REGULARITY RESULTS FOR ROUGH SOLUTIONS OF THE INCOMPRESSIBLE EULER EQUATIONS VIA INTERPOLATION METHODS

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Abstract. Given any solution u of the Euler equations which is assumed to have some regularity in space – in terms of Besov norms, natural in this context – we show by interpolation methods that it enjoys a corresponding regularity in time and that the associated pressure p is twice as regular as u. This generalizes a recent result by Isett [17] (see also Colombo and De Rosa [9]), which covers the case of Hölder spaces.

1. Introduction

In the spatial periodic setting $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, we consider the incompressible Euler equations

$$\begin{cases} \partial_t u + \operatorname{div}(u \otimes u) + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases} \qquad \text{in } (0,T) \times \mathbb{T}^3 \tag{1.1}$$

where $u : (0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ represents the velocity of an incompressible fluid, $p : (0,T) \times \mathbb{T}^3 \to \mathbb{R}$ is the hydrodynamic pressure, with the constraint $\int_{\mathbb{T}^3} p\, dx = 0$, which guaranties its uniqueness.

The interest for low-regularity solutions to the Euler equations is strongly related to Kolmogorov's 1941 theory of turbulence [19] and to the Onsager's conjecture [24]. In recent years, distributional solutions belonging to Hölder spaces were built with convex integration techniques, starting from the works of De Lellis and Székelyhidi [12, 13] and leading to the proof of the Onsager's conjecture by Isett, see [18].

Such techniques were recently adapted by Buckmaster and Vicol to the Navier-Stokes equations by developing a Sobolev (rather than Hölder) based method [5], which in turn appears also in a recent work by Modena and Székelyhidi in [23]. This approach can be used to build solutions of the Euler equation with velocity in $L^\infty((0,T); L^2 \cap W^{1,\theta}(\mathbb{T}^3))$ for some $\theta > 1$. In the context of the physical theory of intermittency it is currently an open problem (see [6, Open Problem 5]) to determine the best exponent θ such that $L^\infty((0,T); H^\theta(\mathbb{T}^3))$ solutions conserve the energy (it is known that for $\theta = 5/6$ conservation holds).

The following theorem provides a regularization property of the Euler equations, for solutions which enjoy some a priori Sobolev or Besov regularity in space. Roughly speaking, we prove that the pressure associated to any such solution enjoys double regularity in space with respect to u, and that both u and p enjoy a corresponding time regularity. In the main theorem below, by $B^\theta_{s,\infty}$ we denote a Besov space, rigorously defined in Section 2. The choice to work in these spaces is motivated to avoid an ε-loss of regularity in time and was previously performed for instance in [1, Chapter 7].

Theorem 1.1. *Let (u,p) be a distributional solution to (1.1) in $(0,T) \times \mathbb{T}^3$, for some $T < \infty$. For any $\theta \in (0,1)$, $s \in [1,\infty]$, $r \in (1,\infty)$, the following implications are true:*

(i) if $u \in L^{2s}((0,T); B^\theta_{2r,\infty}(\mathbb{T}^3))$, then $u \in B^\theta_{s,\infty}((0,T); L^r(\mathbb{T}^3))$ and $p \in L^s((0,T); B^{2\theta}_{r,\infty}(\mathbb{T}^3))$;

(ii) if $u \in L^{3s}((0,T); B^\theta_{4r,\infty}(\mathbb{T}^3))$ and $\theta > 1/2$, then $p \in B^{2\theta-1-\beta}_{s,\infty}((0,T); B^{1+\beta}_{r,\infty}(\mathbb{T}^3))$ for any $\beta \in [0, 2\theta-1)$;

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(iii) *if* $u \in L^{3s}((0,T); B^\theta_{3r,\infty}(\mathbb{T}^3))$ *and if* $\theta \le 1/2$*, then* $p \in B^{2\theta-\varepsilon}_{s,\infty}((0,T); L^r(\mathbb{T}^3))$*, for any* $\varepsilon > 0$*. Moreover in the case* $\theta > 1/2$ *we have* $p \in W^{1,s}((0,T); B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$*;*
(iv) *if* $u \in L^{6s}((0,T); B^\theta_{6r,\infty}(\mathbb{T}^3))$ *and* $\theta > 1/2$*, then* $\partial_t p \in B^{2\theta-1-\varepsilon}_{s,\infty}((0,T); L^r(\mathbb{T}^3))$*, for any* $\varepsilon > 0$*.*

Then we obtain the following corollary on the Sobolev solutions by considering suitable embeddings between Sobolev and Besov spaces.

Corollary 1.2. *Let* (u,p) *be a distributional solution to* (1.1) *in* $(0,T)\times\mathbb{T}^3$*, for some* $T<\infty$*. For any* $\theta \in (0,1)$*,* $s \in [1,\infty]$*,* $r \in (1,\infty)$*, the following implications hold true:*
(i) *if* $u \in L^{2s}((0,T); W^{\theta,2r}(\mathbb{T}^3))$*, then* $u \in W^{\theta-\varepsilon,s}((0,T); L^r(\mathbb{T}^3))$ *and* $p \in L^s((0,T); W^{2\theta-\varepsilon,r}(\mathbb{T}^3))$*;*
(ii) *if* $\theta \le 1/2$ *and* $u \in L^{3s}((0,T); W^{\theta,3r}(\mathbb{T}^3))$*, or if* $\theta > 1/2$ *and* $u \in L^{6s}((0,T); W^{\theta,6r}(\mathbb{T}^3))$*, then* $p \in W^{2\theta-\varepsilon,s}((0,T); L^r(\mathbb{T}^3))$*.*

When $s=r=\infty$, identifying $W^{\theta,\infty}$ with the corresponding Hölder space, the previous theorem corresponds formally to [17, Theorem 1.1] and [9, Theorem 1.1]: roughly speaking, it says that if (u,p) is a distributional solution to (1.1), $\theta\in(0,1)$ and $u \in L^\infty((0,T); C^\theta(\mathbb{T}^3))$, then $u \in C^{\theta-\varepsilon}((0,T); L^\infty(\mathbb{T}^3))$, namely $u \in C^{\theta-\varepsilon}((0,T)\times\mathbb{T}^3)$ and $p \in C^{2\theta-\varepsilon}((0,T)\times\mathbb{T}^3)$. These regularity estimates were observed as a common feature of many convex integration schemes which produce Hölder solutions of the Euler equations [12, 13, 18], at least as far as the double regularity of pressure is concerned. Correspondingly, the regularity properties of Corollary 1.2 must be expected when building nonsmooth solutions in $L^\infty((0,T); H^\theta(\mathbb{T}^3))$ for some $\theta > 1/3$, as expected in [6, Open Problem 5].

Theorem 1.1 follows from two main ingredients: on one side, we obtain the time regularity by estimating, for any time increment h, some norm $||u(t+h)-u(t)||$ by comparison between u and the convolution of u with a mollification kernel at some scale δ, which is then linked to h. On the other side, to obtain the double regularity of the pressure we look at

$$-\Delta p = \operatorname{div}\operatorname{div}(u\otimes u), \tag{1.2}$$

which is the formal equation solved by p. We consider a bilinear operator which associates to two divergence-free vector fields (u,v) the solution to $-\Delta p = \operatorname{div}\operatorname{div}(u\otimes v)$ and we apply an abstract interpolation result for bilinear operators (see Theorem 3.6 below). Previous results on the regularity of the pressure in Hölder spaces (see [10] and [9]) were instead based on suitable representation formulas for the pressure by means of the Green kernel of the Laplacian, while this strategy using real interpolation methods seems to be new in this context.

2. Preliminary tools and notations

Along the paper, we will consider $\mathbb{T}^3$ as spatial domain, identifying it with the 3-dimensional cube $[0,1]^3 \subset \mathbb{R}^3$. Thus for any $f:\mathbb{T}^3\to\mathbb{R}^3$ we will always work with its periodic extension to the whole space.

We will define the norms for a domain $\Omega \subseteq \mathbb{R}^d$, for a general dimension $d \ge 1$, since in this way we can handle both the space and the time regularities. Let $\Omega\subseteq\mathbb{R}^d$ be an open and Lipschitz domain. For $\theta\in(0,\infty)$, $r,s\in[1,\infty]$, the $L^r(\Omega)$ and $W^{\theta,r}(\Omega)$ spaces are the classical Lebesgue and Sobolev-Slobodeckij spaces, with the usual identifications $W^{0,r}(\Omega)=L^r(\Omega)$ and $W^{\theta,\infty}(\Omega)=C^\theta(\Omega)$. We first define the Besov spaces on the whole $\mathbb{R}^d$, then their version on general open sets Ω will be defined by extension. For any $\theta\in(0,\infty)$, let θ^- to be the biggest integer which is strictly less than θ. For any non integer $\theta\in(0,\infty)$, the Besov space $B^\theta_{r,s}(\mathbb{R}^d)$ is the space of functions $f \in W^{\theta^-,r}(\mathbb{R}^d)$ such that

$$[f]_{B^\theta_{r,s}(\mathbb{R}^d)} = \sum_{|\alpha|=\theta^-}\left(\int_{\mathbb{R}^d}\frac{1}{|h|^{d+(\theta-\theta^-)s}}\left(\int_{\mathbb{R}^d}|D^\alpha f(x+h)-D^\alpha f(x)|^r\,dx\right)^{\frac{s}{r}}dh\right)^{\frac{1}{s}}<\infty,$$

with the usual generalization when $r, s = \infty$. The full Besov norm will be then given by

$$\|f\|_{B^\theta_{r,s}(\mathbb{R}^d)} = \|f\|_{W^{\theta-,r}(\mathbb{R}^d)} + [f]_{B^\theta_{r,s}(\mathbb{R}^d)}.$$

