Squeezing Arguments and the Plurality of Informal Notions

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Abstract

In this paper we argue that squeezing arguments *à la* Kreisel fail to univocally capture an informal or intuitive notion of validity. This suggests a form of logical pluralism, at a conceptual level, not only among but also within logical systems.

1 Introduction

Logical practice, understood as what logicians do in their everyday work, attests that there are many different logical systems. This comes as a practical confirmation of logical pluralism: the thesis according to which there is more than one legitimate consequence relation. But, of course, what counts as legitimate and in which sense different logics can be put on a par motivates different forms of logical pluralism. Some formulations of logical pluralism are more permissive than others, in the sense of accepting more or less different notions of logical consequence. In their famous book, Beall & Restall [3] proposed a general framework able to accommodate a plurality of logical systems, by describing logical consequence as a relation which preserves truth from premises to conclusion. In [3], they base their version of logical pluralism on the following definition.

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Definition 1.1. *φ is a* logical consequence *of Σ if and only if in every* case *that the sentences of Σ are true, φ is true.*

Definition 1.1 expresses an informal notion of logical consequence, since it leaves open the interpretation of the word 'case'. Some specifications of 'case' give rise to a formal notion of consequence relation. For example, by considering the concept of "formal construction" (or "tarskian model"), one can obtain intuitionistic logic (or classical first-order logic). Therefore, given an informal concept of validity, one can obtain different notions of formal system. Notice that Definition 1.1 rests on an extensional notion of consequence relation, in the sense that the relation of logical consequence between Σ and φ is completely determined by the notion of preservation of truth between relevant cases. What Definition 1.1 captures is *when* the relation of logical consequence attains and not *what it means* for Σ and φ to be in a relation of logical consequence. Thus, Definition 1.1 is not suitable for capturing the nature of logical consequence and its intensional aspects. This is clearly a limitation of such an account, since this logical relation can be seen to preserve other notions than just truth – as it is the case for intuitionistic logic, which is designed to preserve provability. As a matter of fact, we can find different interpretations of what it means to be logically valid. But then, how is it possible to capture the meaning of a notion of logical consequence? In this paper we argue that there is no obvious answer to this question, showing that logical validity is underdetermined with respect to its intended interpretation, even in the presence of (a variant of) a so called squeezing argument.

Squeezing arguments originate from Kreisel's original formulation [17], which was meant to show that first-order classical logic is able to capture a notion of informal validity that is stronger than the Tarskian model-theoretic one, since it is also able to account for validity in the universe of all sets. Kreisel's argument made essential use of the possibility of bridging syntax and semantics by means of a completeness theorem and, for this reason, this argument has been extended to other logical systems displaying the same degree of completeness. Therefore, the different squeezing arguments we find in the literature are able to match informal notions of logical validity (other than the classical one) with their corresponding formal definitions.

One might wonder whether squeezing argument, thus, can help to mitigate the plurality of notions of validity we find in the literature. In this paper we reckon that this is not the case and we argue for an even stronger form of pluralism, which we call *informal pluralism*. Informal pluralism consists in reckoning that once we fix a notion of logical validity, it is hard to tell what is the corresponding notion of informal validity that the formal notion captures. Then, we show that squeezing arguments cannot squeeze in the uniqueness of the corresponding informal notion, since a complete logical system can be made compatible with different notions of informal validity. In this sense, the perspective we adopt here goes in the opposite direction of Beal & Restall's proposal: while they initially consider an informal notion to obtain different formal systems, we first consider formal systems, and then we look for informal notions that correspond to each of these formal systems.

The paper is organized as follows. In §2, we recall the main structure of Kreisel's squeezing argument, and we advance a first criticism in the context of classical logic. In §3, we review a variant of Kreisel's argument, by [23], and the criticism that it received by [10]. Then in §4, we extend the criticism of §3 to Intuitionistic Logic. We end with a few remarks in §5.

2 Kreisel's squeezing argument

Let us briefly recall the main points of Kreisel's squeezing argument. Its main goal is to capture an informal notion of logical validity by squeezing it between two formal notions. To this aim, let us fix a first-order formula φ. By $Val(\varphi)$ we denote *informal validity*, which is defined as follows.

$Val(\varphi)$: φ is true in every structure.

