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A note on the fractional Hardy inequality

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Abstract
We give a direct proof of fractional Hardy inequality by means of Littlewood–Paley decomposition and properties of singular homogeneous kernels of degree -d. A refinement when $q > 2$ is proved.

Keywords Hardy inequality · Littlewood–Paley decomposition · Fractional Sobolev spaces

Mathematics Subject Classification 46E35 · 39B62

The classical Hardy inequality states that when $d \geq 3$

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla u|^2 dx \tag{0.1}$$

and it is clearly of fundamental importance in analysis. There are of course many different proofs of (0.1), the simplest one consists in restrict by density to $D(\mathbb{R}^d \setminus \{0\})$, to observe that $\frac{1}{|x|^2} = -\frac{1}{2} x \cdot \nabla(\frac{1}{|x|^2})$, then to integrate by parts and eventually to apply Cauchy–Schwarz inequality.

A natural extension of (0.1) is in the framework of fractional Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$. In this setting the following Hardy-type inequality holds

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2s}} dx \leq C||f||^2_{\dot{H}^s(\mathbb{R}^d)}, \tag{0.2}$$

provided that $0 \leq s < \frac{d}{2}$. For a compact and nice proof of (0.2) we quote Theorem 2.57 in [1] and the proof given by Tao in the Appendix of [16] while for an improvement involving Besov spaces we quote [2].

If one is interested in proving an L^q estimate for $\frac{|f|}{|x|^s}$ we need to recall the definition of the homogeneous Sobolev norm $||f||_{\dot{W}^{s,q}(\mathbb{R}^d)}$ which is defined as $|||D|^s f||_{L^q(\mathbb{R}^d)}$ where

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$(\widehat{|D|^s f})(\xi) = |2\pi\xi|^s\widehat{u}(\xi)$. In this note we give a direct proof and a refinement when $q > 2$ for the following class of Hardy-type inequalities that generalize the fractional Hardy inequality (0.2).

Theorem 0.1 (Fractional Hardy inequality) *Let* $0 < s < \frac{d}{q}$, $1 < q < \infty$ *and* $f \in \dot{W}^{s,q}(\mathbb{R}^d)$, *then*

$$\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} \leq C(d,s,q)||f||_{\dot{W}^{s,q}(\mathbb{R}^d)}. \tag{0.3}$$

The explicit value of the constant $C(d,s,q)$ in (0.3) is due to Herbst [11]. The proof of (0.3) goes back to the end of the fifties of the last century thanks to the work of Stein and Weiss [15] who proved an even more general version of (0.3) called Stein–Weiss inequality given by

$$\left(\int_{\mathbb{R}^d} \left(|T_\lambda f(x)||x|^{-\beta}\right)^q dx\right)^{\frac{1}{q}} \leq C(d,q,p,\lambda)\left(\int_{\mathbb{R}^d} \left(|f(x)||x|^{\alpha}\right)^p dx\right)^{\frac{1}{p}} \tag{0.4}$$

where

$$T_\lambda f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^\lambda} dy \quad 0 < \lambda < d,$$

and

$$0 < \lambda < d, 1 < p < \infty, \alpha < \frac{d}{p'}, p \leq q < \infty,\ \beta < \frac{d}{q},\ \alpha + \beta \geq 0,$$
$$\frac{1}{q} = \frac{1}{p} + \left(\frac{\lambda + \alpha + \beta}{d}\right) - 1.$$

The fact that (0.4) implies (0.3) follows by the fact that $T_\lambda f = c|D|^{-s}f$, with $\lambda = d - s$, $c = \frac{\pi^{d/2}\Gamma((d-\lambda)/2)}{\Gamma(\lambda/2)}$ and choosing $p = q$ and $\alpha = 0$, $\beta = s$.

