# Navier-Stokes equations: a new estimate of a possible gap related to the energy equality of a suitable weak solution

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**Abstract** - The paper is concerned with the IBVP of the Navier-Stokes equations. The result of the paper is in the wake of analogous results obtained by the authors in previous articles [4, 5]. The goal is to estimaste the possible gap between the energy equality and the energy inequality deduced for a weak solution.

### 1 Introduction

This note concerns the 3D-Navier-Stokes initial boundary value problem:

$$v_t + v \cdot \nabla v + \nabla \pi_v = \Delta v + f, \ \nabla \cdot v = 0, \ \text{in} \ (0, T) \times \Omega,$$
  
$$v = 0 \ \text{on} \ (0, T) \times \partial \Omega, \ v(0, x) = v_0(x) \ \text{on} \ \{0\} \times \Omega.$$
 (1)

In system (1)  $\Omega \subseteq \mathbb{R}^3$  is assumed bounded or exterior, and its boundary is assumed smooth.

In the two recent papers [4, 5] the authors look for an energy equality for suitable weak solutions. Here, the term suitable is meant in the sense that a new solution is exhibited and not that an improvement is obtained to the one given in [3]. Actually, the crucial result of papers [4, 5], and it seems the first, is the strong convergence in  $L^p(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;L^2(\Omega))$ , for all T > 0 and  $p \in [1,2)$ , of a sequence  $\{v^m\}$  of smooth solutions to the "Leray's approximating Navier-Stokes Cauchy problem" (see (4) below), [7].

Since the strong convergence is not in  $L^2(0,T;W^{1,2}(\Omega))$ , the authors attempt to obtain the energy equality employing the (differential and integral) energy equality of the approximating solutions and some auxiliary functions. Actually, the approaches used so far allow to prove an energy equality which involves other quantities. Here it is proved that a suitable weak solution

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exists and satisfies the following relation

$$\|v(t)\|_{2}^{2} + 2\int_{s}^{t} \|\nabla v(\tau)\|_{2}^{2} d\tau - \|v(s)\|_{2}^{2} - \int_{s}^{t} (f, v) d\tau = -M(s, t) \text{ for all } 0 < s < t \in \mathcal{T},$$
(2)

where

$$\mathcal{T} := \{ t \in (0, T) : \| v^m(t) \|_{1,2} \to \| v(t) \|_{1,2} \}$$

is of full measure in (0, T), and

$$-M(s,t) := -2\lim_{\alpha \to 1^{-}} \lim_{m} \int_{J^{m}(\alpha)} \|\nabla v^{m}(\tau)\|_{2}^{2} d\tau = \lim_{\alpha \to 1^{-}} \lim_{m} \sum_{h \in \mathbb{N}(\alpha,m)} \left[ \|v^{m}(t_{h})\|_{2}^{2} - \|v^{m}(s_{h})\|_{2}^{2} \right]$$

where  $J^m(\alpha)$  is the union of, at most, a countable sequence  $(\mathbb{N}(\alpha, m))$  of disjoint intervals  $(s_h, t_h) \subset (s, t)$  and the following holds:

$$\lim_{\alpha \to 1^{-}} \frac{|J^{m}(\alpha)|}{1-\alpha} \le \frac{1}{\pi} \|v_{0}\|_{2}^{2} + \frac{2}{\pi} \int_{0}^{t} (f, v) d\tau \,, \text{ uniformly in } m \in \mathbb{N}.$$

Instead in the case of s = 0, one obtains

$$\|v(t)\|_{2}^{2} + 2\int_{0}^{t} \|\nabla v(\tau)\|_{2}^{2} d\tau - \|v_{0}\|_{2}^{2} - \int_{0}^{t} (f, v) d\tau = -M(0, t) \text{ for all } t \in \mathcal{T},$$
(3)

where

$$-M(0,T) := \lim_{s_k \to 0} -M(s_k,t), \text{ for any } \{s_k\} \subset \mathcal{T}.$$

Roughly speaking the above intervals seem to contain the possible singular points S of the weak solution that, as is known, has  $\mathcal{H}^{\frac{1}{2}}(S) = 0$  ( $\mathcal{H}^a$  Hausdorff's measure), [11]. Of course, independently of the meaning of the conjecture for the intervals, from a physical view point the energy relation (2) would add a dissipative quantity which is not justifiable. If this is a necessary consequence of an initial datum only in  $L^2$ , then from a physical point of view it is a right reason to reject the  $L^2$ -class as a class of existence.

However, the validity of an energy equality, without requiring extra conditions<sup>1</sup>, is interesting to better delimit the case of validity of possible counterexamples.

Actually, in the papers [2] and [1] two examples of non-uniqueness are furnished.

<sup>&</sup>lt;sup>1</sup> In this connection in paper [9], the so called Prodi-Serrin condition for the energy equality for a weak solution is not required on the whole interval of existence, but just on  $(\varepsilon, T)$ , that is  $L^4(\varepsilon, T; L^4(\Omega))$ , for all  $\varepsilon > 0$ . This means that no extra assumption on the initial datum in  $L^2$  is needed for the validity of the energy equality.

