A NOTE ON THE PERSISTENCE OF MULTIPLICITY OF EIGENVALUES OF FRACTIONAL LAPLACIAN UNDER PERTURBATIONS

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Abstract. We consider the eigenvalue problem for the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, in a bounded domain Ω with Dirichlet boundary condition. A recent result (see *Generic properties of eigenvalues of the fractional Laplacian* by Fall, Ghimenti, Micheletti and Pistoia, CVPDE (2023)) states that, under generic small perturbations of the coefficient of the equation or of the domain Ω, all the eigenvalues are simple. In this paper we give a condition for which a perturbation of the coefficient or of the domain preserves the multiplicity of a given eigenvalue. Also, in the case of an eigenvalue of multiplicity $\nu = 2$ we prove that the set of perturbations of the coefficients which preserve the multiplicity is a smooth manifold of codimension 2 in $C^1(\mathbb{R}^n)$.

1. Introduction

In the last decade, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena. We refer the readers to Di Nezza, Palatucci and Valdinoci's survey paper [3] for a detailed exposition of the function spaces involved in the analysis of the operator, and to the recent Ros-Oton's expository paper [10] for a list of results on Dirichlet problems on bounded domains.

In this paper we will focus on the eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi_s = \lambda \varphi_s & \text{in } \Omega \\ \varphi_s = 0 & \text{in } \Omega^c = \mathbb{R}^n \smallsetminus \Omega \end{cases}, \tag{1}$$

where $(-\Delta)^s$ for $0 < s < 1$ denotes the fractional Laplacian and Ω is an open bounded domain of class C^0 in $\mathbb{R}^n$, with $n > 2s$.

In a weak sense, problem (1) can be formulated as follows. We consider the space

$$\mathcal{H}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^n) \ : \ u \equiv 0 \text{ on } \Omega^c\},$$

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where

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \ : \ \frac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\},$$

and the quadratic form defined on $\mathcal{H}_0^s(\Omega)$ by

$$(u,v) \mapsto \mathcal{E}_s^\Omega(u,v) = \mathcal{E}(u,v) := \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy$$

Then, we say that $\varphi_s \in \mathcal{H}_0^s(\Omega)$ is an eigenfunction corresponding to the eigenvalue λ_s iff

$$\mathcal{E}(\varphi_s, v) = \lambda_s \int_{\mathbb{R}^n} \varphi_s v dx \quad \forall v \in \mathcal{H}_0^s(\Omega).$$

It is well known (see, for instance, [1]) that (1) admits an ordered sequence of eigenvalues

$$0 < \lambda_{1,s} < \lambda_{2,s} \leq \lambda_{3,s} \leq \cdots \leq \lambda_{1,s} \leq \cdots \to +\infty.$$

Since the first eigenvalue is strictly positive, we can also endow $\mathcal{H}_0^s(\Omega)$ with the norm

$$\|u\|^2_{\mathcal{H}_0^s(\Omega)} = \|u\|^2_{L^2(\Omega)} + \mathcal{E}(u,u).$$

We refer to [6] and the references therein for a review of results on eigenvalues of fractional Laplacians and fractional Schrödinger operators.

In the recent paper [5], Fall, Ghimenti, Micheletti and Pistoia prove that there exist arbitrarily small perturbations of the domain, or arbitrarily small perturbations of the coefficient of the linear terms, for which all the eigenvalues of problems (1) (for perturbation of the domain), or

$$\begin{cases} (-\Delta)^s \varphi_s + a(x)\varphi_s = \lambda \varphi_s & \text{in } \Omega \\ \varphi_s = 0 & \text{in } \Omega^c = \mathbb{R}^n \setminus \Omega \end{cases} \tag{2}$$

and

$$\begin{cases} (-\Delta)^s \varphi_s = \lambda a(x) \varphi_s & \text{in } \Omega \\ \varphi_s = 0 & \text{in } \Omega^c = \mathbb{R}^n \setminus \Omega \end{cases} \tag{3}$$

(for perturbation of the coefficients) are simple.

In this paper we want to study the structure of the set of perturbations of the coefficients or of the domain which *preserve* the multiplicity of the eigenvalues[1].