If instead $\theta > 0$ is an integer, the Besov space $B^\theta_{r,s}(\mathbb{R}^d)$ consists of all the functions $f \in W^{\theta,r}(\mathbb{R}^d)$, such that

$$[f]_{B^\theta_{r,s}(\mathbb{R}^d)} = \sum_{|\alpha|=\theta} \left(\int_{\mathbb{R}^d} \frac{1}{|h|^{d+s}} \left(\int_{\mathbb{R}^d} |D^\alpha f(x+2h) - 2D^\alpha f(x+h) + D^\alpha f(x)|^r \, dx \right)^{\frac{s}{r}} dh \right)^{\frac{1}{s}} < \infty,$$

again with the usual generalization when $r, s = \infty$. Thus the full norm will be given by

$$\|f\|_{B^\theta_{r,s}(\mathbb{R}^d)} = \|f\|_{W^{\theta,r}(\mathbb{R}^d)} + [f]_{B^\theta_{r,s}(\mathbb{R}^d)}.$$

For any open and Lipschitz set Ω we then define

$$B^\theta_{r,s}(\Omega) = \left\{ f : \Omega \to \mathbb{R}^d \text{ s.t. } \exists\, \tilde{f} \in B^\theta_{r,s}(\mathbb{R}^d),\ \tilde{f}|_\Omega = f \right\},$$

where the semi-norm is given by

$$[f]_{B^\theta_{r,s}(\Omega)} = \inf \left\{ [\tilde{f}]_{B^\theta_{r,s}(\mathbb{R}^d)},\ \tilde{f}|_\Omega = f \right\}.$$

By the definitions above we have that for any non integer $\theta \in (0, \infty)$, $B^\theta_{r,r}(\Omega) = W^{\theta,r}(\Omega)$ for any $r \in [1, \infty]$, which in the case $r = \infty$ gives $B^\theta_{\infty,\infty}(\Omega) = C^\theta(\Omega)$. Moreover, since the domain Ω is Lipschitz, we always have the existence of a linear extension operator to the whole space. It is well know that this operator turns out to be also continuous between every Sobolev or Besov spaces.

Considering the flat d-dimensional torus $\mathbb{T}^d$, we define the Besov norm as above with $\Omega = [0, 4]^d$ that is, we compute the norm in 4 copies of $\mathbb{T}^d$.

Dealing with time dependent vector fields $u = u(t, x)$, we will use the notations $[u(t)]$ and $\|u(t)\|$ when the spatial semi-norm or norm, respectively, are computed at the fixed time t.

We give the following interpolation result in Besov spaces.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open and Lipschitz set. For any $r \in [1, \infty]$, $\theta, \gamma \in (0, 1)$ with $\theta \geq \gamma$, there exists a constant $C > 0$ such that*

$$[f]_{B^\gamma_{r,\infty}(\Omega)} \leq C \|f\|^{1-\frac{\gamma}{\theta}}_{L^r(\Omega)} \|f\|^{\frac{\gamma}{\theta}}_{B^\theta_{r,\infty}(\Omega)}, \tag{2.1}$$

$$[f]_{B^\theta_{r,\infty}(\Omega)} \leq C \|f\|^{\frac{1-\theta}{1-\gamma}}_{B^\gamma_{r,\infty}(\Omega)} \|f\|^{\frac{\theta-\gamma}{1-\gamma}}_{W^{1,r}(\Omega)}. \tag{2.2}$$

Note that the same inequalities hold if one replaces all the semi-norms with the full norms.

Proof. We start by proving (2.1) and (2.2) in the whole space $\mathbb{R}^d$. Note that for every $f \in B^\theta_{r,\infty}(\mathbb{R}^d)$ and $\theta \geq \gamma$, we have

$$[f]_{B^\gamma_{r,\infty}(\mathbb{R}^d)} \leq 2 \left(\|f\|_{L^r(\mathbb{R}^d)} + [f]_{B^\theta_{r,\infty}(\mathbb{R}^d)} \right). \tag{2.3}$$

By plugging in (2.3) the rescaled function $f(\varepsilon x)$, we also get

$$\varepsilon^\gamma [f]_{B^\gamma_{r,\infty}(\mathbb{R}^d)} \leq 2 \left(\|f\|_{L^r(\mathbb{R}^d)} + \varepsilon^\theta [f]_{B^\theta_{r,\infty}(\mathbb{R}^d)} \right),$$

for every $\varepsilon > 0$. Thus by choosing $\varepsilon = \|f\|^{\frac{1}{\theta}}_{L^r(\mathbb{R}^d)} [f]^{-\frac{1}{\theta}}_{B^\theta_{r,\infty}(\mathbb{R}^d)}$, we get (2.1) for $\Omega = \mathbb{R}^d$. Take now $\lambda \in [0, 1)$ such that $(1-\lambda)\gamma + \lambda = \theta$. We estimate

$$\frac{\|f(\cdot + y) - f(\cdot)\|_{L^r(\mathbb{R}^d)}}{|y|^\theta} = \left(\frac{\|f(\cdot + y) - f(\cdot)\|_{L^r(\mathbb{R}^d)}}{|y|^\gamma} \right)^{1-\lambda} \left(\frac{\|f(\cdot + y) - f(\cdot)\|_{L^r(\mathbb{R}^d)}}{|y|} \right)^{\lambda}$$

$$\leq [f]_{B^\gamma_{r,\infty}(\mathbb{R}^d)}^{1-\lambda} \|\nabla f\|_{L^r(\mathbb{R}^d)}^\lambda,$$

from which, since $\lambda = \frac{\theta-\gamma}{1-\gamma}$, we conclude (2.2) for $\Omega = \mathbb{R}^d$. If $f \in B^\theta_{r,\infty}(\Omega)$ for Ω as in the statement, (2.1) and (2.2) easily follow from their versions in $\mathbb{R}^d$ and the existence of a (continuous) extension operator. □

Let $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ a smooth, nonnegative and compactly supported function with $\|\varphi\|_{L^1} = 1$. For any $\delta > 0$ we define $\varphi_\delta(x) = \delta^{-d}\varphi(x/\delta)$ and we consider, for any vector field $f : \mathbb{T}^d \to \mathbb{R}^d$ its regularization $f_\delta(x) = (f * \varphi_\delta)(x) = \int_{\mathbb{R}^d} f(x-y)\varphi_\delta(y)\,dy$. We conclude this section by recalling some classical estimates. The third one is for instance the one used in [11] to prove the positive statement of the Onsager's conjecture.

Proposition 2.2. *For any $f : \mathbb{T}^d \to \mathbb{R}^d$, $\theta \in (0,1)$, $r \in [1,\infty]$ and any integer $n \geq 0$, we have the following*

$$\|f - f_\delta\|_{L^r(\mathbb{T}^d)} \leq C\delta^\theta \|f\|_{B^\theta_{r,\infty}(\mathbb{T}^d)}, \tag{2.4}$$

$$\|f_\delta\|_{W^{n+1,r}(\mathbb{T}^d)} \leq C\delta^{\theta-n-1} \|f\|_{B^\theta_{r,\infty}(\mathbb{T}^d)}, \tag{2.5}$$

$$\|f_\delta \otimes f_\delta - (f\otimes f)_\delta\|_{W^{n,r}(\mathbb{T}^d)} \leq C\delta^{2\theta-n} \|f\|^2_{B^\theta_{2r,\infty}(\mathbb{T}^d)}, \tag{2.6}$$

for some constant $C > 0$ depending on θ, r, n but otherwise independent of δ.

3. Abstract multilinear interpolation

In this section we provide some estimates for multilinear operators, by means of abstract real interpolation methods. They are the core of the paper, and the proof of Theorem 1.1 relies on them. We start by recalling some definitions and basic facts about interpolation spaces and we refer the reader to the classical monographs [2, 21, 25] for further details.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two real Banach spaces. The couple (X, Y) is said to be an interpolation couple if both X and Y are continuously embedded in a topological Hausdorff vector space. For any interval $I \subseteq (0,\infty)$ we denote by $L^r_*(I)$ the Lebesgue space of r-summable functions with respect to the measure dt/t. Let use notice that in particular $L^\infty(I) = L^\infty_*(I)$. Moreover, we recall the definition of the K-function, by introducing the following notation. Given $x \in X+Y$ we denote $\Omega(x) = \{(a,b) \in X\times Y : a+b = x\} \subset X\times Y$.

Definition 3.1. For every $x \in X+Y$ and $t > 0$, the K-function is defined by

$$K(t,x,X,Y) = \inf_{\Omega(x)} \{\|a\|_X + t\|b\|_Y\}. \tag{3.1}$$

If no confusion can occur, we simply write $K(t,x)$ instead of $K(t,x,X,Y)$.

Definition 3.2. Let $\theta \in (0,1)$ and $r \in [1,\infty]$. We set

$$(X,Y)_{\theta,r} = \left\{x \in X+Y \text{ s.t. } t \mapsto t^{-\theta}K(t,x) \in L^r_*(0,\infty)\right\}$$

endowed with the norm

$$\|x\|_{(X,Y)_{\theta,r}} = \|t^{-\theta}K(\cdot,x)\|_{L^r_*}.$$

For these spaces we have the following inclusions

$$X\cap Y \hookrightarrow (X,Y)_{\theta,r} \hookrightarrow (X,Y)_{\theta,s} \hookrightarrow X+Y, \tag{3.2}$$

for every $\theta \in (0,1)$ and $r,s \in [1,\infty]$ with $r \leq s$. Moreover if $\gamma > \theta$ we also have $(X,Y)_{\gamma,r} \hookrightarrow (X,Y)_{\theta,s}$, for every $r,s \in [1,\infty]$, provided $Y \hookrightarrow X$.