Following [6], under the same quoted assumption, the same result of energy equality holds in the set of very-weak solutions.

The former works for very-weak solutions, which are continuous in  $L^2$ -norm, but do not verify an energy inequality of the kind given by Leray-Hopf, in other words neglecting the term M(s,t)with  $\leq 0$ . Further, in the case of Leray-Hopf weak solutions their counterexample does not work.

The latter works with a homogeneous initial datum. Actually, the non-uniqueness is exhibited for solutions corresponding to a suitable data force, that, among other things, allows an energy equality.

The plan of the paper is the following. In sect. 2 some preliminary lemmas are recalled and some new results of strong convergence are furnished. In sect. 3 the statement and the proof of the chief result are performed.

### 2 Preliminary results

We set  $J^{1,2}(\Omega)$ :=completion of  $\mathscr{C}_0(\Omega)$  in  $W^{1,2}$ -norm, where  $\mathscr{C}_0(\Omega)$  is the set of the test functions of the hydrodynamics.

**Definition 1.** For weak solution to the IBVP (1) we mean a field  $v : (0, \infty) \to \mathbb{R}^3$  such that for all T > 0

- 1.  $v \in L^{\infty}(0,T; L^2(\Omega)) \cap L^2(0,T; J^{1,2}(\Omega))$ ,
- 2. the field v solves the integral equation

$$\int_{s} \int_{s} \left[ (v, \varphi_{\tau}) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) + (\pi_{v}, \nabla \cdot \varphi) \right] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),$$
  
for all  $\varphi \in C_{0}^{1}([0, T] \times \Omega),$ 

3.  $\lim_{t \to 0} \|v(t) - v_0\|_2 = 0$ .

For our goals we consider a mollified Navier-Stokes system. Hence problem (1) becomes

$$v_t^m + J_m[v^m] \cdot \nabla v^m + \nabla \pi_{v^m} = \Delta v^m + f, \ \nabla \cdot v^m = 0, \text{ in } (0,T) \times \Omega,$$
  

$$v^m = 0 \text{ on } (0,T) \times \partial \Omega, \ v^m(0,x) = v_0^m(x) \text{ on } \{0\} \times \Omega,$$
(4)

where  $f \in L^2(0, T, L^2(\Omega)), J_m[\cdot]$  is a mollifier and  $\{v_0^m\} \subset J^{1,2}(\Omega)$  converges to  $v_0$  in  $J^2(\Omega)$ .

**Lemma 1.** For all  $m \in \mathbb{N}$  there exists a unique solution to problem (4) such that for all T > 0

$$v^{m} \in C([0,T); J^{1,2}(\Omega)) \cap L^{2}(0,T; W^{2,2}(\Omega)),$$
  

$$v^{m}_{t}, \nabla \pi^{m} \in L^{2}(0,T; L^{2}(\Omega)).$$
(5)

Moreover, the sequence  $\{v^m\}$  is strong convergent to a limit v in  $L^p(0, T; W^{1,2}(\Omega)) \cap L^2(0, T; L^2(\Omega))$ , for all  $p \in [1, 2)$ , and the limit v is a weak solution to problem (1) with  $(v(t), \varphi) \in C([0, T))$ , for all  $\varphi \in J^2(\Omega)$ . *Proof.* This lemma for data force f = 0 is Theorem 6.1.1 proved in [4]. It is not difficult to image that the proof can be modified without difficulty assuming  $f \neq 0$ . So that we consider as achieved the proof of the lemma.

**Lemma 2.** Let  $\Omega \subseteq \mathbb{R}^n$  and let  $u \in W^{2,2}(\Omega) \cap J^{1,2}(\Omega)$ . Then there exists a constant *c* independent of *u* such that

$$\|u\|_{r} \le c \|P\Delta u\|_{2}^{a} \|u\|_{q}^{1-a}, \quad a\left(\frac{1}{2} - \frac{2}{n}\right) + (1-a)\frac{1}{q} = \frac{1}{r}, \tag{6}$$

provided that  $a \in [0, 1)$ .

*Proof.* See [8, 10].

**Lemma 3.** For any T > 0, there exists a constant M > 0, not depending on m, such that

$$\int_0^T \frac{\left|\frac{d}{dt} \|v^m(t)\|_2^2\right|}{\left(1 + \|\nabla v^m\|_2^2\right)^2} \, dt \le M$$

where  $v^m$  is the solution of problem (4) stated in Lemma 1.