In order to state our result we recall the standard definition for Homogeneous Besov norm $||\cdot||_{\dot{B}^s_{p,q}}$ and Tribel–Lizorkin norm $||\cdot||_{\dot{F}^s_{p,q}}$ (see e.g. [8] for general references). Let f be a tempered distribution such that $\hat{f} \in L^1_{loc}$ and $P_N(f)$ the Littlewood–Paley projector on the dyadic frequency N, i.e. $\widehat{P_N(f)}(\xi) = \psi_N(\xi)\hat{f}(\xi)$ where $\psi_N(\xi) = \psi(\frac{\xi}{N})$ and $\sum_{N\in 2^{\mathbb{Z}}} \psi_N = 1$, then we define

$$||f||_{\dot{B}^s_{p,q}} = \left(\sum_{N\in 2^{\mathbb{Z}}} ||N^s P_N(f)||^q_{L^p}\right)^{\frac{1}{q}},$$

$$||f||_{\dot{F}^s_{p,q}} = \left\| \left(\sum_{N\in 2^{\mathbb{Z}}} |N^s P_N(f)(x)|^q\right)^{\frac{1}{q}} \right\|_{L^p}.$$

Our result is a direct proof of the following

Theorem 0.2 *Let* $0 < s < \frac{d}{q}$, $1 < q < \infty$ *then*

$$\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} \leq C(d,s,q)||f||_{\dot{B}^s_{q,q}(\mathbb{R}^d)}, \tag{0.5}$$

with the following corollary

Corollary 0.1 *Let* $0 < s < \frac{d}{q}$, *if* $1 < q \leq 2$ *then*

$$\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} \leq C(d,s,q)||f||_{\dot{W}^{s,q}(\mathbb{R}^d)}, \tag{0.6}$$

if $q > 2$

$$\left\| \frac{f}{|x|^s} \right\|_{L^q(\mathbb{R}^d)} \leq C(d,s,q)||f||^{\frac{1}{q}}_{\dot{W}^{s,q}(\mathbb{R}^d)}||f||^{\frac{q-1}{q}}_{\dot{F}^s_{q,2(q-1)}(\mathbb{R}^d)}. \tag{0.7}$$

The fact that $||\frac{f}{|x|^s}||_{L^q(\mathbb{R}^d)}$ can be controlled by homogeneous Besov norms is not a novely, a proof of Theorem 0.2 can be found in [18], see also [19]. Here we present a direct proof using the Schur test. We shall remark that our corollary when $q > 2$ is a refinement of Hardy inequality (0.3). Indeed we have when $2(q-1) > 2$

$$||f||^{\frac{q-1}{q}}_{\dot{F}^s_{q,2(q-1)}(\mathbb{R}^d)} \leq ||f||^{\frac{q-1}{q}}_{\dot{F}^s_{q,2}(\mathbb{R}^d)} \sim ||f||^{\frac{q-1}{q}}_{\dot{W}^{s,q}(\mathbb{R}^d)}$$

thanks to square function estimate

$$||f||_{\dot{F}^s_{q,2}} = \left\| \left(\sum_{N\in 2^{\mathbb{Z}}} |N^s P_N(f)(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \sim |||D|^s f||_{L^q(\mathbb{R}^d)}.$$

The case $1 < q < 2$ is proved by duality and it requires proving the L^q continuity for singular homogeneous kernels of degree-d. This fact is well known and is Lemma 2.1 in [15]. We underline however that our strategy in proving Theorem 0.2 permits to skip the more delicate lemmas in the Stein and Weiss paper [15] that are needed to prove (0.3).

As a final comment, recalling that $|D|f = \sum_{j=1}^d R_j(\partial_{x_j} f)$ with R_j the Riesz transform defined as $\widehat{(R_j f)}(\xi) = -i\frac{\xi_j}{|\xi|}\widehat{u}(\xi)$ and that hence $|||D|f||_{L^q(\mathbb{R}^d)} \lesssim ||\nabla f||_{L^q(\mathbb{R}^d)}$ when $1 < q < \infty$, we get

Corollary 0.2 *Let* $2 < q < d$ *then*

$$\left\| \frac{f}{|x|} \right\|_{L^q(\mathbb{R}^d)} \leq C(d,s,q)||\nabla f||^{\frac{1}{q}}_{L^q(\mathbb{R}^d)}||f||^{\frac{q-1}{q}}_{\dot{F}^s_{q,2(q-1)}(\mathbb{R}^d)}. \tag{0.8}$$

