Public debt sustainability in a target zone model with heterogeneous agents

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Abstract

Relying on the assumption of perfect information and agent homogeneity, previous interest rate target zone models could well account for situations of government debt stability ('honeymoon') or instability ('divorce'). In those models, however, the transition from one state to another could only occur with a discrete change in the interest rate and thus could not account for the gradual transition from 'honeymoon' to 'divorce' that occurred in the months leading up to the euro area crisis of 2011-12. The assumption of heterogeneous agents made in this interest rate target zone model, on the other hand, allows that graduality to be represented. Heterogeneous agents are assumed to be characterized by normally distributed beliefs about the maximum sustainable level of the debt-to-GDP ratio. When public debt increases due to an assumed process of stochastic shocks, therefore, the percentage of agents sharing the belief that they have entered a region of instability also increases, thus leading to the gradual transition from 'honeymoon' to 'divorce,' observed in the euro area crisis of 2011-2012.

Keywords: Interest rate target zones, heterogeneous agents, public debt sustainability, speculative attacks, euro area crisis.

JEL Classification: E65, F34, F36

1. Introduction and motivation of the paper[1]

The euro area crisis of 2011-12, led by speculative attacks on government debt and affecting countries such as Greece, Italy, Portugal and Spain, was characterized by a gradual process of transition from a stable situation to its breakdown. The interest rate on the public debt of the above-mentioned countries increased gradually over a period of about two years, from the beginning of 2010 to the beginning of 2012. An initial sudden but temporary stabilization of interest rates occurred around late 2011-early 2012, coinciding with the changed monetary policy environment and the adoption of restrictive fiscal policies. The former was reflected by the interest rate cuts decided by the newly appointed ECB president Mario Draghi, while the latter was represented by the approval of the *Six-Pack* by the European Parliament in September 2011, the *Two Pack* proposal in November 2011, and the approval of the *Fiscal Compact* in March 2012. Interest rates, however, resumed rising gradually and only finally stabilized after Draghi's famous speech in July 2012.[2] His firm commitment to stabilize interest rates was accompanied by the launch of the *Outright Market Transactions* (OMT) program, which led to an immediate downward adjustment in interest rates, followed by a further gradual normalization occurring while the ECB was credibly asserting its role as lender of last resort (Figure 1).

Previous interest rate target zone models have also studied the euro area crisis, but they only succeeded in explaining the abrupt end of the crisis, not its gradual building up. Della Posta (2019), for example, argues that the common knowledge of an upper threshold of the primary surplus determines a corresponding upper limit of the interest rate that ensures the stability of public debt, operating as a target zone. When, given the availability of fiscal resources allowing to run a sufficient primary surplus, the interest rate is expected not to exceed its upper limit, a stable 'honeymoon' situation emerges. In the opposite case, a 'divorce' occurs. Given the constraints that countries adhering to a monetary union are usually subject to, Della Posta (2019) considers the primary surplus as the only instrument to ensure the stability of public debt. Monetary policy, however, can be resumed as an additional stabilizing instrument at the disposal of economic policy, as it was the case during the euro area crisis.

The role of monetary policy in igniting (when the central bank does not intervene) or ending a public debt crisis (when intervening, or even just announcing that she will intervene) is explicitly

[1] I would like to thank three anonymous referees for the valuable comments and suggestions they gave me in order to improve my article. Of course, I remain solely responsible for any remaining errors.

[2] It is unanimously acknowledged that the crisis ended thanks the speech that Mario Draghi, President of the ECB, gave at the Global Investment Conference in London on July 26, 2012. The most quoted part of the speech is: "Within our mandate, the ECB is ready to do whatever it takes to preserve the euro. And believe me, it will be enough."

addressed by Della Posta (2018) in a different interest rate target zone model. He considers De Grauwe's (2012) explanation of the euro area crisis, based on the difference between autonomous ('stand-alone') countries, those who can rely on a national central bank to act as lender of last resort, and countries belonging to a monetary union, which cannot. The presence of a lender of last resort provides a reliable channel for public debt stability, making it possible for a credibility bonus to enjoy a 'honeymoon' of public debt stability.

While still based on an interest rate target zone model, the present work, by eliminating the assumption of agent homogeneity and admitting their heterogeneity, allows to understand not only the abrupt end of the crisis, which previous models already did, but also its gradual buildup.

The paper is structured as follows. Section 2 contains a literature review; in Section 3 I present the equation that determines the interest rate on government debt (Section 3.1) and the equation representing the dynamics of public debt (Section 3.2). Its equilibrium value, based on the availability of stabilizing fiscal and monetary instruments, defines the upper feasibility threshold of the interest rate. Section 4 presents a standard interest rate target zone model, which is set up in Section 4.1 and resolved in Section 4.2, resulting in a 'honeymoon', when the interest rate stability threshold is credibly defended, or a 'divorce', when public debt is expected to become unsustainable. Section 5 introduces the agent heterogeneity hypothesis, allowing the probability of public debt default to be endogenized, thus accounting for the gradual transition from 'honeymoon' to 'divorce'. Some final remarks close the paper in Section 5.

2. Literature review

Some recent work (Della Posta 2018, 2019) has applied exchange rate target zone modeling, which was developed in the 1990s following Krugman's (1991) seminal contribution, to the case of speculative attacks on government debt, and in particular to the euro area crisis.

In Della Posta (2018, 2019), however, the transition from the 'honeymoon' (when the interest rate was below the level that a linear relationship with the level of public debt/GDP would imply), to the 'divorce' (when the interest rate moved above its linear relationship with the public debt/GDP ratio), is necessarily abrupt rather than gradual, given the assumption of common knowledge of the state of the fundamental variable, the public debt/GDP ratio. This implies a sudden upward jump in the interest rate and depends, respectively, on an exogenous change in the state of expectations or, endogenously, on perfect knowledge of fiscal and monetary availability.

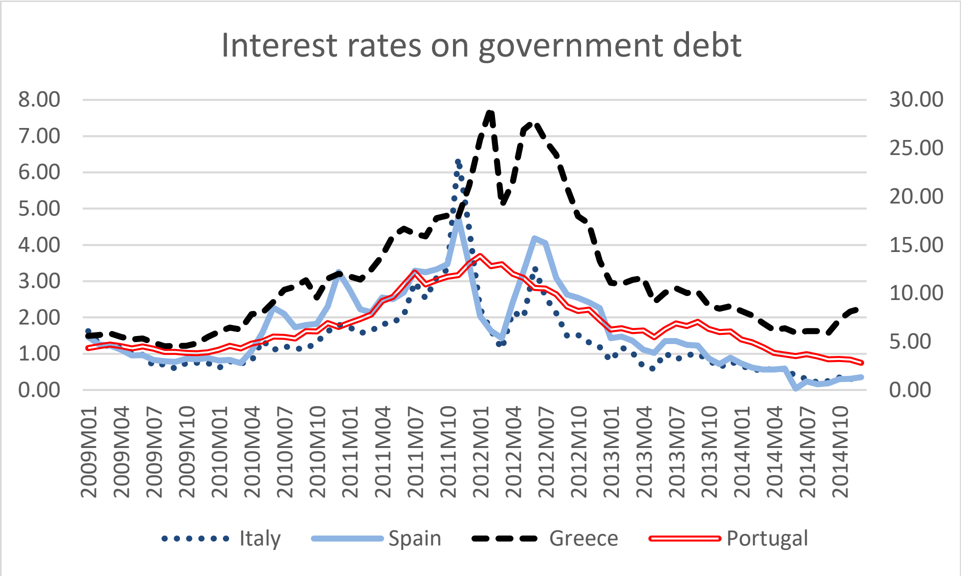

Figure 1: Interest rates on Italian and Spanish Treasury Bills (left scale) and on Greek and Portuguese Government Bonds (right scale). Source: IMF, International Financial Statistics.