*Proof.* By virtue of the regularity of  $(v^m, \pi^m)$  stated in (5), we multiply equation (4)<sub>1</sub> by  $P\Delta v^m - v_t^m$ . Integrating by parts on  $\Omega$ , and applying the Hölder inequality, we get

$$\|P\Delta v^m - v_t^m\|_2^2 \le 2\|v^m \cdot \nabla v^m\|_2^2 + 2\|f\|_2^2, \text{ a.e. in } t > 0.$$
(7)

Applying inequality (6) with  $r = \infty$  and q = 6, by virtue of the Sobolev inequality, we obtain

$$\|v^{m} \cdot \nabla v^{m}\|_{2} \le \|v^{m}\|_{\infty} \|\nabla v^{m}\|_{2} \le c \|P\Delta v^{m}\|_{2}^{\frac{1}{2}} \|\nabla v^{m}\|_{2}^{\frac{3}{2}}.$$
(8)

By inequalities (7) and (8), we get

$$\|P\Delta v^m - v_t^m\|_2^2 \le c \|P\Delta v^m\|_2 \|\nabla v^m\|_2^3 + 2\|f\|_2^2 \le \frac{1}{2} \|P\Delta v^m\|_2^2 + c\|\nabla v^m\|_2^6 + 2\|f\|_2^2,$$
(9)

for all  $m \in \mathbb{N}$  and a.e. in t > 0. Substituting in inequality (9) the identity

$$\frac{d}{dt} \|\nabla v^m\|_2^2 + \|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2 = \|P\Delta v^m - v_t^m\|_2^2 \tag{10}$$

and dividing by  $(1 + \|\nabla v^m(t)\|_2^2)^2$ , we get the following estimate

$$\frac{\frac{d}{dt}\|\nabla v^m\|_2^2}{(1+\|\nabla v^m\|_2^2)^2} + \frac{\frac{1}{2}\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1+\|\nabla v^m\|_2^2)^2} \le c\|\nabla v^m\|_2^2 + \frac{2\|f\|_2^2}{\left(1+\|\nabla v^m\|_2^2\right)^2}.$$

Integrating on (0, T) we have

$$\begin{aligned} \frac{1}{1+\|\nabla v_0^m\|_2^2} &- \frac{1}{1+\|\nabla v^m(T)\|_2^2} + \int_0^T \frac{\frac{1}{2}\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1+\|\nabla v^m\|_2^2)^2} \, dt \\ &\leq c \int_0^T \|\nabla v^m\|_2^2 \, dt + 2 \int_0^T \frac{2\|f\|_2^2}{\left(1+\|\nabla v^m\|_2^2\right)^2} \, dt \leq C. \end{aligned}$$

It follows that

$$\int_{0}^{T} \frac{\|P\Delta v^{m}\|_{2}^{2}}{(1+\|\nabla v^{m}\|_{2}^{2})^{2}} dt \leq 2C+2, \quad \int_{0}^{T} \frac{\|v_{t}^{m}\|_{2}^{2}}{(1+\|\nabla v^{m}\|_{2}^{2})^{2}} dt \leq C+1.$$

Using the identity (10) we get

$$\int_{0}^{T} \frac{\|P\Delta v^{m} - v_{t}^{m}\|_{2}^{2}}{(1 + \|\nabla v^{m}\|_{2}^{2})^{2}} dt = \int_{0}^{T} \frac{\frac{d}{dt} \|\nabla v^{m}\|_{2}^{2}}{(1 + \|\nabla v^{m}\|_{2}^{2})^{2}} dt + \int_{0}^{T} \frac{\|P\Delta v^{m}\|_{2}^{2}}{(1 + \|\nabla v^{m}\|_{2}^{2})^{2}} dt + \int_{0}^{T} \frac{\|v_{t}^{m}\|_{2}^{2}}{(1 + \|\nabla v^{m}\|_{2}^{2})^{2}} dt \\ \leq -\frac{1}{1 + \|\nabla v^{m}(T)\|_{2}^{2}} + \frac{1}{1 + \|\nabla v_{0}^{m}\|_{2}^{2}} + 3C + 3 \leq 3C + 4.$$

Using once again identity (10) we get

$$\int_{0}^{T} \frac{\left\|\frac{d}{dt}\|\nabla v^{m}\|_{2}^{2}\right|}{(1+\|\nabla v^{m}\|_{2}^{2})^{2}} dt \leq \int_{0}^{T} \frac{\|P\Delta v^{m} - v_{t}^{m}\|_{2}^{2}}{(1+\|\nabla v^{m}\|_{2}^{2})^{2}} dt + \int_{0}^{T} \frac{\|P\Delta v^{m}\|_{2}^{2}}{(1+\|\nabla v^{m}\|_{2}^{2})^{2}} dt + \int_{0}^{T} \frac{\|v_{t}^{m}\|_{2}^{2}}{(1+\|\nabla v^{m}\|_{2}^{2})^{2}} dt \leq 6C + 7 =: M.$$

**Lemma 4.** Let  $\{h_m(t)\}$  be a sequence of non-negative functions bounded in  $L^1(0,T)$ . Also, assume that  $h_m(t) \to h(t)$  a.e. in  $t \in (0,T)$  with  $h(t) \in L^1(0,T)$ . Let be  $g: (0,\alpha_0) \longrightarrow \mathbb{R}$  a continuous and strictly increasing function such that  $\lim_{\alpha \to \alpha_0} g(\alpha) = +\infty$  and  $p: [0,1) \times \mathbb{R} \longrightarrow [0,1]$  a continuous function such that  $p(\alpha, \rho) = 1$  if  $0 \le \rho \le g(\alpha)$ ,  $p(\alpha, \cdot)$  is weakly decreasing and  $\lim_{\rho \to +\infty} p(\alpha, \rho) = 0$  for any  $\alpha \in (0, \alpha_0)$ .

