Convergence of second-order in time numerical discretizations for the evolution Navier-Stokes equations

Luigi C. Berselli[1*] and Stefano Spirito[2]

*Correspondence:
luigi.carlo.berselli@unipi.it
[1] Dipartimento di Matematica,
Università degli Studi di Pisa, Via F.
Buonarroti 1/c, I-56127 Pisa, Italy
Full list of author information is
available at the end of the article

Abstract

We prove the convergence of certain second-order numerical methods to weak solutions of the Navier–Stokes equations satisfying, in addition, the local energy inequality, and therefore suitable in the sense of Scheffer and Caffarelli–Kohn–Nirenberg. More precisely, we treat the space-periodic case in three space dimensions and consider a full discretization in which the classical Crank–Nicolson method (θ-method with $\theta = 1/2$) is used to discretize the time variable. In contrast, in the space variables, we consider finite elements. The convective term is discretized in several implicit, semi-implicit, and explicit ways. In particular, we focus on proving (possibly conditional) convergence of the discrete solutions toward weak solutions (satisfying a precise local energy balance) without extra regularity assumptions on the limit problem. We do not prove orders of convergence, but our analysis identifies some numerical schemes, providing alternate proofs of the existence of "physically relevant" solutions in three space dimensions.

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1 Introduction

We consider the homogeneous incompressible 3D Navier–Stokes equations (NSE)

$$\partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } (0, T) \times \mathbb{T}^3,$$
$$\text{div } u = 0 \quad \text{in } (0, T) \times \mathbb{T}^3,$$

(1.1)

in the space periodic setting, with divergence-free initial datum

$$u|_{t=0} = u_0 \quad \text{in } \mathbb{T}^3,$$

(1.2)

where $T > 0$ is arbitrary, $\nu > 0$ is given, and $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$ is the three-dimensional flat torus. Here, the unknowns are the velocity vector field u and the scalar pressure p with zero mean values. The aim of this paper is to consider families of space-time discretization

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of the initial value problem (1.1)–(1.2), which are of the second order in time, and (as the parameters of the discretization vanish) to prove the convergence toward the Leray–Hopf weak solutions, satisfying, in addition, certain estimates on the pressure and the local energy inequality

$$\partial_t\left(\frac{|u|^2}{2}\right) + \mathrm{div}\left(\left(\frac{|u|^2}{2} + p\right)u\right) - \nu\Delta\left(\frac{|u|^2}{2}\right) + \nu|\nabla u|^2 \le 0 \quad \text{in } \mathcal{D}'(]0, T[\times\mathbb{T}^3).$$

Leray–Hopf weak solutions satisfying certain extra properties on pressure and the local energy inequality are known in the literature as *suitable weak solutions*, and they are of fundamental importance from the theoretical point of view since they are those for which partial regularity results hold true, see Scheffer [30] and Caffarelli-Kohn-Nirenberg [11].

Due to possible non-uniqueness of solutions in the 3D case, see, in particular, the recent result in [1] for the case with external forces, it is not ensured that all schemes produce weak solutions with the correct global and local balance. Moreover, from the applied point of view, the local energy inequality is a sort of entropy condition. Even if it is not enough to prove uniqueness, it seems a natural request to select physically relevant solutions, especially for turbulent or convection-dominated problems. For this reason, it is natural to ask, in view of obtaining accurate simulations of turbulent flows, that the above local energy inequality has to be satisfied by solutions constructed by numerical methods. The interplay between suitable weak solutions and numerical computations of turbulent flows has been emphasized starting from the work of Guermond *et al.* [18, 19] and a recent overview can be found in the monograph [4]. In this paper, we continue and extend some previous works in [3, 6–8], and especially [5] to analyze the difficulties arising when dealing with full space-time discretization with second-order schemes in the time variable. The aim of this paper is to extend the case $\theta = 1/2$, which corresponds to the Crank–Nicolson method and could not be treated directly with the same proofs as in [5]. In particular, the case $\theta = 1/2$ requires a coupling between the space and time mesh-size, which is nevertheless common to other second-order models. In fact, besides the Crank–Nicolson scheme (CN) studied in Sect. 5, we will also consider in the final Sect. 6 other schemes involving the Adams-Bashforth or the Linear-Extrapolation for the convective term.

To set up the problem, as in [16], we consider two sequences of discrete approximation spaces $\{X_h\}_h \subset H^1_\#$ and $\{M_h\}_h \subset H^1_\#$, which satisfy – among other properties described in Sect. 3 – an appropriate commutator property, see Definition 3.1. Then, given a net $t_m := m\Delta t$, we consider the following implicit space-time discretization of the problem (1.1)–(1.2): Set $u_h^0 = \pi_h(u_0)$, where π_h is the projection over X_h. For any $m = 1, \dots, N$ and given $u_h^{m-1} \in X_h$ and $p_h^{m-1} \in M_h$, find $u_h^m \in X_h$ and $p^m \in M_h$ such that

$$\begin{aligned}
&\left(d_t u_h^m, v_h\right) + \nu\left(\nabla u_h^{m,1/2}, \nabla v_h\right) + b_h\left(u_h^{m,1/2}, u_h^{m,1/2}, v_h\right) - \left(p_h^m, \mathrm{div}\, v_h\right) = 0, \\
&\left(\mathrm{div}\, u_h^m, q_h\right) = 0,
\end{aligned} \tag{CN}$$

where $d_t u^m := \frac{u_h^m - u_h^{m-1}}{\Delta t}$ is the backward finite-difference approximation for the time-derivative in the interval (t_{m-1}, t_m) of constant length Δt; $u_h^{m,1/2} := \frac{1}{2}(u_h^m + u_h^{m-1})$ is the average of values at consecutive time-steps; $b_h(u_h^{m,1/2}, u_h^{m,1/2}, v_h)$ is a suitable discrete approximation of the non-linear term. Other notations, definitions, and properties regarding (CN) will be given in Sects. 2-3. We refer to Quarteroni and Valli [29] and Thomée [32] for

general properties of θ-schemes (not only for $\theta = 1/2$) for parabolic equations. Recall that for the fully implicit Crank–Nicolson scheme, Heywood and Rannacher [22] proved that it is almost unconditionally stable and convergent. For a two-step scheme with a semi-implicit treatment for the nonlinear term, He and Li [20] gave the convergence condition: $\Delta t h^{-1/2} \leq C_0$. For the Crank–Nicolson/Adams–Bashforth scheme in which the nonlinear term is treated explicitly, Marion and Temam [28] provided the stability condition $\Delta t h^{-2} \leq C_0$, and Tone [33] proved the convergence under the condition $\Delta t h^{-2-3/2} \leq C_0$, and in all cases, $C_0 = C_0(\nu, \Omega, T, u_0, f)$. The situation is different in two space dimensions, cf. He and Sun [21], where more regularity of the solution can be used, but these results are not applicable to genuine (turbulent) weak solutions in the three-dimensional case. We observe that the value $\theta = 1/2$ makes the scheme more accurate in the time variable but, on the other hand, introduces some "natural" or at least expected limitation on the mesh-sizes. Other schemes will also be considered, to adapt the results to different second-order schemes since the proof is rather flexible to handle several different discretizations of the NSE.

As usual in time-discrete problem (see, for instance, [31]), to study the convergence to the solutions of the continuous problem, it is useful to consider $v_h^{\Delta t}$, which is the linear interpolation of $\{u_h^m\}_{m=1}^N$ (over the net $t_m = m\Delta t$), and $u_h^{\Delta t}$ and $p_h^{\Delta t}$, which are the time-step functions such that on the interval $[t_{m-1}, t_m)$ are equal to $u_h^{m,1/2}$ and p_h^m, respectively, see (3.12).

The main result of the paper is the following; we refer to Sect. 2 for further details on the notations.

Theorem 1.1 *Let the finite element spaces (X_h, M_h) satisfy the discrete commutator property and the technical conditions described in Sect. 3.1. Let $u_0 \in H_{\mathrm{div}}^1$ and fix $\Delta t > 0$ and $h > 0$ such that*

$$\frac{\Delta t \|u_0\|_2^3}{\nu h^{1/2}} = o(1), \tag{1.3}$$

Let $\{(v_h^{\Delta t}, u_h^{\Delta t}, p_h^{\Delta t})\}_{\Delta t, h}$ as in (3.12). Then, there exists

$$(u, p) \in L^\infty\big(0, T; L_{\mathrm{div}}^2\big) \cap L^2\big(0, T; H_{\mathrm{div}}^1\big) \times L^{4/3}\big(0, T; L_\#^2\big),$$

such that, up to a sub-sequence, as $(\Delta t, h) \to (0, 0)$,

$$v_h^{\Delta t} \to u \quad \text{strongly in } L^2\big((0, T) \times \mathbb{T}^3\big),$$
$$u_h^{\Delta t} \to u \quad \text{strongly in } L^2\big((0, T) \times \mathbb{T}^3\big),$$
$$\nabla u_h^{\Delta t} \rightharpoonup \nabla u \quad \text{weakly in } L^2\big((0, T) \times \mathbb{T}^3\big),$$
$$p_h^{\Delta t} \rightharpoonup p \quad \text{weakly in } L^{4/3}\big((0, T) \times \mathbb{T}^3\big).$$

Moreover, the couple (u, p) is a suitable weak solution of (1.1)–(1.2) *in the sense of Definition* 2.2.

Remark 1.2 Theorem 1.1 also holds in the presence of an external force f satisfying suitable bounds. For example, $f \in L^2(0, T; L^2(\mathbb{T}^3))$ is enough.

The proof of Theorem 1.1 is given in Sect. 5, and it is based on a compactness argument as we previously developed in [5] and a precise analysis of the quantity $u_h^{m+1} - u_h^m$ using the assumptions linking the time and the spatial mesh size.

