OPEN ACCESS
IOP Publishing
Classical and Quantum Gravity
Class. Quantum Grav. **39** (2022) 215009 (40pp)
https://doi.org/10.1088/1361-6382/ac9108

Kaluza–Klein reductions of maximally supersymmetric five-dimensional Lorentzian spacetimes

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Received 19 July 2022, revised 1 September 2022
Accepted for publication 9 September 2022
Published 1 November 2022

Abstract
A recent study of filtered deformations of (graded subalgebras of) the minimal five-dimensional Poincaré superalgebra resulted in two classes of maximally supersymmetric spacetimes. One class are the well-known maximally supersymmetric backgrounds of minimal five-dimensional supergravity, whereas the other class does not seem to be related to supergravity. This paper is a study of the Kaluza–Klein (KK) reductions to four dimensions of this latter class of maximally supersymmetric spacetimes. We classify the Lorentzian and Riemannian KK reductions of these backgrounds, determine the fraction of the supersymmetry preserved under the reduction and in most cases determine explicitly the geometry of the four-dimensional quotient. Among the many supersymmetric quotients found, we highlight a number of novel non-homogeneous four-dimensional Lorentzian spacetimes admitting $N = 1$ supersymmetry, whose supersymmetry algebra is not a filtered deformation of any graded subalgebra of the four-dimensional $N = 1$ Poincaré superalgebra. Any of these four-dimensional Lorentzian spacetimes may serve as the arena for the construction of new rigidly supersymmetric field theories.

Keywords: space-time symmetries, differential and algebraic geometry, supergravity models

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1. Introduction

It has been just over half a century since the emergence of four-dimensional supersymmetry in a paper [1] of Golfand and Likhtman, which displayed for the first time what is now called the four-dimensional $N = 1$ Poincaré superalgebra. This is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where the even subalgebra $\mathfrak{g}_{\bar{0}}$ is isomorphic to the Poincaré algebra and the odd subspace $\mathfrak{g}_{\bar{1}}$ is isomorphic to the four-dimensional real spinor representation of $\mathfrak{g}_{\bar{0}}$. The notation reflects the fact that $\bar{0}$ and $\bar{1}$ are the residue classes modulo 2 of 0 and 1, respectively, in other words, the parity, and the Lie brackets respect the parity. The Poincaré superalgebra also admits a compatible $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$, where $\mathfrak{g}_0$ is isomorphic to the Lorentz algebra, $\mathfrak{g}_{-2}$ is the translation ideal and $\mathfrak{g}_{-1} = \mathfrak{g}_{\bar{1}}$ is again the four-dimensional real spinor representation. Notice that the parity is simply the reduction modulo 2 of the grading, so that $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-2}$. This grading is an essential ingredient in the story, as we will briefly explain below.

It did not take long for Zumino [2] to exhibit another four-dimensional supersymmetry algebra extending this time the isometry algebra of anti de Sitter spacetime (AdS_4). That Lie superalgebra, $\mathfrak{g} \cong \mathfrak{osp}(1|4)$, with $\mathfrak{g}_{\bar{0}} \cong \mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$ and $\mathfrak{g}_{\bar{1}}$ the four-dimensional vector representation of $\mathfrak{sp}(4,\mathbb{R})$, can be contracted à la Inönü–Wigner to the $N = 1$ Poincaré superalgebra.

For many years, the only (Lorentzian, four-dimensional) spacetimes known to admit $N = 1$ (i.e., $n := \dim \mathfrak{g}_{\bar{1}} = 4$) rigid (i.e., the metric is fixed and not a dynamical field) supersymmetry were Minkowski and anti de Sitter spacetimes. If we drop the condition that the spacetime be Lorentzian, perhaps by allowing it to be a kinematical spacetime [3–5], then one can obtain many supersymmetry algebras (see, e.g. [6–13]), extending the subclass of kinematical Lie algebras of Bacry and Lévy-Leblond [3] which can be obtained by contracting the isometry algebra of AdS_4. In fact, one can obtain a classification [14] of spatially isotropic kinematical supersymmetry algebras (with four real superchanges) and their corresponding homogeneous superspaces.

If we insist on spacetimes being Lorentzian (spin) manifolds, then the first four-dimensional examples beyond Minkowski and anti de Sitter spacetimes were constructed by Festuccia and Seiberg [15] via a decoupling limit of the theory obtained by coupling matter (in the form of a sigma model) to off-shell supergravity and then freezing the gravitational degrees of freedom via a limit in which the Planck mass goes to infinity, resulting in a background admitting rigid supersymmetry. The supersymmetry here is generated by the supergravity Killing spinors: sections of the spinor bundle which are parallel relative to a connection constructed out of the bosonic fields in the supergravity multiplet and uniquely specified by the requirement of local supersymmetry. Let us elaborate briefly.

A supergravity theory is a gauge theory of supersymmetry, where the gauge parameter is a section of a spinor bundle. The corresponding gauge field in supergravity is called the gravitino and just as in standard gauge theory it transforms infinitesimally as the gauge-covariant derivative of the gauge parameter. The form of this gauge-covariant derivative depends on the supergravity theory in question and it is the covariant derivative associated to the so-called gravitino connection on the spin bundle. The gravitino connection may always be written as the spin connection modified by a one-form with values in spinor endomorphisms, which depends *a priori* on all the fields of the supergravity theory. One typically restricts attention to so-called bosonic supergravity backgrounds: those for which the fermionic fields are zero. Due to the nature of supersymmetry transformations, any bosonic field configuration in such a background is automatically invariant (infinitesimally) under a supersymmetry transformation, but this is not necessarily the case for the fermionic fields. On a bosonic supergravity background, the gravitino transforms as the covariant derivative of the spinor gauge parameter. Hence the supersymmetries which preserve the background are generated infinitesimally by spinors which are parallel relative to the gravitino connection. These are the supergravity Killing spinors. If the supergravity theory contains other fermionic fields besides the gravitino, their supersymmetric variations give rise to additional algebraic equations Killing spinors must satisfy.

As shown in the eleven-dimensional context [16] (but true in any dimension) Lie superalgebras generated by parallel spinors (relative to *any* connection on the spinor bundle) have a particular algebraic structure: they are filtered deformations of a graded subalgebra of the Poincaré superalgebra and can be classified by generalised Spencer cohomology [17]. Moreover, the form of the putative gravitino connection can be recovered from the cohomology calculation. This then suggests a method of classification, whose first step is the classification of graded subalgebras of the Poincaré superalgebra. That is not an easy problem, but one can nevertheless classify [18] filtered deformations of maximally supersymmetric graded subalgebras; that is, filtered deformations of graded subalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ with dim $\mathfrak{g}_{-1}$ equal to the rank of the spinor bundle, which is 4 in this case. This recovers the supersymmetric spacetimes of Festuccia and Seiberg [15] with one addition: a conformally flat Cahen–Wallach (CW) [19] Lorentzian symmetric space, whose metric can be recognised as a bi-invariant metric on the Nappi–Witten group [20].

This still leaves open the possibility of obtaining Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with dim $\mathfrak{g}_{\bar{1}} = 4$ which are filtered deformations of graded subalgebras of extended (i.e., $N > 1$) Poincaré superalgebras. These are supersymmetric extensions of the Poincaré algebra, where the odd subspace is now N copies of the four-dimensional spinor representation. Some of these extended Poincaré superalgebras can be obtained by dimensional reduction from minimal Poincaré superalgebras in higher dimensions and all we need to do is break some of the supersymmetry. This is typically achieved via Kaluza–Klein (KK) reduction of higher dimensional supersymmetric backgrounds. Geometrically this corresponds to viewing the higher-dimensional background as a principal bundle over the four-dimensional background: the surviving four-dimensional supersymmetry is then generated by those higher-dimensional Killing spinors which are invariant under the structure group of the principal bundle. Spencer cohomology calculations in five [21] and six [22] dimensions have resulted in a list of supersymmetric backgrounds with eight real supercharges and, intriguingly, not all of them seem to be related to supergravity.

In this paper we concentrate on the five-dimensional supersymmetric backgrounds described in [21]. In that paper, following from the calculation of the generalised Spencer cohomology of the minimal five-dimensional Poincaré superalgebra, two families of maximally supersymmetric backgrounds were obtained. One family consists of the maximally supersymmetric backgrounds of minimal five-dimensional supergravity [23], and the other family consists of five-dimensional Lorentzian locally symmetric spaces (M, g) together with a nontrivial $\mathfrak{sp}(1)$-valued vector field $\Phi = \varphi \otimes r$, with r a fixed element of $\mathfrak{sp}(1)$ and $\varphi \in \mathfrak{X}(M)$ a parallel vector field. Up to covering, (M, g) is one of the following (conformally flat) spacetimes

(a) $-\mathbb{R} \times S^4$ with the Lorentzian product metric, if φ is timelike;
(b) $\mathbb{R} \times \mathrm{AdS}_4$ also with the Lorentzian product metric, if φ is spacelike; and
(c) A certain indecomposable CW Lorentzian symmetric space, if φ is null.

In the first two cases, the Lorentzian norm of φ is related to the curvature of the round metric on S^4 and on AdS_4.

The aim of this paper is to classify four-dimensional KK reductions of these three backgrounds and in particular those which preserve some (nonzero) fraction of the supersymmetry. For each space we will proceed in a similar way: first we classify one-parameter subgroups of the relevant isometry group which lead to quotients which are smooth pseudo-Riemannian manifolds; that is, either Lorentzian or Riemannian. We then find which of these one-parameter subgroups preserve some fraction of supersymmetry. Finally we discuss the geometry of these quotients and comment on the results obtained.

The paper is organised as follows. In section 2 we set out our conventions, introduce the five-dimensional geometries and discuss our methodology: how to classify the one-parameter subgroups of isometries and how to select those which lead to either a Lorentzian or Riemannian quotient; how to determine the fraction of supersymmetry which is preserved by the reduction; and how to work out the quotient metric and its isometries. We then apply this methodology to each of the three five-dimensional spacetimes in turn.

In section 3 we study the reductions of $-\mathbb{R} \times S^4$. The one-parameter subgroups are determined in proposition 1 and the corresponding fraction of supersymmetry is determined in proposition 2. Among the reductions, it is worth mentioning a one-parameter family of $\frac{1}{2}$-BPS quotients (i.e., preserving one half of the supersymmetry) which are Riemannian four-dimensional manifolds admitting rigid Euclidean supersymmetry with four real supercharges. These manifolds are diffeomorphic to S^4 but the metric is far from the round metric, being of cohomogeneity one with $O(3) \times O(2)$ isometry group.