Then we get

$$\lim_{\alpha \to \alpha_0} \lim_{m} \int_{0}^{T} h_m(t) p(\alpha, h_m(t)) dt = \int_{0}^{T} h(t) dt, \qquad (11)$$

*Proof.* We have

$$\int_{0}^{T} h_{m}(t)p(\alpha, h_{m}(t))dt = \int_{0}^{T} (h_{m}(t) - h(t))p(\alpha, h_{m}(t))dt + \int_{0}^{T} h(t)p(\alpha, h_{m}(t))$$
  
=:  $I_{1}(\alpha, m) + I_{2}(\alpha, m)$ .

We fix  $\alpha \in (0, \alpha_0)$  and we consider the first integral. For any  $\varepsilon \in (0, \alpha_0 - \alpha)$  we set

$$J_m^-(\varepsilon) = \{t : h_m(t) \le g(\alpha_0 - \varepsilon)\}, \qquad J_m^+(\varepsilon) = \{t : g(\alpha_0 - \varepsilon) < h_m(t)\}.$$
 (12)

Hence we have

$$I_{1}(\alpha, m) = \int_{0}^{T} \chi_{J_{m}^{-}(\varepsilon)}(t)(h_{m}(t) - h(t))p(\alpha, h_{m}(t))dt + \int_{0}^{T} \chi_{J_{m}^{+}(\varepsilon)}(t)(h_{m}(t) - h(t))p(\alpha, h_{m}(t))dt =: I_{1}^{-}(\alpha, m, \varepsilon) + I_{1}^{+}(\alpha, m, \varepsilon).$$

By (12) we get

$$\chi_{J_m^-(\varepsilon)}(t)(h_m(t) - h(t))p(\alpha, h_m(t))| \le g(\alpha_0 - \varepsilon) + |h(t)|$$

hence, by the dominated convergence theorem, we have

$$\lim_{m} I_1^-(\alpha, m, \varepsilon) = 0, \qquad \forall \alpha, \varepsilon.$$
(13)

Since  $p(\alpha, \cdot)$  is decreasing, we get

$$\left|\chi_{J_m^+(\varepsilon)}(t)(h_m(t) - h(t))p(\alpha, h_m(t))\right| \le p(\alpha, g(\alpha_o - \varepsilon))\left(|h_m(t)| + |h(t)|\right).$$

Using the boundedness of the sequence  $(h_m)$  in  $L^1$  we obtain that

$$|I_1^+(\alpha, m, \varepsilon)| \le cp(\alpha, g(\alpha_0 - \varepsilon)), \qquad \forall m \in \mathbb{N}.$$
(14)

By (13) and (14) we get

$$0 \le \limsup_{m} |I_1(\alpha, m)| \le cp(\alpha, g(\alpha_0 - \varepsilon)), \qquad \forall \, \alpha, \varepsilon$$

Since  $\lim_{\varepsilon \to 0} p(\alpha, g(\alpha_0 - \varepsilon) = 0$  we have that

$$\lim_{m} I_1(\alpha, m) = 0, \qquad \forall \, \alpha.$$

Now we consider the integral  $I_2(\alpha, m)$ . Since  $|p(\alpha, h_m(t))h(t)| \leq 1$  and  $\lim_m h_m(t) = h(t)$  a.e. in  $t \in (0, T)$ , by the dominated convergence theorem, we get

$$\lim_{m} I_2(\alpha, m) = \int_{0}^{T} h(t) p(\alpha, h(t)) dt.$$

Finally, since  $\lim_{\alpha \to \alpha_0} p(\alpha, h(t)) = 1$  we have that

$$\lim_{\alpha \to \alpha_0} \lim_m I_2(\alpha, m) = \int_0^T h(t) dt,$$

and this completes the proof.

## **3** The chief result

We recall the definition

$$\mathcal{T} := \{ t \in (0,T) : \|v^m(t)\|_{1,2} \to \|v(t)\|_{1,2} \}$$

where  $\{v^m\}$  is the sequence of solutions to problem (4). By virtue of the strong convergence stated in Lemma 1, the set  $\mathcal{T}$  is certainly not empty and, as matter of fact, it is of full measure in (0, T).