The following two remarks will be useful in the proof of Theorem 3.6.

Remark 3.3. When $Y \hookrightarrow X$, the definition of K in (3.1) does not change if instead of $\Omega(x)$ we consider the set $\tilde\Omega(x) = \{(a,b) \in \Omega(x) \text{ s.t. } \|a\|_X \leq \|x\|_X\}$; in other words,

$$K(t,x,X,Y) = \inf_{\Omega(x)} \{\|a\|_X + t\|b\|_Y\} = \inf_{\tilde\Omega(x)} \{\|a\|_X + t\|b\|_Y\}.$$

Indeed, since $Y \hookrightarrow X$, one can choose $a = x$ and $b = 0$ in (3.1), obtaining $K(t,x) \le \|x\|_X$. On the other hand, we have that $\|a\|_X + t\|b\|_Y > \|x\|_X$ for all $(a,b) \in \tilde{\Omega}(x)^c$.

Remark 3.4. Consider again the case $Y \hookrightarrow X$. Since $a + b = x$, we have

$$\|a\|_X + \|b\|_X \le 2\|a\|_X + \|x\|_X \le 3\|x\|_X, \qquad \forall\, (a,b) \in \tilde{\Omega}(x).$$

It is well known that $\big((X,Y)_{\theta,r}, \|\cdot\|_{(X,Y)_{\theta,r}}\big)$ is a Banach space. Furthermore, we recall that a linear operator T behaves nicely with respect to interpolation, i.e. if $T \in \mathcal{L}(X_1,Y_1) \cap \mathcal{L}(X_2,Y_2)$, then $T \in \mathcal{L}((X_1,X_2)_{\theta,r},(Y_1,Y_2)_{\theta,r})$ for any $\theta \in (0,1)$ and $r \in [1,\infty]$.

Instead of linear operators, our aim is to treat the case of multilinear operators, in particular bilinear and trilinear ones. It is worth mentioning that there exists a wide literature on Interpolation Theory for multilinear operators, see for example the works [2], [16], [20] and [22], but at the best of our knowledge the following results are new. We also emphasise that they are precisely designed for the applications to incompressible fluid models of the next section. In what follows, a conjugate pair (s,s') is a couple of reals satisfying $s' = \frac{s}{s-1}$.

Theorem 3.5. *Let (X_1,X_2) and (Y_1,Y_2) be two interpolation couples. Let T be a bilinear operator satisfying*

$$\|T(a_1,a_2)\|_{Y_1} \le C_0\|a_1\|_{X_1}\|a_2\|_{X_1}, \tag{3.3}$$

$$\|T(b_1,b_2)\|_{Y_2} \le C_0\|b_1\|_{X_2}\|b_2\|_{X_2}, \tag{3.4}$$

and

$$\|T(a,b)\|_{(Y_1,Y_2)_{\frac12,\infty}} + \|T(b,a)\|_{(Y_1,Y_2)_{\frac12,\infty}} \le C_0\|a\|_{X_1}\|b\|_{X_2}, \tag{3.5}$$

for some constant $C_0 > 0$ independent on $a, a_1, a_2 \in X_1$ and $b, b_1, b_2 \in X_2$, where we implicitly assume that T is well defined between the spaces involved in the previous estimates. Then, for any $\theta,\gamma \in (0,1)$, $r, s, s' \in [1,\infty]$ with s, s' being a conjugate pair,

$$\|T(x_1,x_2)\|_{(Y_1,Y_2)_{\frac{\theta+\gamma}{2},r}} \le C_0\|x_1\|_{(X_1,X_2)_{\gamma,rs}}\|x_2\|_{(X_1,X_2)_{\theta,rs'}} \qquad \forall\, x_1 \in (X_1,X_2)_{\gamma,rs},\ \forall\, x_2 \in (X_1,X_2)_{\theta,rs'}.$$

In particular, for $\gamma = \theta$ and $s = s' = 2$, we get

$$\|T(x,x)\|_{(Y_1,Y_2)_{\theta,r}} \le C_0\|x\|^2_{(X_1,X_2)_{\theta,2r}}, \quad \forall\, x \in (X_1,X_2)_{\theta,2r}.$$

Proof. Let $x_1 \in (X_1,X_2)_{\gamma,sr}$ and $x_2 \in (X_1,X_2)_{\theta,rs'}$. Then we can write $x_1 = a_1 + b_1$ and $x_2 = a_2 + b_2$ for some $a_1, a_2 \in X_1$ and $b_1, b_2 \in X_2$, by definition. Since T is bilinear we have

$$T(x_1,x_2) = T(a_1,a_2) + T(a_1,b_2) + T(b_1,a_2) + T(b_1,b_2).$$

From (3.5) we know that $T(a_1,b_2) \in (Y_1,Y_2)_{\frac12,\infty}$, hence for any $t, \varepsilon > 0$ there exist $T_1 \in Y_1$ and $T_2 \in Y_2$ such that $T(a_1,b_2) = T_1 + T_2$ and

$$\begin{aligned} \|T_1\|_{Y_1} + t\|T_2\|_{Y_2} &\le (1+\varepsilon)K(t, T(a_1,b_2), Y_1, Y_2) \\ &\le (1+\varepsilon)\sqrt{t}\|T(a_1,b_2)\|_{(Y_1,Y_2)_{\frac12,\infty}} \le (1+\varepsilon)C_0\sqrt{t}\|a_1\|_{X_1}\|b_2\|_{X_2}. \end{aligned} \tag{3.6}$$

Similarly, we can decompose $T(b_1,a_2) = U_1 + U_2$ with $U_1 \in Y_1$ and $U_2 \in Y_2$ with estimate

$$\|U_1\|_{Y_1} + t\|U_2\|_{Y_2} \le (1+\varepsilon)C_0\sqrt{t}\|a_2\|_{X_1}\|b_1\|_{X_2}. \tag{3.7}$$

Therefore we can write $T(x_1,x_2) = V + W$, where

$$\begin{aligned} V &= T(a_1,a_2) + T_1 + U_1 \in Y_1, \\ W &= T(b_1,b_2) + T_2 + U_2 \in Y_2. \end{aligned}$$

Summing up (3.3)–(3.7) yields to

$$\begin{aligned}\|V\|_{Y_1}+t\|W\|_{Y_2} &\le (1+\varepsilon)C_0\Big(\|a_1\|_{X_1}\|a_2\|_{X_1}+\sqrt{t}\,(\|a_1\|_{X_1}\|b_2\|_{X_2}+\|a_2\|_{X_1}\|b_1\|_{X_2})+t\|b_1\|_{X_2}\|b_2\|_{X_2}\Big)\\ &=(1+\varepsilon)C_0\Big(\|a_1\|_{X_1}+\sqrt{t}\|b_1\|_{X_2}\Big)\Big(\|a_2\|_{X_1}+\sqrt{t}\|b_2\|_{X_2}\Big),\end{aligned}$$

which in turn implies

$$K(t,T(x_1,x_2),Y_1,Y_2)\le(1+\varepsilon)C_0K(\sqrt{t},x_1,X_1,X_2)K(\sqrt{t},x_2,X_1,X_2). \tag{3.8}$$

Multiplying (3.8) by $t^{-(\gamma+\theta)/2}$ and by taking the $L^r_*(0,\infty)$-norm we get, by means of the Hölder inequality with conjugate exponents s and s',

$$\begin{aligned}\|T(x_1,x_2)\|_{(Y_1,Y_2)_{\frac{\theta+\gamma}{2},r}} &= \|(\cdot)^{-(\theta+\gamma)/2}K(\cdot,T(x_1,x_2))\|_{L^r_*}\\ &\le (1+\varepsilon)C_0\left(\|(\cdot)^{-s\gamma/2}K^s(\sqrt{\cdot},x_1)\|^{1/s}_{L^r_*}\|(\cdot)^{-s'\theta/2}K^{s'}(\sqrt{\cdot},x_2)\|^{1/s'}_{L^r_*}\right)\\ &=(1+\varepsilon)C_0\|x_1\|_{(X_1,X_2)_{\gamma,rs}}\|x_2\|_{(X_1,X_2)_{\theta,rs'}},\end{aligned}$$

and since the last inequality holds true for any $\varepsilon>0$, we are done. □

Let us now focus on trilinear operators, for which a similar result as in Theorem 3.5 can be proved. In what follows, it will be useful to consider interpolation couples (X_1,X_2) such that $X_2\hookrightarrow X_1$. For sake of clarity, we require that the trilinear operator in the statement is symmetric in each variable, even though a suitable adaptation would work without this requirement.