Our first result deals with the perturbation of the coefficients.

Theorem 1.1. *Let λ_0 be an eigenvalue for Problem (2) (respectively Problem (3)) with multiplicity $\nu > 1$, and let $\varphi_1, \ldots, \varphi_\nu$ be an L^2-orthonormal basis for the eigenspace relative to λ_0. Assume that $a \in C^1(\mathbb{R}^n)$ and $\min_{\bar{\Omega}} a > 0$, or $\|a\|_{C^1(\mathbb{R}^n)}$ small (resp. assume that $a \in C^1(\mathbb{R}^n)$ and $\min_{\bar{\Omega}} a > 0$). Let $b \in C^1(\mathbb{R}^n)$ and consider the functionals*

$$b \mapsto \gamma_{ij}(b) := \int_\Omega b \varphi_i \varphi_j, \quad i,j = 1, \ldots, \nu \tag{4}$$

[1]This question was raised by the anonymous referee of our previous paper. We wish to thank them for their interesting suggestion

Then the set $\mathscr{I}$ of the b's close to 0 in $C^1(\mathbb{R}^n)$ such that the perturbed problem

$$(-\Delta)^s\varphi + (a(x) + b(x))\,\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \Omega^c$$

(respectively $(-\Delta)^s\varphi + \varphi = \lambda\,(a(x)+b(x))\,\varphi$ in $\Omega, \varphi = 0$ in Ω^c) admits an eigenvalue λ_b close to λ_0 of the same multiplicity ν is a subset of

$$\mathscr{H} := \left\{ b \in C^1(\mathbb{R}^n) \;:\; \gamma_{ij}(b) = 0 \text{ for } i \neq j,\; \gamma_{11}(b) = \gamma_{22}(b) = \cdots = \gamma_{\nu\nu}(b) \right\}.$$

In addition, if the map

$$\begin{aligned} G :& C^1(\mathbb{R}^n) \to L(\mathbb{R}^\nu, \mathbb{R}^\nu) \\ G(b) &= (\gamma_{ij}(b))_{ij} \end{aligned}$$

is such that the span of $G(b)$ and the Identity map gives all the $\nu\times\nu$ symmetric matrices, then the set $\mathscr{I}$ is a manifold in $C^1(\mathbb{R}^n)$ of codimension $\frac{\nu(\nu+1)}{2} - 1$.
In particular the last claim holds if λ_0 is an eigenvalue of multiplicity $\nu = 2$.

Our second result deals with the perturbation of the domain.

Theorem 1.2. *Assume Ω be an open bounded domain of class $C^{1,1}$. Let λ_0 be an eigenvalue for Problem (1) with multiplicity $\nu > 1$, and let $\varphi_1, \ldots, \varphi_\nu$ be an L^2-orthonormal basis for the eigenspace relative to λ_0. Let $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and consider the functionals*

$$\psi \mapsto \gamma_{ij}(\psi) := \int_{\partial\Omega} \frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\psi\cdot N, \quad i,j = 1,\ldots,\nu \tag{5}$$

where $\delta(x) = \mathrm{dist}(x, \mathbb{R}^n \smallsetminus \Omega)$ and N is the exterior normal of $\partial\Omega$

Then the set $\mathscr{I}$ of the ψ's close to 0 in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ such that the problem

$$(-\Delta)^s\varphi + \varphi = \lambda\varphi \text{ in } \Omega_\psi, \qquad \varphi = 0 \text{ in } \Omega_\psi^c$$

in the perturbed domain $\Omega_\psi = (I+\psi)\Omega$ admits an eigenvalue λ_ψ close to λ_0 of the same multiplicity ν is a subset of

$$\mathscr{H} := \left\{ \psi \in C^1(\mathbb{R}^n, \mathbb{R}^n) \;:\; \gamma_{ij}(\psi) = 0 \text{ for } i \neq j,\; \gamma_{11}(\psi) = \gamma_{22}(\psi) = \cdots = \gamma_{\nu\nu}(\psi) \right\}.$$

In addition, if the map

$$\begin{aligned} G :& C^1(\mathbb{R}^n, \mathbb{R}^n) \to L(\mathbb{R}^\nu, \mathbb{R}^\nu) \\ G(\psi) &= (\gamma_{ij}(\psi))_{ij} \end{aligned}$$

is such that the span of $G(\psi)$ and the Identity map gives all the $\nu\times\nu$ symmetric matrices, then the set $\mathscr{I}$ is a manifold in $C^1(\mathbb{R}^\nu, \mathbb{R}^\nu)$ of codimension $\frac{\nu(\nu+1)}{2} - 1$.