The 2011-12 euro area crisis, however, cannot be explained by those models, since the interest rate increased gradually rather than suddenly.

Several recent works, including Lorenzoni and Werning (2018), Paluszynski (2021), Aguiar et al. (2016), and Ayres et al. (2018, 2019), have been more successful in recognizing and explaining the graduality in crisis construction and the role played by self-fulfilling expectations.

Lorenzoni and Werning (2018) analyze what they call ‘slow crises’ in which fear of default results in higher risk premiums and faster debt accumulation, as opposed to ‘rollover crises’, characterized by an investor rush that precipitates an immediate default.

Paluszynski (2021) also analyzes the role of slow moving crises. He builds a model in which agents slowly learn the fundamentals of the economy due to the presence of information frictions that hide the possibility of the occurrence of rare events.

Aguiar et al. (2016) analyze different sorts of public debt crises in developing countries, although suggesting that their analysis might well apply also to the developed ones. They find that fundamental variables are not so significant in explaining debt crises, thereby suggesting that self-fulfilling expectations and in particular some dynamic time-varying risk may play a more significant role. As we will see, this paper may be interpreted as going in that direction.

Ayres et al. (2018) study the role played by self-fulfilling expectations in the transition between good and bad equilibria. They find that while a situation of inherent fragility emerges when output endowments are assumed, to be normally distributed, as is often the case in the literature, when the assumed distribution of the endowments is bimodal, being characterized by good and bad periods, multiplicity of equilibria emerges only in the presence of intermediate levels of debt, but not when debt is unequivocally low or high.

Ayres et al. (2019) also consider the possibility of self-fulfilling crises in an environment of multiple equilibria *à la* Calvo. In line with previous studies on self-fulfilling speculative attacks, they find that expectations can play a destabilizing role only when the state of economic fundamentals is rather 'weak', that is, in periods of stagnation characterized by low GDP growth. Both results are reminiscent of the well-known distinction between stable area, unstable area and 'gray' area in the exchange rate literature.

On a seemingly unrelated side of research, Tamborini (2015), assuming an uncertain level of the country's maximum sustainable primary budget surplus resulting from heterogeneity in agents' beliefs, provides a rigorous but also very intuitive explanation of the convex nonlinearity of interest rates that characterized the euro area crisis. Focusing on the role played by fiscal authorities, the author shows that the risk premium of the interest rate on government debt increases dramatically when stabilization lacks credibility. This happens when the primary surplus needed to ensure debt stabilization approaches─and risks exceeding─its sustainable upper limit, resulting from the maximum level of revenue a government can collect from its citizens and the lower limit of spending it can cut.

As already noted, therefore, the target zone models have so far left unexplained the gradual transition from the 'honeymoon' that preceded the eurozone crisis to the 'divorce' that emerged as soon as confidence in the stability of government debt waned. Instead, in this article I show that by combining an interest rate target zone model with Tamborini's hypothesis of heterogeneous agents' beliefs, the graduality that characterized the construction of the eurozone crisis emerges, while still leaving open the possibility of explaining the abrupt changes from stability to instability and vice versa.

3. The interest rate on public debt and its upper threshold

3.1 The interest rate on public debt

The interest rate equation for public debt bonds can be represented as follows:

(1) $$i_t = \hat{r} + \alpha(b_t - \hat{b}) + \beta \frac{E[di_t]}{dt}.$$

This equation can be thought of as arising from a basic interest rate arbitrage in which the interest rate on government debt, i_t must equal the sum of the interest rate on a safe asset, $\hat{r}$, and a default risk, which can be thought of as depending on a fundamental component, $\alpha(b_t - \hat{b})$,[3] and a self-fulfilling component, which depends on the expected change in the interest rate itself, ($\beta \frac{E[di_t]}{dt}$) (see Ayres *et al.*, 2018, among others):[4] the lower the degree of expected stability of government debt, that is, the higher the probability of default, the higher the current interest rate on it.[5] This expectation is assumed to influence the current level of interest rates with a weight given by the parameter β.

Equation (1) can be rewritten as:

(1') $$i_t = \bar{r} + \alpha b_t + \beta \frac{E[di_t]}{dt},$$

with $\bar{r} = \hat{r} - \alpha\hat{b}$.

[3] The fundamentals component of the risk premium depends on the difference between the absolute size of the public debt-to-GDP ratio b_t and its expected maximum 'safe' value ($\hat{b}$). A level of public debt above such a safe value implies a higher risk of a future default, for example because of the higher cost of the response to an unexpected negative shock hitting the economy (Corsetti *et al.*, 2014). Alcidi and Gros (2018), IMF (2011) and European Commission (2014), also consider the risk premium as determined by the difference between the public debt-to-GDP ratio and a not risky level, for example the 60% ratio reported in the Maastricht Treaty, the value that could be assigned to $\hat{b}$ in equation (1). It should be stressed, however, that such a value is far from objective and univocally determined but results instead from institutional and possibly also contingent choices. This point, which is quite relevant, will be further extensively discussed below. The sensitivity of the interest rate with respect to $(b_t - \hat{b})$ is measured by the parameter α. The value assigned to α is 0.03 in the case of developed countries (Alcidi and Gros, 2018) and 0.04 in the case of developing countries (IMF, 2011).

[4] This is common to other 'jumping' financial variables, whose current value also depends on their expected future value (it is sufficient to think about the exchange rate, whose current value depends on its expected future variation, the inflation rate, whose current value, in the expectations-augmented Phillips curve, depends on its expected future level, and the stock exchange shares (equities), whose sudden jumps determined by announcements and news often oblige to break off their negotiations). As Shiller puts it: "Economists usually like to model people as calculating optimally their investment decisions based on expectations of future price changes and estimates of the risk in alternative investments" (Shiller, 2000, p. 55).

[5] The presence of a self-fulfilling component can be further justified by observing that the expectation of a default on public debt is sufficient to induce an excess of sales of domestic treasury bonds, causing a reduction of their price. In turn, the lower price of bonds, for a given fixed coupon, determines a higher interest rate.