Theorem 3.6. *Let $C_0>0$, (X_1,X_2) and (Y_1,Y_2) be two interpolation couples with $X_2\hookrightarrow X_1$. Let T be a trilinear and symmetric operator satisfying the following conditions*

$$\|T(a_1,a_2,a_3)\|_{Y_1}\le C_0\|a_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1}, \tag{3.9}$$

$$\|T(b_1,b_2,b_3)\|_{Y_2}\le C_0\Big(\|b_1\|_{X_1}\|b_2\|_{X_2}\|b_3\|_{X_2}+\|b_1\|_{X_2}\|b_2\|_{X_1}\|b_3\|_{X_2}+\|b_1\|_{X_2}\|b_2\|_{X_2}\|b_3\|_{X_1}\Big), \tag{3.10}$$

and

$$\|T(a_1,b_2,b_3)\|_{(Y_1,Y_2)_{\frac12,\infty}}\le C_0\|a_1\|_{X_1}\Big(\|b_2\|_{X_2}\|b_3\|_{X_1}+\|b_2\|_{X_1}\|b_3\|_{X_2}\Big), \tag{3.11}$$

where we implicitly assume that T is well defined between the spaces involved in the previous estimates. Then for any $\gamma,\theta\in(0,1)$ and $r,s\in[1,\infty]$, for every x_1,x_2,x_3 we have

$$\begin{aligned}\|T(x_1,x_2,x_3)\|_{(Y_1,Y_2)_{\frac{\theta+\gamma}{2},r}}\le 3C_0\Bigg(&\|x_1\|_{X_1}\|x_2\|_{(X_1,X_2)_{\gamma,rs}}\|x_3\|_{(X_1,X_2)_{\theta,rs'}}\\ &+\|x_1\|_{(X_1,X_2)_{\gamma,rs}}\left(\|x_2\|_{X_1}\|x_3\|_{(X_1,X_2)_{\theta,rs'}}+\|x_2\|_{(X_1,X_2)_{\theta,rs'}}\|x_3\|_{X_1}\right)\Bigg).\end{aligned} \tag{3.12}$$

In particular, for $\gamma=\theta$ and $s=s'=2$, we get

$$\|T(x,x,x)\|_{(Y_1,Y_2)_{\theta,r}}\le 3C_0\|x\|_{X_1}\|x\|^2_{(X_1,X_2)_{\theta,2r}},\quad \forall x\in(X_1,X_2)_{\theta,2r}.$$

Proof. We assume without loss of generality that $\theta\ge\gamma$. Consider $x_1\in(X_1,X_2)_{\gamma,rs}$ and $x_2,x_3\in(X_1,X_2)_{\theta,rs'}$. For $k=1,2,3$ we write $x_k=a_k+b_k$ with $a_k\in X_1$ and $b_k\in X_2$; therefore we expand

$$T(x_1,x_2,x_3)=U+V+W$$

where

$$\begin{aligned}U&=T(a_1,a_2,a_3)+T(b_1,a_2,a_3)+T(a_1,b_2,a_3)+T(a_1,a_2,b_3),\\ V&=T(b_1,b_2,a_3)+T(b_1,a_2,b_3)+T(a_1,b_2,b_3),\\ W&=T(b_1,b_2,b_3).\end{aligned}$$

Since $X_2 \hookrightarrow X_1$ we have that $b_k \in X_1$ for any $k = 1, 2, 3$, then by (3.9) we can control U as

$$\begin{aligned}\|U\|_{Y_1} \le C_0\Big(&\|a_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1} + \|b_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1}\\ &+ \|a_1\|_{X_1}\|b_2\|_{X_1}\|a_3\|_{X_1} + \|a_1\|_{X_1}\|a_2\|_{X_1}\|b_3\|_{X_1}\Big).\end{aligned} \tag{3.13}$$

The symmetry of the operator T and (3.11) imply that every term defining V belongs to $(Y_1, Y_2)_{\frac{1}{2},\infty}$. Let us consider without loss of generality the term $T(b_1, b_2, a_3)$; as already done in Theorem 3.5, for any $t, \varepsilon > 0$ there exist $T_1 \in Y_1$ and $T_2 \in Y_2$ such that $T(b_1, b_2, a_3) = T_1 + T_2$ and

$$\begin{aligned}\|T_1\|_{Y_1} + t\|T_2\|_{Y_2} &\le (1+\varepsilon)K(t, T(b_1, b_2, a_3), Y_1, Y_2) \le (1+\varepsilon)\sqrt{t}\|T(b_1, b_2, a_3)\|_{(Y_1,Y_2)_{\frac{1}{2},\infty}}\\ &\le (1+\varepsilon)C_0\sqrt{t}\|a_3\|_{X_1}\left(\|b_1\|_{X_1}\|b_2\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\right).\end{aligned}$$

We point out that the elements T_1 and T_2 actually depend on $a_3, b_1, b_2, \varepsilon$ and t as well. The same consideration for the other two terms defining V yields, for any $t, \varepsilon > 0$, to the existence of $V_1 \in Y_1$ and $V_2 \in Y_2$ such that $V = V_1 + V_2$ and

$$\begin{aligned}\|V_1\|_{Y_1} + t\|V_2\|_{Y_2} \le (1+\varepsilon)C_0\sqrt{t}\Big(&\|a_1\|_{X_1}\left(\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_2\|_{X_2}\|b_3\|_{X_1}\right)\\ &+ \|a_2\|_{X_1}\left(\|b_1\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_3\|_{X_1}\right) + \|a_3\|_{X_1}\left(\|b_1\|_{X_1}\|b_2\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\right)\Big).\end{aligned} \tag{3.14}$$

By using (3.10) we also get

$$\|T(b_1, b_2, b_3)\|_{Y_2} \le C_0\Big(\|b_1\|_{X_1}\|b_2\|_{X_2}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_2}\|b_3\|_{X_1}\Big). \tag{3.15}$$

By combining (3.13), (3.14) and (3.15) we obtain, for any $t, \varepsilon > 0$, a decomposition of $T(x_1, x_2, x_3) = (U + V_1) + (V_2 + W)$, with $U + V_1 \in Y_1$ and $V_2 + W \in Y_2$ such that

$$\begin{aligned}\|U + V_1\|_{Y_1} + t\|V_2 + W\|_{Y_2} \le (1+\varepsilon)C_0\Bigg(&\|a_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1} + \|b_1\|_{X_1}\|a_2\|_{X_1}\|a_3\|_{X_1}\\ &+ \|a_1\|_{X_1}\|b_2\|_{X_1}\|a_3\|_{X_1} + \|a_1\|_{X_1}\|a_2\|_{X_1}\|b_3\|_{X_1} + \sqrt{t}\Big(\|a_1\|_{X_1}\left(\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_2\|_{X_2}\|b_3\|_{X_1}\right)\\ &+ \|a_2\|_{X_1}\left(\|b_1\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_3\|_{X_1}\right) + \|a_3\|_{X_1}\left(\|b_1\|_{X_1}\|b_2\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\right)\Big)\\ &+ t\Big(\|b_1\|_{X_1}\|b_2\|_{X_2}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_1}\|b_3\|_{X_2} + \|b_1\|_{X_2}\|b_2\|_{X_2}\|b_3\|_{X_1}\Big)\Bigg)\\ \le (1+\varepsilon)C_0\Big(&\left(\|a_1\|_{X_1} + \|b_1\|_{X_1}\right)\left(\|a_2\|_{X_1} + \sqrt{t}\|b_2\|_{X_2}\right)\left(\|a_3\|_{X_1} + \sqrt{t}\|b_3\|_{X_2}\right)\\ &+ \left(\|a_1\|_{X_1} + \sqrt{t}\|b_1\|_{X_2}\right)\left(\|a_2\|_{X_1} + \|b_2\|_{X_1}\right)\left(\|a_3\|_{X_1} + \sqrt{t}\|b_3\|_{X_2}\right)\\ &+ \left(\|a_1\|_{X_1} + \sqrt{t}\|b_1\|_{X_2}\right)\left(\|a_2\|_{X_1} + \sqrt{t}\|b_2\|_{X_2}\right)\left(\|a_3\|_{X_1} + \|b_3\|_{X_1}\right)\Big) := R(t)\end{aligned}$$

which clearly implies

$$K(t, T(x_1, x_2, x_3), Y_1, Y_2) \le R(t). \tag{3.16}$$

Now, by using Remark 3.3 and Remark 3.4 and by taking the infima over all the sets $\tilde{\Omega}(x_k) = \{(a_k, b_k) \in \Omega(x_k) \text{ s.t. } \|a_k\|_{X_1} \le \|x_k\|_{X_1}\}$ for $k = 1, 2, 3$ in the right-hand side of (3.16), we achieve

$$\begin{aligned}K(t, T(x_1, x_2, x_3), Y_1, Y_2) \le 3(1+\varepsilon)C_0\Big(&\|x_1\|_{X_1}K(\sqrt{t}, x_2)K(\sqrt{t}, x_3)\\ &+ K(\sqrt{t}, x_1)\Big(\|x_2\|_{X_1}K(\sqrt{t}, x_3) + \|x_3\|_{X_1}K(\sqrt{t}, x_2)\Big)\Big).\end{aligned}$$

Multiplying by $t^{-(\theta+\gamma)/2}$ the last inequality, taking the $L^r_*(0,\infty)$-norm and using the Hölder inequality with s, s' as conjugate pair, we obtain (3.12) by letting $\varepsilon \to 0$. $\square$

We recall that interpolation theory also provides the following useful characterization of Besov spaces (see for instance [2, Theorem 6.2.4]).

Proposition 3.7. *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz open set. For any $\theta \in (0,1)$, $r,s \in [1,\infty]$ and $\sigma_1 \neq \sigma_2 \in \mathbb{Z}$,*

$$(W^{\sigma_1,r}(\Omega), W^{\sigma_2,r}(\Omega))_{\theta,s} = B^{(1-\theta)\sigma_1+\theta\sigma_2}_{r,s}(\Omega). \tag{3.17}$$

Moreover, the same holds if we restrict all spaces in (3.17) to the linear subspace of divergence-free vector fields.

Notice that for the sake of simplicity, we did not define, in Section 2, Besov spaces of order less than or equal to 0. However, we will apply Proposition 3.7 only for the Besov spaces of strictly positive θ. The statement for divergence-free vector fields follows instead from the same proof as (3.17), since the construction in the interpolation is based on mollification at a suitable scale, and convolutions preserve the divergence-free structure of the vector fields.