Plan of the paper. In Sect. 2, we fix the notation that will be used in the paper and recall the main definitions and tools applied. In Sect. 3, we introduce and give some details about the space-time discretization methods. Finally, in Sect. 4, we prove the main *a priori* estimates needed to study the convergence, and in Sect. 5, we prove Theorem 1.1. In the final Sect. 6, we adapt the proofs to a couple of different second-order schemes.

2 Notations and preliminaries

In this section, we fix the notation that will be used in the paper; we also recall the main definitions concerning weak solutions of incompressible NSE and a compactness result.

2.1 Notations

We introduce the notations typical of space-periodic problems. We will use the customary Lebesgue spaces $L^p(\mathbb{T}^3)$ and Sobolev spaces $W^{k,p}(\mathbb{T}^3)$, and we will denote their norms by $\| \cdot \|_p$ and $\| \cdot \|_{W^{k,p}}$. We will not distinguish between scalar and vector valued functions since it will be clear from the context, which one has to be considered. In the case $p = 2$, the $L^2(\mathbb{T}^3)$ scalar product is denoted by $(\cdot, \cdot)$, we use the notation $H^s(\mathbb{T}^3) := W^{s,2}(\mathbb{T}^3)$ and define, for $s > 0$, the dual spaces $H^{-s}(\mathbb{T}^3) := (H^s(\mathbb{T}^3))'$. Moreover, we will consider always sub-spaces of functions with zero mean value, and these will be denoted by

$$L_\#^p := \left\{ w \in L^p(\mathbb{T}^3) : \int_{\mathbb{T}^3} w \, dx = 0 \right\}, \quad 1 \le p \le +\infty,$$

and also by

$$H_\#^s := H^s(\mathbb{T}^3) \cap L_\#^2.$$

As usual, we consider spaces of divergence-free vector fields defined as follows:

$$L_{\mathrm{div}}^2 := \left\{ w \in \left(L_\#^2\right)^3 : \mathrm{div}\, w = 0 \right\} \quad \text{and, for } s > 0,\ H_{\mathrm{div}}^s := H_\#^s \cap L_{\mathrm{div}}^2.$$

Finally, given X a Banach space, $L^p(0, T; X)$ denotes the classical Bochner spaces of X valued functions, endowed with its natural norm, denoted by $\| \cdot \|_{L^p(X)}$. We denote by $l^p(X)$ the discrete counterpart for X-valued sequences $\{x^m\}$, defined on the net $\{m\Delta t\}$, and with weighted norm defined by $\|x\|_{l^p(X)}^p := \Delta t \sum_{m=0}^M \|x^m\|_X^p$.

2.2 Weak solutions and suitable weak solutions

We start by recalling the notion of weak solution (as introduced by Leray and Hopf) and adapted to the space periodic setting.

Definition 2.1 The vector field u is a Leray–Hopf weak solution of (1.1)–(1.2) if

$$u \in L^\infty\left(0, T; L_{\mathrm{div}}^2\right) \cap L^2\left(0, T; H_{\mathrm{div}}^1\right),$$

and if u satisfies the NSE (1.1)–(1.2) in the weak sense, namely the integral equality

$$\int_0^T \left[(u, \partial_t \phi) - \nu(\nabla u, \nabla \phi) - \big((u \cdot \nabla)u, \phi\big) \right] dt + \big(u_0, \phi(0)\big) = 0 \tag{2.1}$$

holds true for all smooth, periodic, and divergence-free functions $\phi \in C_c^\infty([0,T]; C^\infty(\mathbb{T}^3))$. Moreover, the initial datum is attained in the strong L^2-sense, that is

$$\lim_{t \to 0^+} \left\| u(t) - u_0 \right\|_2 = 0,$$

and the following *global* energy inequality holds

$$\frac{1}{2} \left\| u(t) \right\|_2^2 + \nu \int_0^t \left\| \nabla u(s) \right\|_2^2 ds \leq \frac{1}{2} \| u_0 \|_2^2, \quad \text{for all } t \in [0,T]. \tag{2.2}$$

Suitable weak solutions are a particular subclass of Leray–Hopf weak solutions and the definition is the following.

Definition 2.2 A pair (u, p) is a suitable weak solution to the Navier–Stokes equation (1.1) if u is a Leray–Hopf weak solution, $p \in L^{4/3}(0, T; L_\#^2)$, and the local energy inequality

$$\nu \int_0^T \int_{\mathbb{T}^3} |\nabla u|^2 \phi \, dx \, dt \leq \int_0^T \int_{\mathbb{T}^3} \left[\frac{|u|^2}{2} (\partial_t \phi + \nu \Delta \phi) + \left(\frac{|u|^2}{2} + p \right) u \cdot \nabla \phi \right] dx \, dt \tag{2.3}$$

holds for all $\phi \in C_0^\infty(0, T; C^\infty(\mathbb{T}^3))$ such that $\phi \geq 0$.

Remark 2.3 The definition of suitable weak solution is usually stated with $p \in L^{5/3}((0,T) \times \mathbb{T}^3)$ while in Definition 2.2 $p \in L^{4/3}(0, T; L^2(\mathbb{T}^3))$. This is not an issue since, of course, we have a bit less integrability in time, but we gain a full L^2-integrability in space. The main property of suitable weak solutions is the fact that they satisfy the local energy inequality (2.3), and weakening the requests on the pressure does not influence the validity of local regularity results; see, for instance, discussion in Vasseur [34].

2.3 A compactness lemma
In this subsection, we recall the main compactness lemma, which allows us to prove the strong convergence of the approximations. We remark that it is a particular case of a more general lemma, whose statement and proof can be found in [27, Lemma 5.1]. Even if it is a tool used for compressible equations most often, it is useful here to recall the special version taken from [5].

Lemma 2.4 *Let $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be uniformly bounded in $L^\infty(0, T; L^2(\mathbb{T}^3))$, and let be given $f, g \in L^\infty(0, T; L^2(\mathbb{T}^3))$ such that*

$$f_n \rightharpoonup f \quad \text{weakly in } L^2\big((0,T) \times \mathbb{T}^3\big),$$
$$g_n \rightharpoonup g \quad \text{weakly in } L^2\big((0,T) \times \mathbb{T}^3\big).$$

Let $p \geq 1$ and assume that

$$\{\partial_t f_n\}_n \subset L^p\big(0, T; H^{-1}(\mathbb{T}^3)\big), \qquad \{g_n\}_n \subset L^2\big(0, T; H^1(\mathbb{T}^3)\big),$$

with uniform (with respect to $n \in \mathbb{N}$) bounds on the norms. Then,

$$f_n g_n \rightharpoonup fg \quad \text{weakly in } L^1\big((0, T) \times \mathbb{T}^3\big).$$

3 Setting of the numerical approximation

In this section, we introduce the space-time discretization of the initial value problem (1.1)–(1.2). We start by introducing the space discretization by finite elements.

3.1 Space discretization

For the space discretization, we strictly follow the setting considered in [16]. Let $\mathcal{T}_h$ be a non-degenerate (shape regular) simplicial subdivision of $\mathbb{T}^3$. Let $\{X_h\}_{h>0} \subset H^1_\#$ be the discrete space for approximate velocity and $\{M_h\}_{h>0} \subset L^2_\#$ be that of approximate pressure. To avoid further technicalities, we assume, as in [16], that $M_h \subset H^1_\#$.

We make the following (technical) assumptions on the spaces X_h and M_h:

1. For any $v \in H^1_\#$ and for any $q \in L^2_\#$, there exists $\{v_h\}_h$ and $\{q_h\}_h$ with $v_h \in X_h$ and $q_h \in M_h$ such that

    $$\begin{aligned} v_h \to v \quad &\text{strongly in } H^1_\# \text{ as } h \to 0, \\ q_h \to q \quad &\text{strongly in } L^2_\# \text{ as } h \to 0; \end{aligned} \tag{3.1}$$

2. Let $\pi_h : L^2(\mathbb{T}^3) \to X_h$ be the L^2-projection onto X_h. Then, there exists $c > 0$ independent of h such that

    $$\forall q_h \in M_h \quad \big\|\pi_h(\nabla q_h)\big\|_2 \geq c\|q_h\|_2; \tag{3.2}$$

3. There is c independent of h such that for all $v \in H^1_\#$

    $$\big\|v - \pi_h(v)\big\|_2 = \inf_{w_h \in X_h} \|v - w_h\|_2 \leq ch\|v\|_{H^1},$$

    $$\big\|\pi_h(v)\big\|_{H^1} \leq c\|v\|_{H^1};$$

4. There exists c independent of h (inverse inequality) such that

    $$\forall v_h \in X_h \quad \|v_h\|_{H^1} \leq ch^{-1}\|v_h\|_2. \tag{3.3}$$

Moreover, we assume that X_h and M_h satisfy the following discrete commutator property.

Definition 3.1 We say that X_h (resp. M_h) has the discrete commutator property if there exists an operator $P_h \in \mathcal{L}(H^1; X_h)$ (resp. $Q_h \in \mathcal{L}(L^2; M_h)$) such that for all $\phi \in W^{2,\infty}$ (resp. $\phi \in W^{1,\infty}$) and all $v_h \in X_h$ (resp. $q_h \in M_h$)

$$\big\|v_h\phi - P_h(v_h\phi)\big\|_{H^l} \leq ch^{1+m-l}\|v_h\|_{H^m}\|\phi\|_{W^{m+1,\infty}}, \tag{3.4}$$

$$\big\|q_h\phi - Q_h(q_h\phi)\big\|_2 \leq ch\|q_h\|_2\|\phi\|_{W^{1,\infty}}, \tag{3.5}$$

for all $0 \leq l \leq m \leq 1$.

Remark 3.2 Explicit and relevant examples of couples (X_h, M_h) of finite element spaces satisfying the commutator property are those employed in the MINI and Hood–Taylor elements with quasi-uniform mesh, see [16]. In addition, similar approximation properties of finite element functions multiplied by a smooth function are known in the literature as superapproximation; see, e.g., Demlow, Guzmán, and Schatz [13].

