Three results on the Energy conservation for the 3D Euler equations

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Abstract

We consider the 3D Euler equations for incompressible homogeneous fluids and we study the problem of energy conservation for weak solutions in the space-periodic case. First, we prove the energy conservation for a full scale of Besov spaces, by extending some classical results to a wider range of exponents. Next, we consider the energy conservation in the case of conditions on the gradient, recovering some results which were known, up to now, only for the Navier-Stokes equations and for weak solutions of the Leray-Hopf type. Finally, we make some remarks on the Onsager singularity problem, identifying conditions which allow to pass to the limit from solutions of the Navier-Stokes equations to solution of the Euler ones, producing weak solutions which are energy conserving.

1. Introduction

The aim of this paper is to extend some nowadays classical results about the energy conservation for the space-periodic 3D Euler equations (here $\mathbb{T}^3 := \big(\mathbb{R}/2\pi\mathbb{Z}\big)^3$)

$$\begin{aligned} \partial_t v^E + (v^E \cdot \nabla)\, v^E + \nabla p^E &= 0 \qquad && \text{in } (0,T)\times\mathbb{T}^3, \\ \operatorname{div} v^E &= 0 && \text{in } (0,T)\times\mathbb{T}^3, \\ v^E(0) &= v_0^E && \text{in } \mathbb{T}^3, \end{aligned} \tag{1.1}$$

to embrace a full space-time range of exponents. Recall that it is known since [9] that weak solutions of the Euler equations such that

$$v^E \in C_w(0,T;L^2(\mathbb{T}^3)) \cap L^3(0,T;B^{\alpha}_{3,\infty}(\mathbb{T}^3)), \qquad \text{with} \quad \alpha > \frac{1}{3}, \tag{1.2}$$

conserve the energy, where $B^{\alpha,\infty}_3(\mathbb{T}^3)$ denotes a standard Besov space and the motivation for this result is the Onsager conjecture [15] following from Kolmogorov K41 theory. The Onsager conjecture (only recently solved also for the negative part, see Isett [14] and De Lellis [6]) suggested the threshold value $\alpha = 1/3$ for energy conservation. Here, we consider a combination of space-time conditions, identifying families of Besov spaces with the range $(1/3, 1)$ for the exponent of regularity balanced by a proper integrability exponent in time. Next, we consider also the limiting case $\alpha = 1$, and finally the connection of energy conservation with the vanishing viscosity limits. We recall that the first rigorous results about Onsager conjecture are probably those of Eyink [12, 13] in the Fourier setting and Constantin, E, and Titi [9] and we are mainly inspired by these references; for the vanishing viscosity limit we follow the same path as in Drivas and Eyink [10].

The original results we prove concern: 1) the extension of (1.2) to a full scale of exponents $\frac{1}{3} < \alpha < 1$, identifying the sharp conditions on the parameters, as previously done in the setting of Hölder continuous functions in [4]; 2) the extension to the case $\alpha = 1$, which means that we look for conditions on the gradient of v^E in standard Lebesgue spaces; 3) we identify hypotheses, uniform in the viscosity, on solutions to the Navier-Stokes equations, which allow us to pass to the limit as $\nu \to 0$ and to construct weak solutions of the Euler equations satisfying the energy equality.

More precisely, concerning point 1) we extend the result of [9] to a wider range of exponents proving the following theorem:

Theorem 1. *Let v^E be a weak solution to the Euler equations such that, for $\frac{1}{3} < \alpha < 1$,*

$$v^E \in L^{1/\alpha}(0,T; B^{\beta}_{\frac{2}{1-\alpha},\infty}(\mathbb{T}^3)), \qquad \text{with} \quad \alpha < \beta < 1. \tag{1.3}$$

Then, v^E conserves the energy.

Similar results have already been proved in the setting of Hölder continuous functions (see [3]) and in both cases, one can see that the limiting case $\alpha \to 1^-$ leads formally to $L^1(0,T;W^{1,\infty})$, which corresponds to the Beale-Kato-Majda criterion. Anyway, working directly with the velocity v^E in a Sobolev space, we obtain the following result:

Theorem 2. *Let v^E be a weak solution to the Euler equations such that, for $q > 2$,*

$$v^E \in L^r(0,T; W^{1,q}(\mathbb{T}^3)), \qquad \text{with} \quad r > \frac{5q}{5q-6}. \tag{1.4}$$

Then, v^E conserves the energy.