We say that $Val(\varphi)$ is informal because it is, intentionally, theoretically vague. The notion of structure present in Val includes the standard set-theoretic notion as well as class-structures. Besides the informal definition of validity, Kreisel presents two formal counterparts of this notion.

$V(\varphi)$: φ is true in all set-theoretic structures.

$D(\varphi)$: φ is deducible by a given set of formal rules (Hilbert Calculus, Natural Deduction, Sequent Calculi) for First-Order Logic (FOL).

V and D are *formal* because they are theoretically precise, in the sense that each one is presented in a well-structured conceptual framework. Then, *informal* notions are defined as non-formal.

Although $Val(\varphi)$ cannot be reduced, in principle, to the other two notions, nonetheless it is directly connected to them as follows.

(1) $D(\varphi) \Rightarrow Val(\varphi)$

(2) $Val(\varphi) \Rightarrow V(\varphi)$

To justify (1), one could provide an argument, similar to an induction on the length of proofs, showing that the axioms of FOL are informally valid and that the rules of FOL preserve informal validity. This means that we use classical principles and inference rules to justify the axioms and rules of FOL. This move is not circular, once Kreisel argues that D is itself a codification of the deductive practice of mathematical reasoning. For this reason, we can assert that (1) holds for Val.[1] To justify (2), the argument runs as follows: the definition of Val encompasses structures whose domains are sets as well as structures whose domains are not sets. So, if φ is valid in the informal sense, then it is so when only set-sized structures are taken into consideration.

Argument 2.1. *Kreisel's squeezing argument for Val.*

(1)	$D(\varphi) \Rightarrow Val(\varphi)$	
(2)	$Val(\varphi) \Rightarrow D(\varphi)$	
(3)	$V(\varphi) \Rightarrow D(\varphi)$	Completeness
(4)	$D(\varphi) \Leftrightarrow Val(\varphi) \Leftrightarrow V(\varphi)$	from (1)-(3)

This squeezing argument offers some philosophical content to the completeness theorem, since, as [1] argues, the informal notion Val can be seen as bridging the gap between the two formal notions of validity.

There are a few important remarks about Kreisel's argument. First, Argument 2.1 does not constitute a formal proof, because Val is itself informal, whereas V and D are formal.[2] Second, as Smith [24] notes, informal validity Val is not, properly speaking, an intuitive notion of validity, but the result of a necessary process of idealization, without which the Argument 2.1 could not work. Instead, it is a rigorously defined notion, still informal, which is close to the model-theoretical construct. This constitutes what Kreisel calls *informal rigour*, the activity of providing a precise analysis of intuitive notions in order "to eliminate the doubtful properties of the intuitive notions when drawing conclusions about them" [17, p.138]. Last, but not least, is the role of the completeness theorem for FOL. Its role is essential,

[1]In Kreisel's original paper [17], D_F stands for Frege's axioms for first-order logic. As Kennedy & Väänänen [16] argue, Kreisel considers D_F as an adequate formalisation of informal validity. So, D_F is correct with respect to Val.

[2]Halbach [11] provides a substitutional analysis of logical validity, defending that such an approach is closer to an informal understanding logical validity. Interestingly, such analysis provides an informal notion for which it is possible, as he argues, to present a formal proof that connects it to the formal definitions of logical validity.

to the extent that in its absence the argument would not work, as it does not for (full) Second-Order Logic (SOL).[3]

That Val may fail to capture our intuitive/pre-theoretical notion of logical validity (which we barely grasp) does not constitute a crucial problem for Kreisel's original argument. As argued by Kennedy & Väänänen [16], the informal notion adopted in natural mathematical language is semantical and it is close to a model-theoretical approach. Then, from this perspective, Kreisel's informal notion seems to capture a notion of validity from mathematical practice. And indeed this seemed to be Kreisel's goal in [17].

A possible response to [16] and to its retraction to a formal interpretation of Val, consists in noting that, even at the formal level, the notion of validity is not completely transparent. The reason being, that a formal notion of validity depends on the choice of the structures that are considered relevant for its definition. But this choice is not always univocal. Indeed, we can easily come up with a distinction between standard and non-standard structures, for example, with respect to the way they interpret equality. For the sake of concreteness, we now analyse a specific instance of this phenomenon, in the case of classical set theory. This example will non only help us to undermine the transparency of a formal notion of validity, but will also show that, even in the classical case, we can find a plurality of notions of validity. The latter, therefore, will be an important hint of a larger phenomenon, that we will later analyse in the broader context of non-classical logic.