**Theorem 1.** Let v be the weak solution stated in Lemma 1. Then v satisfies the relation

$$\|v(t)\|_{2}^{2} + 2\int_{s}^{t} \|\nabla v(\tau)\|_{2}^{2} d\tau - \|v(s)\|_{2}^{2} - \int_{s}^{t} (f, v) d\tau = -M(s, t), \text{ for all } t > s \ge \mathcal{T},$$
(15)

with

$$-M(s,t) := -2\lim_{\alpha \to 1^{-}} \lim_{m} \int_{J^{m}(\alpha)} \|\nabla v^{m}(\tau)\|_{2}^{2} d\tau = \lim_{\alpha \to 1^{-}} \lim_{m} \sum_{h \in \mathbb{N}(\alpha,m)} \left[ \|v^{m}(t_{h})\|_{2}^{2} - \|v^{m}(s_{h})\|_{2}^{2} \right]$$

where  $J^m(\alpha) = \bigcup_{i \in \mathbb{N}(\alpha,m)} (s_i, t_i)$  with  $\mathbb{N}(\alpha, m) \subseteq \mathbb{N}$  which is, at most, a sequence of integers with  $(s_i, t_i) \cap (s_j, t_j) = \emptyset$  for any  $i \neq j$  and

$$\lim_{\alpha \to 1^{-}} \frac{|J^m(\alpha)|}{1-\alpha} \le \frac{1}{\pi} \|v_0\|_2^2 + \frac{2}{\pi} \int_0^t (f, v) d\tau \,, \text{ uniformly with respect to } m \,. \tag{16}$$

Moreover, if s = 0, the relation (15) holds setting s = 0 in the left-hand side, and with the right-hand side replaced by  $\lim_{k} M(s_k, t)$  where  $\{s_k\}$  is any sequence in  $\mathcal{T}$  converging to 0.

*Proof.* We consider the sequence  $\{v^m\}$  of solutions to problem (4) whose existence is ensured by Lemma 1. For all  $m \in \mathbb{N}$  the Reynolds-Orr equation holds:

$$\frac{d}{d\tau} \|v^m(\tau)\|_2^2 + 2\|\nabla v^m(\tau)\|_2^2 = (f, v^m).$$
(17)

We set  $\rho_m(t) := \|\nabla v^m(t)\|_2^2$ , and we consider

$$\alpha \in (0,1), \ p(\alpha,\rho_m) := \begin{cases} 1 & \text{if } \rho_m \in [0,\tan\alpha\frac{\pi}{2}] \\ \frac{\frac{\pi}{2} - \arctan\rho_m}{(1-\alpha)\frac{\pi}{2}} & \text{if } \rho_m \in (\tan\alpha\frac{\pi}{2},\infty) \end{cases}$$
(18)

Let be

$$\mathcal{T} = \{ t \in (0,T) : \|v^m(t)\|_{1,2} \to \|v(t)\|_{1,2} \}$$

and fix  $s, t \in \mathcal{T}$ , with s < t. Let  $\alpha_1$  be such that

$$\max\{\|\nabla v(s)\|_{2}^{2}, \|\nabla v(t)\|_{2}^{2}\} < \tan \alpha \frac{\pi}{2}, \text{ for all } \alpha \in (\alpha_{1}, 1).$$

Hence, by virtue of the pointwise convergence, we claim the existence of  $m_0$  such that

$$\max\{\|\nabla v^{m}(s)\|_{2}^{2}, \|\nabla v^{m}(t)\|_{2}^{2}\} < \tan \alpha \frac{\pi}{2}, \text{ for all } m \ge m_{0} \text{ and } \alpha \in (\alpha_{1}, 1).$$
(19)

We set  $A^m := \max_{[s,t]} \rho_m(t)$ . We denote by

$$J^m(\alpha) := \{\tau : \rho_m(\tau) \in (\tan \alpha \frac{\pi}{2}, A^m]\}.$$

If  $A_m \leq \tan \alpha \frac{\pi}{2}$ , then  $J^m(\alpha)$  is an empty set. If  $A_m > \tan \alpha \frac{\pi}{2}$  holds, since  $\rho_m(s) < \tan \alpha \frac{\pi}{2}$ , there exists the minimum  $\overline{s} > s$  such that  $\rho_m(\overline{s}) = \tan \alpha \frac{\pi}{2}$ , as well, being  $\rho_m(t) < \tan \alpha \frac{\pi}{2}$ , there exists the maximum  $\overline{t} < t$  such that  $\rho_m(\overline{t}) = \tan \alpha \frac{\pi}{2}$ . Thus, if  $J^m(\alpha)$  is a non-empty set, by the regularity of  $\rho_m(t)$ , we get that  $J^m(\alpha)$  is at most the union of a sequence of open interval  $(s_h, t_h)$  such that  $\rho_m(s_h) = \rho_m(t_h) = \tan \alpha \frac{\pi}{2}$ . We justify the claim. The set  $J^m(\alpha)$  is an open set, hence it is at most the countable union of maximal intervals  $(s_h, t_h)$ . We set  $E^m := (s, t) - \bigcup_{h \in \mathbb{N}} (s_h, t_h)$ .