The sharpness of this result comes by observing that we recover for the Euler equations the same results (at least in this range of exponents) which are known for Leray-Hopf weak solutions to the Navier-Stokes equations, see the recent results in [1, 5].

Concerning point 3) we extend results from [10, 4] on the emergence of solutions to the Euler equations satisfying the energy equality as inviscid limits of Leray-Hopf weak solutions to the Navier-Stokes equations ($\nu > 0$)

$$\begin{aligned} \partial_t v^\nu + (v^\nu \cdot \nabla)\, v^\nu - \nu \Delta v^\nu + \nabla p^\nu = 0 \qquad & \text{in } (0,T) \times \mathbb{T}^3, \\ \operatorname{div} v^\nu = 0 \qquad & \text{in } (0,T) \times \mathbb{T}^3, \\ v^\nu(0) = v_0^\nu \qquad & \text{in } \mathbb{T}^3. \end{aligned} \tag{1.5}$$

This result generalizes to a wider range of exponents the result from [10] which deals with $\alpha \sim 1/3$ and also the results in [4], which are in the setting of Hölder continuous functions, but with a more restrictive time-dependence for $\alpha > 1/2$. We have the following result:

Theorem 3. *Let $\{v^\nu\}_{\nu>0}$ be a family of weak solutions of the NSE with the same initial datum $v_0 \in H \cap B^{\beta}_{\frac{2}{1-\alpha},\infty}(\mathbb{T}^3)$, $\beta > \alpha$. Let also assume that for $\frac{1}{3} < \alpha < 1$, and $\alpha < \beta < 1$ there exists a constant $C_{\alpha,\beta} > 0$, independent of $\nu > 0$, such that*

$$\|v^\nu\|_{L^{1/\alpha}(0,T;B^{\beta}_{\frac{2}{1-\alpha},\infty})} \le C_{\alpha,\beta}, \qquad \forall\, \nu \in (0,1]. \tag{1.6}$$

Then, in the limit $\nu \to 0$ the family $\{v^\nu\}$ converges (up to a sub-sequence) to a weak solution v^E in $[0,T]$ of the Euler equations satisfying the energy equality.

The problem of vanishing viscosity and construction of distributional (dissipative) solutions to the Euler equations has a long history and we mainly refer to Duchon and Robert [11] for similar results. We also wish to mention the Fourier-based approach recently developed by Chen and Glimm [7, 8] where spectral properties are used to deduce certain fractional regularity results, suitable to prove the inviscid limit. Our proof uses standard mollification and handling of the commutation terms. Even though the results use elementary techniques, they are new and rather sharp. Note also that Theorem 3 implies that "quasi-singularities" are required in Leray-Hopf weak solutions in order to account for anomalous energy dissipation, see the discussion and interpretation in [10, 4].

Plan of the paper: In Section 2 we set up our notation by giving the definitions of the spaces and the solutions that we use throughout the paper. Moreover, we recall the basic properties of the mollification and the commutator formula that will be used extensively in the proofs of the theorems. In Section 3 we give the proofs of Theorem 1 and 2, investigating minimum regularity conditions for energy conservation, for weak solutions to the Euler equations. Finally, in Section 4, we give the proof of Theorem 3, dealing with the emergence of weak solutions of Euler in the limiting case $\nu \to 0$.

2. Notation

In the sequel we will use the Lebesgue $(L^p(\mathbb{T}^3), \|\,.\,\|_p)$ and Sobolev $(W^{1,p}(\mathbb{T}^3), \|\,.\,\|_{1,p})$ spaces, with $1 \leq p \leq \infty$; for simplicity we denote by $(\,.\,,\,.\,)$ and $\|\,.\,\|$ the L^2 scalar product and norm, respectively, while the other norms are explicitly indicated. By H and V we denote the closure of smooth, periodic, and divergence-free vector fields in $L^2(\mathbb{T}^3)$ or $W^{1,2}(\mathbb{T}^3)$, respectively. Moreover, we will use the Besov spaces $B^{\alpha}_{p,\infty}(\mathbb{T}^3)$, which are the same as Nikol'skiĭ spaces $\mathcal{N}^{\alpha,p}(\mathbb{T}^3)$. They are sub-spaces of L^p for which there exists $c > 0$, such that $\|u(\cdot + h) - u(\cdot)\|_p \leq c|h|^{\alpha}$, and the smallest constant is the semi-norm $[\,.\,]_{B^{\alpha}_{p,\infty}}$.