To give an example of non-standard models, which, however, are standardly used in mathematical practice, one can think of Boolean-valued models for set theory. These are class structures of the form $\mathbf{V}^{\mathbb{B}}$, where $\mathbb{B}$ is a Boolean algebra and where any element $u \in \mathbf{V}^{\mathbb{B}}$ is a function $u : \mathbf{V}^{\mathbb{B}} \to \mathbb{B}$. Although every $\mathbf{V}^{\mathbb{B}}$ is a model of ZFC, whenever $\mathbb{B}$ is a complete[4] Boolean algebra, nonetheless equality receives an *ad hoc* interpretation, recursively intertwined with that of $\in$. By $[\![\cdot]\!]$ we indicate the interpretation function $[\![\cdot]\!] : \mathcal{L}_{\mathbb{B}} \to \mathbb{B}$, where $\mathcal{L}_{\mathbb{B}}$ consists of the language of set theory extended with constants[5] for every element of $\mathbf{V}^{\mathbb{B}}$.

[3]In [16], the authors show that it is possible to provide a squeezing argument for SOL, if we look for its fragment characterized by Henkin's models.

[4][30]: A Boolean algebra is called *complete* whenever each of its subsets possesses a supremum and an infimum.

[5]In order to simplify notation we will use the same symbol for an element u of $\mathbf{V}^{\mathbb{B}}$ and the constant which represents it in the language $\mathcal{L}_{\mathbb{B}}$.

Definition 2.2. *For any two elements $u, v \in \mathbf{V}^{\mathbb{B}}$,*

$$\llbracket u \in v \rrbracket = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge \llbracket x = u \rrbracket),$$
$$\llbracket u = v \rrbracket = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \llbracket x \in v \rrbracket) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow \llbracket y \in u \rrbracket).$$

Even if Boolean-valued models are non-standard, we can stretch the definition of informal validity so to include such structures. But what about when the algebra $\mathbb{B}$ is not complete, or if we replace it by a Heyting algebra $\mathbb{H}$? And what about when we give the same construction with an algebra $\mathbb{P}$ which models a paraconsistent logic?[6]

We do not want to take a stance here on what counts as a proper class-structure, but we notice that the shift from standard to non-standard is not discrete but continuous. And wherever the dividing line is, there will always be two disjoint classes of structures which can give rise to two different notions of informal validity.

One might object that we are here considering a too wide range of structures, to the extent that even classicality results undermined. But this seems to be exactly the point of logical pluralism: considering different classes of structures we end up changing the notion of logical consequence that we consider. Indeed, if we allow such a comprehensive semantics as the algebraic one, we find different classes of structures that determine opposite and incompatible logical notions.

We can distinguish here between two complementary forms of pluralism. We call the first *formal pluralism*. It consists in the absence of a purely logical reason for deciding which is the notion of logical validity that captures the informal notion of validity that we consider appropriate (where thus the appropriateness depends on pre-theoretical reasons).[7] The second may be called *informal pluralism* and consists in reckoning that once we fix a notion of logical validity, it is hard to tell what is the corresponding notion of informal validity that the formal notion captures. These two forms of pluralism point at the same phenomenon; what changes is the perspective one takes: to look at the informal notions from a formal perspective or vice versa.

In the concrete case of the example discussed (i.e. non-standard models of set theory), we see formal pluralism arising when making a choice of the algebra-valued models which correctly capture a pre-theoretic notion of set-theoretic validity. Depending on the choice, we end up with classical FOL, or intuitionistic FOL, or a

[6] This is not just a mental experiment since such models are well-studied in the literature: [4] and [19]

[7] In [20] we can find a similar problem: in the presence of an informal proof, it is hard to tell what it is the best formalization of it.

paraconsistent FOL, or a combination of them. For what concerns informal pluralism, we can witness it in deciding which class-structures we should count for the definition of informal validity, once we have fixed a classical notion of validity: just the universe of all sets **V**, or also the Boolean-valued models?