We underline that Corollary 0.2 is a refinement of the classical Hardy inequality involving ∇f

$$\left\| \frac{f}{|x|} \right\|_{L^q(\mathbb{R}^d)} \leq \left(\frac{q}{d-q} \right) ||\nabla f||_{L^q(\mathbb{R}^d)}. \tag{0.9}$$

by the fact that $||f||_{\dot{F}^s_{q,2(q-1)}(\mathbb{R}^d)} \leq ||f||_{\dot{F}^s_{q,2}(\mathbb{R}^d)} \lesssim ||\nabla f||_{L^q(\mathbb{R}^d)}$. In the literature there is a lot of interest in proving improvements for (0.9), typically such improvement (in bounded or unbounded domains) are in the direction to add a negative term in r.h.s of (0.9), see e.g. [3–7, 9, 10, 13]. Our refinement, although obtained with different techniques, is more in the spirit of [2, 17], i.e. to control r.h.s. of (0.9) with terms that are smaller (up to a multiplicative constant) than the Sobolev norms.

1 Proof of Theorem 0.2

A key argument in our proof is given by the following well known version of Schur test

Proposition 1.1 *Let* $\alpha_{N,R} \geq 0$, *with* $N, R \in 2^{\mathbb{Z}}$, $1 < q < \infty$, *then*

$$\sum_R \left(\sum_N \alpha_{N,R} C_N \right)^q \lesssim \sum_N (C_N)^q$$

provided there exists a sequence of positive numbers p_N *such that*

$$\left(\sum_N \alpha_{N,R} p_N^{\frac{q'}{q}} \right)^{\frac{q}{q'}} \lesssim p_R \tag{1.1}$$

$$\sum_R \alpha_{N,R} p_R \lesssim p_N. \tag{1.2}$$

Proof By Holder's inequality with conjugated exponent (q, q')

$$\sum_N \alpha_{N,R} C_N = \sum_N \alpha_{N,R}^{\frac{1}{q}} \alpha_{N,R}^{\frac{1}{q'}} p_N^{\frac{1}{q}} \frac{C_N}{p_N^{\frac{1}{q}}} \leq \left(\sum_N \alpha_{N,R} p_N^{\frac{q'}{q}} \right)^{\frac{1}{q'}} \left(\sum_N \alpha_{N,R} \frac{C_N^q}{p_N} \right)^{\frac{1}{q}}$$

we get

$$\sum_R \left(\sum_N \alpha_{N,R} C_N \right)^q \leq \sum_R \left(\sum_N \alpha_{N,R} p_N^{\frac{q'}{q}} \right)^{\frac{q}{q'}} \left(\sum_N \alpha_{N,R} \frac{C_N^q}{p_N} \right)$$

that, thanks to (1.1) and Fubini, implies

$$\sum_R \left(\sum_N \alpha_{N,R} C_N \right)^q \lesssim \sum_R p_R \left(\sum_N \alpha_{N,R} \frac{C_N^q}{p_N} \right) = \sum_N \frac{C_N^q}{p_N} \left(\sum_R \alpha_{N,R} p_R \right).$$

Now by (1.2) we conclude

$$\sum_R \left(\sum_N \alpha_{N,R} C_N \right)^q \lesssim \sum_N \frac{C_N^q}{p_N} p_N = \sum_N C_N^q.$$

□

The strategy of the proof for is an adaptation of proof of Hardy inequality in the case $q = 2$ given by Tao [16], i.e. to prove the following estimate

$$\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^{sq}} dx \lesssim \sum_N N^{qs} ||P_N f||_{L^q(\mathbb{R}^d)}^q \tag{1.3}$$

where $P_N f$ are the classical Littlewood–Paley projectors with N a dyadic number.