To properly set the problem we consider, we give the definitions of weak solutions:

Definition 1 (Weak solution to the Euler equations)**.** *Let $v_0 \in H$. A measurable function $v^E : (0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ is called a weak solution to the Euler equations (1.1) if $v^E \in L^\infty(0,T;H)$, solves the equations in the weak sense:*

$$\int_0^T \int_{\mathbb{T}^3} \Big[v^E \cdot \partial_t \phi + (v^E \otimes v^E) : \nabla \phi \Big] \, \mathrm{d}x \, dt = - \int_{\mathbb{T}^3} v_0 \cdot \phi(0) \, \mathrm{d}x, \tag{2.1}$$

for all $\phi \in \mathcal{D}_T := \Big\{ \phi \in C_0^\infty([0,T[\times \mathbb{T}^3) : \ \operatorname{div} \phi = 0 \Big\}$.

We also recall the definition of weak solutions to the Navier-Stokes equations.

Definition 2 (Space-periodic Leray-Hopf weak solution)**.** *Let $v_0^\nu \in H$. A measurable function $v^\nu : (0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ is called a Leray-Hopf weak solution to the space-periodic NSE (1.5) if $v^\nu \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and the following hold true:*
The function v^ν solves the equations in the weak sense:

$$\int_0^T \int_{\mathbb{T}^3} \Big[v^\nu \cdot \partial_t \phi - \nu \nabla v^\nu : \nabla \phi + (v^\nu \otimes v^\nu) : \nabla \phi \Big] \, \mathrm{d}x\mathrm{d}t = - \int_{\mathbb{T}^3} v_0^\nu \cdot \phi(0) \, \mathrm{d}x, \tag{2.2}$$

for all $\phi \in \mathcal{D}_T$;
The (global) energy inequality holds:

$$\frac{1}{2}\|v^\nu(t)\|_2^2 + \nu \int_0^t \|\nabla v^\nu(\tau)\|_2^2 \, d\tau \leq \frac{1}{2}\|v_0^\nu\|_2^2, \qquad \forall\, t \in [0,T]; \tag{2.3}$$

The initial datum is strongly attained: $\lim_{t\to 0^+} \|v^\nu(t) - v_0^\nu\| = 0$.

2.1. Mollification and Sobolev/Besov spaces

We use the classical tools of mollification to justify calculations and to this end we fix $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho(x) = \rho(|x|)$, $\rho \geq 0$, $\operatorname{supp} \rho \subset B(0,1) \subset \mathbb{R}^3$, $\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = 1$, and we define, for $\epsilon \in (0,1]$, the Friedrichs family $\rho_\epsilon(x) := \epsilon^{-3}\rho(\epsilon^{-1}x)$. Then, for any function $f \in L^1_{loc}(\mathbb{R}^3)$ we define by the usual convolution

$$f_\epsilon(x) := \int_{\mathbb{R}^3} \rho_\epsilon(x-y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^3} \rho_\epsilon(y) f(x-y)\,\mathrm{d}y.$$

If $f \in L^1(\mathbb{T}^3)$, then $f \in L^1_{loc}(\mathbb{R}^3)$, and it turns out that $f_\epsilon \in C^\infty(\mathbb{T}^3)$ is 2π-periodic along the x_j-direction, for $j = 1,2,3$. Moreover, if f is a divergence-free vector field, then f_ϵ is a smooth divergence-free vector field. We report now the basic properties of the convolution operator we will use in the sequel, see for instance [9, 2, 4].

Lemma 4. *Let ρ be as above. If $u \in L^q(\mathbb{T}^3)$, then $\exists\, C > 0$ (depending only on ρ) such that*

$$\|u_\epsilon\|_r \leq \frac{C}{\epsilon^{3(\frac{1}{q}-\frac{1}{r})}}\|u\|_q \quad \textit{for all } r \geq q; \tag{2.4}$$