3.2 The public debt equation and the upper threshold on the interest rate granting public debt stability

The debt ratio is assumed to vary over time due to both a deterministic and stochastic component. Its continuous time variation, db_t, can be represented as follows:

(2) $$db_t = -(s_t + m_t)dt + (i_t - g_t)b_t dt + \sigma dz.$$

In Equation (2), s_t is the primary public surplus-to-GDP ratio, and m_t is the percentage central bank's monetization of the public debt-to-GDP ratio.[6] The term $(i_t - g_t)b_t$ is the net real debt service in relation to GDP, with g_t as the nominal GDP growth rate. The stochastic component of public debt-to-GDP growth is assumed to follow an *arithmetic* Brownian motion, σdz.[7] The parameter σ represents the instantaneous standard deviation of the Brownian motion and the term dz is the Brownian motion variation which is so characterized:

(3) $$dz = \chi\sqrt{dt},$$

where χ is a random variable which is independently, identically and normally distributed, with 0 mean and variance equal to 1, and dt is an instantaneous and infinitesimal time variation.[8]

The steady state values of $\bar{\iota}$ and $\bar{b}$ are jointly determined by equation (1'), when $\frac{E[di_t]}{dt} = 0$, and by equation (2), when $db_t = 0$ and $\sigma dz = 0$.

In steady state, equations (1') and (2), become respectively, then:

(1'') $$i_t = \bar{r} + \alpha b_t,$$

and:

(4) $$b_t = \frac{\bar{s} + \bar{m}}{(i_t - \bar{g})},$$

[6] Della Posta (2019) adopted a similar equation, but ignored the possible role played by the monetary authority in stabilizing public debt.

[7] In the literature on the exchange rate target zone, however, the stochastic component was assumed to follow a *geometric* Brownian motion, since the variables were expressed in logs. I thank an anonymous referee for allowing me to clarify this point.

[8] These assumptions are also made in the standard target zone literature initiated by Krugman (1991).

so that $\bar{\imath}$ and $\bar{b}$ take values, respectively, of $\bar{\imath} = \bar{r} + \alpha\bar{b}$, and $\bar{b} = \frac{\bar{s}+\bar{m}}{(\bar{\imath}-\bar{g})}$, where $\bar{s}$, $\bar{m}$ and $\bar{g}$ are, respectively, the primary surplus, monetary creation and nominal GDP growth granting public debt *stability* to a level of $\bar{b}$ for an interest rate taking value $\bar{\imath}$. [9]

As argued by Tamborini (2015), however, the primary surplus that is expected to be feasibly run by a government, s^*, is not unbounded, being determined by equating the cost of solvency (positively related to s_t) with the cost of default (negatively related to s_t) (see, among others, Ghosh *et al*., 2013).

The same applies to the expected size of monetary policy, m^*, which may not be sufficient to stabilize public debt (as in the case of the central bank of a country belonging to a monetary union, as argued by De Grauwe, 2012).

In turn, these maximum expected feasible values of fiscal and monetary policies, s^* and m^*, determine i^*, for a given level of public debt and an exogenously value of $\bar{g}$. This is the highest expected interest rate that a government can afford to pay on its *sustainable* level of public debt-to-GDP ratio, b^*, which it is aiming for together with the central bank.[10]

From equation (4) and the definitions given above, then, it follows that public debt will be *sustainable* (i.e. credibly stable) only if $\bar{s} + \bar{m} \leq s^* + m^*$, thereby implying that:[11]

$$\bar{\imath} = \bar{g} + \frac{\bar{s}+\bar{m}}{\bar{b}} \leq i^* = \bar{g} + \frac{s^*+m^*}{b^*}. \tag{5}$$

The sustainability condition contained in Eq. (5) will be used in the interest rate target zone model that follows, showing the emergence of two opposite types of non-linearities—concave and convex—of the interest rate equation (1'), characterizing respectively the cases of 'honeymoon' and 'divorce'.

4. The interest rate target zone model

[9] The possible strategic game between the two, played in order to reduce the interest rate, is not discussed here and will be the object of future research.

[10] The government could only afford to pay a higher interest rate on b^* if, *ceteris paribus*, there are sufficient fiscal or monetary resources ($s^*$and/or m^*) or there is a sufficiently high level of $\bar{g}$.

[11] As I am considering the case of public debt stabilization, which requires both the availability of a positive primary surplus and/or positive monetary creation, I am going to ignore any lower interest rate threshold.

4.1. The setup of the model

The debt-to-GDP ratio may still rise beyond its steady-state value due to the process of stochastic shocks to which it may be subject:

(6) $$db_t = \sigma dz$$

It follows that the current interest rate, i_t—which depends also on the value of b_t, as clearly stated in equation (1')—would rise when $db_t > 0$. Only the availability of additional fiscal and monetary 'ammunition' will make it possible to defend the stability of public debt and resist the rising interest rate on it. As soon as i_t takes value $\bar{\imath} > i^*$, then, public debt is no longer *sustainable*, and this generates an explosive spiral between interest rate and public debt.

The interest rate target zone model in the case of a fully credible upper target, that is, when $i_t \leq i^*$, then, is composed by the following equations:

(1') $$i_t = \bar{r} + \alpha b_t + \beta \frac{E(di_t)}{dt}$$

(6) $$db_t = \sigma dz$$

This system closely resembles Krugman's (1991) original model of the exchange rate target zone, although here it is applied to a different problem and is not related to variables expressed in logs.

4.2 The 'honeymoon' and 'divorce' solutions.

The solution of equations (1') and (6) follows the standard target zone literature and is given by:

(7) $$i_t = q(b_t) = \bar{r} + \alpha b_t + Ae^{\hat{\lambda} b_t},$$

with $\hat{\lambda} = \sqrt{\frac{2}{\beta\sigma^2}}$ (see the Appendix for the details). If the public debt is expected to stabilize—thanks to available fiscal or monetary space—the interest rate remains below its upper bound due to the stabilizing effect played by market expectations: the higher it rises, the more likely it is to be brought back below the upper interest rate target thanks to available adjustment instruments.

The linear equation representing the reaction of the interest rate to the evolution of the debt-to-GDP ratio then becomes nonlinear, with the nonlinearity represented by the term $Ae^{\lambda b_t}$. The sign of the constant *A* is identified with the help of an end condition. When expectations are such that stabilization of public debt is credible, and therefore believed, *A* will take a negative value and the interest rate curve will be concave. In the case, on the other hand, where there is no confidence in the stabilization of public debt (for example, because there is no confidence in the availability of sufficient fiscal or monetary resources to match the increased supply with the corresponding demand), the interest rate curve will become convex.