We recall from [16] that the coercivity hypothesis (3.2) allows us to define the map $\psi_h : H_\#^2 \to M_h$ such that, for all $q \in H_\#^2$, the function $\psi_h(q)$ is the unique solution to the problem:

$$\big(\pi_h\big(\nabla \psi_h(q)\big), \nabla r_h\big) = (\nabla q, \nabla r_h).$$

This map has the following properties: there exists c, independent of h, such that for all $q \in H_\#^2$,

$$\big\|\nabla\big(\psi_h(q) - q\big)\big\|_2 \le ch\|q\|_{H^2},$$
$$\big\|\pi_h \nabla \psi_h(q)\big\|_{H^1} \le c\|q\|_{H^2}.$$

Let us introduce the space of discretely divergence-free functions

$$V_h = \big\{v_h \in X_h : (\operatorname{div} v_h, q_h) = 0 \ \forall q_h \in M_h\big\}.$$

The most common variational formulation (for the continuous problems) of the convective term $nl(u, v) := (u \cdot \nabla)v$ is

$$b(u, v, w) = \int_{\mathbb{T}^3} (u \cdot \nabla)v \cdot w\, dx,$$

and the fact that $b(u, v, v) = 0$ for $u \in L_{\mathrm{div}}^2$, $v \in H_\#^1$ allows us to deduce, at least formally, the energy inequality (2.2). This cancellation is based on the constraint $\operatorname{div} u = 0$, and this identity is not valid anymore in the case of discretely divergence-free functions in V_h. To have the basic energy estimate, we need to modify the non-linear term since $V_h \nsubseteq H_{\mathrm{div}}^1$ and the choice of the weak formulation becomes particularly relevant in the discrete case since it leads to schemes with very different numerical properties.

To formulate the various schemes, we will consider which one corresponds to the Cases 1-2-3. We define the discrete tri-linear operator $b_h(\cdot, \cdot, \cdot)$ in different (but standard) ways. This permits a sort of unified treatment: for instance, in all the three cases considered below, it holds *at least* that $nl_h(u, v)$ – which is the discrete counterpart of $nl(u, v)$ – satisfies the following estimate

$$\big\|nl_h(u, v)\big\|_{H^{-1}} \le \|u\|_3 \|v\|_{H^1} \quad \forall u, v \in H_\#^1.$$

We will now detail the various different discrete formulations we will use.

Case 1: We use the most common option, which is a "symmetrized" operator

$$nl_h(u, v) := (u \cdot \nabla)v + \frac{1}{2}v \operatorname{div} u, \tag{3.6}$$

for the convective term, which leads to the tri-linear form

$$b_h(u,v,w) := \langle nl_h(u,v), w \rangle_{H^{-1} \times H_\#^1}, \tag{3.7}$$

such that

$$b_h(u,v,v) = 0 \quad \forall u, v \in H_{\text{div}}^1 + V_h.$$

Moreover, this tri-linear operator can also be estimated as follows

$$\begin{aligned}
\left| b_h(u,v,w) \right| &\leq \|u\|_6 \|\nabla v\|_2 \|w\|_3 + \frac{1}{2} \|v\|_6 \|\operatorname{div} u\|_2 \|w\|_3 \\
&\leq C \|\nabla u\|_2 \|\nabla v\|_2 \|w\|_2^{1/2} \|\nabla w\|_2^{1/2},
\end{aligned} \tag{3.8}$$

by means of the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ and of the convex interpolation inequality.

Case 2: Alternatively, we can consider the "rotational form without pressure," as in Layton *et al.* [26], which corresponds to the formulation

$$nl_h(u,v) := (\nabla \times u) \times v, \tag{3.9}$$

and leads to the tri-linear form

$$b_h(u,v,w) := \langle nl_h(u,v), w \rangle_{H^{-1} \times H_\#^1},$$

such that

$$b_h(u,v,v) = 0 \quad \forall u, v \in H_{\text{div}}^1 + V_h.$$

Moreover, this term can be estimated as follows

$$\begin{aligned}
\left| b_h(u,v,w) \right| &\leq \|\nabla \times u\|_2 \|v\|_6 \|w\|_3 \\
&\leq C \|\nabla u\|_2 \|\nabla v\|_2 \|w\|_2^{1/2} \|\nabla w\|_2^{1/2}
\end{aligned} \tag{3.10}$$

again by means of the Sobolev embedding and of the convex interpolation inequality. In this case, one is *hiding* the Bernoulli pressure $\frac{1}{2}|v|^2$ into the kinematic pressure. It is well documented that the scheme is easier to be handled but the *under resolution* of the pressure has some effects on the accuracy; see Horiuti [23], Zang [35], and the discussion in [26].

To overcome the numerical problems arising when using the operator from Case 2, other computationally more expensive methods are considered, as the one below.

Case 3: We consider the rotational form with the approximation of the Bernoulli pressure, as was studied in Guermond [16].

$$nl_h(u,v) := (\nabla \times u) \times v + \frac{1}{2} \nabla \big(\mathcal{K}_h(v \cdot u) \big),$$

where $\mathcal{K}_h$ is the $L^2 \to M_h$ projection operator, which is stable, linear and is defined as $(\mathcal{K}_h u, v_h) = (u, v_h)$, for all $u \in L^2$ and $v_h \in M_h$. In this way, the tri-linear term is such that

$$b_h(u, v, w) := \big\langle nl_h(u, v), w \big\rangle_{H^{-1} \times H^1_\#},$$

such that

$$b_h(u, v, v) = 0 \quad \forall u, v \in H^1_{\mathrm{div}} + V_h.$$

The first estimate, which is proved also in [16], is the following one:

$$\big| b_h(v, v, w) \big| \le c \|v\|_{H^1} \|v\|_3 \|w\|_{H^1}.$$

Here, to gain a better estimate from the presence of the projection of the Bernoulli pressure, we use some improved properties of the L^2-projection operator $\mathcal{K}_h$, which are valid in the case of quasi-uniform meshes. Indeed, for special meshes, one can also show the $W^{1,p}$ stability. The improved stability for the L^2-projection has a long history; see Douglas, Jr., Dupont, and Wahlbin [15], Bramble and Xu [9], and Carstensen [12]. Moreover, in the recent work by Diening, Storn, and Tscherpel [14], it is also analyzed the stability for highly adaptive meshes and the newest vertex bisection (NVB). In three-space dimensions, their theory covers the range of exponents we consider, at least for Taylor–Hood elements. Here, we limit ourselves to consider quasi-uniform meshes, but combining the cited results with our methods more general meshes could be probably handled. A different approach in Hilbert fractional spaces is used in [17], but we do not know whether this applies to the estimates we use. Anyway, we could not find the detailed proof of the required stability, which can be obtained using the L^p-stability of the operator $\mathcal{K}_h$, the inverse inequality, valid for the meshes we consider, and the $W^{1,p}$-stability and approximation of the Scott–Zhang projection operator Π_h (see [10]) valid for $f \in W^{1,p}(\Omega)$, for $p \in [1, \infty[$. Just to sketch the argument, it is enough to use the following inequalities

$$\begin{aligned}
\|\mathcal{K}_h f\|_{W^{1,p}} &\le c \big\| \mathcal{K}_h (f - \Pi_h f) \big\|_{W^{1,p}} + \|\Pi_h f\|_{W^{1,p}} \\
&\le c h^{-1} \big\| \mathcal{K}_h (f - \Pi_h f) \big\|_p + \|\Pi_h f\|_{W^{1,p}} \\
&\le c h^{-1} \| f - \Pi_h f \|_p + \|\Pi_h f\|_{W^{1,p}} \\
&\le c \| f \|_{W^{1,p}}.
\end{aligned}$$

With this stability result, in Case 3, the nonlinear term can be estimated as follows

$$\begin{aligned}
\big| b_h(u, v, w) \big| &\le \|\nabla u\|_2 \|v\|_6 \|w\|_3 + \big\| \nabla \mathcal{K}_h (u \cdot v) \big\|_{3/2} \|w\|_3 \\
&\le \big(C \|\nabla u\|_2 \|\nabla v\|_2 + \big\| \mathcal{K}_h (u \cdot v) \big\|_{W^{1,3/2}} \big) \|w\|_3 \\
&\le \big(C \|\nabla u\|_2 \|\nabla v\|_2 + \|u\|_3 \|v\|_3 + \|\nabla u\|_2 \|v\|_6 + \|u\|_6 \|\nabla v\|_2 \big) \|w\|_3 \\
&\le C \|\nabla u\|_2 \|\nabla v\|_2 \|w\|_2^{1/2} \|\nabla w\|_2^{1/2},
\end{aligned} \tag{3.11}$$

by means of the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ and of the convex interpolation inequality, exactly as in the first two cases.

3.2 Time discretization

We now pass to the description of the time discretization. For the time variable t, we define the mesh as follows: Given $N \in \mathbb{N}$ the time-step $0 < \Delta t \leq T$ is defined as $\Delta t := T/N$. Accordingly, we define the corresponding net $\{t_m\}_{m=1}^N$ by

$$t_0 := 0, \qquad t_m := m\Delta t, \quad m = 1, \dots, N.$$

We consider the Crank–Nicolson method (CN) (cf. [29, §5.6.2]). With a slight abuse of notation, we consider $\Delta t = T/N$ and h, instead of (N, h), as the indexes of the sequences for which we prove the convergence. Then, the convergence will be proved in the limit as $(\Delta t, h) \to (0, 0)$. We stress that this does not affect the proofs since all the convergences are proved up to sub-sequences.