In section 4 we study the reductions of $\mathbb{R} \times \mathrm{AdS}_4$. The one-parameter subgroups are determined in proposition 3 and the corresponding reductions contain both Lorentzian and Riemannian four-dimensional geometries. Those one-parameter subgroups preserving some supersymmetry are determined in proposition 4 and listed in table 2. We see that there are two $\frac{1}{2}$-BPS reductions: one Riemannian and one Lorentzian. The metric of the Lorentzian reduction is given in equation (4.35) and that of the Riemannian reduction by equation (4.37). The former Lorentzian metric is not conformally flat and hence it is not one of the four-dimensional geometries in [18]. It is a novel four-dimensional Lorentzian geometry admitting rigid supersymmetry with four real supercharges. The latter Riemannian metric gives a four-dimensional Riemannian geometry admitting rigid Euclidean supersymmetry with four real supercharges. The (hereditary) isometry Lie algebras of the quotients are listed in table 4. We observe that, in particular, the isometry Lie algebra of the Lorentzian metric in equation (4.35) is isomorphic to $\mathfrak{so}(2) \oplus \mathfrak{so}(2,1)$, while that of the Riemannian metric in equation (4.37) is isomorphic to $\mathfrak{so}(2) \oplus \mathfrak{so}(3)$. Neither metric is homogeneous, but of cohomogeneity one.

In section 5 we study the reductions of the CW spacetime. The possible Killing vector fields (KVFs) are determined in proposition 5. All the resulting quotients are Lorentzian. Those quotients preserving some supersymmetry are determined in proposition 6 and listed, along with the corresponding fraction of supersymmetry, in table 5. We see that there are three families of reductions preserving half the supersymmetry. Upon determining the quotient metric, we see that two of them (labelled X_6 and $X_8^{\pm}$) are isometric to the bi-invariant metric on the Nappi–Witten group, hence they agree with one of the geometries in [18]. The third family of reductions (labelled X_9) contains at least a one-parameter family of novel four-dimensional Lorentzian geometry admitting rigid $N = 1$ supersymmetry. The (hereditary) isometry Lie algebras of the quotients are listed in table 6.

Finally in section 6 we summarise the results and offer some conclusions. The paper contains two appendices: appendix A on the geometry of the CW spaces and appendix B on the explicit form of the gamma matrices used to check the fraction of supersymmetry in the quotients.

2. Setup

2.1. Conventions

We will work with metrics g of mostly plus signature. Accordingly we call a nonzero vector v timelike if $g(v,v) < 0$, spacelike if $g(v,v) > 0$ and null if $g(v,v) = 0$. We denote by $-\mathbb{R}$ the space $\mathbb{R}$ with negative-definite Euclidean inner product, by $\mathbb{R}^{p,q}$ the space $\mathbb{R}^{p+q}$ with flat pseudo-inner product of signature (p,q), where p is the number of negative eigenvalues, and by $\eta^{p,q}$, or just η if the signature is clear from the context, the corresponding inner product. The isometry group of a pseudo-Riemannian manifold M is denoted by $\mathrm{Iso}(M)$. Einstein's summation convention is enforced throughout.

Let V be a finite dimensional vector space V equipped with a non-degenerate inner product g. The Clifford algebra $\mathrm{Cl}(V)$ associated to (V,g) is defined by the relation

$$uv + vu = -2g(u,v)I \tag{2.1}$$

for any $u, v \in V$. For $(p,q) = (1,4)$, $\mathrm{Cl}(V) \simeq \mathrm{End}(\Sigma) \oplus \mathrm{End}(\Sigma')$ where Σ, Σ' are irreducible inequivalent $\mathrm{Cl}(V)$ modules, each of which is isomorphic to $\mathbb{H}^2$ as a (right) vector space. They are distinguished by the action of the volume element which is by I on Σ and $-I$ on Σ'. We will work with Σ from now on. The Clifford algebra $\mathrm{Cl}(V)$ is generated by an orthonormal basis (e_i) of V, and we denote by γ_i the image of e_i under the representation $\mathrm{Cl}(V) \to \mathrm{End}\,\Sigma$. The gamma matrices obey the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\eta^{1,4}_{ij}. \tag{2.2}$$

We also define

$$\gamma_{ij} = \frac{1}{2}(\gamma_i\gamma_j - \gamma_j\gamma_i). \tag{2.3}$$

The Riemann tensor R of a pseudo-Riemannian manifold (M,g) is

$$R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z, \tag{2.4}$$

with components with respect to a local frame $\{e_i\}$ and dual coframe $\{e^i\}$

$$R^i{}_{jkl} = e^i(R(e_k,e_l)e_j). \tag{2.5}$$

The associated Ricci tensor Ric and scalar curvature s are

$$\mathrm{Ric}_{ij} = R^k{}_{ikj}, \quad s = \mathrm{Ric}^i{}_i. \tag{2.6}$$

We also define

$$R_{ijkl} = g_{im}R^m{}_{jkl}. \tag{2.7}$$

2.2. Maximally supersymmetric five-dimensional backgrounds

As mentioned in the introduction, we are going to consider four-dimensional KK reductions of maximally supersymmetric backgrounds associated to filtered deformations of graded subalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ of the minimal five-dimensional Poincaré superalgebra with $\dim\,\mathfrak{g}_{-1} = 8$ (and hence with $\dim\,\mathfrak{g}_{-2} = 5$). It is proved in [21] that these are either the usual maximally supersymmetric backgrounds of minimal supergravity, or locally symmetric spaces (M, g) with Riemann tensor

$$R_{ijkl} = -\frac{1}{2}\big(g_{il}\varphi_j\varphi_k - g_{ik}\varphi_j\varphi_l + g_{jk}\varphi_i\varphi_l - g_{jl}\varphi_i\varphi_k + |\varphi|^2\big(g_{ik}g_{jl} - g_{il}g_{jk}\big)\big), \tag{2.8}$$

where $\varphi \in \mathfrak{X}(M)$ is a certain parallel vector field. The corresponding Ricci tensor and scalar curvature are

$$\mathrm{Ric}_{ij} = \frac{3}{2}(\varphi_i\varphi_j - |\varphi|^2 g_{ij}), \quad s = -6|\varphi|^2. \tag{2.9}$$

It follows that, up to covering, (M, g) is one of three possible spaces depending on the causal character of φ.

If φ is timelike then $M = -\mathbb{R} \times S^4$ with the Lorentzian product metric. We take global coordinates $(x^0, x^1, x^2, x^3, x^4, x^5)$ on $\mathbb{R}^{1,5}$ and embed $S^4 \subset \mathbb{R}^5$ in the usual way, $S^4 = \{(x^1, \ldots, x^5) : x_1^2 + \cdots + x_5^2 = R^2\}$. In this case

$$\varphi = c\partial_0. \tag{2.10}$$

By (2.8), the radius R of S^4 is related to $|\varphi|$ by the equation

$$R^2|\varphi|^2 = -2. \tag{2.11}$$

If φ is spacelike then $M = \mathbb{R} \times \mathrm{AdS}_4$ where AdS_4 is four-dimensional anti de Sitter space. Taking global coordinates $(y, x^1, x^2, x^3, x^4, x^5) \in \mathbb{R} \times \mathbb{R}^{2,3}$, with x^1, x^2 timelike and the other coordinates spacelike, we regard AdS_4 as the universal cover of the quadric in $\mathbb{R}^{2,3}$

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = R^2. \tag{2.12}$$

In this case

$$\varphi = c\partial_y \tag{2.13}$$

and by (2.8) the 'radius' R of AdS_4 is related to $|\varphi|$ by the equation

$$R^2|\varphi|^2 = 2. \tag{2.14}$$

If φ is null then M is an indecomposable CW space. Some general facts about CW spaces are recalled in appendix A. In five dimensions the geometry is that of $\mathbb{R}^5$ with global coordinates $(x^+, x^-, x^1, x^2, x^3)$ and the Lorentzian metric

$$g = 2\mathrm{d}x^+\,\mathrm{d}x^- + \sum_{i,j=1}^{3} A_{ij}x^i x^j(\mathrm{d}x^-)^2 + |\mathrm{d}x|^2, \tag{2.15}$$

where $|\mathrm{d}x|^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$ and A is a bilinear symmetric form on $\mathbb{R}^3$. The only non-vanishing components of the Riemann and Ricci tensors associated to (2.15) are

$$R_{-i-j} = -A_{ij}, \quad \mathrm{Ric}_{--} = -\operatorname{Tr} A. \tag{2.16}$$

In this case

$$\varphi = c\partial_+. \tag{2.17}$$

Comparing (2.16) with (2.8) we find that for the CW spaces arising in [21] A is a scalar matrix,

$$A = -\frac{c^2}{2}\eta^{0,3}. \tag{2.18}$$

For such a choice of A the metric (2.15) is conformally flat, see (A.17). From now on we take $c = \sqrt{2}$ so that

$$g = 2\mathrm{d}x^+\,\mathrm{d}x^- - |x|^2(\mathrm{d}x^-)^2 + |\mathrm{d}x|^2, \tag{2.19}$$

where $|x|^2 = x_1^2 + x_2^2 + x_3^3$.

2.3. One parameters subgroups

Let (M, g) be a pseudo-Riemannian manifold, $G = \mathrm{Iso}(M)$. For $\Gamma \subset G$ a suitable one-parameter subgroup of G, the KK reduction of (M, g) along Γ is the space of orbits M/Γ with the induced metric. In order for M/Γ to have a well defined pseudo-Riemannian metric we require Γ to be generated by a KVF ξ which is either spacelike or timelike, so $|\xi| \neq 0$. The quotient by a null KVF, leading to Newton–Cartan geometries, is also interesting but we defer it to future work.

The condition $|\xi| \neq 0$ automatically ensures that the action generated by ξ is free. In order to ensure that M is a smooth manifold we also want Γ to act properly. In indefinite signature determining if the action of a subgroup of G is proper is a non-trivial problem which we will not address in this paper, but simply assume that parameters are chosen in such a way that the action is proper.

Let Γ be a one-parameter subgroup of G, $g \in G$ an isometry. In classifying KK reductions we want to identify the isometric quotients

$$M/\Gamma \simeq (g \cdot M)/(g\Gamma g^{-1}). \tag{2.20}$$

The G action on M induces a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$ taking values in the space of KVFs, given by

$$X \mapsto \xi_X, \quad \xi_X|_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp(-tX) \cdot p, \tag{2.21}$$

where we act with $-X$ rather than X in order to obtain a Lie algebra homomorphism rather than anti-homomorphism. It follows that if ξ_X is a KVF and $\Gamma = \Gamma_{\xi_X}$ the corresponding one-parameter subgroup of G then

$$g\Gamma_{\xi_X}g^{-1} = \Gamma_{\xi_{\mathrm{Ad}_g X}}. \tag{2.22}$$

Hence for any $g \in G$, $X \in \mathfrak{g}$, for $\Gamma = \Gamma_{\xi_X}$ (2.20) is equivalent to

$$M/\Gamma_{\xi_X} \simeq (g \cdot M)/\Gamma_{\xi_{\mathrm{Ad}_g X}}. \tag{2.23}$$

Moreover for any $\lambda \in \mathbb{R}^\times$, $X \in \mathfrak{g}$ and λX generate the same one-parameter subgroup. Therefore classifying one-parameter subgroups of G is equivalent to classifying one-dimensional Lie subalgebras of $\mathfrak{g}$ under the equivalence relation

$$X \sim \lambda g X g^{-1}, \tag{2.24}$$

for any $\lambda \in \mathbb{R}^\times$, $g \in G$. We also recall that, for any $g \in G$, $X \in \mathfrak{g}$, (2.21) satisfies

$$\xi_{\mathrm{Ad}_g X} = g_* \xi_X, \tag{2.25}$$

where g_* is the induced G-action on TM. Hence conjugation of X by g corresponds to left translation by an isometry of the associated KVF.