The proof of our results follows the strategy developed by Micheletti and Lupo in [8, 9], where an abstract transversality result is applied to a second order elliptic operator under the effect of the perturbations of the domain. The application of the abstract theorem in the case of multiplicity $\nu = 2$, which gives a concrete example of $\mathscr{I}$ being a manifold, relies on the unique continuation property. For nonlocal problem this property has been proved only in particular setting, and it is a challenging field of research. That is why we can prove that $\mathscr{I}$ is a manifold only for problems (2) and (3). Actually in Remark 6.1 we will point out what it would be necessary to complete the proof in the case of the perturbation of the domain for problem (1).

The paper is organized as follows. Firstly we recall the abstract transversality theorem. Then we prove the result for Problem (2) and we sketch the proof for Problem (3), concluding the proof of Thm 1.1. In the last section we prove Thm 1.2.

2. The abstract transversality result

We recall here an abstract result which holds in a Hilbert space X endowed with a scalar product $< \cdot,\cdot >_X$ for a selfadjoint compact operator $T_b : X \to X$ depending smoothly on a parameter b which is defined in some Banach space B. If T_0 admits an eigenvalue $\bar\lambda$ with multiplicity $\nu > 1$, we provide a characterization for the set $\mathscr{I}$ of parameter b for which T_b has an eigenvalue λ_b, with $\lambda_0 = \bar\lambda$, which maintains the same multiplicity ν. In addition the result gives a sufficient condition which ensure that $\mathscr{I}$ is a smooth sub-manifold of B.

Theorem 2.1. *Let $T_b : X \to X$ be a selfadjoint compact operator which depends smoothly on a parameter b belonging to a real Banach space B. Let T_b be Frechet differentiable with respect to b, in $b = 0$. Let λ_b an eigenvalue such that $\lambda_0 = \bar\lambda$ and let $x_1^0, \dots, x_\nu^0$ be an orthonormal basis for the eigenspace relative to $\bar\lambda$. Consider the functionals*

$$b \mapsto \gamma_{ij}(b) :=< T_b'(0)[b]x_j^0, x_i^0 >_X, \quad i, j = 1, \dots, \nu \tag{6}$$

Then the set $\mathscr{I}$ of the b's close to 0 in B such that T_b admits an eigenvalue λ_b of the same multiplicity ν is a subset of

$$\mathscr{H} := \{b \in B \ : \ \gamma_{ij}(b) = 0 \text{ for } i \neq j, \ \gamma_{11}(b) = \gamma_{22}(b) = \cdots = \gamma_{\nu\nu}(b)\}.$$

In addition, if the map

$$G : B \to L(\mathbb{R}^\nu, \mathbb{R}^\nu)$$
$$G(b) = (\gamma_{ij}(b))_{ij}$$

is such that the span of $G(b)$ and the Identity map gives all the $\nu \times \nu$ symmetric matrices, then the set $\mathscr{I}$ is a manifold in B of codimension $\frac{\nu(\nu+1)}{2} - 1$.

The first part ot Theorem 2.1 was firstly proved in [7]. A sketched version of the proof can be found also in [5], since condition (6) was the main tool to prove that the eigenvalues for fractional laplacian are generically simple under perturbation of the domain or of the coefficients. The proof of the second part can be found in [9, ,Th. 1].