Once (CN) is solved, we consider a continuous version useful to study the convergence. To this end, we associate to the triple $(u_h^{m,1/2}, u_h^m, p_h^m)$ the functions

$$\left(v_h^{\Delta t}, u_h^{\Delta t}, p_h^{\Delta t}\right) : [0, T] \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R},$$

defined as follows:

$$
\begin{aligned}
v_h^{\Delta t}(t) &:= \begin{cases} u_h^{m-1} + \frac{t - t_{m-1}}{\Delta t}(u_h^m - u_h^{m-1}) & \text{for } t \in [t_{m-1}, t_m), \\ u_h^N & \text{for } t = t_N, \end{cases} \\[2mm]
u_h^{\Delta t}(t) &:= \begin{cases} u_h^{m,1/2} & \text{for } t \in [t_{m-1}, t_m), \\ u_h^{N,1/2} & \text{for } t = t_N, \end{cases} \\[2mm]
p_h^{\Delta t}(t) &:= \begin{cases} p_h^m & \text{for } t \in [t_{m-1}, t_m), \\ p_h^N & \text{for } t = t_N. \end{cases}
\end{aligned}
\tag{3.12}
$$

Then, the discrete equations (CN) can be rephrased as the following time-continuous system:

$$
\begin{aligned}
&\left(\partial_t v_h^{\Delta t}, w_h\right) + b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, w_h\right) + \nu\left(\nabla u_h^{\Delta t}, \nabla w_h\right) - \left(p_h^{\Delta t}, \operatorname{div} w_h\right) = 0, \\
&\left(\operatorname{div} u_h^{\Delta t}, q_h\right) = 0,
\end{aligned}
\tag{3.13}
$$

for all $w_h \in L^s(0, T; X_h)$ (with $s \geq 4$) and for all $q_h \in L^2(0, T; M_h)$. We notice that the divergence-free condition comes from the fact that u_h^m is such that

$$\left(\operatorname{div} u_h^m, q_h\right) = 0 \quad \text{for } m = 1, \dots, N, \forall q_h \in M_h.$$

4 A priori estimates

In this section, we prove the *a priori* estimates that we need to study the convergence of solutions of (3.13) to suitable weak solutions of (1.1)–(1.2). We start with the following discrete energy equality.

Lemma 4.1 *Let $N \in \mathbb{N}$ and $m = 1, .., N$. Then, for solutions to* (CN), *the following* (*global*) *discrete energy-type equality holds true*:

$$\frac{1}{2}\big(\big\|u_h^m\big\|_2^2 - \big\|u_h^{m-1}\big\|_2^2\big) + \nu\Delta t\big\|\nabla u_h^{m,\frac{1}{2}}\big\|_2^2 = 0. \tag{4.1}$$

Moreover, if $u_0 \in H_{\mathrm{div}}^1$, there exists $C > 0$ depending on $\|u_0\|_{H^1}$ such that

$$\sum_{m=1}^N \big\|u_h^m - u_h^{m-1}\big\|_2^2 \le C\bigg(\Delta t + \frac{1}{h^{1/2}}\bigg). \tag{4.2}$$

Proof We start by proving the (global) discrete energy equality. For any $m = 1, \ldots, N$ take $w_h = \chi_{[t_{m-1}, t_m)} u_h^{m,1/2} \in L^\infty(0, T; X_h)$ as test function in (3.13). Then, it follows

$$\bigg(\frac{u_h^m - u_h^{m-1}}{\Delta t}, u_h^{m,1/2}\bigg) + \nu\big\|\nabla u_h^{m,1/2}\big\|_2^2 = 0,$$

which holds true since $u_h^{m,1/2} \in X_h$ and $p_h^m \in M_h$, we have that

$$b_h\big(u_h^{m,1/2}, u_h^{m,1/2}, u_h^{m,1/2}\big) = 0 \quad \text{and} \quad \big(p_h^m, \mathrm{div}\, u_h^{m,1/2}\big) = 0.$$

The term involving the discretization of the time-derivative reads as follows:

$$\big(u_h^m - u_h^{m-1}, u_h^{m,1/2}\big) = \frac{1}{2}\big(u_h^m - u_h^{m-1}, u_h^m + u_h^{m-1}\big) = \frac{1}{2}\big(\big\|u_h^m\big\|_2^2 - \big\|u_h^{m-1}\big\|_2^2\big).$$

Then, multiplying by $\Delta t > 0$, Eq. (4.1) holds true. In addition, summing over m, we also get

$$\frac{1}{2}\big\|u_h^N\big\|_2^2 + \nu\Delta t \sum_{m=0}^N \big\|\nabla u_h^{m,1/2}\big\|_2^2 = \frac{1}{2}\big\|u_h^0\big\|_2^2,$$

which proves the $l^\infty(L_\#^2) \cap l^2(H_\#^1)$ uniform bound for the sequence $\{u_h^m\}$.

To prove (4.2) take $w_h = \chi_{[t_{m-1}, t_m)}(u_h^m - u_h^{m-1}) \in L^\infty(0, T; X_h)$ in (3.13). Then, after multiplication by Δt, we get

$$\big\|u_h^m - u_h^{m-1}\big\|_2^2 + \frac{\nu\Delta t}{2}\big(\big\|\nabla u_h^m\big\|^2 - \big\|\nabla u_h^{m-1}\big\|^2\big)$$

$$\le \Delta t\big|b_h\big(u_h^{m,\frac{1}{2}}, u_h^{m,\frac{1}{2}}, u_h^m - u_h^{m-1}\big)\big|$$

$$\le C\Delta t\big\|\nabla u_h^{m,\frac{1}{2}}\big\|_2^2\big\|u_h^m - u_h^{m-1}\big\|_2^{1/2}\big\|\nabla\big(u_h^m - u_h^{m-1}\big)\big\|_2^{1/2},$$

where in the last line, (3.8) has been used. Using the inverse inequality (3.3) and summing over $m = 1, \ldots, N$, we get

$$\frac{\nu\Delta t}{2}\big\|\nabla u_h^N\big\|_2^2 + \sum_{m=1}^N \big\|u_h^m - u_h^{m-1}\big\|_2^2$$

$$\le \frac{\nu\Delta t}{2}\big\|\nabla u_h^0\big\|_2^2 + C\frac{\Delta t}{h^{1/2}} \sum_{m=1}^N \big\|\nabla u_h^{m,\frac{1}{2}}\big\|_2^2\big\|u_h^m - u_h^{m-1}\big\|_2$$

$$\leq \frac{\nu \Delta t}{2} \left\| \nabla u_h^0 \right\|_2^2 + C \frac{\Delta t}{h^{1/2}} \sum_{m=1}^{N} \left\| \nabla u_h^{m,\frac{1}{2}} \right\|_2^2 \left(\left\| u_h^m \right\|_2 + \left\| u_h^{m-1} \right\|_2 \right)$$

$$\leq \frac{\nu \Delta t}{2} \left\| \nabla u_h^0 \right\|_2^2 + 2C \left\| u_h^0 \right\|_2 \frac{\Delta t}{h^{1/2}} \sum_{m=1}^{N} \left\| \nabla u_h^{m,\frac{1}{2}} \right\|_2^2$$

$$\leq \frac{\nu \Delta t}{2} \left\| \nabla u_h^0 \right\|_2^2 + \frac{2C \| u_h^0 \|_2^3}{\nu h^{1/2}},$$

where we used the $l^\infty(L_\#^2) \cap l^2(H_\#^1)$ bounds coming from the energy equality and then proving the thesis. □

Remark 4.2 At first glance the inequality (4.2) seems useless, being badly depending on h. Recall that the convergence to zero of $\sum_{m=1}^{N} \| u_h^m - u_h^{m-1} \|_2^2$ is a required step to identify the limits of $v_h^{\Delta t}$ and of $u_h^{\Delta t}$. Nevertheless, the key step in the next section will be that of combining this inequality with a standard restriction on the ratio between time and space mesh-size, to enforce the equality of the two limiting functions.

The following lemma concerns the regularity of the pressure. Following the argument in [16], we notice that we are essentially solving the standard discrete Poisson problem associated with the pressure. It is for this result that the space-periodic setting is needed.

Lemma 4.3 *There exists a constant $c > 0$, independent of Δt and of h, such that*

$$\left\| p_h^m \right\|_2 \leq c \left(\left\| u_h^{m,1/2} \right\|_{H^1} + \left\| u_h^{m,1/2} \right\|_3 \left\| u_h^{m,1/2} \right\|_{H^1} \right) \quad \text{for } m = 1, \dots, N.$$

Proof The proof is exactly the same as in [5, Lemma 4.3]. □

We are now in position to prove the main a priori estimates on the approximate solutions of (3.13).

Proposition 4.4 *Let $u_0 \in L_{\mathrm{div}}^2$ and assume that (1.3) holds. Then, there exists a constant $c > 0$, independent of Δt and of h, such that*

  a) $\| v_h^{\Delta t} \|_{L^\infty(L^2)} \leq c$,
  b) $\| u_h^{\Delta t} \|_{L^\infty(L^2) \cap L^2(H^1)} \leq c$,
  c) $\| p_h^{\Delta t} \|_{L^{4/3}(L^2)} \leq c$,
  d) $\| \partial_t v_h^{\Delta t} \|_{L^{4/3}(H^{-1})} \leq c$,

Moreover, we also have the following estimate

$$\int_0^T \left\| u_h^{\Delta t} - v_h^{\Delta t} \right\|_2^2 dt \leq \frac{\Delta t}{12} \sum_{m=1}^{N} \left\| u_h^m - u_h^{m-1} \right\|_2^2. \tag{4.3}$$

Proof The bound in $L^\infty(0, T; L_\#^2) \cap L^2(0, T; H_\#^1)$ for $v_h^{\Delta t}$ follows from (3.12) and Lemma 4.1, as well as the bounds on $u_h^{\Delta t}$ in b). The bound on the pressure $p_h^{\Delta t}$ follows again from (3.12) and Lemma 4.3. Finally, the bound on the time derivative of $v_h^{\Delta t}$ follows by (3.13) and a standard comparison argument. Concerning (4.3), using the definitions in (3.12), we get for $t \in [t_{m-1}, t_m)$

$$u_h^{\Delta t} - v_h^{\Delta t} = \frac{1}{2} u_h^m + \left(1 - \frac{1}{2} \right) u_h^{m-1} - u_h^{m-1} - \frac{t - t_{m-1}}{\Delta t} \left(u_h^m - u_h^{m-1} \right)$$

$$= \left(\frac{1}{2} - \frac{t - t_{m-1}}{\Delta t} \right) \left(u_h^m - u_h^{m-1} \right).$$

Then, evaluating the integrals, we have

$$\int_0^T \left\| u_h^{\Delta t} - v_h^{\Delta t} \right\|_2^2 dt = \sum_{m=1}^N \left\| u_h^m - u_h^{m-1} \right\|_2^2 \int_{t_{m-1}}^{t_m} \left(\frac{1}{2} - \frac{t - t_{m-1}}{\Delta t} \right)^2 dt$$

$$\leq \frac{\Delta t}{12} \sum_{m=1}^N \left\| u_h^m - u_h^{m-1} \right\|_2^2. \qquad \square$$

5 Proof of the main theorem

In this section, we prove Theorem 1.1. We split the proof into two main steps: The first concerns showing that discrete solutions converge to a Leray–Hopf weak solution, while the second proves the constructed solutions are, in fact, suitable.