We divide $\mathbb{R}^d$ in dyadic shells obtaining

$$\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^{qs}} dx = \sum_{R \in 2^{\mathbb{Z}}} \int_{\frac{R}{2} \leq |x| \leq R} \frac{|f(x)|^q}{|x|^{qs}} dx \lesssim \sum_{R \in 2^{\mathbb{Z}}} \frac{1}{R^{sq}} \int_{\{\frac{R}{2} \leq |x| \leq R\}} |f|^q \, dx. \tag{1.4}$$

such that using the Littlewood-Paley decomposition we get

$$\sum_{R\in 2^{\mathbb{Z}}} \frac{1}{R^{sq}} \int_{\{\frac{R}{2}\le |x|\le R\}} |f|^q \, dx \le \sum_{R\in 2^{\mathbb{Z}}} R^{-sq} \left(\sum_{N\in 2^{\mathbb{Z}}} \left(\int_{\{\frac{R}{2}\le |x|\le R\}} |P_N(f)|^q \right)^{\frac{1}{q}} \right)^q . \tag{1.5}$$

By the Bernstein inequality $||P_N(f)||_{L^\infty(\mathbb{R}^d)} \le N^{\frac{d}{q}} ||P_N(f)||_{L^q(\mathbb{R}^d)}$ it follows that

$$\begin{aligned} \left(\int_{\frac{R}{2}<|x|<R} |P_N(f)|^q \right)^{\frac{1}{q}} &\le R^{\frac{d}{q}} \|P_N(f)\|_{L^\infty} \\ &\le (NR)^{\frac{d}{q}} \|P_N(f)\|_{L^q}, \end{aligned} \tag{1.6}$$

and clearly

$$\left(\int_{\frac{R}{2}<|x|<R} |P_N(f)|^q \right)^{\frac{1}{q}} \le \|P_N f\|_{L^q},$$

such that we get

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^{qs}} dx &\lesssim \sum_R R^{-qs} \left(\sum_N \min\{1, (NR)^{\frac{d}{q}}\} \|P_N f\|_{L^q} \right)^q \\ &= \sum_R \left(\sum_N \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \| N^s \, P_N f\|_{L^q} \right)^q . \end{aligned}$$

The last step is to apply the Schur test given by Proposition 1.1 in order to conclude that

$$\begin{aligned} &\sum_R \left(\sum_N \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \| N^s \, P_N f\|_{L^q} \right)^q \le \sum_{N\in 2^{\mathbb{Z}}} N^{sq} \, \|P_N(f)\|_{L^q}^q \\ &= \sum_{N\in 2^{\mathbb{Z}}} N^{sq} \int_{\mathbb{R}^d} |P_N(f)|^q = \int_{\mathbb{R}^d} \sum_{N\in 2^{\mathbb{Z}}} N^{sq} |P_N(f)|^q . \end{aligned}$$

Notice that

$$\begin{aligned} &\sum_{N>\frac{1}{R}} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} + \sum_{N\le\frac{1}{R}} \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \\ &= R^{-s} \sum_{N>\frac{1}{R}} N^{-s} + R^{\frac{d}{q}-s} \sum_{N\le\frac{1}{R}} N^{\frac{d}{q}-s} \lesssim 1 \end{aligned}$$

such that (arguing in the same way when summing over R)

$$\sum_N \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \lesssim 1 \tag{1.7}$$

$$\sum_R \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\} \lesssim 1. \tag{1.8}$$

The hypoteses for Schur test given by Proposition 1.1 are hence fulfilled by choosing $\alpha_{N,R} = \min\{(NR)^{-s}, (NR)^{\frac{d}{q}-s}\}$ and $p_N = 1$ in Proposition 1.1. This proves (0.3).

2 Proof of Corollary 0.1

In Theorem 0.2 we proved the following estimate

$$\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^{sq}} dx \lesssim \sum_N N^{qs} ||P_N f||^q_{L^q(\mathbb{R}^d)} \tag{2.1}$$

where $P_N f$ are the classical Littlewood–Paley projectors with N a dyadic number. First we prove that (2.1) implies the Fractional Hardy inequality. We have two cases: $q \geq 2$, $q < 2$.

Case $q \geq 2$:

Thanks to (2.1) we derive

$$\sum_N N^{qs} ||P_N f||^q_{L^q(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sum_N N^{sq} |P_N f(x)|^q dx \leq \int_{\mathbb{R}^d} \left(\sum |N^s P_N f(x)|^2 \right)^{\frac{q}{2}} dx$$

from the elementary inequality $\left(\sum_i a_i^{p_1}\right)^{\frac{1}{p_1}} \leq \left(\sum_i a_i^{p_2}\right)^{\frac{1}{p_2}}$ with $p_1 \geq p_2$, obtaining

$$\int_{\mathbb{R}^d} \frac{|f(x)|^q}{|x|^{sq}} dx \lesssim \sum_N N^{qs} ||P_N f||^q_{L^q(\mathbb{R}^d)}$$
$$\leq \int_{\mathbb{R}^d} \left(\sum_N |N^s P_N f(x)|^2 \right)^{\frac{q}{2}} dx \sim |||D|^s f||^q_{L^q(\mathbb{R}^d)}$$

where the last equivalence is nothing but the classical square function estimate, see for instance [14].