2.4. Preserved SUSY

As discussed in the introduction, for the purposes of this paper a Killing spinor is a solution of the parallelism condition $\mathcal{D}\epsilon = 0$ for $\mathcal{D}$ a certain connection on the spinor bundle which in the present case is [21]

$$\mathcal{D} = \nabla - \beta, \tag{2.26}$$

where ∇ is (the lift to the spin bundle of) the Levi-Civita connection and, for ξ a KVF, ϵ a spinor field, $\beta \in \Gamma(T^*M \otimes \mathrm{End}(\Sigma))$ is given by

$$\beta(\xi)(\epsilon) =: \beta_\xi \epsilon = \frac{1}{4}(\xi \cdot \varphi \cdot + 3g(\xi, \varphi)I)r\epsilon = \frac{1}{4}\xi^i \varphi^j (\gamma_i \gamma_j + 3g_{ij})r\epsilon. \tag{2.27}$$

Here r is some fixed non-zero element of $\mathfrak{su}(2)$,

$$r = \begin{pmatrix} i|r^1{}_1| & r^1{}_2 \\ -\overline{r^1{}_2} & -i|r^1{}_1| \end{pmatrix} \tag{2.28}$$

which we are free to rescale as well to conjugate within $SU(2)$. Hence from now on we take

$$r = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \tag{2.29}$$

A given bosonic background M is supersymmetric if $\nu = \dim(S)/n > 0$ for S the space of Killing spinors on M and n the rank of the spinor bundle. If $\nu = 1$ then M is said to be maximally supersymmetric. As discussed in [24], if M is simply connected spin, ξ a KVF, Γ_ξ the corresponding one-parameter group, then M/Γ_ξ is a spin manifold if and only if and only if the Γ_ξ-action on M lifts to an action on the spin bundle of M in a Γ_ξ-equivariant way[3]. This always happens if Γ_ξ has the topology of a line, which is the case for all the examples we will consider.

In standard supergravity, the KK reduction M/Γ_ξ of a $5d$ solution gives a solution of the $4d$ supergravity equations provided that ξ preserves any other field which is turned on in five dimensions: e.g., for minimal supergravity, ξ should also preserve the two-form. A $4d$ Killing spinor lifts to a $5d$ Killing spinor which is invariant under ξ. Conversely, any $5d$ Killing spinor which is invariant under ξ, descends to a Killing spinor on M/Γ_ξ. In our case we prefer not to specify a $4d$ theory on the quotient, but in analogy with standard supergravity we define a Killing spinor on M/Γ_ξ to be a Killing spinor on M which is invariant under Γ_ξ.

A Killing spinor ϵ is invariant under Γ_ξ if and only if $L_\xi \epsilon = 0$ for L_ξ the spinorial Lie derivative. The fraction of SUSY preserved by the quotient M/Γ_ξ is thus

$$\frac{1}{n}\dim\bigl(\mathrm{Ker}\, L_\xi|_S\bigr). \tag{2.30}$$

We recall that the Lie derivative of a spinor s along a KVF ξ can be written

$$L_\xi s = \nabla_\xi s + \frac{1}{4}\mathrm{d}\xi^\flat \cdot s. \tag{2.31}$$

[3] In [24] only the case ξ spacelike is considered, but the argument extends directly to any KVF ξ which never vanishes.

For a Killing spinor ϵ, $\nabla\epsilon = \beta\epsilon$, so Γ_ξ preserves a Killing spinor if

$$L_\xi\epsilon = \beta_\xi\epsilon + \frac{1}{4}\,\mathrm{d}\xi^\flat \cdot \epsilon = 0. \tag{2.32}$$

Since ξ preserves all the bosonic fields which are turned on, $L_\xi \circ \mathcal{D} = 0$, hence ξ preserves the spinor connection and $[L_\xi, \mathcal{D}_X] = \mathcal{D}_{[\xi,X]}$. Therefore the Lie derivative of a Killing spinor is a Killing spinor. Being defined as parallel sections, Killing spinors (on a connected manifold) are fully determined by their value at one point, hence (2.32) only needs to be checked at an arbitrarily selected point, which we can choose so to simplify computations.

Writing a spinor ϵ as a pair $(\epsilon_1, \epsilon_2)^\mathrm{T}$, with $\epsilon_i \in \mathbb{H} \simeq \mathbb{R}^4$ acted upon by the gamma matrices $\gamma_\mu \in \mathrm{Cl}(1,4)$, and with the choice (2.29) for r, (2.32) decouples into the two matrix equations

$$\begin{aligned} &\left[\frac{1}{8}(\mathrm{d}\xi^\flat)_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{4}\xi^\mu\varphi^\nu(\gamma_\mu\gamma_\nu + 3g_{\mu\nu})\right]\epsilon_1 = 0,\\ &\left[\frac{1}{8}(\mathrm{d}\xi^\flat)_{\mu\nu}\gamma^{\mu\nu} - \frac{i}{4}\xi^\mu\varphi^\nu(\gamma_\mu\gamma_\nu + 3g_{\mu\nu})\right]\epsilon_2 = 0. \end{aligned} \tag{2.33}$$

2.5. Geometry of the quotient

We first recall the geometry of a KK quotient. Let (M, g) be a pseudo-Riemannian five-manifold with a connected one-dimensional Lie group Γ acting smoothly and properly by isometries. Let ξ be the KVF generating the Γ action. Provided that $|\xi| \neq 0$ everywhere, the quotient M/Γ is a smooth pseudo-Riemannian four-manifold and

$$\pi : M \to M/\Gamma \tag{2.34}$$

a principal Γ-bundle. The metrics h on M/Γ and g on M are related by

$$g = \pi^*h + \frac{\xi^\flat \otimes \xi^\flat}{g(\xi,\xi)}, \tag{2.35}$$

where $\xi^\flat = g(\xi, \cdot)$. In adapted local coordinates with $\xi = \partial/\partial z$ we have

$$\xi^\flat = \pm \mathrm{e}^{2\tilde{\phi}}(\mathrm{d}z + A), \tag{2.36}$$

with the top (respectively bottom) sign if ξ is spacelike (timelike), $\tilde{\phi} \in C^\infty(M)$ and $A \in \Omega^1(M)$ such that $\xi(\tilde{\phi}) = 0$, $i_\xi A = 0 = i_\xi\,\mathrm{d}A$. It follows that $\tilde{\phi} = \pi^*\phi$ for some $\phi \in C^\infty(M/\Gamma)$ and that $\mathrm{d}A = \pi^*F$ for some $F \in \Omega^2(M/\Gamma)$. In summary, (2.35) can be written locally as

$$g = \pi^*h \pm \mathrm{e}^{2\pi^*\phi}(\mathrm{d}z + A)^2. \tag{2.37}$$

In all the cases that we are going to study M is a trivial principal bundle. In fact the KVFs considered in sections 3–5, see propositions 1, 3 and 5, all generate groups Γ having the topology of a line, and principal $\mathbb{R}$-bundles over paracompact bases are always trivial, see e.g. [25, proposition I.32].

By making a suitable choice of coordinates, for example adapted to the KVF ξ, we can explicitly exhibit the metric on M/Γ, whose isometry group can then be easily determined, at least in some cases. We also recall that if G acts effectively on M by isometries and Γ, H are subgroups of G, then the H action descends to M/Γ if and only if H is a subgroup of the normaliser $N_\Gamma(G)$. Any isometry of M descending to M/Γ is an isometry of M/Γ. Of course Γ itself acts trivially on M/Γ, so we need to quotient by it to obtain an effective action,

$$N_\Gamma(G)/\Gamma \subset \mathrm{Iso}(M/\Gamma). \tag{2.38}$$

We call the Lie algebra of $N_\Gamma(G)$ the hereditary isometry algebra of the quotient and denote it by $\mathfrak{l}$. In general it is a proper subalgebra as M/Γ may have additional 'accidental' isometries.

3. Kaluza–Klein quotients of $-\mathbb{R} \times S^4$

Recall that we take global coordinates $(x^0, x^1, x^2, x^3, x^4, x^5)$ on $\mathbb{R}^{1,5}$ with $S^4 = \{(x^1, \ldots, x^5) : x_1^2 + \cdots + x_5^2 = R^2\}$. The parallel vector field φ is

$$\varphi = c\partial_0, \tag{3.1}$$

with

$$R^2|\varphi|^2 = -R^2c^2 = -2. \tag{3.2}$$

3.1. One parameters subgroups of G

The isometry group of $M = -\mathbb{R} \times S^4$ is the direct product $G = \mathbb{R} \times O(5)$ of time translations and rotations, with Lie algebra $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{so}(5)$. Let $X = (\tau^0, -\omega) \in \mathfrak{g}$. Here

$$\omega = \omega^i{}_j e_i \otimes e^j \tag{3.3}$$

is a skew-adjoint endomorphism, $\omega_{ij} := g_{ia}\omega^a{}_j = -g_{ja}\omega^a{}_i = -\omega_{ji}$. The vector field ξ associated to X by (2.21) is, taking into account that the action on coordinates is the inverse of that on points,

$$\begin{aligned}
\xi_p &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 x^\mu(\exp(-tX)\cdot p)\partial_\mu = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\exp(tX))^\mu{}_\nu x^\nu(p)\partial_\mu \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \Big((x^0(p) + \mathrm{e}^{t\tau^0})\partial_0 + (\mathrm{e}^{-t\omega})^i{}_j x^j(p)\partial_i\Big) = \tau^0\partial_0 - \omega^i{}_j x^j(p)\partial_i \\
&= \tau^0\partial_0 + \frac{1}{2}\omega^{ij}(x_i(p)\partial_j - x_j(p)\partial_i).
\end{aligned} \tag{3.4}$$

Identifying a point with its coordinates, from now on we simply write

$$\xi = \tau + \lambda, \quad \tau = \tau^0\partial_0, \quad \lambda = \frac{1}{2}\omega^{ij}(x_i\partial_j - x_j\partial_i). \tag{3.5}$$

