary differential equations (for the sake of shortness, we do not write explicitly the dependence on t of ρ_1 and θ_1)

$$\rho_1' = \rho_1 \sin\theta_1 \cos\theta_1 \left(\frac{c(t)}{\sqrt{m_1}} - \sqrt{m_1} \right), \qquad \theta_1' = \sqrt{m_1} \sin^2\theta_1 + \frac{c(t)}{\sqrt{m_1}} \cos^2\theta_1.$$

In particular, from the second equation and the estimate from below in (A.4) we deduce that

$$\theta_1'(t) \geqslant \sqrt{m_1} \qquad \forall t \geqslant 0,$$

which implies the estimates from above in (A.5).

Analogously, we can write $u(t)$ and $u'(t)$ in the form

$$u(t) = \frac{1}{\sqrt{m_2}} \rho_2(t) \cos\theta_2(t) \qquad \text{and} \qquad u'(t) = -\rho_2(t) \sin\theta_2(t),$$

where $\rho_2 : [0, +\infty) \to \mathbb{R}$ and $\theta_2 : [0, +\infty) \to \mathbb{R}$ are function of class C^1 such that

$$\rho_2' = \rho_2 \sin\theta_2 \cos\theta_2 \left(\frac{c(t)}{\sqrt{m_2}} - \sqrt{m_2} \right), \qquad \theta_2' = \sqrt{m_2} \sin^2\theta_2 + \frac{c(t)}{\sqrt{m_2}} \cos^2\theta_2,$$

and as before $t \in \mathcal{S}$ if and only if $\theta_2(t) = -\pi/2 + 2k\pi$ for some $k \in \mathbb{Z}$. Moreover, from the second equation and the estimate from above in (A.4), we deduce that

$$\theta_2'(t) \leqslant \sqrt{m_2} \qquad \forall t \geqslant 0,$$

which implies the estimate from below in (A.5).

As for statement (2), it is enough to write $u(t)$ and $u'(t)$ in the form (A.7), and observe that θ_1 is an increasing function, and the times S_n with the property (A.6) are those in which $\theta_1(t) = \pi/2 + 2k\pi$ for some $k \in \mathbb{Z}$.

□

A.1.2. Step two—estimate on the difference with a translated simple mode. Let us consider the function

$$r_n(t) := z_\lambda(t - T_n) - w(t) \qquad \forall t \geqslant 0,$$

and its usual energy

$$E_n(t) := r_n'(t)^2 + \lambda^2 r_n(t)^2 \qquad \forall t \geqslant 0. \tag{A.8}$$

With some calculations we discover that

$$r_n''(t) + \lambda^2 m\left(\lambda^2 z_\lambda(t - T_n)^2\right) r_n(t) = \lambda^2 f_n(t)\, w(t) \qquad \forall t \geqslant 0,$$

where

$$f_n(t) := m\left(v(t)^2 + \lambda^2 w(t)^2\right) - m\left(\lambda^2 z_\lambda(t - T_n)^2\right) \qquad \forall t \geqslant 0,$$

and that (A.2) is equivalent to

$$E_n(t) \leqslant H_2 \exp\left(H_3(t - T_n) - 2A_0 T_n\right) \qquad \forall t \geqslant T_n. \tag{A.9}$$

In order to prove this estimate, we compute the time-derivative

$$E_n'(t) = 2\lambda^2 \left\{1 - m\left(\lambda^2 z_\lambda(t - T_n)^2\right)\right\} r_n'(t)\, r_n(t) + 2\lambda^2 f_n(t)\, r_n'(t)\, w(t). \tag{A.10}$$

From the bound from above in (3.4) we deduce that

$$2\lambda^2 \left\{1 - m\left(\lambda^2 z_\lambda(t - T_n)^2\right)\right\} r_n'(t)\, r_n(t) \leqslant \lambda(1 + \mu_2)\, E_n(t). \tag{A.11}$$

From the Lipschitz continuity of m, and the estimates (3.3) and (3.7), we deduce that

$$\begin{aligned}|f_n(t)| &\leqslant Lv(t)^2 + L\lambda^2 |w(t) + z_\lambda(t - T_n)| \cdot |w(t) - z_\lambda(t - T_n)| \\ &\leqslant Lv(t)^2 + 2H_1 L \cdot \lambda |r_n(t)|,\end{aligned}$$

and therefore

$$\begin{aligned}2\lambda^2 f_n(t)\, r_n'(t)\, w(t) &\leqslant 2L\lambda^2 v(t)^2 |r_n'(t)| \cdot |w(t)| + 2\lambda^2 \cdot 2H_1 L \cdot \lambda |r_n(t)| \cdot |r_n'(t)| \cdot |w(t)| \\ &\leqslant L\lambda\left(\lambda^2 w(t)^2 r_n'(t)^2 + v(t)^4\right) + 2LH_1 \cdot \lambda^2 |w(t)| \cdot E_n(t) \\ &\leqslant 3L\lambda H_1^2 E_n(t) + L\lambda v(t)^4,\end{aligned}$$

so that recalling (2.3) we obtain that

$$2\lambda^2 f_n(t)\, r_n'(t)\, w(t) \leqslant 3L\lambda H_1^2 E_n(t) + L\lambda B_0^2 \exp(-2A_0 t). \tag{A.12}$$