To prove (0.7) we notice that

$$\int_{\mathbb{R}^d} \sum_N N^{sq} |P_N f(x)|^q dx$$
$$\leq \int_{\mathbb{R}^d} \left(\sum_N N^{2s} |P_N f(x)|^2 \right)^{\frac{1}{2}} \left(\sum_N N^{2s(q-1)} |P_N f(x)|^{2(q-1)} \right)^{\frac{1}{2}} dx$$
$$\leq \left(\int_{\mathbb{R}^d} \left(\sum_N N^{2s} |P_N f(x)|^2 \right)^{\frac{q}{2}} dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^d} \left(\sum_N N^{2s(q-1)} |P_N f(x)|^{2(q-1)} \right)^{\frac{q}{2(q-1)}} dx \right)^{\frac{q-1}{q}}$$

by applying twice the Holder's inequality, first in the serie with conjugated exponent (2, 2) and then in the integral with conjugated exponent $(q, \frac{q}{q-1})$. By definition

$$\left(\int_{\mathbb{R}^d} \left(\sum_N N^{2s(q-1)} |P_N f(x)|^{2(q-1)} \right)^{\frac{q}{2(q-1)}} dx \right)^{\frac{q-1}{q}} = ||f||^{q-1}_{\dot{F}^s_{q,2(q-1)}}.$$

Case $q < 2$:

For the case $q < 2$ we use the dual characterization of L^q norms, i.e.

$$\begin{aligned}\left\|\frac{f}{|x|^s}\right\|_{L^q} &= \sup_{\|g\|_{q'}=1}\left\langle\frac{f(x)}{|x|^s},\, g\right\rangle = \sup_{\|g\|_{q'}=1}\left\langle f(x),\, \frac{g(x)}{|x|^s}\right\rangle \\ &= \sup_{\|g\|_{q'}=1}\left\langle |D|^{-s}(|D|^s f(x)),\, \frac{g(x)}{|x|^s}\right\rangle = \sup_{\|g\|_{q'}=1}\left\langle |D|^s f,\, |D|^{-s}\left(\frac{g(x)}{|x|^s}\right)\right\rangle \\ &\leq \left\||D|^s f\|_{L^q}\, \||D|^{-s}\left(\frac{g(x)}{|x|^s}\right)\right\|_{L^{q'}}.\end{aligned}$$

Now we aim to prove that

$$\left\||D|^{-s}\left(\frac{g(x)}{|x|^s}\right)\right\|_{L^{q'}(\mathbb{R}^d)} \lesssim \|g\|_{L^{q'}(\mathbb{R}^d)}, \tag{2.2}$$

for all $g \in L^{q'}$ with $q' > 2$ such that we could conclude that

$$\left\|\frac{f}{|x|^s}\right\|_{L^q(\mathbb{R}^d)} = \sup_{\|g\|_{q'}=1}\left\langle\frac{|f(x)|}{|x|^s},\, g\right\rangle \lesssim \||D|^s f\|_{L^q(\mathbb{R}^d)}.$$

Now we prove (2.2). We recall that $|D|^{-s} f \sim \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}}\, dy$, see Theorem 5.9 in [12], such that we have (renaming q' by q to simplify the notation)