Therefore we have the correspondence

$$(\tau^0, -\omega) \in \mathfrak{g} \quad \longleftrightarrow \quad \tau^0\partial_0 + \frac{1}{2}\omega^{ij}R_{ij} \in \mathfrak{X}(M), \tag{3.6}$$

where

$$R_{ij} = x_i\partial_j - x_j\partial_i \tag{3.7}$$

is the generator of a rotation in the plane (i, j). Its squared norm with respect to $\eta^{1,5}$ is

$$|R_{ij}|^2 = x_i^2 + x_j^2. \tag{3.8}$$

In particular, let (e_i), (e^i) be bases of $\mathbb{R}^5$ and of its dual. Taking generators

$$\epsilon_{ij} = e_j \otimes e^i - e_i \otimes e^j, \quad i, j = 1, \ldots, 5, \tag{3.9}$$

of $\mathfrak{so}(5)$, with bracket

$$[\epsilon_{ij}, \epsilon_{kl}] = \eta^{0,5}_{ik}\epsilon_{jl} + \eta^{0,5}_{jl}\epsilon_{ik} - \eta^{0,5}_{il}\epsilon_{jk} - \eta^{0,5}_{jk}\epsilon_{il}, \tag{3.10}$$

we have

$$\epsilon_{ij} \in \mathfrak{so}(5) \quad \longleftrightarrow \quad R_{ij} \in \mathfrak{X}(S^4). \tag{3.11}$$

Proposition 1. *Let ξ be a KVF of $M = -\mathbb{R} \times S^4$ and assume that $|\xi|$ never vanishes. Then ξ is timelike and there are coordinates such that, up to rescaling,*

$$\xi = \partial_0 + \lambda, \quad \lambda = \beta_1 R_{12} + \beta_2 R_{34}, \tag{3.12}$$

with β_1, β_2 satisfying

$$R|\beta_1| < 1 \quad \text{and} \quad R|\beta_2| < 1. \tag{3.13}$$

Proof. Since rotations fix the origin of $\mathbb{R}^{1,5}$, in order for ξ not to have zeros we need $\tau \neq 0$ and we can always rescale ξ so that $\tau = \partial_0$. As it is well known, by conjugating via a rotation λ can be brought to normal form

$$\lambda = \beta_1 R_{12} + \beta_2 R_{34}, \tag{3.14}$$

so that $|\xi|^2 = -1 + |\lambda|^2 = -1 + \beta_1^2(x_1^2 + x_2^2) + \beta_2^2(x_3^2 + x_4^2)$. Restricting to S^4 we have

$$0 \leqslant |\lambda|^2 \leqslant \max\{\beta_1^2 R^2, \beta_2^2 R^2\}.$$

Hence in order to avoid zeros of $|\xi|$ we need $\max\{\beta_1^2 R^2, \beta_2^2 R^2\} < 1$ and ξ is timelike. □

Note that since (3.12) always has a non-trivial translation part, the associated one-parameter group is non-compact and has the topology of a line.

3.2. Preserved SUSY

Proposition 2. *The KVF*

$$\xi = \partial_0 + \beta_1 R_{12} + \beta_2 R_{34} \tag{3.15}$$

of proposition 1 preserves some SUSY if and only if

$$(\beta_1 \pm \beta_2)^2 = -|\varphi|^2. \tag{3.16}$$

The fraction ν of preserved SUSY is

$$\nu = \begin{cases} \dfrac{1}{2} & \text{if } \beta_2\beta_1 = 0, \\ \dfrac{1}{4} & \text{otherwise.} \end{cases}$$

Proof. By the Poincaré–Hopf theorem any vector field on S^4 has a zero, so let o be a zero of $\lambda = \beta_1 R_{12} + \beta_2 R_{34}$. Then $\xi_o = \partial_0|_o$ and substituting

$$\varphi = c\partial_0 \tag{3.17}$$

in (2.27) and evaluating at o we find

$$\beta_\xi \epsilon|_o = -\frac{c}{2} r\epsilon. \tag{3.18}$$

Since r is invertible, if $\lambda = 0$ no SUSY is preserved. Suppose $\lambda \neq 0$. We calculate

$$\frac{1}{4}\,\mathrm{d}\xi^\flat \cdot \epsilon = \frac{1}{4}\omega_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \cdot \epsilon = \frac{1}{4}\omega_{ij}\gamma^{ij}\epsilon. \tag{3.19}$$

Substituting $\omega_{12} = \beta_1$, $\omega_{34} = \beta_2$, we get

$$\frac{1}{4}\,\mathrm{d}\xi^\flat \cdot \epsilon = \frac{1}{2}(\beta_1\gamma^{12} + \beta_2\gamma^{34})\epsilon.$$

For convenience set

$$a = \beta_1\gamma^{12} + \beta_2\gamma^{34}, \tag{3.20}$$

then

$$L_\xi \epsilon = \frac{1}{2}(a\mathrm{Id} - cr)\epsilon. \tag{3.21}$$

Vanishing of the Lie derivative is thus equivalent to the equations

$$(a - ic)\epsilon_1 = 0 = (a + ic)\epsilon_2. \tag{3.22}$$

One can check, e.g. by using the explicit representation of the gamma matrices given in appendix B, that the eigenvalues of $a - ic$ are

$$i(\pm\beta_1 \pm \beta_2 - c), \tag{3.23}$$

with all four possible sign combinations. Thus

$$D = \det(a - ic) = \det(a + ic) = (\beta_1 + \beta_2 - c)(\beta_1 + \beta_2 + c)(\beta_1 - \beta_2 - c)(\beta_1 - \beta_2 + c), \tag{3.24}$$

and

$$D = 0 \quad \Leftrightarrow \quad (\beta_1 \pm \beta_2)^2 = -|\varphi|^2. \tag{3.25}$$

If $D = 0$ we see from (3.23) that if $\beta_1\beta_2 \neq 0$ then $a \pm ic$ has a one-dimensional kernel, while if $\beta_1\beta_2 = 0$ it has a two-dimensional kernel. Correspondingly the fraction of preserved SUSY is $\frac{1}{4}$ in the former case and $\frac{1}{2}$ in the latter. □

3.3. Geometry of the quotient

The quotient M/Γ, where Γ is the one-parameter group generated by the KVF $\xi = \partial_0 + \lambda$, is diffeomorphic to S^4. Since ξ is timelike, the induced metric is Riemannian. To find an explicit expression for it we first rewrite the Minkowski space metric η on $\mathbb{R}^{1,5} \supset -\mathbb{R} \times S^4$ in terms of more convenient coordinates.

Define $U = \exp(-x^0\lambda) \in SO(5)$, so that

$$\xi(f) = U\partial_0(U^{-1}f). \tag{3.26}$$

Let $y = (y^1, \ldots, y^5)$, $x = (x^1, \ldots, x^5)$, and set

$$y = Ux. \tag{3.27}$$

Then $\xi(y) = 0$, so $(y^1, \ldots, y^5)$, are good coordinates on the quotient. Since $\lambda \in \mathfrak{so}(5)$, its action on $\mathbb{R}^5$ is linear. Denote by B the matrix representing the action of λ with respect to the coordinates $(x^1, \ldots, x^5)$, $\lambda x^i = B^i{}_j x^j$. Then

$$\mathrm{d}x = \mathrm{e}^{x^0 B}\big(\mathrm{d}y + By\,\mathrm{d}x^0\big), \tag{3.28}$$

and

$$\begin{aligned}\eta &= -(\mathrm{d}x^0)^2 + \mathrm{d}x^{\mathrm{T}}\,\mathrm{d}x = -(\mathrm{d}x^0)^2 + (\mathrm{d}y + By\,\mathrm{d}x^0)^{\mathrm{T}}(\mathrm{e}^{x^0B})^{\mathrm{T}}(\mathrm{e}^{x^0B})(\mathrm{d}y + By\,\mathrm{d}x^0)\\ &= -(1 - (By)^{\mathrm{T}}(By))\big(\mathrm{d}x^0 - (1-(By)^{\mathrm{T}}(By))^{-1}(By)^{\mathrm{T}}\,\mathrm{d}y\big)^2\\ &\quad + (1-(By)^{\mathrm{T}}(By))^{-1}\big((By)^{\mathrm{T}}\,\mathrm{d}y\big)^2 + \mathrm{d}y^{\mathrm{T}}\,\mathrm{d}y.\end{aligned} \tag{3.29}$$

Setting

$$\Lambda = 1 - (By)^{\mathrm{T}}By, \quad A = -\Lambda^{-1}(By)^{\mathrm{T}}\mathrm{d}y \tag{3.30}$$

we can write

$$\eta = \mathrm{d}y^{\mathrm{T}}\big(1 + \Lambda^{-1}By(By)^{\mathrm{T}}\big)\mathrm{d}y - \Lambda(\mathrm{d}x^0 + A)^2. \tag{3.31}$$

Comparing with (2.37) we see that the metric on the quotient S^4 is thus

$$h = \phi^* k, \quad k = \mathrm{d}y^{\mathrm{T}}\big(1 + \Lambda^{-1}By(By)^{\mathrm{T}}\big)\mathrm{d}y, \tag{3.32}$$

where ϕ is the standard embedding $S^4 \hookrightarrow \mathbb{R}^5$. Since y is related to x by an orthogonal transformation, restriction to S^4 is achieved simply by imposing $y^{\mathrm{T}}y = R^2$. As a sanity check note that $\lambda = 0 \Rightarrow B = 0, \Lambda = 1$ so that the quotient is S^4 with its round metric.

If ξ has the canonical form (3.12) then

$$B = \begin{pmatrix} 0 & -\beta_1 & 0 & 0 & 0\\ \beta_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\beta_2 & 0\\ 0 & 0 & \beta_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.33}$$

and we get

Table 1. Hereditary isometry algebra $\mathfrak{l}$ of the KK reduction of $-\mathbb{R}\times S^4$ by $\xi = \partial_0 + \beta_1 R_{12} + \beta_2 R_{34}$.