For all  $\tau \in E^m$  we have  $\rho_m(\tau) \leq \tan \alpha \frac{\pi}{2}$ , thus, by continuity of  $\rho_m$ , we get  $\rho_m(s_h) = \tan \alpha \frac{\pi}{2} = \rho_m(t_h)$  for all  $h \in \mathbb{N}$ . For the measure of  $J^m(\alpha)$  we get

$$|J^{m}(\alpha)|\tan\alpha\frac{\pi}{2} \le \int_{J^{m}(\alpha)} \rho_{m}(\tau)d\tau < \frac{1}{2}\|v(s)\|_{2}^{2} + \int_{s}^{t} (f,v)d\tau, \qquad (20)$$

where we took the energy relation (17) into account and the strong convergence of the right-hand side too. Estimate (20) leads to (16). Recalling the definition of  $p(\alpha, \rho_m(t))$ , we have

$$\frac{d}{d\tau}p(\alpha,\rho_m(\tau)) = \begin{cases} 0 & \text{a.e. in } \tau \in E^m, \\ \frac{-2}{(1-\alpha)\pi} \frac{\dot{\rho}_m(\tau)}{1+(\rho_m(\tau))^2} & \text{for all } \tau \in J^m(\alpha), \end{cases}$$
(21)

where we took into account that, for all  $\alpha \in (0, 1)$ , function p is Lipschitz's function in  $\rho_m$ , and  $\rho_m(t)$  is a regular function in t. Hence, we get  $p(\alpha, \rho_m(t))$  is Lipschitz's function with respect to t. We multiply equation (17) for  $p(\alpha, \rho_m(\tau))$ , with  $\alpha > \alpha_1$ , and we integrate by parts on (s, t):

$$\|v^{m}(t)\|_{2}^{2} + 2\int_{s}^{t} p(\alpha, \rho_{m}(\tau))\|\nabla v^{m}(\tau)\|_{2}^{2}d\tau + \frac{2M(t, s, m, \alpha)}{(1-\alpha)\pi} = \|v^{m}(s)\|_{2}^{2} + \int_{s}^{t} (f, v^{m})p(\alpha, \rho_{m}(\tau))d\tau ,$$

where we set

$$M(t, s, m, \alpha) := \int_{J^m(\alpha)} \frac{\|v^m(\tau)\|_2^2}{1 + \rho_m^2(\tau)} \dot{\rho}_m(\tau) d\tau$$

where we took (19) and definition of p into account. Letting  $m \to \infty$  and  $\alpha \to 1$ , by virtue of the pointwise convergence in s and in t, and Lemma 4, we arrive at

$$\|v(t)\|_{2}^{2} + 2\int_{s}^{t} \|\nabla v(\tau)\|_{2}^{2} d\tau - \|v(s)\|_{2}^{2} - \int_{s}^{t} (f, v) d\tau = -M(s, t),$$
(22)

where we set

$$M(s,t) := \lim_{\alpha \to 1^{-}} \overline{\lim_{m}} \frac{2}{(1-\alpha)\pi} \int_{J^{m}(\alpha)} \frac{\|v^{m}(\tau)\|_{2}^{2}}{1+\rho_{m}(\tau)^{2}} \dot{\rho}_{m}(\tau) d\tau \,.$$

Recalling the properties of  $J^m(\alpha)$ , for all  $\alpha$  and m, integrating by parts, we get

$$\begin{split} \int_{J^m(\alpha)} \frac{\|v^m\|_2^2}{1+\rho_m^2} \dot{\rho}_m d\tau &= \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2}{1+\rho_m^2} \dot{\rho}_m d\tau \\ &= \frac{\tan \alpha \frac{\pi}{2}}{1+\tan^2 \alpha \frac{\pi}{2}} \sum_{h \in \mathbb{N}(\alpha,m)} \left[ \|v^m(t_h)\|_2^2 - \|v^m(s_h)\|_2^2 \right] + \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{2\rho_m^2}{1+\rho_m^2} d\tau \\ &- 2 \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\rho_m(f,v_m)}{1+\rho_m^2} d\tau + 2 \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2 \rho_m^2}{(1+\rho_m^2)^2} \dot{\rho}_m d\tau \,. \end{split}$$

Hence we arrive at

$$\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{\|v^{m}\|_{2}^{2}}{1+\rho_{m}^{2}}\dot{\rho}_{m}d\tau - 2\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{\|v^{m}\|_{2}^{2}\rho_{m}^{2}}{(1+\rho_{m}^{2})^{2}}\dot{\rho}_{m}d\tau + \sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{\rho_{m}(f,v_{m})}{1+\rho_{m}^{2}}d\tau$$

$$= \frac{\tan\alpha\frac{\pi}{2}}{1+\tan^{2}\alpha\frac{\pi}{2}}\sum_{h\in\mathbb{N}(\alpha,m)}\left[\|v^{m}(t_{h})\|_{2}^{2} - \|v^{m}(s_{h})\|_{2}^{2}\right] + \sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{2\rho_{m}^{2}}{1+\rho_{m}^{2}}d\tau.$$
(23)

We estimate the last integral. Let be

$$\widetilde{J}(\alpha) := \limsup_{m \to \infty} J^m(\alpha) = \bigcap_{j=0}^{\infty} \bigcup_{m=j}^{\infty} J^m(\alpha).$$
(24)

It results that

$$\tau \in \widetilde{J}(\alpha) \iff \exists m_k \to \infty \text{ s.t. } \tau \in J^{m_k}(\alpha) \ \forall k \in \mathbb{N} \iff \limsup_{m \to \infty} \chi_{J^m(\alpha)}(\tau) = 1.$$

hence, if  $\tau \in \widetilde{J}(\alpha) \cap \mathcal{T}$  we get that

$$\rho(\tau) = \lim_{k \to \infty} \rho_{m_k}(\tau) \ge \tan \frac{\alpha \pi}{2}.$$
(25)