$$\begin{aligned}|D|^{-s}\left(\frac{g(x)}{|x|^s}\right)|^q &\sim \left|\int_{\mathbb{R}^d}\frac{g(y)}{|x-y|^{d-s}\,|y|^s}\,dy\right|^q \leq \left|\int_{\mathbb{R}^d}\frac{|g(y)|}{|y|^s\,|x-y|^{d-s}}\,dy\right|^q \\ &\lesssim \left|\int_{\mathbb{R}^d}\frac{|g(y)|\,\mathbb{1}_{\{|y|>\frac{|x|}{2}\}}(y)}{|y|^s\,|x-y|^{d-s}}dy\right|^q + \left|\int_{\mathbb{R}^d}\frac{|g(y)|\,\mathbb{1}_{\{|y|\leq\frac{|x|}{2}\}}(y)}{|y|^s\,|x-y|^{d-s}}dy\right|^q \\ &\lesssim \frac{1}{|x|^{qs}}\left|\int_{\mathbb{R}^d}\frac{|g(y)|\,\mathbb{1}_{\{|y|>\frac{|x|}{2}\}}(y)}{|x-y|^{d-s}}dy\right|^q + \left|\int_{\mathbb{R}^d}\frac{|g(y)|\,\mathbb{1}_{\{|y|\leq\frac{|x|}{2}\}}(y)}{|y|^s\,|x-y|^{d-s}}dy\right|^q \\ &\lesssim \frac{1}{|x|^{qs}}\left|\int_{\mathbb{R}^d}\frac{|g(y)|}{|x-y|^{d-s}}dy\right|^q + \left|\int_{\mathbb{R}^d}\frac{|g(y)|\,\mathbb{1}_{\{|y|\leq\frac{|x|}{2}\}}(y)}{|y|^s\,|x-y|^{d-s}}dy\right|^q \\ &:= |S_1(g)|^q + |S_2(g)|^q\end{aligned}$$

By previous estimates using Paley–Littlewood decomposition and the square function equivalence we get when $q > 2$

$$\int_{\mathbb{R}^d}|S_1(g)|^q\,dx \sim \int_{\mathbb{R}^d}\left|\frac{|D|^{-s}|g(x)|}{|x|^s}\right|^q\,dx \lesssim \||D|^s(|D|^{-s}|g|)\|^q_{L^q(\mathbb{R}^d)} = \|g\|^q_{L^q(\mathbb{R}^d)}.$$

Concerning $\|S_2(g)\|_{L^q}$ we follow the strategy of Stein and Weiss [15] proving the L^q continuity for singular homogeneous kernels of degree-d. The proof of this fact is Lemma 2.1 in [15] that we show for reader convenience. First notice that $\frac{|y|}{|x|} \leq \frac{1}{2}$ implies

$$|x-y| \geq |x| - |y| \geq \frac{|x|}{2},$$

such that

$$\int_{|y|\leq\frac{|x|}{2}}\frac{|g(y)|}{|x-y|^{d-s}|y|^s}\,dy \lesssim \int_{|y|\leq\frac{|x|}{2}}\frac{|g(y)|}{|y|^s\,|x|^{d-s}}\,dy. \tag{2.3}$$

Now we introduce following [15] the function,

$$K(x,y)=\begin{cases}|y|^s\,|x|^{d-s} & |y|\le\frac{|x|}{2}\\ 0 & \text{otherwise}\end{cases}$$

and

$$Ug(x):=\int_{|y|\le|\frac{x}{2}|}\frac{|g(y)|}{|y|^s\,|x|^{d-s}}\,dy=\int_{\mathbb{R}^d}K(|x|,\ |y|)|g(y)|\,dy.$$

To conclude the proof it suffices hence to show that

$$\int_{\mathbb{R}^d}|Ug|^q\,dx\lesssim\int|g|^q\,dx.$$

Fixing $\eta\in S^{d-1}$ and calling $|x|=R$ we define

$$U_\eta g(R):=\int_0^{+\infty}r^{d-1}\,K(R,r)\cdot|g(r\,\eta)|dr,$$

such that

$$\begin{aligned}Ug(x)&=\int_{\mathbb{R}}K(|x|,|y|)|g(y)|dy=\int_0^{+\infty}\left(\int_{S^{d-1}}K(R,r)|g(r\,\eta)|\,d\sigma_\eta\right)r^{d-1}\,dr\\&=\int_{S^{d-1}}\int_0^{+\infty}K(R,r)|g(r\eta)|\,r^{d-1}\,dr\,d\sigma_\eta=\int_{S^{d-1}}U_\eta g(R)\,d\sigma_\eta.\end{aligned}$$