Proof of Theorem 1.1 We first prove the convergence of the numerical sequence to a Leray–Hopf weak solution, mainly proving the correct balance of the global energy (2.2); then, we prove that the weak solution constructed is suitable, namely that it satisfies the local energy inequality (2.3).

Step 1: convergence toward a Leray–Hopf weak solution. We start by observing that from a simple density argument, the test functions considered in (2.1) can be chosen in the space $L^s(0, T; H^1_{\mathrm{div}}) \cap C^1(0, T; L^2_{\mathrm{div}})$, with $s \geq 4$. In particular, using (3.1) for any $w \in L^s(0, T; H^1_{\mathrm{div}}) \cap C^1(0, T; L^2_{\mathrm{div}})$ such that $w(T, x) = 0$, we can find a sequence $\{w_h\}_h \subset L^s(0, T; H^1_\#) \cap C(0, T; L^2_\#)$ such that

$$w_h \to w \quad \text{strongly in } L^s\left(0, T; H^1_\#\right) \text{ as } h \to 0,$$

$$w_h(0) \to w(0) \quad \text{strongly in } L^2_\# \text{ as } h \to 0, \tag{5.1}$$

$$\partial_t w_h \rightharpoonup \partial_t w \quad \text{weakly in } L^2\left(0, T; L^2_\#\right) \text{ as } h \to 0.$$

Let $\{(v_h^{\Delta t}, v_h^{\Delta t}, p_h^{\Delta t})\}_{(\Delta t, h)}$, defined as in (3.12), be a family of solutions of (3.13). By Proposition 4.4a), we have that

$$\left\{ v_h^{\Delta t} \right\}_{(\Delta t, h)} \subset L^\infty\left(0, T; L^2_\#\right), \quad \text{with uniform bounds on the norms.}$$

Then, by standard compactness arguments, there exists $v \in L^\infty(0, T; L^2_\#)$ such that (up to a sub-sequence)

$$v_h^{\Delta t} \rightharpoonup v \quad \text{weakly in } L^2\left(0, T; L^2_\#\right) \text{ as } (\Delta t, h) \to (0, 0). \tag{5.2}$$

Again using Proposition 4.4 b), there exists $u \in L^\infty(0, T; L^2_\#)$ such that (up to a sub-sequence)

$$u_h^{\Delta t} \overset{*}{\rightharpoonup} u \quad \text{weakly* in } L^\infty\left(0, T; L^2_\#\right) \text{ as } (\Delta t, h) \to (0, 0),$$

$$u_h^{\Delta t} \rightharpoonup u \quad \text{weakly in } L^2\left(0, T; H^1_\#\right) \text{ as } (\Delta t, h) \to (0, 0). \tag{5.3}$$

Moreover, using (3.1), for any $q \in L^2(0, T; L^2_\#)$, we can find a sequence $\{q_h\}_h \subset L^2(0, T; L^2_\#)$ such that $q_h \in L^2(0, T; M_h)$ and

$$q_h \to q \quad \text{strongly in } L^2\big(0, T; L^2_\#\big) \text{ as } h \to 0.$$

Then, using (5.3) and (3.13), we have that

$$0 = \int_0^T \big(\operatorname{div} u_h^{\Delta t}, q_h\big)\, dt \to \int_0^T (\operatorname{div} u, q)\, dt \quad \text{as } (\Delta t, h) \to (0, 0),$$

hence u is divergence-free, since it belongs to H^1_{div}. Let us consider (4.3), then

$$\int_0^T \big\| v_h^{\Delta t} - u_h^{\Delta t} \big\|_2^2\, dt \leq \frac{\Delta t}{12} \sum_{m=1}^N \big\| u_h^m - u_h^{m-1} \big\|_2^2 \leq C\bigg((\Delta t)^2 + \frac{\Delta t}{h^{1/2}}\bigg), \tag{5.4}$$

where in the last inequality we used again Proposition 4.4 and the estimate (4.2). Then, the integral $\int_0^T \| v_h^{\Delta t} - u_h^{\Delta t} \|_2^2\, dt$ vanishes as $\Delta t \to 0$ if $\Delta t = o(h^{1/2})$, that is if (1.3) is satisfied. Then, from (5.2) and (5.3), it easily follows that $v = u$.

The rest of the proof follows the same as in [5], so we just sketch out the proof, referring to [5] for complete details.

By Lemma 2.4 and the fact that $u = v$, we get that $u_h^{\Delta t} v_h^{\Delta t} \rightharpoonup |u|^2$ weakly in $L^1((0, T) \times \mathbb{T}^3)$ as $(\Delta t, h) \to (0, 0)$. In particular, using (5.4), we have that

$$v_h^{\Delta t}, u_h^{\Delta t} \to u \quad \text{strongly in } L^2\big(0, T; L^2_\#\big) \text{ as } (\Delta t, h) \to (0, 0). \tag{5.5}$$

Concerning the pressure term the uniform bound in Proposition 4.4 d) ensures the existence of $p \in L^{4/3}(0, T; L^2_\#)$ such that (up to a sub-sequence)

$$p_h^{\Delta t} \rightharpoonup p \quad \text{weakly in } L^{4/3}\big(0, T; L^2_\#\big) \text{ as } (\Delta t, h) \to (0, 0). \tag{5.6}$$

Then, using (5.1) and (5.2), we have that

$$\lim_{(\Delta t, h) \to (0,0)} \int_0^T \big(\partial_t v_h^{\Delta t}, w_h\big)\, dt = -\int_0^T (u, \partial_t w)\, dt - \big(u_0, w(0)\big),$$

Next, using (5.3), (5.1), and (5.6), we also get

$$\lim_{(\Delta t, h) \to (0,0)} \int_0^T \big(\nabla u_h^{\Delta t}, \nabla w_h\big)\, dt = \int_0^T (\nabla u, \nabla w)\, dt,$$

$$\int_0^T \big(p_h^{\Delta t}, \operatorname{div} w_h\big)\, dt \to 0 \quad \text{as } (\Delta t, h) \to (0, 0).$$

Concerning the non-linear term, let $s \geq 4$, with a standard compactness argument

$$nl_h\big(u_h^{\Delta t}, u_h^{\Delta t}\big) \rightharpoonup u \cdot \nabla u, \quad \text{in } L^{s'}\big(0, T; H^{-1}\big) \text{ as } (\Delta t, h) \to (0, 0).$$

Then, also from (5.1), it follows that

$$\int_0^T b_h\big(u_h^{\Delta t}, u_h^{\Delta t}, w_h\big)\, dt \to \int_0^T \big((u \cdot \nabla)u, w\big)\, dt \quad \text{as } (\Delta t, h) \to (0, 0). \tag{5.7}$$

Finally, the energy inequality follows by Lemma 4.1, using the lower semi-continuity of the L^2-norm with respect to the weak convergence, since the estimate (4.1) can be rewritten as

$$\frac{1}{2}\left\|v_h^{\Delta t}(T)\right\|_2^2 + \nu \int_0^T \left\|\nabla u_h^{\Delta t}(t)\right\|_2^2 dt \leq \frac{1}{2}\|u_0\|_2^2.$$

The treatment of the Case 2 and Case 3 can be done with minor changes, concerning the tri-linear term, just using the estimate already proved in the previous section. The other terms are unchanged, and the energy estimate remains the same.

Remark 5.1 The results for Case 2 and Case 3 can be easily adapted also to cover the θ-scheme for $\theta > 1/2$, hence completing the results in [5], which were focusing only on the treatment of Case 1.

Note that the tri-linear term based on the rotational formulation from Case 2 and Case 3 $(\nabla \times u_h^{\Delta t}) \times u_h^{\Delta t}$ converges exactly as in the previous step, since

$$\left(\nabla \times u_h^{\Delta t}\right) \times u_h^{\Delta t} \rightharpoonup (\nabla \times u) \times u, \text{ in } L^{s'}\left(0, T; H^{-1}\right) \quad \text{as } (\Delta t, h) \to (0,0),$$

which implies (5.7), ending the proof in Case 2.