4. Regularity of Euler equation

The following result about elliptic equations follows by a direct application of Theorem 3.5 and Theorem 3.6 of the previous section. The reader can compare the following proposition with [9, Proposition 3.1] obtained for Hölder spaces through estimates on a representation formula for p and q.

Proposition 4.1. *Let $\gamma, \theta \in (0,1)$ and $r \in (1,\infty)$. Let $u,w,z : \mathbb{T}^3 \to \mathbb{R}^3$ be divergence-free vector fields and let $p,q : \mathbb{T}^3 \to \mathbb{R}^3$ be the unique 0-average solutions of*

$$-\Delta p = \operatorname{div}\operatorname{div}(u \otimes w), \tag{4.1}$$

$$-\Delta q = \operatorname{div}\operatorname{div}\operatorname{div}(u \otimes w \otimes z). \tag{4.2}$$

Then, for any $s \in [1,\infty]$, we have

$$\|p\|_{B^{\gamma+\theta}_{r,s}} \leq C\|u\|_{B^{\gamma}_{2r,2s}}\|w\|_{B^{\theta}_{2r,2s}}. \tag{4.3}$$

Furthermore, if $\theta+\gamma>1$

$$\|q\|_{B^{\gamma+\theta-1}_{r,s}} \leq C\bigg(\|u\|_{L^{3r}}\|w\|_{B^{\gamma}_{3r,2s}}\|z\|_{B^{\theta}_{3r,2s}} + \|u\|_{B^{\gamma}_{3r,2s}}\Big(\|w\|_{L^{3r}}\|z\|_{B^{\theta}_{3r,2s}} + \|w\|_{B^{\theta}_{3r,2s}}\|z\|_{L^{3r}}\Big)\bigg). \tag{4.4}$$

Proof. We denote by $W^{1,r}_{\mathrm{div}}$ the linear subspace of $W^{1,r}$ made by divergence-free vector fields (and similarly for $B^{\theta}_{r,s,\mathrm{div}}$). Let $T(u,w)$ be the operator that for each couple (u,w) associate the unique 0-average solution of (4.1). By the Calderón-Zygmund theory, we have

$$\|T(u,w)\|_{L^r} \leq C\|u\|_{L^{2r}}\|w\|_{L^{2r}}.$$

Moreover since $\operatorname{div} u = \operatorname{div} w = 0$ the right-hand side of (4.1) can be rewritten as

$$\operatorname{div}\operatorname{div}(u\otimes w) = \partial^2_{ij}(u^i w^j) = \partial_j(u^i\partial_i w^j) = \partial_j u^i \partial_i w^j,$$

thus we can use again Calderón-Zygmund to get

$$\|T(u,w)\|_{W^{1,r}} \leq C\|u\|_{L^{2r}}\|w\|_{W^{1,2r}}$$

and

$$\|T(u,w)\|_{W^{2,r}} \leq C\|u\|_{W^{1,2r}}\|w\|_{W^{1,2r}}.$$

Since, by Proposition 3.7, we have the embedding $W^{1,r}_{\mathrm{div}} \hookrightarrow B^1_{r,\infty,\mathrm{div}} = (L^r_{\mathrm{div}}, W^{2,r}_{\mathrm{div}})_{\frac{1}{2},\infty}$, we can apply Theorem 3.5 with $X_1 = L^{2r}_{\mathrm{div}}$, $X_2 = W^{1,2r}_{\mathrm{div}}$, $Y_1 = L^r$, $Y_2 = W^{2,r}$, hence obtaining (4.3). Note that it is important that all the spaces above consist of divergence-free vector fields.

The proof of (4.4) follows similarly as a consequence of Calderón-Zygmund and Theorem 3.6, with $X_1 = L^{3r}_{\mathrm{div}}$, $X_2 = W^{1,3r}_{\mathrm{div}}$, $Y_1 = W^{-1,r}$ and $Y_2 = W^{1,r}$ once one notices that the solenoidal nature of u, w, z implies that

$$\begin{aligned}\operatorname{div}\operatorname{div}\operatorname{div}(u\otimes w\otimes z) &= \partial^3_{ijk}(u^i w^j z^k) = \partial^2_{ij}(\partial_k u^i w^j z^k) + \partial^2_{ij}(u^i \partial_k w^j z^k)\\ &= \partial_j(\partial_k u^i \partial_i w^j z^k + \partial_k u^i w^j \partial_i z^k) + \partial_i(\partial_j u^i \partial_k w^j z^k + u^i \partial_k w^j \partial_j z^k). \quad \square\end{aligned}$$

Remark 4.2. The regularity estimates for the pressure of the proposition above are also a generalization of previously known results contained in [1, Lemma 7.9, 7.10 and 7.14], where some Lipschitz regularity of the vector fields is assumed. Proposition 4.1 is however more general, both because it proves the double regularity of the pressure based only on the Besov regularity of the vector field and because it does not require boundedness or Lipschitz assumptions on the vector field, which are not satisfied for instance by the solutions built by convex integration methods.

Moreover, the above double regularity results on the pressure do not depend on the specific structure given by the Laplacian but also apply to more general elliptic operators. Indeed the Calderón-Zygmund estimates in the extremal spaces L^r and $W^{1,r}$ is enough to apply our abstract interpolation theorems.

We consider now a weak solution (u, p) of the incompressible Euler equations (1.1). Taking the divergence of the first equation in (1.1), using the incompressibility constraint $\operatorname{div} u = 0$, the pressure p solves

$$-\Delta p = \operatorname{div}\operatorname{div}(u\otimes u), \tag{4.5}$$

thus it can be uniquely determined if one imposes that $\int_{\mathbb{T}^3} p(t,x)\,dx = 0$, for any time $t\in(0,T)$. For every $\theta\in(0,1)$ and $r\in(1,\infty)$, a direct application of Calderón-Zygmund leads to

$$\|p(t)\|_{B^\theta_{r,\infty}} \le C\|u(t)\|^2_{B^\theta_{2r,\infty}}. \tag{4.6}$$

Since our solutions are just weak solutions, we will need to mollify (1.1) in order to justify some computations; moreover, we will tune the convolution parameter in terms of the time increment h (a similar approach was used for instance in [14]). By regularizing (in space) the equations (1.1), one gets that the couple $(u_\delta, p_\delta) = (u*\varphi_\delta, p*\varphi_\delta)$ solves

$$\begin{cases}\partial_t u_\delta + \operatorname{div}(u_\delta\otimes u_\delta) + \nabla p_\delta = \operatorname{div} R_\delta\\ \operatorname{div} u_\delta = 0\end{cases}, \tag{4.7}$$

where $R_\delta = u_\delta\otimes u_\delta - (u\otimes u)_\delta$. We can now prove our main theorem.

Proof of Theorem 1.1. Let $h>0$ be a time increment. When it will help the readability we will also put in the constants C all the norms of u and p which are already known to be finite. We prove the theorem for $s<\infty$, since the case $s=\infty$ is a simple adaptation and it is easier using the identification $B^\theta_{\infty,\infty} = C^\theta$. In the following, given an interval I, the function $\chi(\cdot)_I$ will denote the usual characteristic function on the set I.

Proof of (i). Assume that $u\in L^{2s}((0,T);B^\theta_{2r,\infty}(\mathbb{T}^3))$, for some $s\in[1,\infty)$. We split

$$\|u(t+h)-u(t)\|_{L^r} \le \|u(t+h)-u_\delta(t+h)\|_{L^r} + \|u_\delta(t+h)-u_\delta(t)\|_{L^r} + \|u_\delta(t)-u(t)\|_{L^r}. \tag{4.8}$$

Using (2.4) we have $\|u_\delta(t)-u(t)\|_{L^r} \le C\delta^\theta\|u(t)\|_{B^\theta_{r,\infty}}$ for every $t\in(0,T)$, from which we deduce

$$\begin{aligned}\left(\int_0^{T-h}\|u(t+h)-u_\delta(t+h)\|^s_{L^r}\,dt\right)^{\frac1s} + \left(\int_0^{T-h}\|u(t)-u_\delta(t)\|^s_{L^r}\,dt\right)^{\frac1s} &\le C\delta^\theta\|u\|_{L^s(B^\theta_{r,\infty})}\\ &\le C\delta^\theta\|u\|_{L^{2s}(B^\theta_{2r,\infty})}.\end{aligned}$$

In the last inequality we used the fact that both the time and spatial domains are bounded. We are left with the second term in the right-hand side of (4.8). Since u_δ solves (4.7), using also (2.5) and (4.6) we get

$$\begin{aligned}\|u_\delta(t+h)-u_\delta(t)\|_{L^r} &\le \int_t^{t+h}\|\partial_t u_\delta(\tau)\|_{L^r}\,d\tau \le \int_t^{t+h}\Big(\|\operatorname{div}(u\otimes u)_\delta(\tau)\|_{L^r}+\|\nabla p_\delta(\tau)\|_{L^r}\Big)\,d\tau\\ &\le C\delta^{\theta-1}\int_t^{t+h}\big(\|u\otimes u(\tau)\|_{B^\theta_{r,\infty}}+\|p(\tau)\|_{B^\theta_{r,\infty}}\big)\,d\tau \le C\delta^{\theta-1}\int_t^{t+h}\|u(\tau)\|^2_{B^\theta_{2r,\infty}}\,d\tau.\end{aligned}$$