Conditions	$\mathfrak{l}$	Generators
$\beta_1\beta_2 \neq 0, \beta_1 \neq \pm\beta_2$	$\mathfrak{so}(2)\oplus\mathfrak{so}(2)$	$\epsilon_{12}; \epsilon_{34}$
$\beta_1\beta_2 \neq 0, \beta_1 = \pm\beta_2$	$\mathfrak{so}(2)\oplus\mathfrak{so}(3)$	$\epsilon_{12}\pm\epsilon_{34}; \epsilon_{12}\mp\epsilon_{34}, \epsilon_{13}\pm\epsilon_{24}, \epsilon_{14}\mp\epsilon_{23}$
$\beta_2 = 0$	$\mathfrak{so}(2)\oplus\mathfrak{so}(3)$	$\epsilon_{12}; \epsilon_{34}, \epsilon_{35}, \epsilon_{45}$

$$\begin{aligned}\Lambda &= 1-\beta_1^2(y_1^2+y_2^2)-\beta_2^2(y_3^2+y_4^2),\\ [(By)^{\mathrm{T}}\,\mathrm{d}y]^2 &= \beta_1^2(y^2\,\mathrm{d}y^1 - y^1\,\mathrm{d}y^2)^2+\beta_2^2(y^4\,\mathrm{d}y^3-y^3\,\mathrm{d}y^4)^2.\end{aligned} \tag{3.34}$$

Note that $\Lambda > 0$ provided that β_1, β_2 satisfy (3.13). Switching to double polar coordinates $y^1 = r\cos\alpha$, $y^2 = r\sin\alpha$, $y^3 = \rho\cos\gamma$, $y^4 = \rho\sin\gamma$ in (3.34) and (3.32) we get

$$h = \phi^* k, \quad k = \mathrm{d}r^2 + \mathrm{d}\rho^2 + \mathrm{d}y_5^2 + \frac{(1-\beta_2^2\rho^2)r^2\,\mathrm{d}\alpha^2 + (1-\beta_1^2 r^2)\rho^2\,\mathrm{d}\gamma^2 + 2\beta_1\beta_2 r^2\rho^2\,\mathrm{d}\alpha\mathrm{d}\gamma}{1-\beta_1^2 r^2 - \beta_2^2\rho^2}. \tag{3.35}$$

The restriction $y^{\mathrm{T}}y = R^2$ is now $y_5^2 + \rho^2 + r^2 = R^2$. The geometry described by (3.35) is that of a two-torus fibration over the interior of the round closed quarter two-sphere

$$Q = \{(y^5, \rho, r) : r^2 + \rho^2 + y_5^2 = R^2,\ \rho \geqslant 0,\ r \geqslant 0\} \tag{3.36}$$

attached to the boundary ∂Q where the circle fibres collapse to zero size. The fibres are parametrised by (α, γ). Because of the constraint (3.13) the factors $1-\beta_1^2 r^2$, $1-\beta_2^2\rho^2$, $1-\beta_1 r^2 - \beta_2\rho^2$ never vanish, so the α-circles have maximum radius $\frac{R}{\sqrt{1-\beta_1^2 R^2}}$ for $r = R$ and collapse to zero size for $r = 0$. The γ-circles have a similar behaviour with r replaced by ρ. Independently translating and reflecting along the circle fibres gives an $O(2)^2$ subgroup of the isometry group. There is an additional $\mathbb{Z}_2$ isometry generated by $y^5 \mapsto -y^5$ which fixes the base Q. It is clear that the symmetry is enhanced if $\beta_2 = \pm\beta_1$, in fact we see from table 1 below that then $\mathfrak{l} = \mathfrak{so}(2)\oplus\mathfrak{so}(3)$.

If $\beta_1\beta_2 = 0$ (3.32) it is convenient to introduce polar coordinates in one plane only. Up to relabeling the coordinates, we can assume $\beta_2 = 0$ and set $\beta_1 = \beta$, $y^1 = r\cos\alpha$, $y^2 = r\sin\alpha$. Then

$$k = g_{E^3} + \mathrm{d}r^2 + \frac{r^2\,\mathrm{d}\alpha^2}{1-\beta^2 r^2}, \tag{3.37}$$

where

$$g_{E^3} = (\mathrm{d}y^3)^2 + (\mathrm{d}y^4)^2 + (\mathrm{d}y^5)^2 \tag{3.38}$$

is the Euclidean metric on $\mathbb{R}^3$. The geometry described by (3.37) is that of a $U(1)$ fibration over the interior of the round closed half three-sphere

$$S = \{(y^3, y^4, y^5, r) : y_3^2 + y_4^2 + y_5^2 + r^2 = R^2,\ r \geqslant 0\} \tag{3.39}$$

attached to the boundary ∂S where the circle fibres collapse to zero size. The angular coordinate α parametrises the circle fibres. The circle size is maximal for $r = R$ and decreases with r reaching zero for $r = 0$. It is clear from this description that the isometry group of $\phi^* k$ is $O(3) \times O(2) \subset O(5)$, with $O(3)$ arising as the isometry group of S.

By (2.38) the hereditary isometry algebra of the metric (3.32) on S^4 is

$$\mathfrak{l} = \frac{N_{X_\xi}(\mathbb{R} \oplus \mathfrak{so}(5))}{\mathbb{R}X_\xi}, \tag{3.40}$$

where $X_\xi = (1, -\omega) \in \mathbb{R} \oplus \mathfrak{so}(5)$ is the element corresponding to the KVF ξ, $N_{X_\xi}(\mathbb{R} \oplus \mathfrak{so}(5))$ is the normaliser of X_ξ in $\mathbb{R} \oplus \mathfrak{so}(5)$ and $\mathbb{R}X_\xi$ is the one-dimensional Lie algebra generated by X_ξ. Denote by $N_\omega(\mathfrak{so}(5))$ the normaliser of ω in $\mathfrak{so}(5)$. Then $N_{X_\xi}(\mathbb{R} \oplus \mathfrak{so}(5)) = \mathbb{R} \oplus N_\omega(\mathfrak{so}(5))$ and we can use the quotient by $\mathbb{R}X_\xi$ to fix the $\mathbb{R}$ part to zero so that we only have to calculate $N_\omega(\mathfrak{so}(5))$. Using (3.11) we find the result in table 1.

4. Kaluza–Klein quotients of $\mathbb{R} \times \mathrm{AdS}_4$

Recall that we take global coordinates $(y, x^1, x^2, x^3, x^4, x^5) \in \mathbb{R} \times \mathbb{R}^{2,3}$, with x^1, x^2 timelike and the other coordinates spacelike. We consider AdS_4 as the universal cover of the quadric in $\mathbb{R}^{2,3}$

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = R^2. \tag{4.1}$$

The parallel vector field φ is

$$\varphi = c\partial_y \tag{4.2}$$

with

$$R|\varphi| = Rc = \sqrt{2}. \tag{4.3}$$

4.1. One parameters subgroups of G

The isometry group of $M = \mathbb{R} \times \mathrm{AdS}_4$ is the direct product $\mathbb{R} \times O(2, 3)$ with Lie algebra $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{so}(2, 3)$. Let $(\tau^y, -\omega) \in \mathfrak{g}$. A computation similar to (3.4) gives the corresponding KVF,

$$\xi = \tau + \lambda, \quad \tau = \tau^y \partial_y, \quad \lambda = \frac{1}{2}\omega^{ij}(x_i\partial_j - x_j\partial_i). \tag{4.4}$$

Therefore we have the correspondence

$$(\tau^y, -\omega) \in \mathfrak{g} \quad \longleftrightarrow \quad \tau^y\partial_y + \frac{1}{2}\omega^{ij}R_{ij} \in \mathfrak{X}(M), \tag{4.5}$$

where, for $i, j = 1, \ldots, 5$,

$$R_{ij} = x_i\partial_j - x_j\partial_i. \tag{4.6}$$

In particular, taking generators (ϵ_{ij}) of $\mathfrak{so}(2,3)$, $\epsilon_{ji} = -\epsilon_{ij}$,

$$\epsilon_{ij} = \begin{cases} e_1 \otimes e^2 - e_2 \otimes e^1, & \text{if } (i,j) = (1,2), \\ -(e_i \otimes e^j + e_j \otimes e^i), & \text{if } i = 1,2,\ j = 3,4,5, \\ e_j \otimes e^i - e_i \otimes e^j, & \text{if } 3 \leqslant i < j \leqslant 5, \end{cases} \tag{4.7}$$

with bracket

$$[\epsilon_{ij}, \epsilon_{kl}] = \eta^{2,3}_{ik}\epsilon_{jl} + \eta^{2,3}_{jl}\epsilon_{ik} - \eta^{2,3}_{il}\epsilon_{jk} - \eta^{2,3}_{jk}\epsilon_{il}, \tag{4.8}$$

we get

$$\epsilon_{ij} \in \mathfrak{so}(2,3) \quad \longleftrightarrow \quad R_{ij} \in \mathfrak{X}(\mathrm{AdS}_4). \tag{4.9}$$

Proposition 3. *Let ξ be a KVF on $M = \mathbb{R} \times \mathrm{AdS}_4$ and assume that $|\xi|$ never vanishes. Then there are coordinates such that, up to rescaling,*

$$\xi = \partial_y + \lambda, \tag{4.10}$$

with $\lambda \in \mathfrak{X}(\mathrm{AdS}_4)$ one of the following KVFs:

$$\lambda_4 = \beta R_{34}, \quad \beta > 0, \tag{4.11}$$

$$\lambda_5 = R_{13} - R_{34}, \tag{4.12}$$

$$\lambda_6 = R_{24} + R_{34} - R_{12} - R_{13}, \tag{4.13}$$

$$\lambda_{11} = \beta(R_{13} + R_{24}), \quad \beta > 0, \tag{4.14}$$

$$\lambda_1 = \beta R_{12}, \quad \beta > 0, \quad |\varphi| < \sqrt{2}\beta, \tag{4.15}$$

$$\lambda_8 = R_{24} - R_{13} + (1+\beta)R_{34} - (1-\beta)R_{12}, \quad \beta < 0, \quad |\varphi| \leqslant -\sqrt{2}\beta, \tag{4.16}$$

$$\lambda_{10} = \beta_1 R_{12} + \beta_2 R_{34}, \quad \beta_1 > \beta_2 > 0, \quad |\varphi| < \sqrt{2}\beta_1, \tag{4.17}$$

$$\lambda_{10*} = \beta(R_{12} + R_{34}), \quad \beta > 0, \quad |\varphi| < \sqrt{2}\beta. \tag{4.18}$$

The KVF $\xi = \partial_y + \lambda$ is spacelike for λ given by (4.11)–(4.14); timelike for λ given by (4.15)–(4.18).

Proof. To avoid zeros the translation part τ of ξ must be non-zero and we can rescale it so that $\tau = \partial_y$. Then $|\xi|^2 = 1 + |\lambda|^2$ so we need either $|\lambda|^2 > -1$, leading to a spacelike KVF, or $|\lambda|^2 < -1$, leading to a timelike one.

The KVFs on AdS_5, up to conjugation, are listed in [24, section 4.2.2]. We report the list here, following the same numbering.