In Case 3, the term, which needs some care, is the projected Bernoulli pressure. In this case, note that for $\frac{1}{s^*} + \frac{1}{2} = \frac{1}{s'}$,

$$\int_0^T \left\|\mathcal{K}_h\left(\left|u_h^{\Delta t}\right|^2\right) - \mathcal{K}_h\left(|u|^2\right)\right\|_2^{s'} dt \leq \int_0^T \left\|\left|u_h^{\Delta t}\right|^2 - |u|^2\right\|_2^{s'} dt$$

$$\leq \int_0^T \left\|\left|u_h^{\Delta t} - u\right|\left|u_h^{\Delta t} + u\right|\right\|_2^{s'} dt$$

$$\leq \left\|u_h^{\Delta t} - u\right\|_{L^{s^*}(L^3)}\left(\left\|u_h^{\Delta t}\right\|_{L^2(L^6)} + \|u\|_{L^2(L^6)}\right).$$

This shows that

$$\mathcal{K}_h\left(\left|u_h^{\Delta t}\right|^2\right) \to \mathcal{K}_h\left(|u|^2\right) \quad \text{in } L^{s'}\left(0, T; L^2(\mathbb{T}^3)\right).$$

Moreover, since $\mathcal{K}_h(|w|^2) \to |w|^2$ in $L^2(\mathbb{T}^3)$ for a.e. $t \in (0, T)$ and $\|\mathcal{K}_h(|w|^2)\|_2 \leq \|w\|_2^2$, by Lebesgue dominated convergence, we have $\mathcal{K}_h(|w|^2) \to |w|^2$ in $L^{s'}(0, T; L^2(\mathbb{T}^3))$, finally showing that

$$\int_0^T \left(\mathcal{K}_h\left(\left|u_h^{\Delta t}\right|^2\right), \operatorname{div} v\right) dt \to \int_0^T \left(|u|^2, \operatorname{div} v\right) dt.$$

This proves, after integration by parts, that

$$\int_0^T b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, w\right) dt \to \int_0^T (u \cdot \nabla)u, w) dt.$$

Note also that since in all cases, it holds that

$$\frac{1}{2}\left\|v_h^{\Delta t}(T)\right\|_2^2 + \nu \int_0^T \left\|\nabla u_h^{\Delta t}(t)\right\|_2^2 dt \leq \frac{1}{2}\|u_0\|_2^2, \tag{5.8}$$

a standard lower semi-continuity argument is enough to infer that the weak solution

$$u = v = \lim_{(h,\Delta t)\to(0,0)} u_h^{\Delta t} = \lim_{(h,\Delta t)\to(0,0)} v_h^{\Delta t}$$

also satisfies the global energy inequality (2.2).

Step 2: proof of the local energy inequality. In order to conclude the proof of Theorem 1.1, we need to prove that the Leray–Hopf weak solution constructed in Step 1 is suitable. According to Definition 2.2, this requires just to prove the local energy inequality. To this end, let us consider a smooth, periodic in the space variable function $\phi \geq 0$, vanishing for $t = 0, T$; we use $P_h(u_h^{\Delta t}\phi)$ as test function in the momentum equation in (3.13).

The term involving the time-derivative is treated as in [5].

$$\int_0^T \left(\partial_t v_h^{\Delta t}, P_h\left(u_h^{\Delta t}\phi\right)\right) dt$$
$$= \int_0^T \left(\partial_t v_h^{\Delta t}, u_h^{\Delta t}\phi\right) dt + \int_0^T \left(\partial_t v_h^{\Delta t}, P_h\left(u_h^{\Delta t}\phi\right) - u_h^{\Delta t}\phi\right) dt =: I_1 + I_2.$$

Concerning the term I_1, we have that

$$= \int_0^T \left(\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi\right) dt + \int_0^T \left(\partial_t v_h^N, \left(u_h^{\Delta t} - v_h^{\Delta t}\right)\phi\right) dt =: I_{11} + I_{12}.$$

Let us first consider I_{11}. By splitting the integral over $[0, T]$ as the sum of integrals over $[t_{m-1}, t_m]$ and by integration by parts, we get

$$\int_0^T \left(\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi\right) dt$$
$$= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi\right) dt = \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\frac{1}{2}\partial_t\left|v_h^{\Delta t}\right|^2, \phi\right) dt$$
$$= \frac{1}{2}\sum_{m=1}^N \left(\left|u_h^m\right|^2, \phi(t_m, x)\right) - \left(\left|u_h^{m-1}\right|^2, \phi(t_{m-1}, x)\right) - \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\frac{1}{2}\left|v_h^{\Delta t}\right|^2, \partial_t\phi\right) dt,$$

where we used that $\partial_t v_h^{\Delta t}(t) = \frac{u_h^m - u_h^{m-1}}{\Delta t}$, for $t \in [t_{m-1}, t_m[$. Next, since the sum telescopes and ϕ is with compact support in $(0, T)$, we get

$$\int_0^T \left(\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi\right) dt = -\int_0^T \left(\frac{1}{2}\left|v_h^{\Delta t}\right|^2, \partial_t\phi\right) dt.$$

By the strong convergence of $v_h^{\Delta t} \to u$ in $L^2(0, T; L_\#^2)$, we can conclude that

$$\lim_{(\Delta t, h)\to(0,0)} \int_0^T \left(\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi\right) dt = -\int_0^T \left(\frac{1}{2}|u|^2, \partial_t\phi\right) dt.$$

Then, we consider the term I_{12}. Since $u_h^{\Delta t}$ is constant on the interval $[t_{m-1}, t_m[$, we can write

$$\int_0^T \left(\partial_t v_h^{\Delta t}, \left(u_h^{\Delta t} - v_h^{\Delta t}\right)\phi\right) dt = -\sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\partial_t\left(v_h^{\Delta t} - u_h^{\Delta t}\right), \left(v_h^{\Delta t} - u_h^{\Delta t}\right)\phi\right) dt$$

$$= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\frac{|v_h^{\Delta t} - u_h^{\Delta t}|^2}{2}, \partial_t\phi\right) dt,$$

since the sum telescopes. Hence, we have that $u_h^{\Delta t} - v_h^{\Delta t}$ vanishes (strongly) in $L^2(0, T; L_\#^2)$, provided that $\Delta t = o(h^{1/2})$. Then, $I_{12} \to 0$ as $(\Delta t, h) \to (0, 0)$.

Remark 5.2　It is at this point that the coupling between h and Δt plays a role. For the convergence of the other terms, the discrete commutator property is needed. We are skipping some details from the other proofs because they are very close to that in the cited references.

We have that the $I_2 \to 0$ as $(\Delta t, h) \to (0, 0)$. Indeed, by the discrete commutator property (3.4), Proposition 4.4, and the inverse inequality (3.3), we can infer

$$|I_2| \leq \int_0^T \left\|\partial_t v_h^{\Delta t}\right\|_{H^{-1}} \left\|P_h\left(u_h^{\Delta t}\phi\right) - u_h^{\Delta t}\phi\right\|_{H^1} dt$$

$$\leq ch^{1/2} \left\|\partial_t v_h^{\Delta t}\right\|_{L^{4/3}(H^{-1})} \left\|u_h^{\Delta t}\right\|_{L^\infty(L^2)}^{1/2} \left\|u_h^{\Delta t}\right\|_{L^2(H^1)}^{1/2} \leq ch^{1/2}.$$

Hence, this term also vanishes as $h \to 0$, ending the analysis of the term involving the time-derivative.

Concerning the viscous term, we write

$$\left(\nabla u_h^{\Delta t}, \nabla P_h\left(u_h^{\Delta t}\phi\right)\right) = \left(\left|\nabla u_h^{\Delta t}\right|^2, \phi\right) - \left(\frac{1}{2}\left|u_h^{\Delta t}\right|^2, \Delta\phi\right) + R_{visc},$$

with the "viscous remainder" $R_{visc} := (\nabla u_h^{\Delta t}, \nabla[P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi])$. Since $u_h^{\Delta t}$ converges to u weakly in $L^2(0, T; H_\#^1)$ and strongly in $L^2(0, T; L_\#^2)$,

$$\liminf_{(\Delta t, h) \to (0, 0)} \int_0^T \left(\left|\nabla u_h^{\Delta t}\right|^2, \phi\right) dt \geq \int_0^T \left(\left|\nabla u\right|^2, \phi\right) dt,$$

$$\frac{1}{2}\int_0^T \left(\left|u_h^{\Delta t}\right|^2, \Delta\phi\right) dt \to \frac{1}{2}\int_0^T \left(\left|u\right|^2, \Delta\phi\right) dt.$$

For the remainder R_{visc}, using again the discrete commutator property from Definition 3.1, we have that

$$\left|\int_0^T R_{visc}\, dt\right| \leq ch \int_0^T \left\|\nabla u_h^{\Delta t}\right\|_2^2 dt \to 0 \quad \text{as } (\Delta t, h) \to (0, 0).$$

We consider now the nonlinear term b_h. We have

$$b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, P_h\left(u_h^{\Delta t}\phi\right)\right) = b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi\right) + R_{nl}. \tag{5.9}$$

The "nonlinear remainder" $R_{nl} := b_h(u_h^{\Delta t}, u_h^{\Delta t}, P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi)$ can be estimated using the discrete commutator property, (3.3), and (3.8), (3.10), (3.11) for the choices of the nonlinear term approximation in Case 1, Case 2, and Case 3, respectively. Indeed, we have

$$|R_{nl}| \le \left\| nl_h\left(u_h^{\Delta t}, u_h^{\Delta t}\right) \right\|_{H^{-1}} \left\| P_h\left(u_h^{\Delta t}\phi\right) - u_h^{\Delta t}\phi \right\|_{H^1} \le ch^{1/2} \left\| u_h^{\Delta t} \right\|_2 \left\| u_h^{\Delta t} \right\|_{H^1}^2, \qquad (5.10)$$

hence, by integrating over time

$$\int_0^T R_{nl}\, dt \to 0 \quad \text{as } (\Delta t, h) \to (0,0).$$

The last term we consider is that involving the pressure. By integrating by parts, we have

$$\left(p_h^{\Delta t}, \operatorname{div} P_h\left(u_h^{\Delta t}\phi\right)\right) = \left(p_h^{\Delta t} u_h^{\Delta t}, \nabla\phi\right) + R_{p1} + R_{p2}.$$

where the two "pressure remainders" are defined as follows

$$R_{p1} := \left(p_h^{\Delta t}, \operatorname{div}\left(P_h\left(u_h^{\Delta t}\phi\right) - u_h^{\Delta t}\phi\right)\right) \quad \text{and} \quad R_{p2} := \left(\phi p_h^{\Delta t}, \operatorname{div} u_h^{\Delta t}\right).$$