Plugging (A.11) and (A.12) into (A.10) we deduce that

$$E_n'(t) \leqslant \lambda\left(1 + \mu_2 + 3LH_1^2\right) E_n(t) + L\lambda B_0^2 \exp(-2A_0 t).$$

Integrating this differential inequality we conclude that

$$E_n(t) \leqslant \left\{ E_n(T_n) + \frac{L\lambda B_0^2}{2A_0 + H_3} \exp(-2A_0 T_n) \right\} \exp(H_3(t - T_n)) \qquad \forall t \geqslant T_n, \tag{A.13}$$

where for the sake of shortness we set

$$H_3 := \lambda \left(1 + \mu_2 + 3LH_1^2\right). \tag{A.14}$$

It remains to estimate $E_n(T_n)$. To begin with, we observe that

$$r_n(T_n) = z_\lambda(0) - w(T_n) = 0,$$

and

$$r_n'(T_n) = z_\lambda'(0) - w'(T_n) = H_0 - w'(T_n) > 0,$$

so that it is enough to estimate $w'(T_n)$ from below. To this end, we consider the energy equality (3.1) with $t = T_n$. Recalling that $w(T_n) = 0$ and $M(\sigma) \leqslant \mu_2 \sigma$ for every $\sigma \geqslant 0$, we deduce that

$$\begin{aligned} w'(T_n)^2 &= H_0^2 - M\left(v(T_n)^2\right) - v'(T_n)^2 \\ &\geqslant H_0^2 - \mu_2 v(T_n)^2 - v'(T_n)^2 \\ &\geqslant H_0^2 - \max\{1, \mu_2\} \left(v'(T_n)^2 + v(T_n)^2\right) \\ &\geqslant H_0^2 - \max\{1, \mu_2\} B_0 \exp(-A_0 T_n). \end{aligned}$$

It follows that

$$|r_n'(T_n)| = H_0 - w'(T_n) = \frac{H_0^2 - w'(T_n)^2}{H_0 + w'(T_n)} \leqslant \frac{\max\{1, \mu_2\} B_0}{H_0} \exp(-A_0 T_n),$$

and hence

$$E_n(T_n) = r_n'(T_n)^2 \leqslant \left(\frac{\max\{1, \mu_2\} B_0}{H_0}\right)^2 \exp(-2A_0 T_n).$$

Substituting this relation into (A.13) we conclude that (A.2) holds true with H_3 given by (A.14) and

$$H_2 := \left(\frac{\max\{1, \mu_2\} B_0}{H_0}\right)^2 + \frac{L\lambda B_0^2}{2A_0 + H_3}.$$

A.1.3. Step three—asymptotic behavior of zeroes with positive derivative. The strategy of the proof of (A.3) is the following.

- To begin with, we observe that (A.2) implies that

$$r_n'(t)^2 + \lambda^2 r_n(t)^2 \leqslant H_2 \exp(2H_3\pi_\lambda)\exp(-2A_0T_n) \qquad \forall t \in [T_n, T_n + 2\pi_\lambda] \tag{A.15}$$

  for every $n \geqslant 0$.
- Then we consider the constant

$$C := \frac{2\sqrt{H_2}}{H_0\lambda}\exp(H_3\pi_\lambda),$$

  and the two sequences

$$A_n := T_n + \pi_\lambda - C\exp(-A_0T_n), \qquad B_n := T_n + \pi_\lambda + C\exp(-A_0T_n).$$

  We show that $w(A_n) < 0$ and $w(B_n) > 0$ when n is large enough. This implies that there exists $\widehat{T}_n \in (A_n, B_n)$ such that $w(\widehat{T}_n) = 0$. Then we show also that $w'(\widehat{T}_n) \to H_0$, which implies that $w'(\widehat{T}_n) > 0$ when n is large enough.
- We show that, when n is large enough, there are no points in $(T_n, \widehat{T}_n)$ where w vanishes and w' is positive. This proves that $T_{n+1} = \widehat{T}_n$, and therefore

$$T_{n+1} = T_n + \pi_\lambda + R_n \tag{A.16}$$

  for a suitable R_n such that

$$|R_n| \leqslant C\exp(-A_0T_n) \tag{A.17}$$

  for n large enough. Up to changing the value of C, we can always assume that (A.17) holds true for every $n \geqslant 0$.
- Finally, we show that (A.16) and (A.17) imply (A.3) for suitable values of S_∞ and H_4.