λ	$\lvert\lambda\rvert^2$
$\lambda_1 = \beta R_{12}, \beta > 0$	$-\beta^2(x_1^2 + x_2^2)$
$\lambda_2 = \beta R_{13}, \beta > 0$	$\beta^2(x_1^2 - x_3^2),$
$\lambda_3 = R_{12} - R_{23}$	$-(x_1 + x_3)^2$
$\lambda_4 = \beta R_{34}, \beta > 0$	$\beta^2(x_3^2 + x_4^2),$
$\lambda_5 = R_{13} - R_{34}$	$(x_1 + x_4)^2,$
$\lambda_6 = R_{24} + R_{34} - R_{12} - R_{13}$	0
$\lambda_7 = R_{24} + R_{34} - R_{12} - R_{13} + \beta(R_{14} - R_{23}), \beta > 0$	$4\beta(x_2 + x_3)(x_1 + x_4) + \beta^2(R^2 + x_5^2),$
$\lambda_8 = R_{24} + R_{34} - R_{12} - R_{13} + \beta(R_{12} + R_{34}), \beta \neq 0$	$2\beta\left[(x_1 + x_4)^2 + (x_2 + x_3)^2\right] - \beta^2(R^2 + x_5^2),$
$\lambda_9 = \beta_1(R_{12} - R_{34}) + \beta_2(R_{14} - R_{23}), \beta_i > 0$	$(\beta_2^2 - \beta_1^2)(R^2 + x_5^2) - 4\beta_1\beta_2(x_1x_3 + x_2x_4)$
$\lambda_{10} = \beta_1 R_{12} + \beta_2 R_{34}, \beta_i > 0$	$\beta_2^2(x_3^2 + x_4^2) - \beta_1^2(x_1^2 + x_2^2),$
$\lambda_{11} = \beta_1 R_{13} + \beta_2 R_{24}, \beta_1 \geqslant \beta_2 > 0$	$\beta_1^2(x_1^2 - x_3^2) + \beta_2^2(x_2^2 - x_4^2),$
$\lambda_{12} = \beta_1 R_{13} + \beta_2 R_{45}, \beta_i > 0$	$\beta_1^2(x_1^2 - x_3^2) + \beta_2^2(x_4^2 + x_5^2),$
$\lambda_{13} = R_{12} + R_{13} + R_{15} - R_{24} - R_{34} - R_{45}$	$(x_1 - x_4)^2 - 4(x_2 + x_3)x_5,$
$\lambda_{14} = R_{12} - R_{23} + \beta R_{45}, \beta > 0$	$\beta^2(x_4^2 + x_5^2) - (x_1 + x_3)^2,$
$\lambda_{15} = R_{13} - R_{34} + \beta R_{25}, \beta > 0$	$(x_1 + x_4)^2 + \beta^2(x_2^2 - x_5^2).$

The parameters β, β_i can always be chosen to satisfy the listed constraints by conjugating within $SO(2,3)$. The norms of the vectors λ_1–λ_{15} can be computed by working on $\mathbb{R}^{2,3}$ and imposing the AdS_5 constraint $x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_5^2 = R^2$. We find the following bounds.

$$\lambda_1\colon \quad -\infty < |\lambda_1|^2 \leqslant -\beta^2 R^2,$$

$$\lambda_2\colon \quad -\infty < |\lambda_2|^2 < \infty,$$

$$\lambda_3\colon \quad -\infty < |\lambda_3|^2 \leqslant 0,$$

$$\lambda_4\colon \quad 0 \leqslant |\lambda_4|^2 < \infty,$$

$$\lambda_5\colon \quad 0 \leqslant |\lambda_5|^2 < \infty,$$

$$\lambda_6\colon \quad |\lambda_6|^2 = 0,$$

$$\lambda_7\colon \quad -\infty < |\lambda_7|^2 < \infty,$$

$$\lambda_8\colon \quad \begin{cases} -\infty < |\lambda_8|^2 < \infty & \beta > 0, \\ -\infty < |\lambda_8|^2 < -\beta^2 R^2 & \beta < 0, \end{cases}$$

$$\lambda_9\colon \quad -\infty < |\lambda_9|^2 < \infty,$$

$$\lambda_{10}\colon \quad \begin{cases} -\infty < |\lambda_{10}|^2 < \infty & \beta_2^2 > \beta_1^2, \\ -\infty < |\lambda_{10}|^2 \leqslant -\beta_1^2 R^2 & \beta_2^2 \leqslant \beta_1^2, \end{cases}$$

$$\lambda_{11}\colon \quad \begin{cases} -\infty < |\lambda_{11}|^2 < \infty & \beta_1 > \beta_2, \\ \beta^2 R^2 \leqslant |\lambda_{11}|^2 < \infty & \beta_1 = \beta_2 = \beta, \end{cases}$$

$$\lambda_{12}\colon \quad -\infty < |\lambda_{12}|^2 < \infty,$$

$$\lambda_{13}\colon \quad -\infty < |\lambda_{13}|^2 < \infty,$$

Table 2. Fraction $\nu > 0$ of SUSY preserved by the KVFs of proposition 3.

λ	Condition	ν
λ_4	$0 < \beta = \|\varphi\|$	$1/2$
λ_1	$0 < \beta = \sqrt{2}\|\varphi\|$	$1/2$
λ_6	$\|\varphi\| = 2^{3/4}$	$1/4$
λ_8	$\|\varphi\| = \varphi_1$ or $\|\varphi\| = \varphi_2$	$1/4$
λ_{10}	$(\beta_1 + \sqrt{2}\beta_2 = \sqrt{2}\|\varphi\|, \beta_1 > \sqrt{2}\beta_2 > 0)$ or $(\|\beta_1 - \sqrt{2}\beta_2\| = \sqrt{2}\|\varphi\|, \beta_1 > \beta_2 > 0, \beta_1 \neq \sqrt{2}\beta_2)$	$1/4$
λ_{10*}	$0 < (\sqrt{2}-1)\beta = \sqrt{2}\|\varphi\|$	$1/4$

$$\lambda_{14}\colon \quad -\infty < |\lambda_{14}|^2 < \infty,$$

$$\lambda_{15}\colon \quad -\infty < |\lambda_{15}|^2 < \infty.$$

In particular note that there are points with arbitrarily small $|x_1 + x_4|, |x_2 + x_3|$ and arbitrarily large $R^2 + x_5^2 = (x_1 + x_4)(x_1 - x_4) + (x_2 + x_3)(x_2 - x_3)$. It follows that $|\lambda_8|^2$ is unbounded below and, if $\beta < 0$, $|\lambda_8|^2 < -\beta^2 R^2$, while if $\beta > 0$ then $|\lambda_8|^2$ is also unbounded above. The vectors $\lambda_4, \lambda_5, \lambda_6$ satisfy $|\lambda_i|^2 > -1$. The vector λ_{11} satisfies $|\lambda_{11}|^2 > -1$ provided that

$$\lambda_{11}: \quad \beta_1 = \beta_2 = \beta.$$

The vectors $\lambda_1, \lambda_8, \lambda_{10}$ satisfy $|\lambda_i|^2 < -1$ provided that

$$\lambda_1\colon \quad \beta R > 1,$$

$$\lambda_8\colon \quad \beta < 0, \ -\beta R \geqslant 1,$$

$$\lambda_{10}\colon \quad \beta_1 \geqslant \beta_2, \ \beta_1 R > 1.$$

Rewriting the conditions in terms of $|\varphi|$ using (4.3) gives the stated result. □

Note that since (4.10) always has a non-trivial translation part, the associated one-parameter group is non-compact and has the topology of a line.

4.2. Preserved SUSY

Proposition 4. *The KVFs of proposition 3 preserving a fraction $\nu > 0$ of SUSY are given by table 2.*

For $\lambda = \lambda_8$ the values of φ_1, φ_2 are given by (4.24) and (4.25) and the corresponding range of β by (4.26) and (4.27). The conditions listed in table 2 take into account both the constraints coming from proposition 3 and those arising from SUSY preservation.

Proof. φ is the spacelike vector field

$$\varphi = c\partial_y \tag{4.19}$$

with $c = |\varphi| = \sqrt{2}/R$. Substituting in (2.27) we get

$$\beta_\xi \epsilon = \frac{|\varphi|}{4} \xi^i (\gamma_i \gamma_y + 3\eta_{iy}) r\epsilon = \frac{|\varphi|}{2} \left(1 + \frac{1}{2}\lambda^i \gamma_i \gamma_y\right) r\epsilon, \tag{4.20}$$

so

$$L_\xi \epsilon = \frac{1}{2}\left[|\varphi|\left(1 + \frac{1}{2}\lambda^i \gamma_i \gamma_y\right) r + \frac{1}{2}\omega_{ij}\gamma^{ij}\right]\epsilon = \frac{1}{2}[br + a]\epsilon = 0, \tag{4.21}$$

Table 3. Values of $a|_o$, $b|_o$ for the KVFs of proposition 3.

λ	$a\|_o$	$\|\varphi\|^{-1}b\|_o$
λ_4	$\beta\gamma_{34}$	1
λ_5	$-\gamma_{13}-\gamma_{34}$	1
λ_6	$\gamma_{34}-\gamma_{24}$	$1-\frac{1}{\sqrt{2}\|\varphi\|}(\gamma_2+\gamma_3)\gamma_y$
λ_{11}	$-\beta\gamma_{24}$	$1+\frac{\beta}{\sqrt{2}\|\varphi\|}\gamma_3\gamma_y$
λ_1	0	$1+\frac{\beta}{\sqrt{2}\|\varphi\|}\gamma_2\gamma_y$
λ_8	$(1+\beta)\gamma_{34}-\gamma_{24}$	$1-\frac{1}{\sqrt{2}\|\varphi\|}(\gamma_3+(1-\beta)\gamma_2)\gamma_y$
λ_{10}	$\beta_2\gamma_{34}$	$1+\frac{\beta_1}{\sqrt{2}\|\varphi\|}\gamma_2\gamma_y$
λ_{10*}	$\beta\gamma_{34}$	$1+\frac{\beta}{\sqrt{2}\|\varphi\|}\gamma_2\gamma_y$

having defined

$$b=|\varphi|\left(1+\frac{1}{2}\lambda^i\gamma_i\gamma_y\right),\quad a=\frac{1}{2}\omega_{ij}\gamma^{ij}. \tag{4.22}$$

In order to compute $L_\xi\epsilon$ for ξ one of the KVFs given in proposition 3 we work at some convenient point o. For $\lambda=\lambda_5$ we take o to have coordinates $x^2=R\Rightarrow x^1=x^3=x^4=x^5=0$ so that $(\partial_1,\partial_y,\partial_3,\partial_4,\partial_5)$ is a local orthonormal frame for $\mathbb{R}\times \mathrm{AdS}_4$ at o. Since $x^2=R$, $\mathrm{d}x^2|_o=0$ hence ∂_2 acts by zero on spinors while $(\partial_1,\partial_y,\partial_3,\partial_4,\partial_5)$ map to a representation $(\gamma_1,\gamma_y,\gamma_3,\gamma_4,\gamma_5)$ of Cl(1, 4) with $\gamma_1^2=1$, $\gamma_y^2=\gamma_i^2=-1$, $i=3,4,5$. For all the other KVFs we take o to have coordinates $x^1=R\Rightarrow x^2=x^3=x^4=x^5=0$ so that $(\partial_2,\partial_y,\partial_3,\partial_4,\partial_5)$ is an orthonormal frame for $\mathbb{R}\times\mathrm{AdS}_4$ at o. In this case ∂_1 acts trivially on spinors while $(\partial_2,\partial_y,\partial_3,\partial_4,\partial_5)$ map to a representation $(\gamma_2,\gamma_y,\gamma_3,\gamma_4,\gamma_5)$ of Cl(1, 4) with $\gamma_2^2=1$, $\gamma_y^2=\gamma_i^2=-1$, $i=3,4,5$. An explicit choice of gamma matrices is given in appendix B. For $\lambda=\lambda_4$ and $\lambda=\lambda_5$ we have $\lambda|_o=0$ but in all the other cases $\lambda|_o\neq 0$. With these choices, and substituting $R=\sqrt{2}/|\varphi|$, the values of $a|_o$, $b|_o$ are listed in table 3.