By the Hölder inequality with conjugate exponents s and $\frac{s}{s-1}$ we deduce

$$\|u_\delta(t+h)-u_\delta(t)\|^s_{L^r}\le C\delta^{(\theta-1)s}h^{s-1}\int_0^T\chi(\tau)_{(t,t+h)}\|u(\tau)\|^{2s}_{B^\theta_{2r,\infty}}\,d\tau,$$

from which, by integrating in time, we conclude

$$\begin{aligned}\int_0^{T-h}\|u_\delta(t+h)-u_\delta(t)\|^s_{L^r}\,dt &\le C\delta^{(\theta-1)s}h^{s-1}\int_0^{T-h}\int_0^T\chi(\tau)_{(t,t+h)}\|u(\tau)\|^{2s}_{B^\theta_{2r,\infty}}\,d\tau\,dt\\ &\le C\delta^{(\theta-1)s}h^s\|u\|^{2s}_{L^{2s}(B^\theta_{2r,\infty})},\end{aligned}$$

where in the last inequality we also used $\int_0^{T-h}\chi(t)_{(\tau-h,\tau)}\,dt\le h$. By choosing $\delta=h$, we achieve

$$\left(\int_0^{T-h}\|u(t+h)-u(t)\|^s_{L^r}\,dt\right)^{\frac1s}\le Ch^\theta\left(\|u\|_{L^{2s}(B^\theta_{2r,\infty})}+\|u\|^2_{L^{2s}(B^\theta_{2r,\infty})}\right),$$

from which, by taking the supremum all over $h\in(0,T)$, we conclude $u\in B^\theta_{s,\infty}((0,T);L^r(\mathbb{T}^3))$. Since p solves (4.5), we can use (4.3) with $u=w=u(t)$, $\gamma=\theta$, $s=\infty$, getting

$$\|p(t)\|_{B^{2\theta}_{r,\infty}}\le C\|u(t)\|^2_{B^\theta_{2r,\infty}}. \tag{4.9}$$

Taking the $L^s(0,T)$-norm, we deduce that $p\in L^s((0,T);B^{2\theta}_{r,\infty}(\mathbb{T}^3))$, namely that (i) holds.

Proof of (ii). Let $\theta>1/2$ and $\beta\in[0,2\theta-1)$. Note that

$$-\Delta(p(t+h)-p(t))=\operatorname{div}\operatorname{div}\Big(\big(u(t+h)-u(t)\big)\otimes u(t+h)+u(t)\otimes\big(u(t+h)-u(t)\big)\Big).$$

Thus, by using (4.3) with $\gamma=1-\theta+\beta$, $s=\infty$, we get

$$\|p(t+h)-p(t)\|_{B^{1+\beta}_{r,\infty}}\le C\|u(t+h)-u(t)\|_{B^{1-\theta+\beta}_{2r,\infty}}\left(\|u(t+h)\|_{B^\theta_{2r,\infty}}+\|u(t)\|_{B^\theta_{2r,\infty}}\right), \tag{4.10}$$

and taking the $L^s(0,T-h)$-norm in time, by also using the Hölder inequality, we achieve

$$\left(\int_0^{T-h}\|p(t+h)-p(t)\|^s_{B^{1+\beta}_{r,\infty}}\,dt\right)^{\frac1s}\le C\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{B^{1-\theta+\beta}_{2r,\infty}}\,dt\right)^{\frac{2}{3s}}\|u\|_{L^{3s}(B^\theta_{2r,\infty})}. \tag{4.11}$$

By the interpolation inequality (2.1), the Hölder inequality, and since $u\in B^\theta_{\frac{3s}{2},\infty}((0,T);L^{2r}(\mathbb{T}^3))$ by (i), we can estimate

$$\begin{aligned}\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{B^{1-\theta+\beta}_{2r,\infty}}\,dt &\le \int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}\frac{2\theta-1-\beta}{\theta}}_{L^{2r}}\|u(t+h)-u(t)\|^{\frac{3s}{2}\frac{1-\theta+\beta}{\theta}}_{B^\theta_{2r,\infty}}\,dt\\ &\le\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{L^{2r}}\,dt\right)^{\frac{2\theta-1-\beta}{\theta}}\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{\frac{3s}{2}}_{B^\theta_{2r,\infty}}\,dt\right)^{\frac{1-\theta+\beta}{\theta}}\\ &\le Ch^{\frac{3s}{2}(2\theta-1-\beta)}\|u\|^{\frac{3s}{2}\frac{2\theta-1-\beta}{\theta}}_{B^\theta_{\frac{3s}{2},\infty}(L^{2r})}\|u\|^{\frac{3s}{2}\frac{1-\theta+\beta}{\theta}}_{L^{\frac{3s}{2}}(B^\theta_{2r,\infty})}\le Ch^{\frac{3s}{2}(2\theta-1-\beta)}.\end{aligned}$$

By plugging this last estimate in (4.11), we conclude that $p \in B^{2\theta-1-\beta}_{s,\infty}((0,T);B^{1+\beta}_{r,\infty}(\mathbb{T}^3))$, since we get

$$\left(\int_0^{T-h} \|p(t+h)-p(t)\|^s_{B^{1+\beta}_{r,\infty}}\,dt\right)^{\frac{1}{s}} \le Ch^{2\theta-1-\beta}.$$

Proof of (iii). In order to prove the Besov regularity in time of the pressure, we split

$$\|p(t+h)-p(t)\|_{L^r} \le \|p(t+h)-p_\delta(t+h)\|_{L^r} + \|p_\delta(t+h)-p_\delta(t)\|_{L^r} + \|p_\delta(t)-p(t)\|_{L^r}. \tag{4.12}$$

Using (2.4) and (4.9), we have, for every $t \in (0,T)$,

$$\|p_\delta(t)-p(t)\|_{L^r} \le C\delta^{2\theta}\|p(t)\|_{B^{2\theta}_{r,\infty}} \le C\delta^{2\theta}\|u(t)\|^2_{B^\theta_{2r,\infty}} \le C\delta^{2\theta}\|u(t)\|^2_{B^\theta_{3r,\infty}},$$

from which we deduce

$$\int_0^{T-h}\|p(t+h)-p_\delta(t+h)\|^s_{L^r}\,dt + \int_0^{T-h}\|p(t)-p_\delta(t)\|^s_{L^r}\,dt \le C\delta^{2\theta s}\|u\|^{2s}_{L^{3s}(B^\theta_{3r,\infty})}.$$

It remains to prove the estimate for the middle term $\|p_\delta(t+h)-p_\delta(t)\|_{L^r}$ in the right-hand side of (4.12). Notice that $p_\delta(t+h)-p_\delta(t)$ solves

$$\begin{aligned}
-\Delta(p_\delta(t+h)-p_\delta(t)) &= \operatorname{div}\operatorname{div}\Big(R_\delta(t)-R_\delta(t+h)+u_\delta(t+h)\otimes u_\delta(t+h)-u_\delta(t)\otimes u_\delta(t)\Big)\\
&= \operatorname{div}\operatorname{div}\Big(R_\delta(t)-R_\delta(t+h)+\int_t^{t+h}\Big(\frac{d}{d\tau}u_\delta(\tau,x)\otimes u_\delta(\tau,x)+u_\delta(\tau,x)\otimes\frac{d}{d\tau}u_\delta(\tau,x)\Big)\,d\tau\Big)\\
&= \operatorname{div}\operatorname{div}\Big(R_\delta(t)-R_\delta(t+h)+\int_t^{t+h}\Big((\operatorname{div}(u_\delta\otimes u_\delta)-\nabla p_\delta-\operatorname{div}R_\delta)\otimes u_\delta\\
&\quad + u_\delta\otimes(\operatorname{div}(u_\delta\otimes u_\delta)-\nabla p_\delta-\operatorname{div}R_\delta)\Big)\,d\tau\Big).
\end{aligned}$$

Thus $p_\delta(t+h)-p_\delta(t) = q^1+q^2+q^3$, where q^1,q^2,q^3 are the unique 0-average solutions to

$$\begin{aligned}
-\Delta q^1 &= \operatorname{div}\operatorname{div}(R_\delta(t,x)-R_\delta(t+h,x)),\\
\Delta q^2 &= 2\int_t^{t+h}\operatorname{div}\operatorname{div}((\operatorname{div}R_\delta+\nabla p_\delta)\otimes u_\delta)\,d\tau,\\
-\Delta q^3 &= \int_t^{t+h}\operatorname{div}\operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta\otimes u_\delta)\,d\tau.
\end{aligned}$$

By Calderón-Zygmund, (2.6) and (2.5) we have that

$$\|q^1(t)\|_{L^r} \le C\big(\|R_\delta(t+h)\|_{L^r}+\|R_\delta(t)\|_{L^r}\big) \le C\delta^{2\theta}\big(\|u(t+h)\|^2_{B^\theta_{3r,\infty}}+\|u(t)\|^2_{B^\theta_{3r,\infty}}\big),$$

and

$$\|q^2(t)\|_{L^r} \le C\int_t^{t+h}\Big(\|\operatorname{div}R_\delta(\tau)\|_{L^{\frac{3r}{2}}}+\|\nabla p_\delta(\tau)\|_{L^{\frac{3r}{2}}}\Big)\|u_\delta(\tau)\|_{L^{3r}}\,d\tau \le C\delta^{2\theta-1}\int_t^{t+h}\|u(\tau)\|^3_{B^\theta_{3r,\infty}}\,d\tau.$$

Hence, by taking the $L^s(0,T-h)$-norm, we deduce

$$\|q^1\|_{L^s(L^r)} \le C\delta^{2\theta}\|u\|^2_{L^{3s}(B^\theta_{3r,\infty})}. \tag{4.13}$$

and, similarly to above, by the Hölder inequality we have

$$\begin{aligned}
\int_0^{T-h}\|q^2(t)\|^s_{L^r}\,dt &\le C\delta^{(2\theta-1)s}h^{s-1}\int_0^{T-h}\left(\int_0^T\chi_{(t,t+h)}(\tau)\|u(\tau)\|^{3s}_{B^\theta_{3r,\infty}}\,d\tau\right)dt\\
&\le C\delta^{(2\theta-1)s}h^s\|u\|^{3s}_{L^{3s}(B^\theta_{3r,\infty})}.
\end{aligned}\tag{4.14}$$