While formal pluralism has been already recognised and discussed in [29],[8] in connection with a sufficiently expressive semantics, this is not the case for the informal one. The connection between the latter and Kreisel's squeezing argument is clear. If any squeezing argument is meant to capture an informal notion validity, informal pluralism says that there are many informal notions and the corresponding formal system is not able to uniquely point to one of them. Further, we will argue that this form of pluralism does not necessarily depend on a sufficiently expressive formal semantics, like the algebraic one. Indeed, we will argue that even at the level of an intuitive notion of validity, a squeezing argument is not able to deliver uniqueness.

From the observations above, we can only conclude that Argument 2.1 establishes the extensional equivalence between notions V, D and Val, a highly theorized informal notion, when first-order formulas are considered. It does not mean that Val is the only informal notion of validity. Even if it was not Kreisel's objective in determining whether Val is unique or not, we think it to be relevant to ask whether it is the case. If such notion is not the only notion captured by V and D, there may be other informal notions, still theorized, which may provide a more intuitive understanding of the formal notions of validity. And, interestingly, it may suggest that the formal notions of validity are underdetermined by its informal counterparts. In what follows, we develop this idea by investigating variations of Kreisel's argument presented in the literature.

3 Variants of Kreisel's argument

Kreisel's squeezing argument is not meant to establish that $Val(\varphi)$ is *the* intuitive notion of validity. The point of the argument is only to show that an informal notion corresponds to a formal one. However, because of its simple form, the squeezing argument has been proposed to capture the intuitive notions of validity from natural language. In order to see how this reduction works, let F and $Prem$ be the counterparts of the formula φ and the set of formulas Γ in natural language. Shapiro [23] defines a notion of consequence as follows.

[8] In [29], it is argued that this is an instance of relativism. We do not take a stance on this matter.

Definition 3.1. *The relation $Val_B(Prem, F)$ holds whenever F is logical consequence of $Prem$ in the* blended *sense; that is, it is not possible to every member of $Prem$ to be true and F be false, and this impossibility holds in virtue of the meaning of the logical terms.*

By $Val_B(F)$ we mean that F is informally valid in the blended sense (i.e. considering an empty set of premises). Shapiro argues that this blended notion captures the formality and necessity of the model-theoretical consequence relation. Now, it is clear that every φ provable in FOL is valid in the sense of Val_B. So, we say that the deductive system of FOL is *faithful* with respect to Val_B. It is also clear that every valid F, which is a natural language correspondent of φ, has a valid formalization φ in FOL. In this sense, we say that V is *adequate* to Val_B. Indeed, assuming that F is the natural language counterpart of a FOL formula φ, it is possible to apply a squeezing argument to Val_B as follows.[9]

Argument 3.2. *Shapiro's squeezing argument for Val_B.*

(1)	$D(\varphi) \Rightarrow Val_B(F)$	Faithfulness
(2)	$Val_B(F) \Rightarrow V(\varphi)$	Adequacy
(3)	$V(\varphi) \Rightarrow D(\varphi)$	Completeness
(4)	$D(\varphi) \Leftrightarrow Val_B(F) \Leftrightarrow V(\varphi)$	from (1)-(3)

The argument is meant to show that blended validity in natural language extensionally coincides both with the Tarskian model-theoretic notion of validity and with its proof-theoretical counterpart.

Interestingly, a similar argument can be applied to a more syntactic notion of informal validity.

Definition 3.3. *The relation $Val_{Ded}(Prem, F)$ holds whenever F is logical consequence of $Prem$ in the* deductive *sense; that is, there is a deduction of F from $Prem$ by a chain of legitimate gap-free (self-evident) rules of inference.*

By $Val_{Ded}(F)$ we mean that F is informally valid in the deductive sense (i.e. considering an empty set of premises). The arguments for faithfulness and adequacy of V and D with respect to Val_{Ded} are similar to the arguments for Val_B. Then we also have a corresponding squeezing argument for Val_{Ded}.

[9] In [23], one finds a version of the Argument 3.2. But we follow here Griffiths's version [10] for the sake of simplicity.