To demonstrate these intuitive conclusions, we need to consider what happens when i_t reaches the interest rate threshold that ensures public debt stability, $i_t = i^*$. Following Bertola and Caballero (1992a) in the different context of an exchange rate target zone, we can assumed that the public debt-to-GDP ratio moves between 0 and the highest level of sustainable public debt (b^*) by linearly determining, in the absence of any expectation effect, the maximum bearable interest rate (i^*). The center of such a public debt-to-GDP ratio floating band is, then, $b^*/2$.

A non-arbitrage argument provides the closing equation. When the interest rate reaches the upper threshold that guarantees government debt stability, i^*, it must equal the expected value of the weighted probabilities of two complementary events that may occur.

The first is the probability p that neither the government nor the central bank has sufficient resources to ensure the stability of government debt, that is, to prevent i_t from moving beyond i^*. In other words, one possibility is that b_t moves above the maximum sustainable level, b^*. The size of the increase above b^*can be taken to be between 0 and δ, and its initial value can be assumed to be $\delta/2$. In such a case, the interest rate will have to increase to compensate for the higher risk of loss.

However, there is also the complementary probability (1 - p) that no government debt default will occur and that, instead, when b_t reaches b^*, a reduction will be achieved (e.g., due to monetization or repayment with a primary surplus). As a result, the risk premium would decrease as a function of the expected reduction in public debt, which can range from 0 to ε, and with an initial magnitude of $\varepsilon/2$.

The no-arbitrage equation when the interest rate hits the upper threshold level, then, is as follows:

$$(8) \qquad i_t(b^*, \frac{b^*}{2}) = p\, i_t(b^* + \frac{\delta}{2}, b^* + \frac{\delta}{2}) + (1-p) i_t(b^* - \frac{\varepsilon}{2}, b^* - \frac{\varepsilon}{2}),$$

(where, in the generic expression $i_t(b_t, c)$, b_t refers to the actual value taken by the economic fundamental, and c refers to the value it takes at the center of the floating band). Considering a symmetric fluctuation band centered on point c, and recalling that we are ignoring the lower band, equation (7) becomes:

$$(7') \qquad i_t(b_t, c) = \bar{r} + \alpha b_t + A e^{\lambda(b_t - c)}.$$

Using equation (7') into equation (8) we have, then, that:

(9) $$\bar{r} + \alpha b^* + Ae^{\lambda\frac{b^*}{2}} = p[\bar{r} + \alpha\left(b^* + \frac{\delta}{2}\right) + A] + (1-p)\,[\,\bar{r} + \alpha\left(b^* - \frac{\varepsilon}{2}\right) + A],$$

from which it follows that:

(10) $$A = \frac{\alpha[p\left(\frac{\delta+\varepsilon}{2}\right) - \frac{\varepsilon}{2}]}{e^{\lambda\frac{b^*}{2}} - 1}.$$

This also means that $A \geq 0$ iff $[p\left(\frac{\delta+\varepsilon}{2}\right) - \frac{\varepsilon}{2}] \geq 0$, that is iff:

(11) $$p \;\geq \frac{\varepsilon}{\delta+\varepsilon},$$

which corresponds to the case made by Bertola and Caballero (1992a) of $p \geq \frac{1}{2}$, given their assumption that $\delta = \varepsilon = b^*$.

The value of A, then, can be positive or negative, depending on the value taken by the exogenous probability, p, that the interest rate does not exceed the maximum feasible level that ensures public debt stability. There will be a 'divorce,' then, when the sum of the fiscal surplus and money creation that would be needed to ensure public debt stability, $\bar{s}$+$\bar{m}$, is greater than the maximum and feasible level expected by the country, $s^* + m^*$.

Calculating equation (7') respectively at the point at which the sustainable public debt takes the highest possible value, which I identify again with b^*, and at the corresponding value of public debt moving linearly with the interest rate, which I identify with $\bar{b}$, we have, respectively:

(12) $$i^*\left(b^*, \frac{b^*}{2}\right) = \bar{r} + \alpha b^* + Ae^{\lambda\frac{b^*}{2}}$$

and

(13) $$\bar{\imath}(\bar{b}, \bar{b}\,) = \;\bar{r} + \alpha\bar{b}.$$

From the equation above it follows that, at the top of the band, when $i^* = \bar{\imath}$:

(14) $$b^* = \bar{b} - \frac{Ae^{\lambda\frac{b^*}{2}}.}{\alpha}.$$

As it is easy to understand, when $A < 0$ we are in the case of 'honeymoon' and $b^* \geq \;\bar{b}$. In the case of 'divorce', in which $A > 0$, instead, $b^* < \bar{b}$, i.e., the largest possible level reached by a stable public debt is lower than the one obtained by the linear relationship with the interest rate.

As it is also clear, since considering equation (5) we have that $b^* = \frac{s^*+m^*}{i^*-\bar{g}}$, coherently we have:

(5') $$\bar{b} = \frac{\bar{s}+\bar{m}}{\bar{\imath}-\bar{g}}.$$

recalling that at the top of the band $i^* = \bar{\imath}$, it follows that:

$$(15) \qquad \frac{(s^*+m^*)-(\bar{s}+\bar{m})}{\bar{\imath}-\bar{g}} = -\frac{Ae^{\lambda\frac{b^*}{2}}}{\alpha}$$

The conclusion is rather intuitive: public debt stabilization (implying a negative value of A) will only be possible if there is enough additional fiscal and monetary space, $(s^* + m^*)$, above the level resulting from a linear relationship between public debt and interest rate, $(\bar{s} + \bar{m})$. Draghi's credible 'whatever it takes' statement made it known that this was the case, thus determining a sudden downward interest rate jump from an unstable 'divorce' path to a stable 'honeymoon' trajectory.

Figure 2 depicts different paths of 'honeymoon', characterized by an interest rate that increases less than proportionally with the level of public debt (the concave curves), and 'divorce', in which it increases more than proportionally (the convex curves).[12]

Until Draghi's statement, however, the hypothesis of agent heterogeneity regarding the highest expected sustainable value of the debt-to-GDP ratio, b^*, was the most appropriate. This is what I will discuss in the next section, which is the real unprecedented contribution of this paper to the literature.

5. Agents' heterogeneity and the endogenization of the probability of a public debt default

In the approach considered so far, the maximum interest rate target and the corresponding maximum value of the debt-to-GDP ratio were assumed to be known. In addition, the probability of instability of public debt, p, was assumed to be exogenous, so that while we can explain in which cases the interest rate is higher or lower than the value it would be in the case of a linear relationship with the debt-to-GDP ratio, we cannot rationalize what determines such situations, nor what explains the transition from one situation to the other.

Assuming agents' heterogeneity about the expected value of b^*, i.e. about the values of s^* and/or m^*, makes it possible to endogenize the probability of public debt instability, p, and to account for the gradual transition from 'honeymoon' to 'divorce' and vice versa.[13]

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[12] The scripts of this graph and of the following ones, that are run with Matlab, are available upon request.