Equation (4.21) gives the two linear equations

$$(a+ib)\epsilon_1=0=(a-ib)\epsilon_2. \tag{4.23}$$

It is clear that there are no non-trivial solutions if $\lambda=0$. For $\lambda=\lambda_5$, $a^2=0$ and b is not nilpotent so no SUSY is preserved. The case $\lambda=\lambda_{11}$ also does not preserve any SUSY. In the other cases computing the rank of the matrices making use of an explicit representation, such as the one given in appendix B, gives the following result.

λ	Condition	Fraction of SUSY
λ_4	$\|\beta\|=\|\varphi\|$	1/2
λ_1	$\|\beta\|=\sqrt{2}\|\varphi\|$	1/2
λ_{10}	$\|\beta_1\pm\sqrt{2}\beta_2\|=\sqrt{2}\|\varphi\|$	1/4
λ_{10^*}	$(\sqrt{2}\pm 1)\|\beta\|=\sqrt{2}\|\varphi\|$	1/4
λ_6	$\|\varphi\|=2^{3/4}$	1/4
λ_8	See (4.24) and (4.25)	1/4

For $\lambda = \lambda_8$ some SUSY is preserved if and only if $|\varphi| = \varphi_1$ or $|\varphi| = \varphi_2$, with

$$\varphi_1 = \sqrt{\beta + \frac{3}{2}\beta^2 + \sqrt{2}|\beta^2 - 2|}, \tag{4.24}$$

$$\varphi_2 = \sqrt{\beta + \frac{3}{2}\beta^2 - \sqrt{2}|\beta^2 - 2|}, \tag{4.25}$$

where the value of β is constrained by the condition that φ_1, φ_2 are real.

Combining the conditions listed above with those, listed in proposition 3, coming from imposing that the KK reduction results in a smooth manifold gives the stated result. In the case of λ_8 the combined conditions read

$$|\varphi| = \varphi_1 : -\left(\frac{\sqrt{17 - 4\sqrt{2}} + 1}{2\sqrt{2} - 1}\right) \leqslant \beta \leqslant -\left(\frac{\sqrt{17 + 4\sqrt{2}} - 1}{2\sqrt{2} + 1}\right), \tag{4.26}$$

$$|\varphi| = \varphi_2 : -\left(\frac{\sqrt{17 - 12\sqrt{2}} + 1}{3 - 2\sqrt{2}}\right) < \beta < -\left(\frac{\sqrt{17 + 12\sqrt{2}} + 1}{3 + 2\sqrt{2}}\right). \tag{4.27}$$

□

4.3. Geometry of the quotient

The quotient $(\mathbb{R} \times \mathrm{AdS}_4)/\Gamma_\xi$, for ξ one of the KVFs listed in proposition 3, is diffeomorphic to $\mathbb{R}^4$ equipped with a Lorentzian or Riemannian metric depending on the causal character of ξ. In order to find the quotient metric we proceed similarly as we did in section 3.3, first working on $\mathbb{R} \times \mathbb{R}^{2,3}$ and then restricting to $\mathbb{R} \times \mathrm{AdS}_4$.

Let

$$\eta^{2,4} = \mathrm{d}z^2 - \mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 + \mathrm{d}x_5^2 \tag{4.28}$$

be the flat metric on $\mathbb{R} \times \mathbb{R}^{2,3}$, $x = (x^1, x^2, \ldots, x^5) \in \mathbb{R}^{2,3}$, z a global coordinate on the $\mathbb{R}$ factor. Define $M^\dagger = M^\mathrm{T}\eta^{2,3}$. The KVF $\xi = \partial_z + \lambda$ can be written

$$\xi = U\partial_z U^{-1} \tag{4.29}$$

with $U = \exp(-z\lambda)$. The coordinates (y^i) defined by $y = Ux$ are good coordinates on the orbit space. The action of $\mathfrak{so}(2,3)$ on $\mathbb{R}^{2,3}$ is linear so let B be the matrix representing λ with respect to the x coordinates, $\lambda x^i = B^i{}_j x^j$. Then

$$\eta^{2,4} = \Lambda(\mathrm{d}z + A)^2 + \mathrm{d}y^\dagger\left(1 - \Lambda^{-1}By(By)^\dagger\right)\mathrm{d}y,$$

with

$$\Lambda = 1 + (By)^\dagger By, \quad A = \Lambda^{-1}(By)^\dagger\,\mathrm{d}y. \tag{4.30}$$

The quotient metric is

$$h = \phi^* k, \quad k = \mathrm{d}y^\dagger\left(1 - \Lambda^{-1}By(By)^\dagger\right)\mathrm{d}y, \tag{4.31}$$

where ϕ^* is the restriction to $\mathrm{AdS}_4 \subset \mathbb{R}^{2,3}$, which is achieved by imposing $y^\mathrm{T}\eta^{2,3}y = x^\mathrm{T}\eta^{2,3}x = -R^2$.

Let us consider some cases in more detail. If $\lambda = R_{kl}$ then B has components

$$B^i{}_j = \eta^{2,3}_{kj}\delta^i_l - \eta^{2,3}_{lj}\delta^i_k. \tag{4.32}$$

For $\lambda = \lambda_{10}$, the quotient is a Riemannian manifold which is the hyperbolic equivalent of (3.35). Introducing double polar coordinates $y^1 = r\cos\alpha,\ y^2 = r\sin\alpha,\ y^3 = \rho\cos\gamma,\ y^4 = \rho\sin\gamma$ we get

$$k = -\mathrm{d}r^2 + \mathrm{d}\rho^2 + \mathrm{d}y_5^2 + \frac{(1+\beta_2^2\rho^2)r^2\,\mathrm{d}\alpha^2 + (\beta_1^2 r^2 - 1)\rho^2\,\mathrm{d}\gamma^2 + 2\beta_1\beta_2 r^2\rho^2\,\mathrm{d}\alpha\,\mathrm{d}\gamma}{-1+\beta_1^2 r^2 - \beta_2^2\rho^2}. \tag{4.33}$$

Because of the constraints $\beta_1 > \beta_2$, $\beta_1 R > 1$, we have $\beta_1 r^2 > 1$, $-1 + \beta_1^2 r^2 - \beta_2^2\rho^2 > 0$. The geometry described by (4.33) is that of a two-torus fibration over the interior of Q, where Q is the portion of hyperbolic two-space given by

$$Q = \{(r,\rho,y^5) \in \mathbb{R}^3 : -r^2 + \rho^2 + y_5^2 = -R^2,\ \rho \geqslant 0\}, \tag{4.34}$$

collapsing to a circle fibration on the boundary ∂Q. The circles are parametrised by (α,γ) and while the α fibres always have non-zero length, the γ fibres collapse on the boundary $\rho = 0$ of Q.

If $\beta_1 = 0$, so that $\lambda = \lambda_4$, it is convenient to introduce polar coordinates in the (y_3, y_4)-plane only. The KVF is now spacelike so we get the Lorentzian metric

$$k = -\mathrm{d}y_1^2 - \mathrm{d}y_2^2 + \mathrm{d}y_5^2 + \mathrm{d}r^2 + \frac{r^2\,\mathrm{d}\alpha^2}{1+\beta^2 r^2} \tag{4.35}$$

which describes a circle bundle over the interior of S, for S 'half' AdS_3,

$$S = \{(y^1, y^2, y^5, r) : -y_1^2 - y_2^2 + y_5^2 + r^2 = -R^2,\ r \geqslant 0\}, \tag{4.36}$$

with the circle fibres always of finite length and collapsing to zero on the boundary $r = 0$. The isometry group of S is $O(2,1)$.

If $\beta_2 = 0$, so that $\lambda = \lambda_1$, it is convenient to introduce polar coordinates in the (y_1, y_2)-plane only. The KVF is timelike and we get the Riemannian metric

$$k = -\mathrm{d}r^2 + \mathrm{d}y_3^2 + \mathrm{d}y_4^2 + \mathrm{d}y_5^2 + \frac{r^2\,\mathrm{d}\alpha^2}{\beta^2 r^2 - 1} \tag{4.37}$$

which describes a circle bundle over the interior of S, for S 'half' hyperbolic three-space

$$S = \{(y^3, y^4, y^5, r) : y_3^2 + y_4^2 + y_5^2 - r^2 = -R^2,\ r \geqslant 0\}, \tag{4.38}$$

with the circle fibres always of finite length and collapsing to zero on the boundary $r = 0$. The isometry group of S is $O(3)$.

The hereditary isometry algebra is

$$\mathfrak{l} = \frac{N_{X_\xi}(\mathbb{R}\times\mathfrak{so}(2,3))}{\mathbb{R}X_\xi}, \tag{4.39}$$

where X_ξ is the Lie algebra element corresponding to the vector field ξ and $N_{X_\xi}(\mathbb{R}\times\mathfrak{so}(2,3))$ its normaliser in $\mathbb{R}\times\mathfrak{so}(2,3)$. Taking the generators (4.7) and using (4.9) one finds the result given in table 4.

The cases of λ_5 and λ_6 warrant some additional discussion.

Table 4. Hereditary isometry algebra $\mathfrak{l}$ of the KK reductions of $\mathbb{R}\times \mathrm{AdS}^4$ by the KVFs of proposition 3.