On the complement of the set  $\mathcal{T}$  we can set  $\rho = 0$ , since the value on a null measure set does not

change the estimates. Since  $0 \le \chi_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} \le 1$ , by Fatou's lemma, it follows that

$$\frac{1}{1-\alpha} \limsup_{m} \int_{J^{m}(\alpha)} \frac{\rho_{m}^{2}}{1+\rho_{m}^{2}} d\tau = \frac{1}{1-\alpha} \limsup_{m} \int_{s}^{t} \chi_{J^{m}(\alpha)} \frac{\rho_{m}^{2}}{1+\rho_{m}^{2}} d\tau$$
$$\leq \frac{1}{1-\alpha} \int_{s}^{t} \limsup_{m} \chi_{J^{m}(\alpha)} \frac{\rho_{m}^{2}}{1+\rho_{m}^{2}} d\tau = \frac{1}{1-\alpha} \int_{s}^{t} \chi_{\widetilde{J}(\alpha)} \frac{\rho^{2}}{1+\rho^{2}} d\tau = \frac{1}{1-\alpha} \int_{\widetilde{J}(\alpha)} \frac{\rho^{2}}{1+\rho^{2}} d\tau$$
$$\leq \frac{1}{1-\alpha} \frac{1}{\tan \frac{\alpha\pi}{2}} \int_{\widetilde{J}(\alpha)} \rho(\tau) d\tau$$

Since  $\rho \in L^1$  and, by (25),

$$\left|\widetilde{J}(\alpha)\right| \le \frac{\|\rho\|_1}{\tan\frac{\alpha\pi}{2}} \tag{26}$$

the last integral vanishes as  $\alpha$  tends to 1<sup>-</sup>. Moreover

$$\lim_{\alpha \to 1^{-}} \frac{1}{(1-\alpha)\tan \alpha \frac{\pi}{2}} = \frac{\pi}{2}$$
(27)

hence

$$\lim_{\alpha \to 1^{-}} \overline{\lim_{m}} \frac{1}{1-\alpha} \int_{J^{m}(\alpha)} \frac{\rho_{m}^{2}}{1+\rho_{m}^{2}} d\tau = 0.$$
(28)

Concerning the force term we have

$$\begin{aligned} \left| \int_{J^{m}(\alpha)} \frac{\rho_{m}(f, v^{m})}{1 + \rho_{m}^{2}} d\tau \right| &\leq \left( \int_{J^{m}(\alpha)} \frac{\rho_{m}^{2}}{(1 + \rho_{m}^{2})^{2}} d\tau \right)^{\frac{1}{2}} \left( \int_{J^{m}(\alpha)} \|f(\tau)\|_{2}^{2} \|v^{m}(\tau)\|_{2}^{2} d\tau \right)^{\frac{1}{2}} \\ &\leq \left( \frac{1}{1 + (\tan\frac{\alpha\pi}{2})^{2}} \right)^{\frac{1}{2}} \left( \int_{J^{m}(\alpha)} \frac{\rho_{m}^{2}}{1 + \rho_{m}^{2}} d\tau \right)^{\frac{1}{2}} \sup_{m,t} \|v^{m}(t)\|_{2} \left( \int_{J^{m}(\alpha)} \|f(\tau)\|_{2}^{2} d\tau \right)^{\frac{1}{2}} \\ &\leq \frac{C}{\tan\frac{\alpha\pi}{2}} \left( \frac{(\tan\frac{\alpha\pi}{2})^{2}}{1 + (\tan\frac{\alpha\pi}{2})^{2}} \right)^{\frac{1}{2}} |J^{m}(\alpha)|^{\frac{1}{2}} \leq \frac{C}{\tan\frac{\alpha\pi}{2}} \left( \frac{c}{\tan\frac{\alpha\pi}{2}} \right)^{\frac{1}{2}}. \end{aligned}$$

It follows that

$$\lim_{\alpha \to 1^{-}} \frac{1}{1 - \alpha} \overline{\lim}_{m} \int_{J^{m}(\alpha)} \frac{\rho_{m}(f, v^{m})}{1 + \rho_{m}^{2}} d\tau = 0.$$
(29)

Using algebraic manipulation we obtain the following relation:

$$\begin{split} \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2}{1 + \rho_m^2} \dot{\rho}_m d\tau &- 2 \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2 \rho_m^2}{(1 + \rho_m^2)^2} \dot{\rho}_m d\tau \\ &= - \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2}{1 + \rho_m^2} \dot{\rho}_m d\tau + 2 \sum_{h \in \mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2}{(1 + \rho_m^2)^2} \dot{\rho}_m d\tau \,. \end{split}$$