[13] Endogenization was also obtained by Krugman and Rotemberg (1992) and Bertola and Caballero (1992b) in the context of an exchange rate target zone model and by Della Posta (2018) in the context of public debt speculative attacks. In those cases, however, the endogenization is obtained by just shifting the assumption of certainty to a different fundamental variable (m^*, in the case considered by Della Posta, 2018).

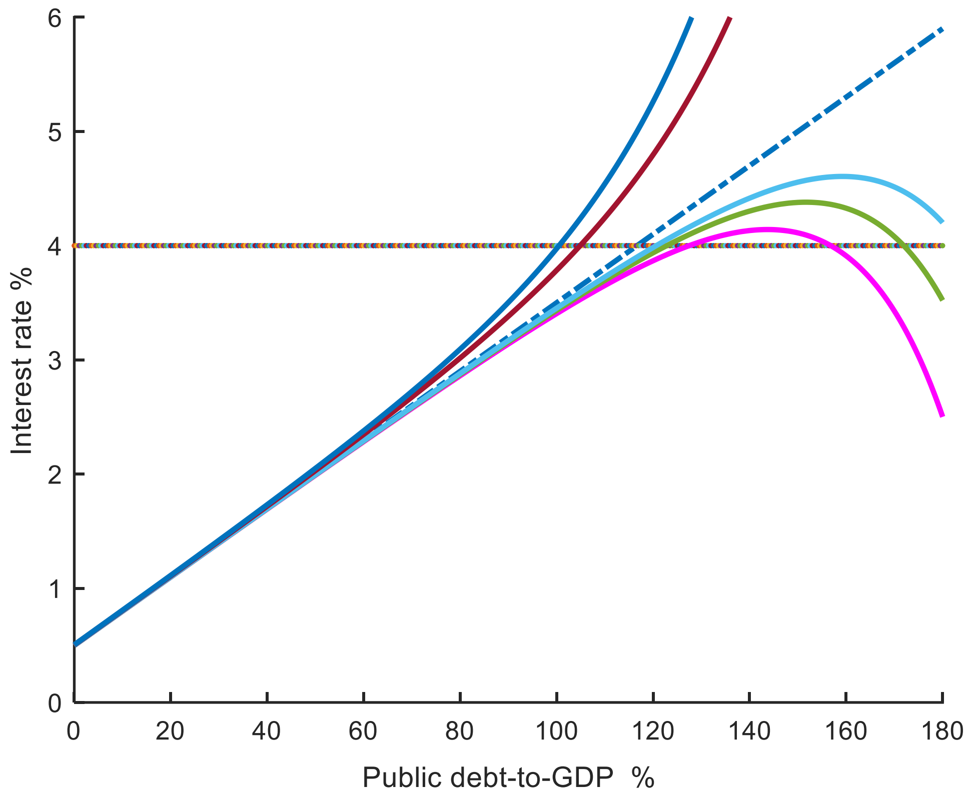

Figure 2: 'Honeymoon' and 'divorce' cases, resulting from different default probabilities

In the model that follows, therefore, we assume that b^* is a normally distributed random variable, $\tilde{b}$, between two extremes $\tilde{b}_{inf}$ and $\tilde{b}_{sup}$, with a given mean and variance.[14]

Consequently, following the methodology adopted by Tamborini (2015), it is possible to argue that the market probability of a government debt default, i.e., the probability that b_t exceeds the (unknown) upper bound assumed by the random variable $\tilde{b}$, depends on the proportion of heterogeneous agents sharing this belief. The higher the level of the public debt-to-GDP ratio, b_t, the higher the proportion of heterogeneous agents who will believe that the primary surplus and money creation, necessary to stabilize it, have already exceeded their maximum feasible level. The higher, therefore, will be the probability p that the market assigns to the event of a government debt default, leading to a 'divorce' from stability.

Before Draghi's 'whatever it takes', it was known that the Maastricht Treaty did not allow the ECB's monetary policy to stabilize public debt and that the only option would be to run a sufficiently

[14] Tamborini (2015) refers explicitly to the uncertain value taken by the feasible primary surplus, rather than public debt.

large primary surplus (assuming GDP growth as constant). Thus, given the existence of a ceiling on the sustainable primary surplus needed to avoid a default, the probability of public debt instability was relatively high. Draghi's firm statement removed the existing constraint on budget financing and created a common understanding of a much higher level of sustainable public debt.

The above can be represented with the following equation, which introduces the assumption of agent heterogeneity in the target zone model we have analyzed so far:

$$p(b_t) = F(b_t) = \int_{\tilde{b}_{inf}}^{b_t} f(\tilde{b})d(\tilde{b}). \tag{16}$$

In Eq. (16), $F(b_t)$ is the cumulative distribution function of the normally distributed threshold level of sustainable public debt, $\tilde{b}$, between $\tilde{b}_{inf}$ and $\tilde{b}_{sup}$. As should be clear by now, $0 \leq F(b_t) \leq 1$ represents, then, the endogenously determined fraction of people, increasing with b_t, according to which $b_t > \hat{b}_{inf}$. When $b_t < \tilde{b}_{inf}$, it turns out that $F(b_t) = p(b_t) = 0$, i.e. no agent believes that public debt is unstable, and the resulting probability of instability is zero. When $b_t \geq \tilde{b}_{sup}$, on the other hand, all agents believe that it is unstable, that is $F(b_t) = p(b_t) = 1$.

Having endogenized both the probability $p(b_t)$ that the current interest rate exceeds the sustainable level that ensures the stability of public debt, and the complementary probability $(1 - p(b_t))$ that it remains below it, I am now able to use it in the arbitrage equation introduced with equation (8):

$$i_t(b_t, c) = F(b_t)\, i_t(b_t + \tfrac{\delta}{2}, b_t + \tfrac{\delta}{2}) + (1 - F(b_t)) i_t(b_t - \tfrac{\varepsilon}{2}, b_t - \tfrac{\varepsilon}{2}). \tag{17}$$

The equation above, then, can be considered as the key equation of this paper, since it combines target zone modeling and heterogeneous agents' approach. Considering equation (9) above, we have:

$$\alpha b_t + Ae^{\lambda \frac{b_t}{2}} = F(b_t)[\alpha \left(b_t + \tfrac{\delta}{2}\right) + A] + (1 - F(b_t)) \left[\alpha \left(b_t - \tfrac{\varepsilon}{2}\right) + A\right], \tag{18}$$

from which it follows that $A = \frac{\alpha[\, F(b_t)\left(\frac{\delta+\varepsilon}{2}\right) - \frac{\varepsilon}{2}]}{e^{\lambda \frac{b_t}{2}} - 1}$, meaning that $A \gtreqless 0$ iff :

$$F(b_t) \gtreqless \frac{\varepsilon}{\delta+\varepsilon} \tag{19}$$

The sign of A will be negative for low values of b_t (given the resulting low value of $F(b_t) < \frac{\varepsilon}{\delta+\varepsilon}$, unless $\varepsilon = 0$ and/or $\delta \to \infty$) and will become positive as soon as the latter overtakes the critical value $F_{CR}(b_t) = \frac{\varepsilon}{\delta+\varepsilon}$. When b_t is low, the proportion of heterogeneous agents expecting it to have already exceeded the unknown stability threshold is also low, while the opposite will be the case for high values of b_t.[15]

The relationship between public debt-to-GDP ratio and interest rate in the case of heterogeneous agents is depicted in Figure 3, which shows that for low values of b_t the interest rate will be lower than that resulting from a linear relationship with the public debt-to-GDP ratio ($A < 0$), while for higher values it will be higher ($A > 0$).[16] The first case is the 'honeymoon' case, in which the 'good' state of expectations allows the government to pay a lower interest rate on its debt than should be paid based on the actual level of public debt. When the interest rate exceeds the dotted line we are instead in the case of a 'bad' state of expectations ('divorce'), which occurs when the government debt exceeds the critical level at which $F_{CR}(b_t) = \frac{\varepsilon}{\delta+\varepsilon}$.