3. The case of Problem (2)

We consider on $\mathcal{H}_0^s(\Omega)$ the quadratic form

$$\mathcal{B}^a(u, v) = \mathcal{E}(u, v) + \int_{\mathbb{R}^n} au^2 dx.$$

Since $\min_{\bar\Omega} a > 0$ or $\|a\|_{C^1(\mathbb{R}^n)}$ is small, the first eigenvalue λ_1^a is positive, $\mathcal{B}^a(u, v)$ is a positive definite scalar product, and we can consider on $\mathcal{H}_0^s(\Omega)$ the equivalent norm

$$\|u\|^2_{\mathcal{H}_0^s(\Omega)} = \mathcal{B}^a(u, u) = \mathcal{E}(u, u) + \int_{\mathbb{R}^n} au^2 dx. \tag{7}$$

Given the continuous and compact embedding $i : \mathcal{H}_0^s(\Omega) \to L^2(\Omega)$ we can consider its adjoint operator with respect to the scalar product $\mathcal{B}^a$,

$$i^* : L^2(\Omega) \to \mathcal{H}_0^s(\Omega).$$

The composition $(i^* \circ i)_a : \mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^s(\Omega)$ is selfadjoint, compact, injective with dense image in $\mathcal{H}_0^s(\Omega)$ and it holds

$$\mathcal{B}^a\left((i^* \circ i)_a u, v\right) = \mathcal{E}\left((i^* \circ i)_a u, v\right) + \int_\Omega a u (i^* \circ i)_a v = \int_\Omega uv. \tag{8}$$

We call $\varphi^a \in \mathcal{H}_0^s(\Omega)$ an eigenfunction of $((-\Delta)^s + a)$ corresponding to the eigenvalue λ^a if

$$\mathcal{E}(\varphi^a, v) + \int_{\mathbb{R}^n} a\varphi^a v dx = \lambda^a \int_{\mathbb{R}^n} \varphi v dx \quad \forall v \in \mathcal{H}_0^s(\Omega).$$

Notice that if $\varphi_k^a \in \mathcal{H}_0^s(\Omega)$ is an eigenfunction of the fractional Laplacian with eigenvalue λ_k^a, then φ_k^a is an eigenfunction of $(i^* \circ i)_a$ with eigenvalue $\mu_k^a := 1/\lambda_k^a$. In fact, it holds, for all $v \in \mathcal{H}_0^s(\Omega)$

$$\mathcal{B}^a(\varphi_k^a, v) = \lambda_k^a \int_{\mathbb{R}^n} \varphi_k^a v dx = \int_{\mathbb{R}^n} \lambda_k^a \varphi_k^a v dx = \mathcal{B}^a\left(\lambda_k^a (i^* \circ i)_a \varphi_k^a, v\right),$$

thus $(i^* \circ i)_a \varphi_k^a = 1/\lambda_k^a \varphi_k^a$. For eigenvalues μ_k^a relative to the operator $(i^* \circ i)_a$, there are two equivalent min-max characterizations: we have

$$\mu_1^a := \sup_{u \in \mathcal{H}_\Omega^s \setminus \{0\}} \frac{\int_\Omega u^2 dx}{\mathcal{B}^a(u,u)}; \qquad \mu_\nu^a := \sup_{\substack{u \in \mathcal{H}_\Omega^s \setminus \{0\} \\ \mathcal{B}^a(u, e_t) = 0 \\ t = 1, \dots \nu - 1}} \frac{\int_\Omega u^2 dx}{\mathcal{B}^a(u,u)};$$

where $(i^* \circ i)_a e_t = \mu_t^a e_t$; equivalently,

$$\mu_\nu^a := \inf_{V = \{v_1, \dots, v_{\nu-1}\}} \sup_{\substack{u \in \mathcal{H}_\Omega^s \setminus \{0\} \\ \mathcal{B}^a(u, v_t) = 0 \\ t = 1, \dots \nu - 1}} \frac{\int_\Omega u^2 dx}{\mathcal{B}^a(u,u)}.$$

By the min-max characterization of the eigenvalues of $(i^* \circ i)_a$, one can prove that, for all k, μ_k^a depends continuously on a.

Recall, finally, that $((-\Delta)^s + a)$ admits an ordered sequence of eigenvalues

$$0 < \lambda_1^a < \lambda_2^a \le \lambda_3^a \le \dots \le \lambda_k^a \le \dots \to +\infty.$$

and the eigenvalues λ_k^a depend continuously on a (since μ_k^a are continuous).