Using again the discrete commutator property (3.5) and (3.3), we easily get

$$|R_{p1}| \le ch \left\| p_h^{\Delta t} \right\|_2 \left\| u_h^{\Delta t} \right\|_{H^1}$$

which implies

$$\int_0^T R_{p1}\, dt \to 0 \quad \text{as } (\Delta t, h) \to (0,0).$$

The term R_{p2} can be treated in the same way but now using the discrete commutator property for the projector over Q_h

$$|R_{p2}| \le c \left\| Q_h\left(p_h^{\Delta t}\phi\right) - p_h^{\Delta t}\phi \right\|_2 \left\| u_h^{\Delta t}\phi \right\|_{H^1} \le ch^{1/2} \left\| p_h^{\Delta t} \right\|_{L^{4/3}(L^2)} \left\| u_h^{\Delta t} \right\|_{L^2(H^1)}^{1/2} \left\| u_h^{\Delta t} \right\|_{L^\infty(L^2)}^{1/2},$$

and finally this implies that

$$\int_0^T R_{p2}\, dt \to 0 \quad \text{as } (\Delta t, h) \to (0,0).$$

The convergence

$$\int_0^T \left(p_h^{\Delta t} u_h^{\Delta t}, \nabla\phi\right) \to \int_0^T (pu, \nabla\phi)$$

is an easy consequence of (5.5), (5.6) and Proposition 4.4 b). These steps are common to the three cases.

We now treat the inertial term. In Case 1, the definition of nl_h in (3.6) allows us to handle the first term on the right-hand side in (5.9) with some integration by parts as follows:

$$b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi\right) = -\left(u_h^{\Delta t}\frac{1}{2}\left|u_h^{\Delta t}\right|^2, \nabla\phi\right).$$

By arguing as in [5], it can be proved that

$$u_h^{\Delta t} \frac{1}{2}\big|u_h^{\Delta t}\big|^2 \to u \frac{1}{2}|u|^2 \quad \text{strongly in } L^1\big(0,T;L^1\big), \text{ as } (\Delta t, h) \to (0,0),$$

and one shows that

$$\int_0^T b_h\big(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi\big)\,dt \to -\int_0^T \left(u \frac{1}{2}|u|^2, \nabla\phi\right)dt \quad \text{as } (\Delta t, h) \to (0,0).$$

In Case 2, the result is much simpler since by direct computations, one shows that for smooth enough w, we have (by a point-wise equality, where ϵ_{ijk} the totally anti-symmetric tensor)

$$\big[(\nabla \times w) \times w\big] \cdot (\phi w) = \sum_{i,j,k,l} (\epsilon_{jki} - \epsilon_{jlm}) \partial_l w_m w_k w_i \phi$$

$$= \phi \sum_{i,k} w_k \partial_k w_i w_i - w_i \partial_i w_k w_k = 0.$$

Hence, we get

$$b_h\big(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi\big) = 0,$$

and there are no terms to be estimated.

In Case 3, we get instead (cf. [16, Lemma 4.1])

$$b_h\big(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi\big) = -\frac{1}{2}\big(\mathcal{K}_h\big(\big|u_h^{\Delta t}\big|^2\big), \text{div}\big(\phi u_h^{\Delta t}\big)\big)$$

$$= -\frac{1}{2}\big(u_h^{\Delta t}\big|u_h^{\Delta t}\big|^2, \nabla\phi\big) + R_1 + R_2$$

with

$$R_1 := -\frac{1}{2}\big(u_h^{\Delta t}\mathcal{K}_h\big(\big|u_h^{\Delta t}\big|^2\big) - u_h^{\Delta t}\big|u_h^{\Delta t}\big|^2, \nabla\phi\big) \quad \text{and} \quad R_2 := -\frac{1}{2}\big(\phi\mathcal{K}_h\big(\big|u_h^{\Delta t}\big|^2\big), \text{div}\, u_h^{\Delta t}\big),$$

The strong $L^{s'}(0,T;L^2)$-convergence of $\mathcal{K}_h(|u_h^{\Delta t}|^2$ implies that $\int_0^T |R_1|\,dt \to 0$. While using the discrete commutator property for R_2, we estimate

$$|R_2| = \frac{1}{2}\big(\phi\mathcal{K}_h\big(\big|u_h^{\Delta t}\big|^2\big) - Q_h\big(\phi\mathcal{K}_h\big(\big|u_h^{\Delta t}\big|^2\big)\big), \text{div}\, u_h^{\Delta t}\big)|$$

$$\le ch\big\|\mathcal{K}_h\big(\big|u_h^{\Delta t}\big|^2\big)\big\|_2 \big\|u_h^{\Delta t}\big\|_{H^1} \le ch\big\|u_h^{\Delta t}\big\|_4^2 \big\|u_h^{\Delta t}\big\|_{H^1}$$

$$\le ch^{1/2}\big\|u_h^{\Delta t}\big\|_2^2 \big\|u_h^{\Delta t}\big\|_{H^1}^2,$$

which shows that $\int_0^T |R_2|\,dt \to 0$.　　　　　　　　　　　　　　□

6 Extension to other second-order schemes

The techniques developed in the previous sections are general enough to be used to handle with minor changes, also some more general second-order schemes, as, for instance, the Crank–Nicolson with Linear Extrapolation (CNLE) and the Crank-Nicholson/Adams-Bashforth (CNAB), as reported below. We have the following result.

Theorem 6.1 *Let the same assumptions of Theorem* 1.1 *be satisfied, and let replace assumption* (1.3) *with* (6.2) *for the* (CNLE) *algorithm and replace assumption* (1.3) *with* (6.4) *for the* (CNAB) *algorithm. Then, solutions of both schemes converge to a suitable weak solution of the NSE.*

The proofs are in the same spirit as in the previous section, after the corresponding estimates (independent of m) have been proved. For this reason, we simply include the changes related to the proofs of the other cases previously treated.

Crank–Nicolson with Linear Extrapolation (CNLE). Another scheme that is similar to the Crank–Nicolson (CN) in terms of theory, but performs better in terms of numerical properties is the Crank–Nicolson with Linear Extrapolation as introduced by Baker [2] and studied by Ingram [24], especially in the context of non-homogeneous Dirichlet problems.

In this case, the scheme is defined, for $m \geq 2$, by

$$
\begin{aligned}
&\left(d_t u_h^m, v_h\right) + \nu\left(\nabla u_h^{m,1/2}, \nabla v_h\right) \\
&\quad + \frac{1}{2} b_h\left(3u_h^{m-1} - u_h^{m-2}, u_h^{m+1/2}, v_h\right) - \left(p_{h,}^m, \operatorname{div} v_h\right) = 0, \\
&\left(\operatorname{div} u_h^m, q_h\right) = 0,
\end{aligned}
\tag{CNLE}
$$

where the operator $b_h(\cdot, \cdot, \cdot)$ is the same as in Case 1 (see the previous section), and for $m = 1$, the scheme is replaced by (CN) to be consistent with second-order time-discretization. Scheme (CNLE) is linearly implicit, unconditionally, and nonlinearly stable, and second-order accurate [2, 24, 25]. In [25], it is shown that no time-step restriction is required for the convergence (but with mild assumptions on the pressure), and additionally, it is proved the optimal convergence for smoother solutions.

Here, we prove the following result that is not assuming any extra-assumption neither on the Leray–Hopf weak solution u nor on the pressure p and can be used to prove the local energy inequality, reasoning as in the previous sections.

Lemma 6.2 *Let $N \in \mathbb{N}$ and $m = 1, .., N$. Then, for* (CNLE) *the following discrete energy-type equality holds true*:

$$
\frac{1}{2}\left(\left\|u_h^m\right\|_2^2 - \left\|u_h^{m-1}\right\|_2^2\right) + \nu \Delta t\left\|\nabla u_h^{m,\frac{1}{2}}\right\|_2^2 = 0.
\tag{6.1}
$$

Moreover, if $u_0 \in H_\#^1$, there exists $C > 0$ such that if

$$
\Delta t \leq \frac{\nu}{16} \min\left\{h^2, \frac{h^3\|u_0\|^2}{4C^2}\right\},
\tag{6.2}
$$

then

$$
\sum_{m=2}^{N}\left\|u_h^m - u_h^{m-1}\right\|_2^2 \leq C.
$$

Proof The first part of Lemma 6.2 can be proved in a direct way simply using $u_h^{m+1/2}$ as test function, obtaining (6.1); the proof of the second estimate requires some additional work

in the spirit of [28, Sect. 19]. To this end, let us define

$$\delta_h^m := \frac{u_h^m - u_h^{m-1}}{2}.$$

Using $2\Delta t\, u_h^m$ as test function in (CNLE), we get

$$\left\| u_h^m \right\|_2^2 - \left\| u_h^{m-1} \right\|_2^2 + \left\| \delta_h^m \right\| + 2\nu\Delta t \left\| \nabla u_h^m \right\|_2^2$$
$$= -2\nu\Delta t\left(\nabla\delta_h^m, \nabla u^m\right) - \Delta t\, b_h\left(3u_h^{m-1} - u_h^{m-2}, u_h^{m+1/2}, u_h^m\right),$$
$$= -2\nu\Delta t\left(\nabla\delta_h^m, \nabla u^m\right) - \Delta t\, b_h\left(3u_h^{m-1} - u_h^{m-2}, \frac{-u_h^m + u_h^{m-1}}{2}, u_h^m\right),$$