Argument 3.4. *Shapiro's squeezing argument for Val_{Ded}.*

(1)	$D(\varphi) \Rightarrow Val_{Ded}(F)$	Faithfulness
(2)	$Val_{Ded}(F) \Rightarrow V(\varphi)$	Adequacy
(3)	$V(\varphi) \Rightarrow D(\varphi)$	Completeness
(4)	$D(\varphi) \Leftrightarrow Val_{Ded}(F) \Leftrightarrow V(\varphi)$	from (1)-(3)

Before discussing the relevance of these arguments for the present discussion, let us briefly review the main criticism that these arguments received.

Following Griffiths's argument [10] against Shapiro's squeezing argument, we can argue that Val_B and Val_{Ded}, although defined for natural language, do not account for the totality of such language, but just for a non-ambiguous fragment of it: the one that can be formalized in FOL. Indeed, F is obtained as the *reading* of a first-order formula where reading is understood as the reverse process of formalization. Then, Griffiths objects to Shapiro's argument by arguing that (1) and (2) of the Argument 3.2 hold only in virtue of the connection between F and φ and not because of the definition of Val_B. Therefore there is nothing special about the blended notion of validity because Val_B is coextensive with the Val relation used in Kreisel's argument. And of course the same works for Val_{Ded}.

We can now advance a further objection to Shapiro's argument(s). We agree with Griffiths that the three notions Val, Val_B, and Val_{Ded}, are coextensive. But then, if the formal notions of classical FOL capture the three of them, which one can be seen as the informal or the intuitive content of the formal notions? This is a relevant question due to the fact that the notions Val, Val_B, and Val_{Ded} are meant to be intensional objects: properties of formulas.[10] If these squeezing arguments are able to show that we can capture these notions by means of extensional concepts (V and D), however, we are left in the dark with respect to which one of these represents the intensional concept we associate to logical (classical FOL) validity. In this sense, logical validity is therefore underdetermined by its formal counterparts, even if these manifest a perfect correspondence between syntax and (formal) semantics.

To counter our point, one could simply accept that FOL captures informal notions of validity which have semantic or syntactic aspects. Then, the completeness theorem shows that these notions are extensionally equivalent, despite their intensional difference. This response, however, misses our main point: Argument 3.2 and

[10] One could say that the objections also work against Kreisel. It would do so if Kreisel's interests were natural language. But, as we highlighted before, his interests were only mathematical.

Argument 3.4 only hold in virtue of the correspondence between F and φ, not in virtue of the intrinsic characteristics of Val_B and Val_{Ded}. To make our point clear, consider the following notion presented in [10]:

Definition 3.5. *The relation $Val_{Nec}(Prem, F)$ holds whenever F is logical consequence of $Prem$ in the* modal *sense if and only if necessarily, every member of $Prem$ is true, F is true.*

Griffiths presents a squeezing argument for Val_{Nec} in order to show that there is nothing distinctive about Val_B, since Val_{Nec} holds by the same reason as the validity in the blended sense. Given this abundance of options, and since formal logic is mute on this topic, any judgement about which notion is more appropriated is moved by pre-theoretical reasons, which therefore suggests a form of informal pluralism with respect to our pre-formal notion of validity.

A consequence of this phenomenon is that the formal consequence relation of FOL is not able to capture the intuitive notion of logical consequence. Kreisel's argument and its variants neither capture such intuitive notion nor do they capture a unique one, even if they capture relevant informal notions, which regulate our inferential practice. Moreover, and following Griffiths's analysis of Shapiro's argument, we cannot hold that these informal notions capture the whole of our inferential practices, but only a small, formalisable fragment of natural language inferences. Probably, to capture all inferences of natural language in a system like FOL, we should extend this system to the point of doubting that it remains formal.[11]

So far, we maintain that all that is safe to infer from Kreisel's and Shapiro's arguments is that formal validity is able to capture a fragment of the intuitive validity of natural language which deals with preservation of truth from premises to the conclusion. From this perspective, we can say that the axioms and rules of FOL capture general principles for correct truth-preserving reasoning. Therefore, we can follow a pragmatic vindication of the logical principles of FOL (as [7] does) in arguing that FOL is the correct logic to adopt in the case we want to yield true conclusions from true premises.

However, given the plurality of the logical systems used in our formal practice, one can ask whether the pluralism found with respect to the informal notions captured by classical FOL can be extended to other formal notions of logical validity. In other terms, we wonder whether the underdetermination of the informal classical notions is a weakness of classical FOL, or else an intrinsic limitation of any squeezing argument. It is, therefore, to a pluralist perspective on logic that we now turn.