For $b \in C^1(\mathbb{R}^n)$ with $\|b\|_{C^1}$ small enough consider $\mathcal{B}^{a+b}$ and $(i^* \circ i)_{a+b}$ and set

$$B_b := \mathcal{B}^{a+b} \text{ and } E_b := (i^* \circ i)_{a+b}. \tag{9}$$

We want to apply the abstract Theorem 2.1 to the operator E_b to check when a perturbation b preserves the multiplicity of an eigenvalue μ_k^a. Thus, in light of previous consideration, we get the persistence result for the operator $((-\Delta)^s + a)$. Since we endowed $\mathcal{H}_0^s(\Omega)$ with the scalar product $\mathcal{B}^a = B_0$, to check condition (6) we need to compute $B_0\left(E'(0)[b]u, v\right)$.

By the identity (8), differentiating along the coefficient $a(x)$ we get (see [5, Lemma 20]) that $B_0\left(E'(0)[b]u,v\right)+B'(0)[b]\left(E_0u,v\right)=0$, so, by (7) and by direct computation (see also [5, Remarks 21 and 22])

$$-B_0\left(E'(0)[b]u,v\right)=B'(0)[b]\left(E_0u,v\right)=\int_\Omega b(E_0u)v=\int_\Omega b\left[(i^*\circ i)_a u\right]v.$$

So, if f μ^a is an eigenvalue of the map $E_0=(i^*\circ i)_a$ with multiplicity $\nu>1$, and $\varphi_1^a,\dots,\varphi_\nu^a$ are its L^2-orthonormal eigenvectors we have

$$\left(B'(0)[b]E_0\varphi_i^a,\varphi_j^a\right)=\int_\Omega bE_0(\varphi_i^a)\varphi_j^a=-\mu^a\int_\Omega b\varphi_i^a\varphi_j^a,$$

for all $i,j=1,\dots,\nu$. In this case, considering (6), we have to deal with

$$b\mapsto\gamma_{ij}(b)=\gamma_{ij}:=\int_\Omega b(x)\varphi_i^a\varphi_j^a d\sigma.$$

4. The case of Problem (3)

As in the previous section, we want to see how equation (6) translates in the setting of Problem (2). Since we assumed $a>0$ on $\bar\Omega$, we endow the space $L^2(\Omega)$ with scalar product and norm given, respectively, by

$$\langle u,v\rangle_{L^2}=\int_\Omega auv;\qquad \|u\|_{L^2}^2=\int_\Omega au^2,$$

while on $\mathcal{H}_0^s$ we consider the usual scalar product $\mathcal{E}(u,v)$. Again we consider the embedding $i:\mathcal{H}_0^s\to L^2$ and its adjoint operator $i^*:L^2\to\mathcal{H}_0^s$. Then we have

$$\mathcal{E}((i^*\circ i)_a v,u)=\int_\Omega auv\quad\forall u,v\in\mathcal{H}_0^s.$$

As before, the map $(i^*\circ i)_a$ is selfadjoint, continuous and compact from $\mathcal{H}_0^s$ in itself, and if φ^a is an eigenfunction with eigenvalue λ^q for the problem (2), then it is also an eigenfunction for $(i^*\circ i)_a$ associated to the eigenvalue $\mu^a=1/\lambda^a$.

In this case we have to compute $\mathcal{E}(E'(0)[b]u,v)$. This can be computed directly (see [5, Lemma 26] and we have

$$\mathcal{E}(E'(0)[b]u,v)=\int_\Omega buv.$$

So, also in this case, if f μ^a is an eigenvalue of the map $(i^*\circ i)_a$ with multiplicity $\nu>1$, and $\varphi_1^a,\dots,\varphi_\nu^a$ are its L^2-orthonormal eigenvectors, in the end we have to consider the same function

$$b\mapsto\gamma_{ij}(b)=\gamma_{ij}:=\int_\Omega b(x)\varphi_i^a\varphi_j^a d\sigma.$$

In the next section we will check the conditions on $b\mapsto\gamma_{ij}(b)$ to prove Thm 1.1.