For q^3 we can use, for any $\varepsilon > 0$, (4.4) with $\theta = \gamma = (1+\varepsilon)/2$, $s = \infty$, $u = w = z = u_\delta(t)$, getting

$$\|q^3(t)\|_{L^r} \le \|q^3(t)\|_{B^\varepsilon_{r,\infty}} \le C \int_t^{t+h} \|u_\delta(\tau)\|_{L^{3r}} \|u_\delta(\tau)\|^2_{B^{\frac{1+\varepsilon}{2}}_{3r,\infty}} \, d\tau. \tag{4.15}$$

By (2.2) and the estimate (2.5), we have

$$\|u_\delta(t)\|_{B^{\frac{1+\varepsilon}{2}}_{3r,\infty}} \le \|u_\delta(t)\|^{\frac{1-\varepsilon}{2(1-\theta)}}_{B^\theta_{3r,\infty}} \|u_\delta(t)\|^{\frac{1+\varepsilon-2\theta}{2(1-\theta)}}_{W^{1,3r}} \le C\delta^{\theta - \frac{1+\varepsilon}{2}} \|u(t)\|_{B^\theta_{3r,\infty}}.$$

Plugging this last estimate in (4.15), we achieve

$$\|q^3(t)\|_{L^r} \le C\delta^{2\theta-1-\varepsilon} \int_t^{t+h} \|u(\tau)\|^3_{B^\theta_{3r,\infty}} \, d\tau,$$

from which we deduce

$$\|q^3\|_{L^s(L^r)} \le C\delta^{2\theta-1-\varepsilon} h \|u\|^3_{L^{3s}(B^\theta_{3r,\infty})}. \tag{4.16}$$

Choosing $\delta = h$, from (4.13), (4.14) and (4.16), we conclude

$$\left(\int_0^{T-h} \|p_\delta(t+h) - p_\delta(t)\|^s_{L^r} \, dt \right)^{\frac{1}{s}} \le Ch^{2\theta-\varepsilon} \left(\|u\|^2_{L^{3s}(B^\theta_{3r,\infty})} + \|u\|^3_{L^{3s}(B^\theta_{3r,\infty})} \right),$$

which implies that $p \in B^{2\theta-\varepsilon}_{s,\infty}((0,T); L^r(\mathbb{T}^3))$. If now $\theta > 1/2$, we have to prove that $p \in W^{1,s}((0,T); B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$. It is enough to show that $\partial_t p \in L^s((0,T); B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$. Indeed by point (i) of the Theorem 1.1 $p \in L^{\frac{3s}{2}}((0,T); B^{2\theta}_{\frac{3r}{2},\infty}(\mathbb{T}^3)) \hookrightarrow L^s((0,T); B^{2\theta-1}_{r,\infty}(\mathbb{T}^3))$. Thus we can write, by using (4.19), $\partial_t p = q^1 + q^2$ where q^1, q^2 are the unique 0-average solutions of

$$\Delta q^1 = \operatorname{div}\operatorname{div}\operatorname{div}(u \otimes u \otimes u),$$
$$\Delta q^2 = 2 \operatorname{div}\operatorname{div}(\nabla p \otimes u).$$

Since, by (4.9),

$$\|\nabla p(t)\|_{B^{2\theta-1}_{\frac{3r}{2},\infty}} \le C\|u(t)\|^2_{B^\theta_{3r,\infty}},$$

by Calderón-Zygmund we get

$$\|q^2(t)\|_{B^{2\theta-1}_{r,\infty}} \le C\|(\nabla p \otimes u)(t)\|_{B^{2\theta-1}_{r,\infty}} \le C\|\nabla p(t)\|_{B^{2\theta-1}_{\frac{3r}{2},\infty}} \|u(t)\|_{B^\theta_{3r,\infty}} \le C\|u(t)\|^3_{B^\theta_{3r,\infty}}.$$

Moreover, by (4.4) with $\gamma = \theta$, $s = \infty$ and $u = w = z = u(t)$,

$$\|q^1(t)\|_{B^{2\theta-1}_{r,\infty}} \le C\|u(t)\|^3_{B^\theta_{3r,\infty}}.$$

Hence, by taking the $L^s(0,T)$-norm we obtain

$$\|\partial_t p\|_{L^s(B^{2\theta-1}_{r,\infty})} \le \|q^1\|_{L^s(B^{2\theta-1}_{r,\infty})} + \|q^2\|_{L^s(B^{2\theta-1}_{r,\infty})} \le C\|u\|^3_{L^{3s}(B^\theta_{3r,\infty})},$$

which concludes the proof of (iii).

Proof of (iv). By Lemma 4.3 we have that $\partial_t p$ solves (4.19). Therefore $\partial_t p(t+h) - \partial_t p(t) = q^1 + q^2$ where

$$\begin{aligned}
\Delta q^1 &= \operatorname{div}\operatorname{div}\operatorname{div}(u(t+h) \otimes u(t+h) \otimes u(t+h) - u(t) \otimes u(t) \otimes u(t)) \\
&= \operatorname{div}\operatorname{div}\operatorname{div}\big((u(t+h) - u(t)) \otimes u(t+h) \otimes u(t+h) + u(t) \otimes (u(t+h) - u(t)) \otimes u(t+h) \\
&\quad + u(t) \otimes u(t) \otimes (u(t+h) - u(t))\big),
\end{aligned}$$

$$\Delta q^2 = 2 \operatorname{div}\operatorname{div}\big(\nabla p(t+h) \otimes u(t+h) - \nabla p(t) \otimes u(t)\big).$$

To estimate q^1, for any small $\varepsilon > 0$, we apply (4.4) with $\gamma = 1-\theta+\varepsilon$ and $s = \infty$, in such a way that the factor $u(t+h)-u(t)$ gets only the $B^{1-\theta+\varepsilon}_{3r,\infty}$-norm and not the $B^{\theta}_{3r,\infty}$-norm. Thus we get

$$\|q^1(t)\|_{L^r} \le \|q^1(t)\|_{B^{\varepsilon}_{r,\infty}} \le C\|u(t+h)-u(t)\|_{B^{1-\theta+\varepsilon}_{3r,\infty}}\big(\|u(t+h)\|^2_{B^{\theta}_{3r,\infty}} + \|u(t)\|^2_{B^{\theta}_{3r,\infty}}\big).$$

Integrating in time on $(0, T-h)$ yields to

$$\int_0^{T-h}\|q^1(t)\|^s_{L^r}\,dt \le C\int_0^{T-h}\|u(t+h)-u(t)\|^s_{B^{1-\theta+\varepsilon}_{3r,\infty}}\big(\|u(t+h)\|^{2s}_{B^{\theta}_{3r,\infty}} + \|u(t)\|^{2s}_{B^{\theta}_{3r,\infty}}\big)\,dt$$

and by the Cauchy-Schwarz inequality we get

$$\int_0^{T-h}\|q^1(t)\|^s_{L^r}\,dt \le C\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{2s}_{B^{1-\theta+\varepsilon}_{3r,\infty}}\,dt\right)^{\frac12}\|u\|^{2s}_{L^{4s}(B^{\theta}_{3r,\infty})}.$$

Now, by (2.1) together with the Hölder inequality in time, we have

$$\begin{aligned}\int_0^{T-h}\|u(t+h)-u(t)\|^{2s}_{B^{1-\theta+\varepsilon}_{3r,\infty}}\,dt &\le \int_0^{T-h}\|u(t+h)-u(t)\|^{2s\frac{2\theta-1-\varepsilon}{\theta}}_{L^{3r}}\|u(t+h)-u(t)\|^{2s\frac{1-\theta+\varepsilon}{\theta}}_{B^{\theta}_{3r,\infty}}\,dt\\ &\le \left(\int_0^{T-h}\|u(t+h)-u(t)\|^{2s}_{L^{3r}}\,dt\right)^{\frac{2\theta-1-\varepsilon}{\theta}}\left(\int_0^{T-h}\|u(t+h)-u(t)\|^{2s}_{B^{\theta}_{3r,\infty}}\,dt\right)^{\frac{1-\theta+\varepsilon}{\theta}}\\ &\le Ch^{2s(2\theta-1-\varepsilon)}\|u\|^{2s\frac{2\theta-1-\varepsilon}{\theta}}_{B^{\theta}_{2s,\infty}(L^{3r})}\|u\|^{2s\frac{1-\theta+\varepsilon}{\theta}}_{L^{2s}(B^{\theta}_{3r,\infty})} \le Ch^{2s(2\theta-1-\varepsilon)},\end{aligned}$$

where in the last inequality we used $u \in B^{\theta}_{3s,\infty}((0,T);L^{3r}(\mathbb{T}^3)) \hookrightarrow B^{\theta}_{2s,\infty}((0,T);L^{3r}(\mathbb{T}^3))$, that comes from (i). Thus we conclude with

$$\int_0^{T-h}\|q^1(t)\|^s_{L^r}\,dt \le Ch^{s(2\theta-1-\varepsilon)}. \tag{4.17}$$