If $u \in B^\beta_{q,\infty}(\mathbb{T}^3)$, then

$$\|u(\cdot + y) - u(\cdot)\|_q \leq [u]_{B^\beta_{q,\infty}} |y|^\beta, \tag{2.5}$$

$$\|u - u_\epsilon\|_q \leq [u]_{B^\beta_{q,\infty}}\, \epsilon^\beta, \tag{2.6}$$

$$\|\nabla u_\epsilon\|_q \leq C[u]_{B^\beta_{q,\infty}}\, \epsilon^{\beta-1}, \tag{2.7}$$

while if $u \in W^{1,q}(\mathbb{T}^3)$, then

$$\|u(\cdot + y) - u(\cdot)\|_q \leq \|\nabla u\|_q |y|, \tag{2.8}$$

$$\|u - u_\epsilon\|_q \leq \|\nabla u\|_q\, \epsilon, \tag{2.9}$$

$$\|\nabla u_\epsilon\|_q \leq C\|u\|_q\, \epsilon^{-1}. \tag{2.10}$$

In the sequel, the following well-known commutator formula derived in [9] and known as the "Constantin-E-Titi commutator" will be crucial:

$$(u \otimes u)_\epsilon = u_\epsilon \otimes u_\epsilon + r_\epsilon(u,u) - (u - u_\epsilon) \otimes (u - u_\epsilon), \tag{2.11}$$

with

$$r_\epsilon(u,u) := \int_{\mathbb{T}^3} \rho_\epsilon(y)(\delta_y u(x) \otimes \delta_y u(x))\,\mathrm{d}y, \qquad \text{for} \quad \delta_y u(x) := u(x-y) - u(x).$$

3. On the conservation of energy for ideal fluids

We prove Theorem 1 and, for $\beta \in (\frac{1}{3}, 1)$, we investigate the minimum Besov regularity that is needed, so that weak solutions of the Euler equations conserve their kinetic energy.

Proof of Theorem 1. We test the Euler equations against $\varphi = (v^E_\epsilon)_\epsilon$ to obtain

$$\begin{aligned} \frac{1}{2}\|v^E_\epsilon(T)\|^2_{L^2(\Omega_T)} = \frac{1}{2}\|v^E_\epsilon(0)\|^2_2 &- \int_0^T \int_{\mathbb{T}^3} r_\epsilon(v^E, v^E) : \nabla v^E_\epsilon \,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{T}^3} (v^E - v^E_\epsilon) \otimes (v^E - v^E_\epsilon) : \nabla v^E_\epsilon \,\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.1}$$

since $\int_0^T \int_{\mathbb{T}^3} v^E_\epsilon \otimes v^E_\epsilon : \nabla v^E_\epsilon \,\mathrm{d}x\mathrm{d}t = 0$, due to v^E_ϵ being smooth and divergence-free. We now estimate the last two terms from the right-hand side, using the properties of Besov spaces. Indeed, for $0<\eta<2$ and $q>1$, such that $q-(1+\eta)>0$ we write:

$$\begin{aligned}
I_1 &:= \left| \int_0^T \int_{\mathbb{T}^3} (v^E - v^E_\epsilon) \otimes (v^E - v^E_\epsilon) : \nabla v^E_\epsilon \,\mathrm{d}x\mathrm{d}t \right| \\
&\leq \int_0^T \int_{\mathbb{T}^3} |v^E - v^E_\epsilon|^\eta |v^E - v^E_\epsilon|^{2-\eta} |\nabla v^E_\epsilon| \,\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \|v^E - v^E_\epsilon\|_q^\eta \, \|v^E - v^E_\epsilon\|^{2-\eta}_{\frac{(2-\eta)q}{q-(1+\eta)}} \|\nabla v^E_\epsilon\|_q \,\mathrm{d}t,
\end{aligned}$$

and in the second line we used Hölder's inequality. Since a weak solution v^E is in $L^\infty(0,T;H)$, if $\frac{(2-\eta)q}{q-(1+\eta)}=2$, we can use the energy bound to infer

$$\|v^E - v^E_\epsilon\|_2 \leq \|v^E\|_2 + \|v^E_\epsilon\|_2 \leq 2\|v^E\|_2 \leq 2\,\mathrm{esssup}_{t\in(0,T)}\|v^E\|_2 \leq C,$$

and thus by (2.6)-(2.7)

$$I_1 \leq C \int_0^T \|v^E - v^E_\epsilon\|_q^\eta \,\|\nabla v^E_\epsilon\|_q \,\mathrm{d}t \leq C\epsilon^{\beta\eta+\beta-1} \int_0^T [v^E(t)]^{\eta+1}_{B^\beta_{q,\infty}} \,\mathrm{d}t,$$

where $C>0$ does not depend on $\epsilon>0$.