[11]It has been argued by [8] that logical consequence is not determined by natural language.

4 Informal notions and Logical pluralism

We want here to explore squeezing arguments for propositional Intuitionistic Logic (IL). As in the classical case, we will show that the squeezing arguments for IL do not determine a unique informal validity. Then we will argue that the multiplicity of informal notions corresponding to a particular formal system suggests that formal systems, in general, do not have a canonical interpretation.

The squeezing arguments for FOL presented in §3 show that V and D do not manage to squeeze in a unique notion of classical informal validity. We now investigate whether a similar phenomenon occurs in other logical contexts.

4.1 Intuitionistic Logic and BHK

Intuitionists generally agree that the classical notion of truth is not as tractable as the notion of proof, because it validates some principles which are not constructively valid, such as the *Principle of Excluded Middle* (PEM). Consider, for example, a mathematical conjecture such as Goldbach conjecture. Since we do not have an available proof of either it or its negation, therefore PEM cannot be considered as a valid logical principle. From this perspective, a notion that seems to harmonize better with intuitionism is offered by the *Brouwer-Heyting-Kolmogorov interpretation* (for short, BHK interpretation), which captures a well-justified notion of *constructibility*. Moreover, in [27], one can find an informal justification of all the axioms and rules of IL with respect to BHK. Thus we can define a notion of informal validity with respect to BHK as follows; where φ is a propositional formula.

$Val_I(\varphi)$: φ is constructively provable.

The informality of Val_I stems from the absence of a specification of the methods of construction. And indeed different interpretations of constructibility may lead us to different conceptions of constructivism. For example, under Markov's [28] interpretation of constructivism, every algorithm must terminate, whereas Brouwer's intuitionism allows the construction of infinite sequences of objects.

> (...) the ideal mathematician may construct longer and longer initial segments $\alpha(0), \ldots, \alpha(n)$ of an infinite sequence of natural numbers a where a is not a priori determined by some fixed process of producing the values, so the construction of a is never finished: a is an example of a choice sequence. ([28, p.5])

Let D_I and V_I now stand for deductibility in an intuitionistic proof system and

V_I for structures whose internal logic is IL. We now argue for the coextensivity of D_I, Val_I and V_I.

Although Val_I is theoretically irreducible to both D_I and V_I, nonetheless Val_I is sound with respect to IL (as argued in [27]), and also the constructions allowed by Val_I can clearly be carried out in structures whose internal logic is IL. Indeed, the methods of constructions codified by IL represent a qualification of the (in principle) more general notion of constructibility. Therefore, we can argue that D_I captures a more restricted version of constructibility than Val_I. We can, therefore, run an analogous version of Kreisel's squeezing argument.

Argument 4.1. *First version of squeezing argument for IL.*

(1)	$D_I(\varphi) \Rightarrow Val_I(\varphi)$	Soundness
(2)	$Val_I(\varphi) \Rightarrow V_I(\varphi)$	Adequacy
(3)	$V_I(\varphi) \Rightarrow D_I(\varphi)$	Completeness
(4)	$D_I(\varphi) \Leftrightarrow Val_I(\varphi) \Leftrightarrow V_I(\varphi)$	from (1)-(3)

As for the classical notion of validity, we therefore seem to have squeezed Val_I within the syntax and semantics of Intuitionistic Logic.

4.2 Intuitionistic Logic and $S4$

The philosophical debate about logics and their interpretations has its origins in the proper development of mathematical logic, especially with respect to non-classical logics. In [2], we can find a distinction between *pure logics* and their *philosophical interpretations*. Pure logics, understood as languages equipped with consequence relations, have no intrinsic connections with their possible philosophical interpretations. Of course, there are interpretations more interesting than others and any judgement about the adequacy of a notion to the detriment of others is moved by pre-theoretical reasons and not only by the formal aspect of a pure logic.

For example, although the BHK interpretation is considered the intended informal notion captured by IL, nothing prevents us from proposing alternative conceptions associated to an intuitionistic notion of logical validity.

Let us consider a concrete case. It is a well-known fact that IL is modally characterized as the logic $S4$ ([15]), whose modality $\Box$ has a well-justified epistemic interpretation as true introspective (and deductively closed) knowledge; let us call this specific notion *knowledge**. The logic $S4$ is normally presented as follows.