Figure 3 can be considered as the key figure of this paper, since it is obtained using an interest rate target zone model with heterogeneous agents, and accounts for the gradual passage from the stability ('honeymoon') to the instability ('divorce') characterizing the months preceding the euro area crisis, as resulting from the actual data reported in Figure 1.

Figures 4 and 5, on the other hand, represent the two separate effects of Draghi's 'whatever it takes' speech, which this time leads from instability to stability (notice the inverted *x*-axis). The first effect, then, was to remove uncertainty about the availability of additional debt stabilization instruments, thus transforming the *s*-shaped dashed relationship between the debt-to-GDP ratio and the interest rate into the continuous, linear one, as shown in Figure 4. The second effect refers to the widening of the area of public debt stability, as a result of news of a significantly large availability of stabilizing monetary instrument (see Figure 5).[17] Therefore, if a country were on the upper left solid line of Figure 5 (corresponding to the unstable situation of a relatively high interest rate—e.g. 7%—and a high level of public debt—e.g. 150%), Draghi's speech would have the immediate stabilizing effect of bringing the interest rate down to the dotted line at the bottom—e.g. at 2%— despite the unchanged high level of public debt, as was clearly the case in Italy.

[15] Two limiting cases, corresponding to the two opposite 'one-way bet' situations in which $\varepsilon = 0$ and $\delta = 0$, are easily identified. In the first, corresponding to the case in which it is believed that the public debt cannot be stabilized, it turns out that $A > 0$ and a 'divorce' emerges. In the second case, characterized by a 'honeymoon', $A < 0$.

[16] Della Posta (2020) proposes a similar figure, although in the different context of economic globalization and referring, therefore, to different variables.

[17] Notice that, differently from Figure 3, both Figure 4 and Figure 5 have been represented with an inverted (decreasing) horizontal scale, in order to better adapt the graphical representations to the sudden passage from instability to stability resulting from the Draghi speech that I have been referring to above.

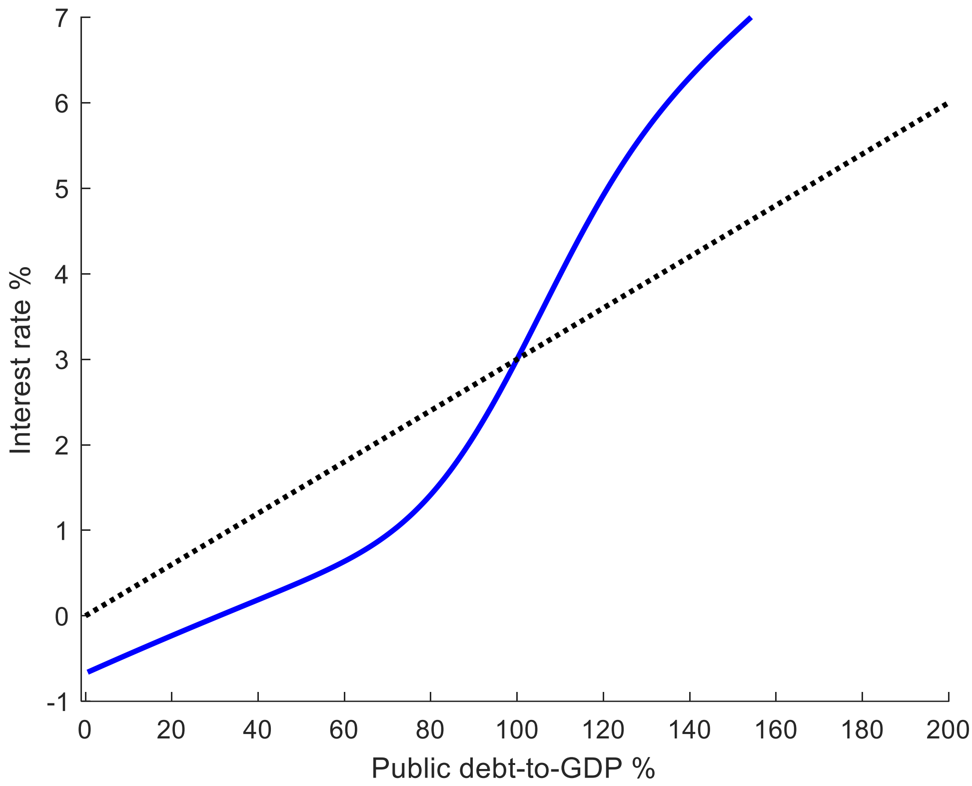

Figure 3: Gradual transition from 'honeymoon' (with low levels of public debt) to 'divorce' (with high levels of public debt).

Concluding remarks

This paper combines the two different strands of literature on interest rate target zones and heterogeneous agents. Differently from what done in the seminal contributions of Krugman (1991), Bertola and Caballero (1992a) and Della Posta (2018), this paper assumes that the upper level of the target zone is not exogenously given, nor that it is known with certainty. Following Tamborini's (2015) heterogeneous agent model, the maximum sustainable level of public debt to GDP is assumed to be a normally distributed random variable, since it depends on the uncertainty of the size of the monetary and fiscal policy instruments available for stabilization purposes.

This makes possible the endogenization of the probability of a speculative attack on public debt, which determines the dynamics of the interest rate equation within an interest rate fluctuation band. This dynamics follows a different path from those traditionally obtained in the target zone literature: as public debt increases due to the stochastic shocks it is assumed to be subject to, the expectation component (which depends on the value of expected future levels of fundamentals, monetary and fiscal policy) shifts gradually, rather than abruptly, from stabilization to destabilization.

The greater the size of public debt, the greater the availability of fiscal and/or monetary instruments needed to ensure its stability. Given the probability distribution assigned by heterogeneous agents to the sustainable level of public debt, the higher its level, the higher the percentage of economic agents who believe in its default.