5. Proof of Theorem 1.1

The first part of the Theorem is the translation of Theorem 2.1 in our setting, and in the previous two sections we showed that both for Problem (2) and for Problem (3) the operator $b \mapsto \gamma_{ij}(b) < T'(0)[b]x_j^0, x_i^0 >_X$ is $\gamma_{ij} = \int_\Omega b(x)\varphi_i^a\varphi_j^a d\sigma$, so the set $\mathscr{I}$ of the b near 0 such that an eigenvalue λ_0 maintains the same multiplicity is a subset of $\mathscr{H} = \left\{b \in C^1(\mathbb{R}^n) : \gamma_{ij}(b) = \rho \mathrm{Id} \text{ for some } \rho \neq 0\right\}$ and it is a smooth submanifold of $C^1(\mathbb{R}^n)$ if the set $\left\{\mathrm{Id}, (\gamma_{ij}(b))_{ij} \text{ for } b \in C^1(\mathbb{R}^n), \|b\|_{C^1} \text{ small}\right\}$ generates all the symmetric $\nu \times \nu$ matrices. It remains to show that this last condition is fulfilled when $\nu = 2$.

Suppose that λ_0 is an eigenvalue for Problem (2) or for Problem (3) with multiplicity $\nu = 2$. Let φ_1 and φ_2 be the two L^2-orthogonal eigenfunctions relative to λ_0. We want to show that

$$b \mapsto \left(\int_\Omega b\varphi_i\varphi_j\right)_{ij=1,2}$$

generates all the symmetric 2×2 matrices. To do so, it is sufficient to prove that

$$b \mapsto \left(\int_\Omega b\varphi_1^2, \int_\Omega b\varphi_2^2, \int_\Omega b\varphi_1\varphi_2\right)$$

generates $\mathbb{R}^3$. Let us suppose, by contradiction, that there exists a $v = (v_1, v_2, v_3) \neq 0$ which is orthogonal to all $\gamma(b)$. Thus it holds

$$0 = v_1 \int_\Omega b\varphi_1^2 + v_2 \int_\Omega b\varphi_2^2 + v_3 \int_\Omega b\varphi_1\varphi_2 = \int_\Omega b(v_1\varphi_1^2 + v_2\varphi_2^2 + v_3\varphi_1\varphi_2)$$

for all $b \in C^1$. This would imply $v_1\varphi_1^2 + v_2\varphi_2^2 + v_3\varphi_1\varphi_2 = 0$ almost everywhere on Ω. At this point showing that $\{\varphi_i\varphi_j\}_{ij=1,2}$ are independent as functions on Ω, ends the proof. We follow the strategy of Micheletti Lupo [9], using as a crucial tool the following result ([4, Teorema 1.4]).

Lemma 5.1. *Let $u \in D^{s,2}(\mathbb{R}^n)$ be a weak solution to (2) or (3) in a bounded domain Ω with $s \in (0,1)$ with $a(x)$ a C^1 function. If $u \equiv 0$ on a set $E \subset \Omega$ of positive measure, then $u \equiv 0$ in Ω.*

Let us set $\tau = \varphi_1$ and $t = \varphi_2$, to simplify notation. We know that τ and t are independent functions on Ω. So τ^2, τt, and t^2 are pairwise independent. We want to rule out that, for some $A, B \in \mathbb{R}$,

$$\tau^2 = At^2 + B\tau t. \tag{10}$$

By Lemma 5.1 an eigenvalue vanish at most on a zero measure set on Ω. If x is such that $\tau(x) \neq 0$ we can divide (10) by τ^2 and solve, obtaining

$$\frac{t}{\tau} = \begin{cases} c_1 := \frac{-B+\sqrt{B^2+4A}}{2} \\ c_2 := \frac{-B-\sqrt{B^2+4A}}{2} \end{cases}.$$

So, there exists a set $E \subset \Omega$ such that

$$t = \begin{cases} c_1\tau & \text{on } E \\ c_2\tau & \text{on } \Omega \setminus E \end{cases}.$$

At least one set between E and $\Omega \setminus E$ has positive measure. So, we can suppose that E has it. At this point we construct an eigenfunction $\varphi_2 - c_1\varphi_1$ which is zero on E. This contradicts Lemma 5.1 , so φ_1^2, φ_2^2 and $\varphi_1\varphi_2$ are independent as functions on Ω and we had completed the proof.