So let us start by proving that $w(B_n) > 0$ for n large enough. To this end we observe that

$$w(B_n) = z_\lambda(B_n - T_n) - r_n(B_n) = z_\lambda(C\exp(-A_0T_n)) - r_n(B_n). \tag{A.18}$$

Moreover, for n large enough it turns out that $B_n \leqslant T_n + 2\pi_\lambda$, and therefore from (A.15) we deduce that

$$|r_n(B_n)| \leqslant \frac{\sqrt{H_2}}{\lambda}\exp(H_3\pi_\lambda)\exp(-A_0T_n). \tag{A.19}$$

Now from the initial data (3.6) we deduce that

$$\lim_{t\to 0}\frac{z_\lambda(t)}{t} = z_\lambda'(0) = H_0,$$

and hence from (A.18) and (A.19) we conclude that

$$\liminf_{n\to+\infty} w(B_n)\exp(A_0T_n) \geqslant CH_0 - \frac{\sqrt{H_2}}{\lambda}\exp(H_3\pi_\lambda) > 0,$$

which proves that $w(B_n) > 0$ when n is large enough. In an analogous way we obtain that $w(A_n) < 0$ when n is large enough. This proves that $\widehat{T}_n$ exists and $\widehat{T}_n - T_n \to \pi_\lambda$. Applying again (A.15) we conclude that

$$\lim_{n\to+\infty} w'\left(\widehat{T}_n\right) = \lim_{n\to+\infty}\left(z'_\lambda\left(\widehat{T}_n - T_n\right) - r'_n\left(\widehat{T}_n\right)\right) = z'_\lambda(\pi_\lambda) = H_0.$$

Now we need to show that $\widehat{T}_n = T_{n+1}$ for n large enough. Let us assume by contradiction that $T_{n+1} \in (T_n, \widehat{T}_n)$ for infinitely many indices n. From statement (2) of lemma A.1 we know that between any two zeroes of w with positive derivative there exists at least one zero of w with negative derivative. This means that for infinitely many positive integers n there exists C_n and D_n, with

$$T_n < C_n < T_{n+1} < D_n < \widehat{T}_n < T_n + \frac{9}{8}\pi_\lambda,$$

such that

$$w(C_n) = w(D_n) = 0, \qquad w'(C_n) < 0, \qquad w'(D_n) < 0.$$

From the energy equality (3.1), letting $n \to +\infty$ on this subsequence (not relabeled), we obtain that

$$\lim_{n\to+\infty} w'(C_n) = \lim_{n\to+\infty} w'(D_n) = -H_0,$$

and analogously

$$\lim_{n\to+\infty} w'(T_n) = \lim_{n\to+\infty} w'(T_{n+1}) = \lim_{n\to+\infty} w'\left(\widehat{T}_n\right) = H_0.$$

As a consequence, since $z'_\lambda(t - T_n) = r'_n(t) + w'(t)$, from (A.15) we deduce that

$$\lim_{n\to+\infty} z'_\lambda(C_n - T_n) = \lim_{n\to+\infty} z'_\lambda(D_n - T_n) = -H_0,$$

and

$$\lim_{n\to+\infty} z'_\lambda(0) = \lim_{n\to+\infty} z'_\lambda(T_{n+1} - T_n) = \lim_{n\to+\infty} z'_\lambda\left(\widehat{T}_n - T_n\right) = H_0.$$

This shows in particular that, for n large enough, $z'_\lambda(t)$ is positive for t equal to 0, $T_{n+1} - T_n$, $\widehat{T}_n - T_n$, and negative for t equal to $C_n - T_n$ and $D_n - T_n$. But this implies that z'_λ has at least four zeroes in the interval $[0, 9\pi_\lambda/8]$, which is absurd.