Next, we estimate the remainder term in the commutator as follows:

$$\begin{aligned}
r_\epsilon(v^E, v^E) &= \int_{\mathbb{T}^3} \rho_\epsilon(y)(v^E(x-y) - v^E(x)) \otimes (v^E(x-y) - v^E(x)) \,\mathrm{d}y \\
&\overset{y=\epsilon z}{=} \int_{\mathbb{T}^3} \rho(z)(v^E(x-\epsilon z)) - v^E(x)) \otimes (v^E(x-\epsilon z)) - v^E(x)) \,\mathrm{d}z \\
&\leq \int_{\mathbb{T}^3} |v^E(x-\epsilon z) - v^E(x)|^2 \,\mathrm{d}z.
\end{aligned}$$

Then, as above, we can write for $0<\eta<2$ such that $\frac{(2-\eta)q}{q-(1+\eta)}=2$:

$$\begin{aligned}
I_2 &:= \left| \int_0^T \int_{\mathbb{T}^3} r_\epsilon(v^E, v^E) : \nabla v^E_\epsilon \,\mathrm{d}x\mathrm{d}t \right| \\
&\leq \int_0^T \int_{\mathbb{T}^3} \int |v^E(x-\epsilon z) - v^E(x)|^2 |\nabla v^E_\epsilon| \,\mathrm{d}z\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{T}^3} \int_{\mathbb{T}^3} |v^E(x-\epsilon z) - v^E(x)|^\eta |v^E(x-\epsilon z) - v^E(x)|^{2-\eta} |\nabla v^E_\epsilon| \,\mathrm{d}z\mathrm{d}x\mathrm{d}t \\
&\leq C \int_0^T \int_{\mathbb{T}^3} \|\nabla v^E_\epsilon\|_q \, \|v^E(\cdot-\epsilon z) - v^E(\cdot)\|_q^\eta \|v^E(\cdot-\epsilon z) - v^E(\cdot)\|^{2-\eta}_{\frac{(2-\eta)q}{q-(1+\eta)}} \,\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

and using (2.6)-(2.7) we arrive at

$$I_2 \leq C\epsilon^{\beta\eta+\beta-1} \int_0^T \int_{\mathbb{T}^3} |z|^{\eta\beta} [v^E]^{\eta+1}_{B^\beta_{q,\infty}} \,\mathrm{d}z\mathrm{d}t \leq C\epsilon^{\beta\eta+\beta-1} \int_0^T \|v^E\|^{\eta+1}_{B^\beta_{q,\infty}} \,\mathrm{d}t,$$

with $C>0$ independent of ϵ.

Hence, for fixed $\beta\in(\frac{1}{3},1)$, we want to find $(q,\eta)\in(1,+\infty)\times(0,2)$ such that $\eta+1$ (the exponent of the Besov semi-norm) is the smallest possible, subject to the following set of constraints:

$$\begin{cases} \beta\eta + \beta - 1 > 0 \\ \eta < q - 1 \\ \frac{(2-\eta)q}{q-(1+\eta)} = 2 \end{cases} .$$

Hence, we set $\eta = \frac{1-\alpha}{\alpha}$ and $q = \frac{2}{1-\alpha}$. The last two constraints are satisfied for $\alpha > 1/3$ (leading to $q > 3$); next, if $\beta > \alpha$, then $\beta\eta + \beta - 1 = \frac{\beta-\alpha}{\alpha} > 0$ and consequently

$$0 \leq I_1 + I_2 \leq C\epsilon^{\frac{\beta-\alpha}{\alpha}} \int_0^T [v^E]^\beta_{B^\beta_{\frac{2}{1-\alpha},\infty}} \,\mathrm{d}t \overset{\epsilon\to 0}{\longrightarrow} 0$$

and letting $\epsilon \to 0$ in (3.1) gives

$$\frac{1}{2}\|v^E(t)\|^2_{L^2} = \frac{1}{2}\|v^E(0)\|^2_{L^2}. \tag{3.2}$$

Therefore, for $\alpha \in (\frac{1}{3}, 1)$, the "critical" space for energy conservation is $L^{1/\alpha}(0,T; B^\alpha_{\frac{2}{1-\alpha},\infty})$. □

We now prove the second theorem, corresponding to conditions on the gradient of v^E, which would, formally, be the same with $\alpha = 1$, but in fact the result here is much stronger, since the bound on the gradient allows us to make sense of the convective term in a more precise manner.