In this context of heterogeneous agents, therefore, the stabilizing effect of a target zone operates only for sufficiently low values of public debt, when a very small percentage of agents are expected to have already exceeded the level of stability. In contrast, a target zone will have a destabilizing effect when the value of public debt is large enough to convince a large percentage of heterogeneous agents that it is already in the unstable region. This conclusion can be seen as a confirmation of Aguiar *et al.*'s (2016) insight that self-fulfilling expectations and in particular some time-varying dynamic risks can play a significant role in explaining public debt crises.

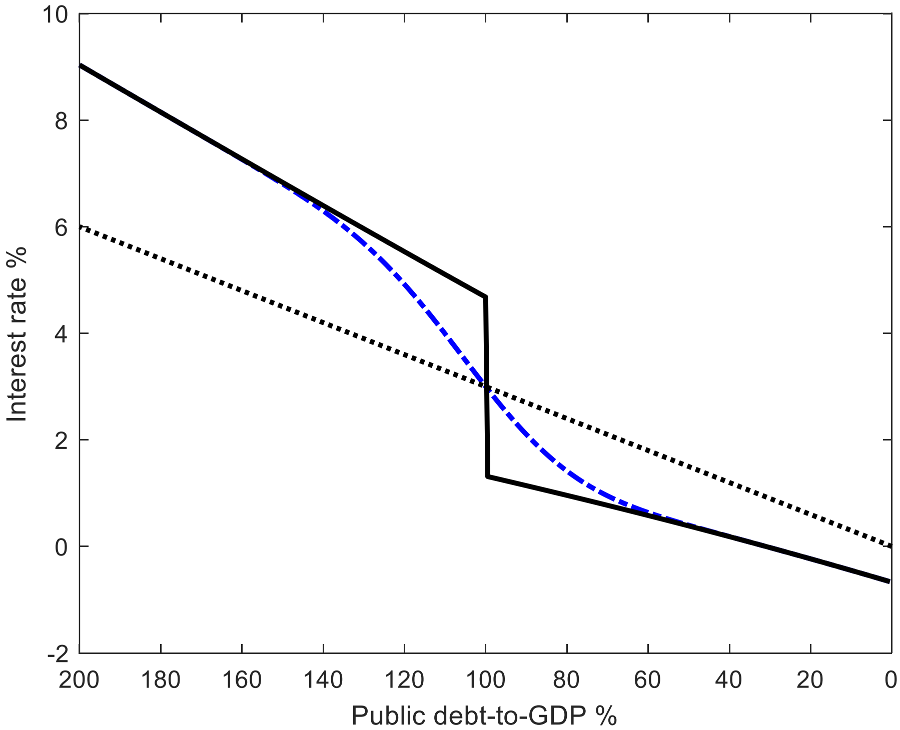

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Figure 4: From graduality to abrupt interest rate adjustment (inverted scale)

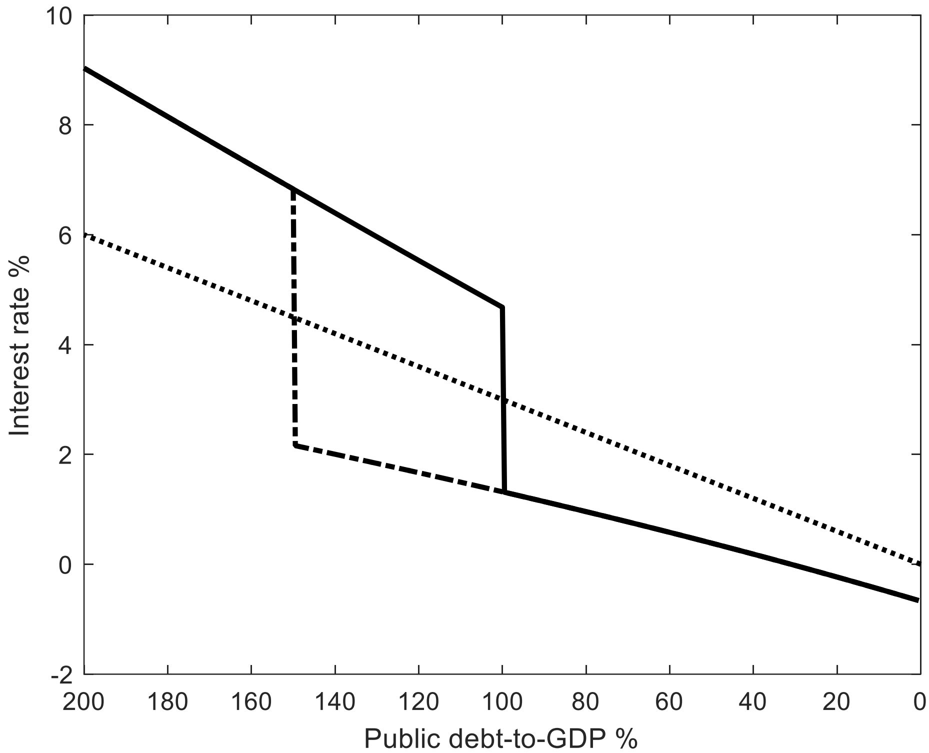

Figure 5: The full effect of Draghi's 'whatever it takes' (enlargement of the public debt stability region and sudden transition from 'divorce' to 'honeymoon') (inverted scale).

Uncertainty about the real value of the maximum level a stable public debt can reach thus weakens the role played by the presence of a target zone. On the other hand, when the availability of public debt stabilization instruments is certain, the conclusion reached in the case of heterogeneous agents becomes irrelevant and the target zone model resumes functioning. As soon as the monetary availability was reaffirmed with Draghi's 'whatever it takes' statement (accompanied and made credible shortly thereafter by the OMT program), a twofold effect emerged: first, the stabilizing effect of the target zone resumed, as the upper limit of the level of public debt stability became credible and, as a result, the 'honeymoon' region of public debt and stability increased. Second, the *s*-shaped gradual transition region that characterizes the case of an uncertain upper bound gave way to a clear separation between stable and unstable region, implying a sudden downward jump in the interest rate.

The results of this work not only provide a bridge between the literature on interest rate target zones and the literature on heterogeneous agent behavior, but also help remove the limitations arising from the fact that the modeling of interest rate target zones did not include the possibility of a smooth transition between 'honeymoon' and 'divorce' or vice versa.

In this interest rate target zone model I have removed the assumption, made in previous studies, of homogenous agents sharing the same belief about the value the upper target of the fluctuation band takes. Following Tamborini (2015), I have considered, instead, the assumption of heterogeneous agents, characterized by normally distributed beliefs about the maximum sustainable level of the debt-to-GDP ratio. As a result, when public debt increases due to an assumed process of stochastic shocks, the percentage of agents sharing the belief that they have entered a region of instability also increases, thus leading to a gradual transition from 'honeymoon' to 'divorce,' as observed in the euro area crisis of 2011-2012.