Hence, the right-hand side can be estimated as follows

$$\left| 2\nu\Delta t\left(\nabla\delta_h^m, \nabla u^m\right) + \Delta t\, b_h\left(3\nabla u_h^{m-1} - \nabla u_h^{m-2}, \delta_h^m, u_h^m\right) \right|$$
$$\leq 2\nu\Delta t \left\| \nabla\delta_h^m \right\|_2 \left\| \nabla u^m \right\|_2$$
$$\quad + C\Delta t\left(\left\| 3\nabla u_h^{m-1} \right\|_2 + \left\| \nabla u_h^{m-2} \right\|_2\right)\left\| \nabla u_h^m \right\|_2 \left\| \delta_h^m \right\|_2^{1/2} \left\| \nabla\delta_h^m \right\|_2^{1/2}$$
$$\leq \frac{2\Delta t}{h}\left\| \delta_h^m \right\|_2 \left\| \nabla u^m \right\|_2 + \frac{C\Delta t}{h^{3/2}}\left(\left\| 3u_h^{m-1} \right\|_2 + \left\| u_h^{m-2} \right\|_2\right)\left\| \nabla u_h^m \right\|_2 \left\| \delta_h^m \right\|_2$$
$$\leq \frac{1}{4}\left\| \delta_h^m \right\|_2^2 + \frac{8(\Delta t)^2}{h^2}\left\| \nabla u^m \right\|_2^2 + \frac{1}{4}\left\| \delta_h^m \right\|_2^2$$
$$\quad + \frac{8C^2(\Delta t)^2}{h^3}\left(\left\| 3u_h^{m-2} \right\|_2 + \left\| u_h^{m-2} \right\|_2\right)^2 \left\| \nabla u_h^m \right\|_2^2.$$

Next, using the uniform estimate on $\| u_h^m \|_2$ coming from the previous step, we get

$$\left\| u_h^m \right\|_2^2 - \left\| u_h^{m-1} \right\|_2^2 + \frac{1}{2}\left\| \delta_h^m \right\|_2^2 + \nu\Delta t \left\| \nabla u_h^m \right\|_2^2\left(2 - \frac{8\Delta t}{\nu h^2} - \frac{32C^2\Delta t}{\nu h^3}\left\| u_0 \right\|_2^2\right) \leq 0,$$

and under the restriction on Δt and h from (6.2), we obtain

$$\left\| u_h^m \right\|_2^2 - \left\| u_h^{m-1} \right\|_2^2 + \frac{1}{2}\left\| \delta_h^m \right\|_2^2 + \nu\Delta t \left\| \nabla u_h^m \right\|_2^2 \leq 0,$$

which ends the proof by summation over m. □

The convergence to a weak solution satisfying the global and the local energy inequality follows in the same manner as in [5] and the results from the previous section. Once the estimated are proven, one has just to rewrite word-by-word the proof in Case 1.

Crank–Nicolson/Adams–Bashforth In the same spirit of Case 1, we can also consider the Crank–Nicolson scheme for the linear part and the Adams–Bashforth for the inertial one, as it is studied, for instance, in [28, Sect. 19]. The algorithm reads as follows: solve for $m \geq 2$

$$\left(d_t u_h^m, v_h\right) + \nu\left(\nabla u_h^{m,1/2}, \nabla v_h\right) + \frac{3}{2}b_h\left(u_h^{m-1}, u_h^{m-1}, v_h\right)$$
$$\quad - \frac{1}{2}b_h\left(u_h^{m-2}, u_h^{m-2}, v_h\right) - \left(p_{h,}^m, \mathrm{div}\, v_h\right) = 0, \qquad\qquad \text{(CNAB)}$$
$$\left(\mathrm{div}\, u_h^m, q_h\right) = 0,$$

where $b_h(\cdot, \cdot, \cdot)$ is defined by means of (3.6)–(3.7), while again u_h^1 is obtained by an iteration of (CN). This method is explicit in the nonlinear term and only conditionally stable [21, 28]. The (CNAB) method is popular for approximating Navier–Stokes flows because it is fast and easy to implement. For example, it is used to model turbulent flows induced by wind turbine motion, turbulent flows transporting particles, and reacting flows in complex geometries; see Ingram [25].

First, observe that it is possible to prove, with a direct argument, a sort of energy balance for the scheme, namely an inequality of this kind

$$\frac{1}{2}\left\|u_h^1\right\|_2^2 + \nu\Delta t\left\|\nabla u_h^{1/2}\right\|_2^2 \leq \frac{1}{2}\|u_0\|_2^2,$$

but nevertheless, from the above estimate, one can also obtain by means of the inverse inequality

$$\frac{1}{2}\left\|u_h^1\right\|_2^2 + \frac{\nu\Delta t}{4}\left\|\nabla u_h^1\right\|_2^2 \leq \frac{1}{2}\|u_0\|_2^2 + \frac{\nu\Delta t}{4}\left\|\nabla u_h^0\right\|_2^2 \leq \left(\frac{1}{2} + \frac{\nu\Delta t}{4h^2}\right)\|u_0\|_2^2 := K_3, \qquad (6.3)$$

with K_3 independent of Δt and h. The estimate for $m > 1$ is obtained by a induction argument in [28, Lemma 19.1]. The proved result is the following.

Lemma 6.3 *Assume that $u_0 \in L_{\mathrm{div}}^2$, and (6.3) holds. Then, there exists K_4 independent of Δt and h such that if*

$$\Delta t \leq \frac{4c_1^2}{\nu} \quad and \quad \frac{\Delta t}{h^3} \leq \max\left\{\frac{1}{32\nu}, \frac{c\nu}{K_4}\right\}, \qquad (6.4)$$

then

$$\left\|u_h^n\right\|_2^2 \leq K_4,$$
$$\sum_{m=1}^{N}\left\|u_h^m - u_h^{m-1}\right\|_2^2 \leq 32K_4, \qquad (6.5)$$
$$\Delta t\sum_{m=1}^{N}\left\|\nabla u_h^m\right\|_2^2 \leq 4K_4.$$

We just comment that the proof is obtained by showing (with the same estimates employed in the previous case) that

$$\left(1 + \frac{\Delta t}{2c_1^2}\right)\xi^n \leq \xi^{n-1} \quad \text{where } \xi^m := \left\|u_h^m\right\|_2^2 + \frac{1}{4}\left\|u_h^m - u_h^{m-1}\right\|_2^2,$$

and then applying an inductive argument. This is enough to prove the standard result $u_h^m \in l^\infty(L^2) \cap l^2(H^1)$ from which one deduces the estimates also on the pressure. Next passage to the limit is again standard showing that the linear interpolated sequence converges to a distributional solution of the NSE.

A nontrivial point is to justify the global energy inequality because, in this case, the estimate (5.8) does not hold. The functions $v_h^{\Delta t}$ and $u_h^{\Delta t}$ have the requested regularity but

do not satisfy the correct energy balance, since

$$\frac{3}{2}b_h\big(u_h^{m-1},u_h^{m-1},u_h^m\big) - \frac{1}{2}b_h\big(u_h^{m-2},u_h^{m-2},u_h^m\big) \neq 0.$$

The correct energy balance is satisfied only in the limit $(h, \Delta t) \to (0, 0)$, but this cannot be deduced at this stage. As usual the global energy inequality cannot be proved by means of testing with the solution itself, but only after a limiting process, cf. [4].

The way of obtaining it passes through the verification that (u, p) is a suitable weak solution. The validity of the local energy inequality can be done as in [5] and results in Case 1, once the (conditional) estimate in $(6.5)_2$ is obtained. Note that in this case, the restriction on the relative size of Δt and h is needed already for the first a priori estimate.

Next, by adapting a well-known argument in [11, Section 2C], we can deduce it from (2.3). In fact, it is enough to replace ϕ by the product of ϕ and χ_ϵ (which is a mollification of $\chi_{[t_1,t_2]}(t)$, the characteristic function of $[t_1, t_2]$) and pass to the limit as $\epsilon \to 0$ to get

$$\int_{\mathbb{T}^3} \big|u(t_2)\big|^2 \phi(t_2)\,dx + \nu \int_0^T \int_{\mathbb{T}^3} |\nabla u|^2 \phi \,dx\,dt$$
$$\leq \int_{\mathbb{T}^3} \big|u(t_1)\big|^2 \phi(t_1)\,dx + \int_0^T \int_{\mathbb{T}^3} \left[\frac{|u|^2}{2}(\partial_t \phi + \nu \Delta \phi) + \left(\frac{|u|^2}{2} + p \right) u \cdot \nabla \phi \right] dx\,dt,$$

and the above formula is particularly significant if $\phi(\tau, x) \neq 0$ in (t_1, t_2). Next, in the above formula, one can take a sequence ϕ_n of smooth functions converging to the function $\phi \equiv 1$ and at least in the whole space or in the space periodic setting, one gets the global energy inequality (2.2) as is explained at the beginning of [11, Sect. 8]. Moreover, the same argument applied to arbitrary time intervals also shows that

$$\frac{1}{2}\big\|u(t_2)\big\|_2^2 + \nu \int_{t_1}^{t_2} \big\|\nabla u(s)\big\|_2^2\,ds \leq \frac{1}{2}\big\|u(t_1)\big\|_2^2 \quad \text{for all } 0 \leq t_1 \leq t_2 \leq T.$$

Hence, the strong global energy inequality holds true.

7 Conclusion

In this paper, we analyzed several second-order (in time) numerical methods for the unsteady Navier–Stokes equations. We proved that numerical solutions converge to physically relevant solutions under mild assumptions on the discretization parameters (in space and time). The analysis is performed for several different discretizations of the convective term, and the methods applied, for their generality, can also be adapted to different situations provided that simple stability estimates are at hand.

Abbreviations
NSE, Navier–Stokes equations; CN, Crank–Nicolson; CNLE, Crank–Nicolson with Linear Extrapolation; CNAB, Crank–Nicholson/Adams–Bashforth Extrapolation.

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All authors read and approved the final manuscript.

Author details
[1]Dipartimento di Matematica, Università degli Studi di Pisa, Via F. Buonarroti 1/c, I-56127 Pisa, Italy. [2]DISIM - Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio, I-67100 L'Aquila, Italy.

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