Proof of Theorem 2. In the case $\alpha = 1$, the required regularity for energy conservation is $\nabla u \in L^r(0,T; L^q(\Omega))$, for $r > \frac{5q}{5q-6}$, as follows from the following computations. The approach is similar as before and we just need to control the commutator terms, after testing the equations by $(v^E_\epsilon)_\epsilon$ (we make explicit computations only for this one, since the remainder can be handled as we have done before). We get in fact

$$\begin{aligned} I_1 :&= \left| \int_0^T \int_{\mathbb{T}^3} (v^E - v^E_\epsilon) \otimes (v^E - v^E_\epsilon) : \nabla v^E_\epsilon \,\mathrm{d}x\mathrm{d}t \right| \\ &\leq \int_0^T \int_{\mathbb{T}^3} |v^E - v^E_\epsilon|^2 |\nabla v^E_\epsilon| \,\mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T \|v^E - v^E_\epsilon\|^2_{2p} \,\|\nabla v^E_\epsilon\|_{p'} \,\mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T \|v^E - v^E_\epsilon\|^{2\theta}_2 \,\|v^E - v^E_\epsilon\|^{2(1-\theta)}_q \|\nabla v^E_\epsilon\|_{p'} \,\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where in the second step we used Hölder's inequality with conjugate exponents p and p' (to be determined), and in the third one convex interpolation such that $\frac{1}{2p} = \frac{\theta}{2} + \frac{1-\theta}{q}$, with $2p < q$. Now, using the fact that $v^E \in L^\infty(0,T; L^2(\mathbb{T}^3))$ and inequality (2.4) for the gradient of v^E

$$\|\nabla v^E_\epsilon\|_{p'} \leq C\epsilon^{-3\left(\frac{1}{q}-\frac{1}{p'}\right)} \|\nabla v^E\|_q, \qquad \text{for } p' > q,$$

we obtain:

$$I_1 \leq C\epsilon^{-3\left(\frac{1}{q}-\frac{1}{p'}\right)} \int_0^T \|v^E - v^E_\epsilon\|_q^{\frac{2q(p-1)}{p(q-2)}} \|\nabla v^E\|_q \,\mathrm{d}t$$

and (2.9)-(2.10) yield:

$$I_1 \leq C\epsilon^{\frac{2q(p-1)}{p(q-2)}-3\left(\frac{1}{q}-\frac{1}{p'}\right)} \int_0^T \|\nabla v^E\|_q^{\frac{2q(p-1)}{p(q-2)}+1} \,\mathrm{d}t.$$

Now, we want to choose p such that the exponent of ϵ is non-negative, corresponding to

$$p > \frac{(5q-6)q}{5q^2-9q+6},$$

and notice that the expression on the denominator is strictly positive. Moreover, the constraints $2p < q$ and $p' > q$ imply that $p < \min\left\{\frac{q}{2}, \frac{q}{q-1}\right\}$ and thus the range of allowed p is

$$\frac{(5q-6)q}{5q^2-9q+6} < p < \min\left\{\frac{q}{2}, \frac{q}{q-1}\right\}.$$

The second term on the right-hand side of (3.1) is handled the same way and yields the same range for p. Then, for the infimum value of p that makes the exponent of ϵ non-negative, we get that the exponent r of the L^q norm of the gradient becomes $r = \frac{5q}{5q-6}$ and thus the "critical" space for energy conservation (in the case $\alpha = 1$) is $\nabla v^E \in L^{\frac{5q}{5q-6}}(0,T;L^q(\mathbb{T}^3))$. $\square$

4. Inviscid limit from Navier-Stokes to Euler

In this section we prove Theorem 3, dealing with the inviscid (singular) limit $\nu \to 0$ and identifying sufficient conditions to construct weak solutions of the Euler equations conserving the kinetic energy.