By the substitution $r=tR$ we obtain

$$\begin{aligned}U_\eta g(R)&=\int_0^{+\infty}K(R,Rt)\,|g(t\,R\,\eta)|\,R^{d-1}\,t^{d-1}\,R\,dt\\&=\int_0^{+\infty}K(1,t)\,|g(t\,R\,\eta)|t^{d-1}\,dt,\end{aligned}$$

thanks to the fact that K is homogeneous of degree $-d$, i.e. that

$$K(\lambda x,\ \lambda y)=|\lambda|^{-d}K(|x|,|y|).$$

Let h be the function in $L^{q'}((0,+\infty);\,R^{d-1}\,dR)$ of unitary norm such that

$$\begin{aligned}&\left(\int_0^{+\infty}|U_\eta g(R)|^q\,R^{d-1}\,dR\right)^{\frac{1}{q}}=\int_0^{+\infty}U_\eta g(R)h(R)\,R^{d-1}\,dR\\&=\int_0^{+\infty}\left\{\int_0^{+\infty}K(1,t)\,|g(t\,R\,\eta)|\,t^{d-1}\,dt\right\}R^{d-1}\,h(R)\,dR\\&=\int_0^{+\infty}K(1,t)\,t^{d-1}\left\{\int_0^{+\infty}|g(tR\eta)|\,h(R)\,R^{d-1}\,dR\right\}dt\\&\le\int_0^{+\infty}K(1,t)\,t^{d-1}\left\{\int_0^{+\infty}|g(t\,R\,\eta)|^q\,R^{d-1}\,dR\right\}^{\frac{1}{q}}dt\\&=\left(\int_0^{+\infty}K(1,t)t^{d-1-\frac{d}{q}}\,dt\right)\cdot\left\{\int_0^{+\infty}|g(R\eta)|^q\,R^{d-1}\,dR\right\}^{\frac{1}{q}}\\&=\left(\int_0^1 t^{d-\frac{d}{q}-1-s}\,dt\right)\cdot\left\{\int_0^{+\infty}|g(R\eta)|^q\,R^{d-1}\,dR\right\}^{\frac{1}{q}}=:J\cdot\left\{\int_0^{+\infty}|g(R\eta)|^q\,R^{d-1}\,dR\right\}^{\frac{1}{q}},\end{aligned}$$

where the last integral J converges due to the fact that by our assumptions $s < \frac{d}{q'}$ (remember that we skipped q' with q).

Now we estimate $L^q(\mathbb{R}^d)$ norm of Ug. By Jensen inequality

$$|Ug(R)|^q = \left| \int_{S^{d-1}} |U_\eta g(R)| \, d\sigma_\eta \right|^q \leq \{|S^{d-1}|\}^{q-1} \int_{S^{d-1}} |U_\eta g|^q \, d\sigma_\eta,$$

such that integrating with respect to the measure $R^{d-1} dR$ we get

$$\begin{aligned}
&\int_0^{+\infty} |Ug(R)|^q R^{d-1} \, dR \\
&\leq J^q \, |S^{d-1}|^{q-1} \left(\int_0^{+\infty} \left\{ \int_{S^{d-1}} |U_\eta g(R)|^q \, d\sigma_\eta \right\} R^{d-1} \, dR \right) \\
&= J^q \, |S^{d-1}|^{q-1} \int_{S^{d-1}} \int_0^{+\infty} |U_\eta g(R)|^q \, R^{d-1} \, dR \, d\sigma_\eta \\
&\leq J^q \, |S^{d-1}|^{q-1} \int_{S^{d-1}} \int_0^{+\infty} |g(R\,\eta)|^q \, R^{d-1} \, dR \, d\sigma = J^q \, |S^{d-1}|^{q-1} \int_{\mathbb{R}^d} |g(x)|^q \, dx.
\end{aligned}$$

By the fact that $Uf(x)$ is radial we can conclude that

$$\int_{\mathbb{R}^d} |Ug(x)|^q \, dx = |S^{d-1}| \cdot \int_0^{+\infty} |Ug(R)|^q \, R^{d-1} \, dR \leq J^q |S^{d-1}|^q \int |g(x)|^q \, dx.$$

This concludes the proof in the case $q < 2$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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