6. Proof of Theorem 1.2

As anticipated in the introduction, we consider a perturbed domain as $\Omega_\psi := (I+\psi)\Omega$, with $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\|\psi\|_{C^1}$ small enough to ensure that $(I+\psi)$ is invertible. We denote J_ψ as the Jacobian determinant of the mapping $I + \psi$.

By the change of variables given by the mapping $(I + \psi)$, and denoted $\tilde{u}(\xi) := u(\xi + \psi(\xi))$, we obtain the bilinear form $\mathcal{B}_s^\psi$ on $\mathcal{H}_0^s(\Omega)$ defined in the following formula:

$$\begin{aligned}\mathcal{E}_s^{\Omega_\psi}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\tilde{u}(\xi)-\tilde{u}(\eta))(\tilde{v}(\xi)-\tilde{v}(\eta))}{|\xi-\eta+\psi(\xi)-\psi(\eta)|^{n+2s}} J_\psi(\xi)J_\psi(\eta)d\xi d\eta \\ &=: \frac{1}{2}\mathcal{B}_s^\psi(\tilde{u},\tilde{v}). \qquad (11)\end{aligned}$$

Here $\tilde{u}, \tilde{v} \in \mathcal{H}_0^s(\Omega)$ and $u, v \in \mathcal{H}_0^s(\Omega_\psi)$. Notice that $\mathcal{B}_s^0(\tilde{u},\tilde{v}) = \mathcal{E}_s^\Omega(\tilde{u},\tilde{v})$.

To simplify notation, we define the map

$$\begin{aligned}\gamma_\psi &: \mathcal{H}_0^s(\Omega_\psi) \to \mathcal{H}_0^s(\Omega); \\ \gamma_\psi(u) &:= \tilde{u}(\xi) = u(\xi + \psi(\xi)).\end{aligned}$$

We recall that the map γ_ψ is invertible since $\|\psi\|_{C^1}$ is small.

As before, given a bounded domain D, we consider the embedding $i : \mathcal{H}_0^s(D) \to L^2(D)$ and its adjoint operator i^* with respect to the scalar product $\mathcal{E}_s^D$. Again, the composition $E_D := (i^* \circ i)_D : \mathcal{H}_0^s(D) \to \mathcal{H}_0^s(D)$ is a selfadjoint compact operator and

$$\mathcal{E}_s^D(E_D u, v) = \int_D uv. \qquad (12)$$

In addition, if $\varphi_k \in \mathcal{H}_0^s(D)$ is an eigenfunction of the fractional Laplacian with eigenvalue λ_k, then it is also an eigenfunction of $E_D = (i^* \circ i)_D$ with eigenvalue $\mu_k := 1/\lambda_k$.

Now, on Ω_ψ, we consider $E_\psi := E_{\Omega_\psi}$ and we recast (8) as

$$\mathcal{B}_s^\psi(\gamma_\psi E_\psi u, \tilde{v}) = \mathcal{E}_s^{\Omega_\psi}(E_\psi u, v) = \int_{\Omega_\psi} uv = \int_\Omega \tilde{u}\tilde{v}J_\psi,$$

and, set

$$T_\psi = \gamma_\psi E_\psi \gamma_\psi^{-1}\tilde{u},$$

we have, for $\tilde{u}, \tilde{v} \in \mathcal{H}_0^s(\Omega)$

$$\mathcal{B}_s^\psi(T_\psi \tilde{u}, \tilde{v}) = \int_\Omega \tilde{u}\tilde{v}J_\psi.$$

We want to apply Theorem 2.1 to the selfadjoint compact operator $T_\psi : \mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^s(\Omega)$.