At this point we know that we can write T_{n+1} in the form (A.16), with R_n that satisfies (A.17) for every $n \geqslant 0$. Now we observe that the estimate from below in (A.1) implies that

$$T_i \geqslant \frac{2\pi i}{\lambda\sqrt{\mu_2}} \qquad \text{and} \qquad T_i - T_n \geqslant \frac{2\pi(i-n)}{\lambda\sqrt{\mu_2}}$$

for every pair of integers $0 \leqslant n \leqslant i$. Due to (A.17), this implies in particular that the series of R_i's converges, and

$$\sum_{i=n}^{\infty}|R_i| \leqslant C\sum_{i=n}^{\infty}\exp\left(-A_0T_i\right)$$
$$= C\exp\left(-A_0T_n\right)\sum_{i=n}^{\infty}\exp\left(-A_0\left(T_i-T_n\right)\right) \leqslant H_4\exp\left(-A_0T_n\right)$$

for a suitable constant H_4 that does not depend on n. At this point we can set

$$S_\infty := T_0 + \sum_{i=0}^{\infty}R_i,$$

and observe that

$$T_n = n\pi_\lambda + T_0 + \sum_{i=0}^{n-1}\left(T_{i+1}-T_i-\pi_\lambda\right) = n\pi_\lambda + S_\infty - \sum_{i=n}^{\infty}R_i,$$

and hence

$$|T_n - n\pi_\lambda - S_\infty| \leqslant \sum_{i=n}^{\infty}|R_i| \leqslant H_4\exp\left(-A_0T_n\right),$$

which establishes (A.3).

A.1.4. Step four—conclusion. Thanks to (A.3), for every $n \geqslant 0$ we can write T_n in the form

$$T_n = n\pi_\lambda + S_\infty + \widehat{R}_n \qquad \text{with} \qquad |\widehat{R}_n| \leqslant H_4\exp\left(-A_0T_n\right). \tag{A.20}$$

Now we observe that z_λ is a periodic function of class C^2, and therefore there exists a positive real number Λ such that

$$\left|z_\lambda'\left(\tau+\widehat{R}_n\right)-z_\lambda'\left(\tau\right)\right|^2+\lambda^2\left|z_\lambda\left(\tau+\widehat{R}_n\right)-z_\lambda\left(\tau\right)\right|^2 \leqslant \Lambda\widehat{R}_n^2 \qquad \forall\tau\in\mathbb{R}. \tag{A.21}$$

Since

$$|z_\lambda\left(t-S_\infty\right)-w\left(t\right)|^2 = |z_\lambda\left(t-T_n+\widehat{R}_n\right)-w\left(t\right)|^2$$
$$\leqslant 2|z_\lambda\left(t-T_n+\widehat{R}_n\right)-z_\lambda\left(t-T_n\right)|^2+2|z_\lambda\left(t-T_n\right)-w\left(t\right)|^2,$$

and analogously

$$|z_\lambda'\left(t-S_\infty\right)-w'\left(t\right)|^2 \leqslant 2|z_\lambda'\left(t-T_n+\widehat{R}_n\right)-z_\lambda'\left(t-T_n\right)|^2+2|z_\lambda'\left(t-T_n\right)-w'\left(t\right)|^2,$$

from (A.21), (A.20), (A.8), and (A.9) we obtain that

$$|z_\lambda'\left(t-S_\infty\right)-w'\left(t\right)|^2+\lambda^2|z_\lambda\left(t-S_\infty\right)-w\left(t\right)|^2 \leqslant 2\Lambda\widehat{R}_n^2+2E_n\left(t\right)$$
$$\leqslant \left\{2\Lambda H_4^2+2H_2\exp\left(H_3\left(t-T_n\right)\right)\right\}\exp\left(-2A_0T_n\right) \tag{A.22}$$

for every $t \geqslant 0$. When $t \in [T_n, T_{n+1}]$, from (A.1) we know that

$$\exp\left(H_3\left(t - T_n\right)\right) \leqslant \exp\left(\frac{2\pi H_3}{\lambda\sqrt{\mu_1}}\right), \tag{A.23}$$

and similarly

$$\exp\left(-2A_0 T_n\right) = \exp\left(-2A_0 t\right)\exp\left(2A_0\left(t - T_n\right)\right) \leqslant \exp\left(-2A_0 t\right)\exp\left(\frac{4A_0\pi}{\lambda\sqrt{\mu_1}}\right). \tag{A.24}$$

Plugging (A.23) and (A.24) into (A.22) we conclude that

$$|z_\lambda'\left(t - S_\infty\right) - w'\left(t\right)|^2 + \lambda^2 |z_\lambda\left(t - S_\infty\right) - w\left(t\right)|^2 \leqslant H_5 \exp\left(-2A_0 t\right) \qquad \forall t \in [T_n, T_{n+1}]$$

for a suitable constant H_5 that does *not* depend on n.

Since $T_n \to +\infty$, this completes the proof of (3.9) with $A_1 = 2A_0$ and $B_1 = H_5$.

□

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