Proof of Theorem 3. In the weak formulation (2.2) of the NSE we set $\varphi = (v^\nu_\epsilon)_\epsilon$. Note that since $v^\nu_0 \in L^2(\Omega)$, we deduce, being v^ν a Leray-Hopf solution, that

$$\|u^\nu\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} + \|\sqrt{\nu}\nabla u^\nu\|_{L^2((0,T)\times\mathbb{T}^3)} \leq C.$$

Moreover, since $v^\nu \in L^{1/\alpha}(0,T;B^\beta_{2/(1-\alpha),\infty})$, it has a derivative in the sense of distributions in the space $L^{1/2\alpha}(0,T;B^{\beta-2}_{1/(1-\alpha),\infty})$, given by $\frac{dv^\nu}{dt} = -\mathbb{P}\operatorname{div}(v^\nu\otimes v^\nu) + \nu\Delta v^\nu$, where $\mathbb{P}$ is the Leray projector. Indeed, by comparison, (the subscript "σ" means divergence-free)

$$\Big\langle \int_0^T \partial_t v^\nu, \phi \,\mathrm{d}t\Big\rangle = \Big\langle \int_0^T -\operatorname{div}(v^\nu\otimes v^\nu) + \nu\Delta u^n, \phi\, dt\Big\rangle, \qquad \forall\,\phi \in C^\infty_\sigma(\mathbb{T}^3),$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between elements of $\mathcal{D}^*(\mathbb{T}^3)$ and $\mathcal{D}(\mathbb{T}^3) = C^\infty(\mathbb{T}^3)$. Choosing $\phi(x,t) = \psi(t)\varphi(x)$, with $\psi \in C^\infty_{0,\sigma}(0,T)$ and $\varphi \in C^\infty_\sigma(\mathbb{T}^3)$ we obtain

$$\Big\langle \int_0^T \partial_t v^\nu\psi, \varphi\,\mathrm{d}t\Big\rangle = \int_0^T \psi(t)\Big\langle\left[-\mathbb{P}\operatorname{div}(v^\nu\otimes v^\nu) + \nu\Delta u^n\right], \varphi\Big\rangle\,\mathrm{d}t$$

and note that

$$\begin{aligned}
\|\mathbb{P}\operatorname{div}(v^\nu\otimes v^\nu)\|_{L^{1/2\alpha}(0,T;B^{\beta-2}_{1/(1-\alpha),\infty})} &\leq C\|v^\nu\otimes v^\nu\|_{L^{1/2\alpha}(0,T;B^{\beta-1}_{1/(1-\alpha),\infty})}\\
&\leq C\|v^\nu\|^2_{L^{1/\alpha}(0,T;B^{\beta-1}_{2/(1-\alpha),\infty})}\\
&\leq C\|v^\nu\|^2_{L^{1/\alpha}(0,T;B^{\beta}_{2/(1-\alpha),\infty})}.
\end{aligned}$$

Moreover, since $\mathbb{T}^3$ is bounded and $T < +\infty$, the embeddings

$$L^{1/\alpha}(0,T;B^{\beta-2}_{2/(1-\alpha),\infty}) \hookrightarrow L^{1/2\alpha}(0,T;B^{\beta-2}_{2/(1-\alpha),\infty}) \hookrightarrow L^{1/2\alpha}(0,T;B^{\beta-2}_{1/(1-\alpha),\infty}),$$

are continuous, thus

$$\begin{aligned}\|\Delta v^\nu\|_{L^{1/2\alpha}(0,T;B^{\beta-2}_{1/(1-\alpha),\infty})} &\le C\|\Delta v^\nu\|_{L^{1/2\alpha}}(0,T;B^{\beta-2}_{2/(1-\alpha),\infty})\\ &\le C\|\Delta v^\nu\|_{L^{1/\alpha}}(0,T;B^{\beta-2}_{2/(1-\alpha),\infty})\\ &\le C\|v^\nu\|_{L^{1/\alpha}(0,T;B^{\beta}_{2/(1-\alpha),\infty})},\end{aligned}$$

and we conclude that $\partial_t v^\nu \in L^{1/2\alpha}(0,T;B^{\beta-2}_{1/(1-\alpha),\infty})$.

Therefore, by the Aubin-Lions' lemma, there exists a sub-sequence (which is not relabeled) such that:

$$\begin{aligned} v^\nu &\to v \text{ strongly in } L^q(0,T;H) \quad \forall\, q\in(1,\infty)\\ \sqrt{\nu}\nabla v^\nu &\rightharpoonup 0 \text{ weakly in } L^2(0,T;H)\\ \partial_t v^\nu &\rightharpoonup \partial_t v \text{ weakly in } L^{1/2\alpha}(0,T;B^{\beta-2}_{1/(1-\alpha),\infty}),\end{aligned}$$

which is enough to pass to the limit as $\nu\to 0$ in (2.2), proving that v is a solution of the Euler equations.