$Taut$: all axioms and inference rules of classical propositional logic;

K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;

T: $\Box\varphi \to \varphi$;

4: $\Box\varphi \to \Box\Box\varphi$;

Nec: From $\vdash \varphi$ we obtain $\vdash \Box\varphi$.

Under such epistemological interpretation, the axiom K says that knowledge is closed under logical consequence. That is, if an agent knows* φ, then they know* all the consequences of φ.[12] The axiom T says that knowledge* is factual. The axiom 4 is an introspection principle. And Nec says that the agent knows* all logical validities.

Theorem 4.2. *There is a translation $T : \mathcal{L} \to \mathcal{L}^{\Box}$ from the language of propositional logic to the language of modal propositional logic, such that $IL \vdash \varphi$ iff $S4 \vdash T(\varphi)$.*

Given the specificity of the notion of knowledge we defined, knowledge*, by definition, is such that (A) every theorem of $S4$ is a thesis about knowability* and (B) every thesis about knowability* is $S4$-valid. We can, therefore, state the second intuitive notion we can associate to IL.

$Val^*_I(\varphi)$: $T(\varphi)$ is known*

Despite knowledge and provability being different concepts, nonetheless we can provide an analogous version of Argument 4.1, using the informal notion Val^*_I. Therefore, we can squeeze Val^*_I in between the syntax and the semantics of IL.

By D_{S4} we mean the predicate of derivability in $S4$ and by V_{S4} the predicate of validity for $S4$, for example in reflexive and transitive Kripke frames.

Argument 4.3. *Second version of squeezing argument for IL.*

[12]The principle K is usually taken as implausible due to the problem of *logical omniscience* ([25],[22]). Here, the logical omniscience is not a problem because we are supposing an idealized notion of knowledge.

(1)	$D_I(\varphi) \Rightarrow D_{S4}(T(\varphi))$	Theorem 4.2
(2)	$D_{S4}(T(\varphi)) \Rightarrow Val^*_I(\varphi)$	(A)
(3)	$Val^*_I(\varphi) \Rightarrow V_{S4}(T(\varphi))$	(B)
(4)	$V_{S4}(T(\varphi)) \Rightarrow V_I(\varphi)$	Theorem 4.2
(5)	$V_I(\varphi) \Rightarrow D_I(\varphi)$	Completeness of IL
(6)	$D_I(\varphi) \Leftrightarrow Val^*_I(\varphi) \Leftrightarrow V_I(\varphi)$	from (1)-(5)

One might object that the choice of this informal notion: knowledge*, is too *ad hoc* or even that $S4$ does not capture the notion of knowledge.[13]

To counter this objection there are two possible moves. On the one hand, we can argue that the issues one can raise on the epistemic completeness of $S4$ are the result of an intrinsic and non-eliminable gap between formality and informality; a gap that is also present in Kreisel's original argument. That is to say that any formalization of an informal notion has non-trivial elements that, similar to a Carnapian explication, have the effect of normatively modifying the (necessarily vague) informal notion.

On the other hand, we can provide another example of an informal notion that we can associate to $S4$; one for which the identification of the informal notion and its formalization is less controversial. This second move does not eliminate the qualms raised by the objection, but it only advances a dialectical strategy: for any well-justified informal notions we can associate to IL, it is the task of the proponent of the objection to show that this association fails.

We can then define a third informal notion for IL.

$Val'_I(\varphi)$: $T(\varphi)$ is informally provable.

By informally provable we mean provability by any correct mathematical/logical means, not being tied to a particular formal system. Indeed this was Gödel's original interpretation [9] of the $S4$ operator $\Box$. According to this interpretation of $S4$, the axiom K says that provability is preserved under modus ponens. The axiom T says that whatever is provable is true. The axiom 4 says that if φ is provable, then it is provable that φ is provable. The latter axiom is a kind of introspection principle. The necessitation rule says that all logical validities are informally provable. Thus, the characteristic modal axioms of $S4$ suggest that this logic captures the concept of informal provability.[14]

[13] See for example the discussion in [26]. In this paper, $S4$ is presented as the logic which captures knowledge without doxastic elements.