One has, by direct computation, that

$$\left(\mathcal{B}_s^{\psi}\right)'(0)[\psi](T_0\tilde{u},\tilde{v})+\mathcal{B}_s^0(T_\psi'(0)[\psi]\tilde{u},\tilde{v})=\int_\Omega \tilde{u}\tilde{v}\mathrm{div}\psi. \tag{13}$$

Since Ω is of class $C^{1,1}$, we can use the results of [5, Lemma 15 and Corollary 16] (see also [2, Thm 1.3]), to obtain that, if $\varphi_i,\varphi_j\in\mathcal{H}_0^s(\Omega)$ are two eigenfunctions with the same eigenvalue λ_0 for the fractional laplacian (in other words, such that $T_0\varphi_i=\frac{1}{\lambda_0}\varphi_i$, and $T_0\varphi_j=\frac{1}{\lambda_0}\varphi_j$), it holds

$$\left(\mathcal{B}_s^{\psi}\right)'(0)[\psi](T_0\varphi_i,\varphi_j)=-\frac{\Gamma^2(1+s)}{\lambda_0}\int_{\partial\Omega}\frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\,\psi\cdot N\,d\sigma+\int_\Omega \varphi_i\varphi_j\mathrm{div}(\psi)dx$$

and, by (13)

$$\mathcal{B}_s^0(T_\psi'(0)[\psi]\varphi_i,\varphi_j)=\frac{\Gamma^2(1+s)}{\lambda_0}\int_{\partial\Omega}\frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\,\psi\cdot N\,d\sigma,$$

so the operator in formula (5) in Thm 1.2 is indeed

$$\psi\mapsto\gamma_{ij}(\psi):=\int_{\partial\Omega}\frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\psi\cdot N$$

as claimed, and the proof of theorem follows.

Remark 6.1. We notice that, also for an eigenvalue of multiplicity $\nu=2$, repeating the same strategy of the proof of Thm 4, one could construct an eigenvector $\bar\varphi$ for which $\frac{\bar\varphi}{\delta^s}=0$ on a subset of the boundary $\partial\Omega$ which has positive measure. We remark that in some sense $\frac{\bar\varphi}{\delta^s}$ could be the analogous of $\partial_N\bar\varphi$ in the local case, and if $\bar\varphi$ is an eigenvector for the local laplacian with Dirichlet boundary condition for which $\partial_N\bar\varphi=0$ on a set of $\partial\Omega$ of positive measure, then $\bar\varphi\equiv0$ on Ω, by an application of the unique continuation principle, that $\bar\varphi\equiv0$ on Ω. Unfortunately, the extension of this result to the fractional case seems very challenging and it is, as far as we know, far from being proved.

References

[1] G. M. Bisci, V. D. Radulescu, R. Servadei. *Variational methods for nonlocal fractional problems.* Vol. 162. Cambridge University Press, (2016).

[2] S. M. Djitte, M. M. Fall, T. Weth, *A generalized fractional Pohozaev identity and applications* Adv. Calc. Var.

[3] E. Di Nezza, G. Palatucci, E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces.* Bull. Sci. Math.**136**, (2012), 521–573.

[4] M.M. Fall, V. Felli *Unique Continuation Property and Local Asymptotics of Solutions to Fractional Elliptic Equations*, Communications in Partial Differential Equations, **39**, (2014) 354–39,

[5] M.M. Fall, M. Ghimenti, A. M. Micheletti, A. Pistoia, *Generic properties of eigenvalues of the fractional Laplacian*, Calc. Var. Partial Differential Equations 62 (2023), no. 8, Paper No. 233, 17 pp.

[6] R. Frank, *Eigenvalue bounds for the fractional Laplacian: a review. Recent developments in nonlocal theory*, 210–235, De Gruyter, Berlin, 2018.

[7] A. M. Micheletti, *Perturbazione dello spettro di un operatore ellittico di tipo variazionale in relazione ad una variazione di campo (II)*, Recherche Math. **25** (1976), 187–200

[8] A.M. Micheletti, D. Lupo, *On multiple eigenvalues of selfadjoint compact operators*, J. Math An. Appl., **172** (1993) 106–116

[9] A.M. Micheletti, D. Lupo, *On the persistence of the multiplicity of eigenvalues for some variational elliptic operator depending on the domain*, J. Math An. Appl., **193** (1995) 990–1002

[10] X. Ros-Oton, *Boundary regularity, Pohozaev identities and nonexistence results.* Recent developments in nonlocal theory, 335–358, De Gruyter, Berlin, 2018.

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