Similarly, we obtain

$$\begin{aligned}\int_0^{T-h}\|q^2(t)\|^s_{L^r}\,dt &\le C\int_0^{T-h}\|(\nabla p\otimes u)(t+h)-\nabla p\otimes u)(t)\|^s_{L^r}\,dt \le Ch^{s(2\theta-1-\varepsilon)}\|\nabla p\otimes u\|^s_{B^{2\theta-1-\varepsilon}_{s,\infty}(L^r)}\\ &\le Ch^{s(2\theta-1-\varepsilon)}\left(\|\nabla p\|_{B^{2\theta-1-\varepsilon}_{2s,\infty}(L^{2r})}\|u\|_{B^{2\theta-1-\varepsilon}_{2s,\infty}(L^{2r})}\right)^s\\ &\le Ch^{s(2\theta-1-\varepsilon)}\left(\|\nabla p\|_{B^{2\theta-1-\varepsilon}_{2s,\infty}(L^{2r})}\|u\|_{B^{\theta}_{2s,\infty}(L^{2r})}\right)^s \le Ch^{s(2\theta-1-\varepsilon)},\end{aligned} \tag{4.18}$$

where we used that $u \in B^{\theta}_{2s,\infty}((0,T);L^{2r}(\mathbb{T}^3))$ by (i), and $\nabla p \in B^{2\theta-1-\varepsilon}_{2s,\infty}((0,T);L^{2r}(\mathbb{T}^3))$ by (ii). Summing up (4.17) and (4.18) we obtain $\partial_t p \in B^{2\theta-1-\varepsilon}_{s,\infty}((0,T);L^r(\mathbb{T}^3))$, as desired. □

Lemma 4.3. *Let $u \in L^{3s}((0,T);B^{\theta}_{3r,\infty}(\mathbb{T}^3))$ for some $r,s \in [1,\infty]$ and $\theta \in (1/2,1)$. Then $\partial_t p$ solves*

$$\Delta\partial_t p = \operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u) + 2\operatorname{div}\operatorname{div}(\nabla p\otimes u), \tag{4.19}$$

in the distributional sense.

Proof. For every $\delta > 0$, we denote by p^δ the unique 0-average solution of

$$-\Delta p^\delta = \operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta).$$

Note that by Calderón-Zygmund, $p^\delta \to p$ in $L^{\frac{3s}{2}}((0,T);L^{\frac{3r}{2}}(\mathbb{T}^3))$ as $\delta \to 0$. Thus $\partial_t p^\delta \to \partial_t p$ in distribution. Since $\partial_t u_\delta \in L^{\frac{3s}{2}}((0,T);C^\infty(\mathbb{T}^3))$ from (4.7), we can compute

$$\begin{aligned}\partial_t\operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta) = 2\operatorname{div}\operatorname{div}(\partial_t u_\delta\otimes u_\delta) &= -\operatorname{div}\operatorname{div}\operatorname{div}(u_\delta\otimes u_\delta\otimes u_\delta)\\ &\quad - 2\operatorname{div}\operatorname{div}(\nabla p_\delta\otimes u_\delta) + 2\operatorname{div}\operatorname{div}(\operatorname{div} R_\delta\otimes u_\delta).\end{aligned}$$

Obviously $u_\delta \to u$ in $L^{3s}((0,T);L^{3r}(\mathbb{T}^3))$. By (2.6), since $\theta > 1/2$ we have that $\operatorname{div} R_\delta \to 0$ in $L^{\frac{3s}{2}}((0,T);L^{\frac{3r}{2}}(\mathbb{T}^3))$. Moreover by (i) in Theorem 1.1 we also have $\nabla p_\delta \to \nabla p$ in $L^{\frac{3s}{2}}((0,T);L^{\frac{3r}{2}}(\mathbb{T}^3))$. Thus we conclude that in the distributional sense

$$\partial_t \operatorname{div}\operatorname{div}(u_\delta \otimes u_\delta) \to -\operatorname{div}\operatorname{div}\operatorname{div}(u\otimes u\otimes u) - 2\operatorname{div}\operatorname{div}(\nabla p \otimes u). \qquad \square$$

Remark 4.4. In the above proof, one can make explicit quantitative estimates on the quantities which appear in the statement of Theorem 1.1. For instance, as regards (i) we have

$$\|u\|_{B^\theta_{s,\infty}(L^r)} \le C\left(\|u\|_{L^s(B^\theta_{r,\infty})} + \|u\|^2_{L^{2s}(B^\theta_{2r,\infty})}\right),$$
$$\|p\|_{L^s(B^{2\theta}_{r,\infty})} \le C\|u\|^2_{L^{2s}(B^\theta_{2r,\infty})}$$

for a constant $C > 0$ depending only on r, s, θ.

Remark 4.5 (The case $r = 1$). When $r = 1$, the statements (i) and (ii) of Theorem 1.1 on the pressure may not be true in general. On the positive side, if $u \in L^{3s}((0,T);W^{1,1}(\mathbb{T}^3))$, the compensated compactness methods [7] give that the pressure belongs to $L^{\frac{3s}{2}}((0,T);W^{2,1}(\mathbb{T}^3))$ (namely, the result with $r = 1$ and $\theta = 1$ would hold). On the other side, however, if $r = 1$ and $\theta = 0$, the lack of the Calderón-Zygmund theory gives us that a solution p to (1.2) is in general not more than in the weak-$L^1(\mathbb{T}^3)$ space. Trying to repeat the proof of the abstract interpolation result of Theorem 3.5, as we did in Proposition 4.1 for $r = 1$, this constitutes a problem because we would need to apply the interpolation result with $Y_1 = L^1_{\text{weak}}$, $Y_2 = W^{2,1}$. Hence, Theorem 3.5 would only give us that $p(t) \in (L^1_{\text{weak}}(\mathbb{T}^3), W^{2,1}(\mathbb{T}^3))_{\theta,1}$ and it is unclear if such space would coincide with a suitable Besov-type space.

Proof of Corollary 1.2. The proof is just a consequence of (i), (ii) and (iv) of Theorem 1.1 together with the embeddings $W^{\theta,r} \hookrightarrow B^\theta_{r,\infty} \hookrightarrow W^{\gamma,r}$, that hold true for any $r \in [1,\infty]$ and $\theta,\gamma \in (0,1)$ with $\theta > \gamma$. $\square$

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References

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011.

[2] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[3] T. Buckmaster, C. De Lellis, P. Isett, and L. Székelyhidi Jr., *Anomalous dissipation for 1/5-Hölder Euler flows*, Ann. of Math. (2) **182** (2015), no. 1, 127–172.

[4] T. Buckmaster, C. De Lellis, L. Székelyhidi Jr., and V. Vicol, *Onsager's conjecture for admissible weak solutions*, Comm. Pure Appl. Math. **72** (2019), no. 2, 229–274.

[5] T. Buckmaster and V. Vicol, *Nonuniqueness of weak solutions to the Navier-Stokes equation*, Ann. of Math. (2) **189** (2019), no. 1, 101–144.

[6] ———, *Convex integration and phenomenologies in turbulence*, EMS Surv. Math. Sci. **6** (2019), no. 1, 173–263.

[7] R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl. (9) **72** (1993), no. 3, 247–286 (English, with English and French summaries).

[8] M. Colombo, C. De Lellis, and L. De Rosa, *Ill-posedness of Leray solutions for the hypodissipative Navier-Stokes equations*, Comm. Math. Phys. **362** (2018), no. 2, 659–688.

[9] M. Colombo and L. De Rosa, *Regularity in time of Hölder solutions of Euler and hypodissipative Navier-Stokes equations*, SIAM J. Math. Anal. **52** (2020), no. 1, 221–238.

[10] P. Constantin, *Local formulas for hydrodynamic pressure and their applications*, Uspekhi Mat. Nauk **69** (2014), no. 3(417), 3–26 (Russian, with Russian summary); English transl., Russian Math. Surveys **69** (2014), no. 3, 395–418.

[11] P. Constantin, W. E, and E. S. Titi, *Onsager's conjecture on the energy conservation for solutions of Euler's equation*, Comm. Math. Phys. **165** (1994), no. 1, 207–209.

[12] C. De Lellis and L. Székelyhidi Jr., *Dissipative continuous Euler flows*, Invent. Math. **193** (2013), no. 2, 377–407.

[13] C. De Lellis and L. Székelyhidi Jr., *Dissipative Euler flows and Onsager's conjecture*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 7, 1467–1505.

[14] F. Duzaar, J. Kristensen, and G. Mingione, *The existence of regular boundary points for non-linear elliptic systems*, J. Reine Angew. Math. **602** (2007), 17–58.

[15] G. L. Eyink, *Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer*, Phys. D **78** (1994), no. 3-4, 222–240.

[16] L. Grafakos and M. Mastyło, *Interpolation of bilinear operators between quasi-Banach spaces*, Positivity **10** (2006), no. 3, 409–429.

[17] P. Isett, *Regularity in time along the coarse scale flow for the incompressible Euler equations*, arXiv preprint, https://arxiv.org/abs/1307.0565.

[18] P. Isett, *A proof of Onsager's conjecture*, Ann. of Math. (2) **188** (2018), no. 3, 871–963.

[19] A. Kolmogoroff, *The local structure of turbulence in incompressible viscous fluid for very large Reynold's numbers*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **30** (1941), 301–305.

[20] J.-L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. **19** (1964), 5–68 (French).

[21] A. Lunardi, *Interpolation theory*, 2nd ed., Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)], Edizioni della Normale, Pisa, 2009.

[22] M. Mastyło, *Bilinear interpolation theorems and applications*, J. Funct. Anal. **265** (2013), no. 2, 185–207.

[23] S. Modena and L. Székelyhidi Jr., *Non-uniqueness for the transport equation with Sobolev vector fields*, Ann. PDE **4** (2018), no. 2, Art. 18, 38.

[24] L. Onsager, *Statistical hydrodynamics*, Nuovo Cimento (9) **6** (1949), no. Supplemento, 2 (Convegno Internazionale di Meccanica Statistica), 279–287.

[25] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.

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