Substituting the above relation in equation (23) we get

$$-\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{\|v^{m}\|_{2}^{2}}{1+\rho_{m}^{2}}\dot{\rho}_{m}d\tau + 2\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{\|v^{m}\|_{2}^{2}}{(1+\rho_{m}^{2})^{2}}\dot{\rho}_{m}d\tau + \sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{\rho_{m}(f,v_{m})}{1+\rho_{m}^{2}}d\tau$$

$$= \frac{\tan\alpha\frac{\pi}{2}}{1+\tan^{2}\alpha\frac{\pi}{2}}\sum_{h\in\mathbb{N}(\alpha,m)}\left[\|v^{m}(t_{h})\|_{2}^{2} - \|v^{m}(s_{h})\|_{2}^{2}\right] + \sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_{h}}^{t_{h}}\frac{2\rho_{m}^{2}}{1+\rho_{m}^{2}}d\tau$$
(30)

At last we estimate the integral

$$\left| \int_{J^{m}(\alpha)} \frac{\|v^{m}\|_{2}^{2}}{(1+\rho_{m}^{2})^{2}} \dot{\rho}_{m} d\tau \right| \leq \int_{J^{m}(\alpha)} \frac{\|v^{m}\|_{2}^{2}}{(1+\rho_{m}^{2})^{2}} |\dot{\rho}_{m}| d\tau \leq \sup_{t,m} \|v^{m}(t)\|_{2}^{2} \int_{J^{m}(\alpha)} \frac{|\dot{\rho}_{m}|}{(1+\rho_{m}^{2})^{2}} d\tau$$

$$\leq c \frac{1}{1+(\tan\frac{\alpha\pi}{2})^{2}} \int_{J^{m}(\alpha)} \frac{|\dot{\rho}_{m}|}{1+\rho_{m}^{2}} d\tau \leq \frac{2c}{1+(\tan\frac{\alpha\pi}{2})^{2}} \int_{J^{m}(\alpha)} \frac{|\dot{\rho}_{m}|}{(1+\rho_{m})^{2}} d\tau \leq \frac{2cM}{1+(\tan\frac{\alpha\pi}{2})^{2}}$$
(31)

where the last inequality follows by Lemma 3. Hence, by (27), we get

$$\lim_{\alpha \to 1^{-}} \limsup_{m} \left| \frac{2}{1-\alpha} \int_{J^{m}(\alpha)} \frac{\|v^{m}\|_{2}^{2}}{(1+\rho_{m}^{2})^{2}} \dot{\rho}_{m} d\tau \right| \leq \lim_{\alpha \to 1^{-}} \frac{2}{1-\alpha} \frac{2cM}{1+(\tan\frac{\alpha\pi}{2})^{2}} = 0.$$

Multiplying equation (30) by  $\frac{2}{(1-\alpha)\pi}$  and passing to the limit using (31), (29) and (28), we get

$$-M(s,t) = \lim_{\alpha \to 1^{-}} \overline{\lim_{m}} \sum_{h \in \mathbb{N}(\alpha,m)} \Big[ \|v^{m}(t_{h})\|_{2}^{2} - \|v^{m}(s_{h})\|_{2}^{2} \Big].$$

By equation (17) we get

$$\sum_{h \in \mathbb{N}(\alpha,m)} \left[ \|v^m(t_h)\|_2^2 - \|v^m(s_h)\|_2^2 \right] = -2 \int_{J^m(\alpha)} \|\nabla v^m(\tau)\|_2^2 d\tau + \int_{J^m(\alpha)} (f, v^m) d\tau.$$

Let us consider the last integral. Since  $|(f(\tau), v^m(\tau))| \le ||f(\tau)||_2 ||v^m(\tau)||_2 \le c ||f(\tau)||_2$  we can apply the Fatou's lemma to get

$$\begin{split} \limsup_{m} \left| \int_{J^{m}(\alpha)} \left( f, v^{m} \right) d\tau \right| &= \limsup_{m} \left| \int_{s}^{t} \chi_{J^{m}(\alpha)}(f, v^{m}) \, d\tau \right| \leq \int_{s}^{t} \limsup_{m} \left| \chi_{J^{m}(\alpha)}(f, v^{m}) \right| \, d\tau \\ &\leq c \int_{s}^{t} \|f\|_{2} \limsup_{m} \chi_{J^{m}(\alpha)} \, d\tau \leq c \int_{s}^{t} \|f\|_{2} \chi_{\widetilde{J}(\alpha)} \, d\tau \end{split}$$

with  $\widetilde{J}(\alpha)$  defined in (24). Since  $||f(\tau)||_2$  is summable, considering (26), we get

$$\lim_{\alpha \to 1^{-}} \limsup_{m} \left| \int_{J^{m}(\alpha)} (f, v^{m}) \, d\tau \right| = 0$$

and this completes the proof in the case of  $s, t \in \mathcal{T}$ . In order to complete the proof of the theorem, we limit ourselves to remark that, letting  $s \to 0$ , the left-hand side tends to values in 0, in particular on any sequence  $\{s_k\} \subset \mathcal{T}$  letting to 0, and as a consequence the limit on  $\{s_k\}$  of the right hand side is well posed.

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#### Declarations

**Conflict of interests** The authors declare that they have no conflict of interest.

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