Such a modeling, however, also allows encompassing the case of sudden changes from stability to instability and vice-versa, when the degree of heterogeneity of agents' beliefs goes to zero, therefore allowing to account also for the sudden jump from instability to stability experienced after Draghi's 'whatever it takes' speech.

Appendix

Following the target zone literature, in order to find a closed form solution of the differential equation (1'), given the assumption that the state variable b_t evolves according to a Brownian motion, as from Eq. (6), we need to assume a generic functional form for i_t as a function of the debt-to-GDP ratio, b_t:

(A1) $$i_t = q(b_t)$$

We can now use this equation to calculate the expected interest rate variation in Eq. (1'). In order to do so, let's expand our (stochastic) equation in a Taylor-type series, by calculating Ito's differential:

(A2) $$di_t = q'(b_t)E(db_t) + \frac{1}{2}q''(b_t)E(db_t)^2$$

From the definition of db_t in (6), and of dz in (3) it turns out that $(db_t)^2 = \sigma^2\chi^2 dt$. By dividing by the infinitesimal temporal variation, taking expectations, and considering that $E[db_t/dt] = 0$ and $E[db_t]^2/dt] = \sigma^2$, we obtain Ito's Lemma:

(A3) $$\frac{E(di_t)}{d_t} = \frac{1}{2}q''(b_t)\sigma^2,$$

By replacing (A3) into (1') we have, then:

(A4) $$i_t = q(b_t) = \bar{r} + \alpha b_t + \beta \frac{1}{2} q''(b_t)\sigma^2$$

This is a differential equation of the second order, whose generic solution (recalling that we are ignoring the lower band) is of the kind (with still an undetermined value for both A and λ):

(A5) $$i_t = q(b_t) = \bar{r} + \alpha b_t + Ae^{\lambda b_t}$$

As it's easy to understand, the value of i_t, then, depends on b_t both on a linear term (αb_t) and on a nonlinear term ($Ae^{\lambda b_t}$).

Let's take the second order derivative of equation (A5) above in order to obtain a value for $q''(b_t)$:

(A6) $$q''(b_t) = \lambda^2 Ae^{\lambda b_t},$$

so that by replacing it into equation (A4), it gives:

(A7) $$i_t = q(b_t) = \bar{r} + \alpha b_t + \beta \frac{\sigma^2}{2}(\lambda^2 A_1 e^{\lambda b_t}).$$

By comparing (A7) with (A5), we have:

(A8) $$\left(\lambda^2 \beta \frac{\sigma^2}{2} - 1\right) = 0,$$

whose (positive, given that we are ignoring the lower band) solution is:

(A9) $$\hat{\lambda} = \sqrt{\frac{2}{\beta\sigma^2}}$$

We have, then, the general solution:

(A10) $$i_t = q(b_t) = \bar{r} + \alpha b_t + Ae^{\hat{\lambda} b_t},$$

corresponding to equation (7) in the text, with $\hat{\lambda}$ defined as above. Note that in (A10) the variable *A* is still indeterminate, so an end-point condition is needed to identify it and define the nonlinear (convex or concave) interest rate trajectory. This is where the target zone assumption begins to play a role, as described in the text.

References

Aguiar, M., S. Chatterjee, H. Cole, and Z. Stangebye (2016): "Quantitative Models of Sovereign Debt Crises," *Handbook of Macroeconomics* Volume 2.

Alcidi, C. and D. Gros (2018), “Debt Sustainability Assessments: The state of the art: Euro area scrutiny”, *In-Depth Analysis, Requested by ECON Committee,* European Parliament.

Ayres, J., Navarro, G., Nicolini, J. P., and P. Teles (2018). “Sovereign default: The role of expectations”. *Journal of Economic Theory*, *175*, 803-812.

Ayres, J., G. Navarro, J. P. Nicolini, and P. Teles (2019): “Self-Fulfilling Debt Crises with Long Stagnations,” *Minneapolis Fed Working Paper* 757.

Bertola, G., and R. Caballero (1992a), “Target Zones and Realignments”, *The American Economic Review, 82*(3), pp. 520-536. Retrieved from http://www.jstor.org/stable/2117319

Bertola, G., and R. Caballero (1992b), “Sustainable intervention policies and exchange rate dynamics”, pp. 186-205, in Kugman, P. and M. Miller (eds) (1992), *Exchange Rate Targets and Currency Bands,* Cambridge Univesity Press.

Corsetti, G., K. Kuester, A. Meier, and G. J. Müller (2014), “Sovereign risk and belief-driven fluctuations in the euro area”, *Journal of Monetary Economics*, Vol. 61, Pages 53–73, January.

De Grauwe, P. (2012), “The Governance of a Fragile Eurozone.” *Australian Economic Review,* 45 (3), pp. 255–68.bas

Della Posta, P. (2018), "Central bank intervention, public debt and interest rate target zones", *Journal of Macroeconomics*, Vol. 56, June, Pages 311-323, June. 10.1016/j.jmacro.2018.04.001, 2018.

Della Posta, P. (2019), “Interest rate targets and speculative attacks on public debt”, *Macroeconomic Dynamics*, Vol. 23, N. 7, pp. 2698-2716, October, https://doi.org/10.1515/gej-2017-0097, 2019. Published online: 16 March 2018.

Della Posta, P. (2020), “An analysis of the current backlash of economic globalization in a model with heterogeneous agents”, *Metroeconomica*, https://doi.org/10.1111/meca.12312, First Published, 2 September 2020.

European Commission (2014), “Assessing Public Debt Sustainability in EU Member States: A Guide”, *European Economy*, Occasional Papers 200 | September 2014.

Ghosh, A. R., Kim, J. L., Mendoza, E., Ostry, J. D. and Qureshmi, M. S. (2013). “Fiscal fatigue, fiscal space and debt sustainability in advanced economies”. *Economic Journal*, 123, F4–F30.

IMF (2011), *Modernizing the Framework for Fiscal Policy and Public Debt Sustainability Analysis*. Prepared by the Fiscal Affairs Department and the Strategy, Policy, and Review Department. August 5.

Krugman, P. (1991), "Target Zones and Exchange Rate Dynamics". *The Quarterly Journal of Economics*, Vol. 106, No. 3. (August), pp. 669-682.

Krugman, P. and J. Rotemberg (1992), "Speculative attacks on target zones", pp. 117-132, in Krugman, P. and M. Miller (eds) (1992), *Exchange Rate Targets and Currency Bands*, Cambridge Univesity Press.

Lorenzoni, Guido, and Ivan Werning (2018), "Slow Moving Debt Crises", *American Economics Review*, 109 (9).

Paluszynski, Radoslaw (2021), "Learning about Debt Crises," *American Economic Journal: Macroeconomics* (forthcoming).

Tamborini, R. (2015), "Heterogeneous Market Beliefs, Fundamentals and the Sovereign Debt Crisis in the Eurozone", *Economica*, Volume 82, Issue s1, December 2015, Pages 1153–1176, DOI: 10.1111/ecca.12155.