[14] Gödel's interpretation of $S4$ was well-received in the literature. In [21], [5] and [12] one can find

Since T is an $S4$ axiom, $\Box$ cannot stand for provability in a consistent formal system which contains arithmetic, on pain of contradicting the second incompleteness theorem. Then, the argument runs as follows.

Similarly to Val_I^*, (A') stands for the assumption that every theorem of $S4$ is a thesis about informal provability and (B') that every thesis about informal provability is $S4$-valid.

Argument 4.4. *Third version of squeezing argument for IL.*

(1)	$D_I(\varphi) \Rightarrow D_{S4}(T(\varphi))$	Theorem 4.2
(2)	$D_{S4}(T(\varphi)) \Rightarrow Val'_I(\varphi)$	(A')
(3)	$Val'_I(\varphi) \Rightarrow V_{S4}(T(\varphi))$	(B')
(4)	$V_{S4}(T(\varphi)) \Rightarrow V_I(\varphi)$	Theorem 4.2
(5)	$V_I(\varphi) \Rightarrow D_I(\varphi)$	Completeness of IL
(6)	$D_I(\varphi) \Leftrightarrow Val'_I(\varphi) \Leftrightarrow V_I(\varphi)$	from (1)-(5)

We are therefore in the position to argue that IL can equally capture, by means of squeezing arguments, three informal notions: being constructively provable, being knowable*, and being informally provable. Therefore we notice that also in the case of IL a squeezing argument is not able to univocally capture an informal notion that can thus be presented as the informal content of a formal notion of logical validity.

5 Conclusion

The underdetermination of informal notions with respect to logical systems has the effect of suggesting that logical systems have no canonical interpretations. As formal systems, logics allow different interpretations. Therefore, IL cannot be seen only as a system that captures constructive reasoning. Indeed, because of Arguments 4.3 and 4.4 we can interpret IL in epistemological terms or as capturing a notion of informal provability. Modal logic displays a paradigmatic example of this phenomenon. Not

arguments in defense of the informal provability interpretation of $S4$. On the other hand, Leitgeb [18] argues that the completeness of $S4$ with respect to informal provability is still an open problem because there is the question whether statements about unprovability should be taken as axioms of informal provability. Given that this suspicion is an open problem, we take Gödel's original interpretation because it is not controversial and it seems to capture the minimal principles of such notion.

only $S4$, but also a logic like $S4.2$ can be seen as an epistemic logic ([26]) or one that codifies set-theoretical notions ([13]). This is, of course, coherent with the views that a formal notion of logical consequence stands with respect to an informal notion of validity in a relation akin to a Carnapian explication (as suggested in [10]) and that this kind of relation is imprecise in nature (as argued in [6]).

To realise that a logical system have no canonical interpretations, thus, amount to recognise that a squeezing arguments does not really help in capturing an informal or intuitive notion of validity. This should not come as a surprise, given the modern-Hilbertian-axiomatic perspective widely accepted on formal systems. What on the other hand is interesting to notice is that Griffiths's criticism to squeezing arguments extends from the informal to the formal context. Not only Kreisel's squeezing argument is not able to capture an intuitive notion of validity, but it also fails to univocally capture an informal idealized notion of validity definable in a mathematical context. Moreover, this phenomenon extends to other logical systems, thus showing a form of pluralism not only between different logics, but also within a fixed one.

The informal pluralism discussed here also suggests that formal systems cannot really conflict over a given interpretation. Indeed, if we undermine the link between a formal system and its interpretation, we realise that any competition between different logical systems is only apparent, since none can really claim to fully capture a given notion of informal validity. Different logics preserve validity in different cases, which in turn refer to a multiplicity of informal notions. Consider again the case of classical and intuitionistic logics. While Val talks about truth in all structures, Val_I talks about constructive provability. Thus, the rivalry between classical and intuitionistic logics is apparent, once one recognizes that their informal interpretations talk about different notions and that these cannot be completely reduced to their formal counterparts. Heyting seemed to have already held a similar position ([14]). For him, a classical mathematician can maintain that mathematical entities exist autonomously, while, at the same time, recognizing that the notion of existence do not play any role, when dealing with proofs. Thus, the plurality of informal notions may offer a more tolerant perspective on logical pluralism, thus vindicating the plurality of formal systems in mathematical practice.

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