As a final step, we show that the dissipation in the energy equation goes to zero as $\nu\to 0$, yielding an energy equation for the limiting solution v, as well. Indeed, for $\beta>\alpha$ and $\alpha\in(\frac{1}{3},\frac{1}{2}]$:

$$\begin{aligned}\nu\int_0^T\|\nabla v^\nu_\epsilon\|_2^2\,\mathrm{d}t &\le C\nu\int_0^T\|\nabla v^\nu_\epsilon\|^2_{\frac{2}{1-\alpha}}\,\mathrm{d}t \le C\nu\epsilon^{2(\beta-1)}\int_0^T\|v^\nu\|^2_{B^\beta_{\frac{2}{1-\alpha},\infty}}\,\mathrm{d}t\\ &\le C\nu\epsilon^{2(\beta-1)}\|v^\nu\|^2_{L^{\frac{1}{\alpha}}(0,T;B^\beta_{\frac{2}{1-\alpha},\infty})},\end{aligned}$$

where $C>0$ does not depend on ϵ or ν and we need to choose ν going to zero faster than $\epsilon^{2(1-\beta)}$. This is the extension of the result in [4] for the Hölder case. In the case $\alpha>1/2$ we prove here a slightly better result, since

$$\begin{aligned}\nu\int_0^T\|\nabla v^\nu_\epsilon\|_2^2\,\mathrm{d}t &= \nu\int_0^T\|\nabla v^\nu_\epsilon\|_2^{2-\frac{1}{\beta}}\|\nabla v^\nu_\epsilon\|_2^{\frac{1}{\beta}}\,\mathrm{d}t \le \nu\int_0^T\left(\frac{1}{\epsilon}\|v_\epsilon\|_2\right)^{2-\frac{1}{\beta}}\left(\epsilon^{\beta-1}\|v_\epsilon\|_{B^\beta_{\frac{2}{1-\alpha},\infty}}\right)^{\frac{1}{\beta}}\,\mathrm{d}t\\ &\le C\nu\epsilon^{-1}\|v^\nu\|^2_{L^{\frac{1}{\alpha}}(0,T;B^\beta_{\frac{2}{1-\alpha},\infty})},\end{aligned}$$

with $C>0$ independent of ϵ and ν. One needs to choose ν going to zero faster than ϵ.

So, in the case $\frac{1}{3}<\beta\le\frac{1}{2}$ we have $\nu\int_0^T\|\nabla v^\nu_\epsilon\|_2^2\,\mathrm{d}t = O(\nu\epsilon^{2(\beta-1)})$, and thus

$$\frac{1}{2}\|v^\nu(T)\|_2^2-\frac{1}{2}\|u^\nu_0\|_2^2 = O(\epsilon^{\frac{\beta-\alpha}{\alpha}})+O(\nu\epsilon^{2(\beta-1)});$$

on the other hand in the case $\frac{1}{2}<\beta<1$ we have $\nu\int_0^T\|\nabla v^\nu_\epsilon\|_2^2\,\mathrm{d}t = O(\nu\epsilon^{-1})$, and thus

$$\frac{1}{2}\|v^\nu(T)\|_2^2-\frac{1}{2}\|u^\nu_0\|_2^2 = O(\epsilon^{\frac{\beta-\alpha}{\alpha}})+O(\nu\epsilon^{-1}).$$

Since $\epsilon>0$ is arbitrary, we can optimize the upper bound, the same way it was performed in [10], by balancing the contribution of the nonlinear flux with the one of the dissipation. Choosing $\epsilon\sim\nu^{\alpha/(\alpha+\beta-2\alpha\beta)}$ in the first case and $\epsilon\sim\nu^{\alpha/\beta}$ in the second one, yields the upper bounds

$$\frac{1}{2}\|v^\nu_\epsilon(T)\|_2^2-\frac{1}{2}\|u^\nu_0\|_2^2 = O(\nu^{\frac{\beta-\alpha}{\alpha-2\alpha\beta+\beta}}),$$

and

$$\frac{1}{2}\|v^\nu_\epsilon(T)\|_2^2-\frac{1}{2}\|u^\nu_0\|_2^2 = O(\nu^{\frac{\beta-\alpha}{\beta}}),$$

respectively, hence showing that as $\epsilon, \nu \to 0$ with the above rates, the kinetic energy is conserved. □

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