λ	$\mathfrak{l}$	Generators
λ_4	$\mathfrak{so}(2)\oplus\mathfrak{so}(2,1)$	$\epsilon_{34};\epsilon_{15},\epsilon_{25},\epsilon_{12}$
λ_5	$\mathbb{R}^2\ltimes\mathbb{R}^3$	$\epsilon_{14};\epsilon_{25};\epsilon_{34}-\epsilon_{13},\epsilon_{24}-\epsilon_{12};\epsilon_{54}-\epsilon_{15}$
λ_6	$\mathfrak{co}(2,1)\ltimes\mathfrak{h}$	$\epsilon_{14}-\epsilon_{23},\epsilon_{13}+\epsilon_{24},\epsilon_{12}+\epsilon_{34},\epsilon_{14}+\epsilon_{23};\epsilon_{25}+\epsilon_{35},\epsilon_{15}+\epsilon_{45},X_{\lambda_6}$
λ_{11}	$\mathfrak{so}(2)\oplus\mathfrak{so}(2,1)$	$\epsilon_{13}+\epsilon_{24};\epsilon_{12}-\epsilon_{34},\epsilon_{13}-\epsilon_{24},\epsilon_{14}+\epsilon_{23}$
λ_1	$\mathfrak{so}(2)\oplus\mathfrak{so}(3)$	$\epsilon_{12};\epsilon_{34},\epsilon_{35},\epsilon_{45}$
λ_8	$\mathfrak{so}(2)\oplus\mathfrak{so}(2)$	$\epsilon_{12}+\epsilon_{34};\epsilon_{13}-\epsilon_{24}-2\epsilon_{34}$
λ_{10}	$\mathfrak{so}(2)\oplus\mathfrak{so}(2)$	$\epsilon_{12};\epsilon_{34}$
λ_{10^*}	$\mathfrak{so}(2)\oplus\mathfrak{so}(2,1)$	$\epsilon_{12}+\epsilon_{34};\epsilon_{12}-\epsilon_{34},\epsilon_{13}-\epsilon_{24},\epsilon_{14}+\epsilon_{23}$

If $\lambda=\lambda_5$ define

$$N_i=\epsilon_{i4}-\epsilon_{1i},\quad i=2,3,5. \tag{4.40}$$

Then

$$\begin{aligned}&[N_i,N_j]=[\epsilon_{14},\epsilon_{25}]=[\epsilon_{25},N_3]=0,\\&[\epsilon_{14},N_i]=-N_i,\\&[\epsilon_{25},N_2]=-N_5,\\&[\epsilon_{25},N_5]=-N_2.\end{aligned} \tag{4.41}$$

Therefore the Lie algebra has the structure

$$\mathfrak{l}=\mathbb{R}^2\ltimes\mathbb{R}^3, \tag{4.42}$$

with $\mathbb{R}^2=\mathrm{Span}_{\mathbb{R}}(\epsilon_{14},\epsilon_{25})$, $\mathbb{R}^3=\mathrm{Span}_{\mathbb{R}}(N_2,N_3,N_5)$.

If $\lambda=\lambda_6$ then $\mathfrak{l}$ is generated by e.g. $\epsilon_{14},\epsilon_{23},\epsilon_{13}-\epsilon_{34},\epsilon_{13}+\epsilon_{24},\epsilon_{34}+\epsilon_{12}$, $\epsilon_{15}+\epsilon_{45}$ and $\epsilon_{25}+\epsilon_{35}$. Defining

$$\begin{aligned}&x_1=\epsilon_{14}-\epsilon_{23},\quad x_2=\epsilon_{13}+\epsilon_{24},\quad x_3=-(\epsilon_{12}+\epsilon_{34}),\\&y_1=\epsilon_{15}+\epsilon_{45},\quad y_2=\epsilon_{25}+\epsilon_{35},\quad a=-(\epsilon_{14}+\epsilon_{23}),\quad b=-X_{\lambda_6},\end{aligned} \tag{4.43}$$

we find

$$\begin{aligned}&[x_1,x_2]=-2x_3,\quad [x_2,x_3]=2x_1,\quad [x_1,x_3]=-2x_2,\\&[y_1,y_2]=b,\quad [b,y_1]=[b,y_2]=0,\\&[x_1,y_1]=-y_1,\quad [x_1,y_2]=y_2,\quad [x_2,y_1]=-y_2,\quad [x_2,y_2]=-y_1,\\&[x_3,y_1]=y_2,\quad [x_3,y_2]=-y_1,\quad [x_i,b]=0,\\&[a,x_i]=0,\quad [a,y_1]=y_1,\quad [a,y_2]=y_2,\quad [a,b]=2b.\end{aligned} \tag{4.44}$$

Hence $\mathfrak{l}=\mathfrak{co}(2,1)\ltimes\mathfrak{h}$ with $\mathfrak{h}=\mathrm{Span}(y_1,y_2,b)$ the three-dimensional Heisenberg algebra, and $\{x_1,x_2,x_3,a\}$ spanning the Lie algebra $\mathfrak{co}(2,1)=\mathbb{R}\oplus\mathfrak{so}(2,1)$ of isometries and dilations of $\mathbb{R}^{1,2}$[4]. Note that $\mathfrak{l}$ is a graded algebra with $\{x_i, a\}$ in degree 0, $\{y_i\}$ in degree 1, b in degree 2 and a acting as a grading element.

5. Kaluza–Klein quotients of the Cahen–Wallach background

Some details on CW spaces are given in appendix A. We recall that a five-dimensional CW space has the topology of $\mathbb{R}^5$ with global coordinates (x^+, x^-, x^i), $i = 1, 2, 3$, and the Lorentzian metric

$$g = 2\mathrm{d}x^+\,\mathrm{d}x^- + \sum_{i,j=1}^{3} A_{ij}x^i x^j (\mathrm{d}x^-)^2 + |\mathrm{d}x|^2. \tag{5.1}$$

Here $|\mathrm{d}x|^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$ and A is a symmetric bilinear form on $\mathbb{R}^3$ which in our case is simply the Euclidean inner product,

$$A = \eta^{0,3}, \tag{5.2}$$

so that

$$g = 2\mathrm{d}x^+\,\mathrm{d}x^- + |x|^2(\mathrm{d}x^-)^2 + |\mathrm{d}x|^2. \tag{5.3}$$

The parallel vector field φ is

$$\varphi = \sqrt{2}\partial_+. \tag{5.4}$$

5.1. One parameters subgroups of G

As discussed in appendix A, the isometry algebra of a CW space with symmetric bilinear form $A = \eta^{0,3}$ is

$$\mathfrak{g} \rtimes \mathfrak{so}(3) \tag{5.5}$$

where $\mathfrak{so}(3)$ is the Lie algebra of the isometry group of $\eta^{0,3}$, whose generators (V_i) satisfy

$$[V_i, V_j] = -\epsilon_{ijk}V_k, \tag{5.6}$$

and $\mathfrak{g}$ is the eight-dimensional Lie algebra with generators (e_i, e_i^*, e_+, e_-), $i = 1, 2, 3$, and non-trivial brackets

$$[e_-, e_i] = e_i^*, \quad [e_-, e_i^*] = e_i, \quad [e_i^*, e_j] = \delta_{ij}e_+. \tag{5.7}$$

The action of $\mathfrak{so}(3)$ on $\mathfrak{g}$ in (5.5) is the natural action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ on $\mathrm{Span}(e_i)$, the adjoint action on $\mathrm{Span}(e_i^*)$, and the trivial action on $\mathrm{Span}(e_+, e_-)$,

$$[V_i, e_j] = V_i e_j = -\epsilon_{ijk}e_k, \quad [V_i, e_j^*] = V_i e_j^* = -\epsilon_{ijk}e_k^*, \quad [V_i, e_\pm] = 0. \tag{5.8}$$

[4] That is the Lie algebra of the group $\{A \in GL(3, \mathbb{R}) : A^{\mathrm{T}}\eta^{2,1}A = c\eta^{2,1}, c \in \mathbb{R}^\times\}$.

Note that (e_i, e_i^*, e_+) form a representation of the seven-dimensional Heisenberg algebra. An explicit matrix representation of (5.7) is given by

$$e_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$e_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$e_- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$V_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

As shown in appendix A.1, in terms of the coordinates (x^i, x^+, x^-) used in (5.1), the KVFs corresponding to (e_i, e_i^*, e_+, e_-) are

$$\xi_{e_\pm} = \partial_\pm, \tag{5.10}$$

$$\xi_{e_i} = \cosh(x^-)\partial_i - \sinh(x^-)x^i\partial_+, \tag{5.11}$$

$$\xi_{e_i^*} = \cosh(x^-)x^i\partial_+ - \sinh(x^-)\partial_i. \tag{5.12}$$

We choose the generators (V_1, V_2, V_3) of $\mathfrak{so}(3)$ so that the associated KVFs are the usual generators of rotations in $\mathbb{R}^{0,3}$,

$$\xi_{V_1} = R_{23}, \quad \xi_{V_2} = R_{31}, \quad \xi_{V_3} = R_{12}. \tag{5.13}$$

Proposition 5. *Let ξ be a KVF of $M = \mathrm{CW}_5$ with the metric (5.3) and assume that $|\xi|$ never vanishes. Then there are coordinates such that, up to rescaling, ξ is the KVF associated to one of the following elements $X_i \in \mathfrak{g} \rtimes \mathfrak{so}(3)$,*

$$X_1 = e_- + bV_3 + \gamma e_+, \quad \gamma > 0, \tag{5.14}$$

$$X_2 = V_3 + ce_3, \quad c \neq 0, \tag{5.15}$$

$$X_4^\pm = V_3 + c(e_3 \pm e_3^*), \quad c \neq 0, \tag{5.16}$$

$$X_6 = e_3, \tag{5.17}$$

$$X_8^{\pm} = e_3 \pm e_3^*, \tag{5.18}$$

$$X_9 = e_3 + d_3 e_3^* + d_1 e_1^*, \quad d_1 \neq 0. \tag{5.19}$$

The KVFs corresponding to (5.14)–(5.19) *are all spacelike.*

Proof. Let X be a generic element of $\mathfrak{g} \rtimes \mathfrak{so}(3)$,

$$X = \alpha e_- + b_i V_i + c_i e_i + d_i e_i^* + \gamma e_+. \tag{5.20}$$

We now act by conjugation using the equations given in appendix A.2 to simplify the form of X as much as possible.

Suppose first $\alpha \neq 0$ and rescale so that $\alpha = 1$. Conjugating first by $\exp(x_k e_k^*)$ and then by $\exp(y_k e_k)$ brings X to the form

$$X \mapsto e_- + b_i V_i + (c + b \times y - x)_i e_i + (d + b \times x - y)_i e_i^* + \tilde{\gamma} e_+, \tag{5.21}$$

where we do not need the explicit expression of $\tilde{\gamma}$. Imposing the coefficients of e_i and e_i^* to vanish we thus get the vectorial equations

$$x = c + b \times y, \tag{5.22}$$

$$y = d + b \times x, \tag{5.23}$$

which are solved by taking

$$x = \frac{c + b \times d + (b \cdot c)b}{1 + |b|^2}, \qquad y = \frac{d + b \times c + (b \cdot d)b}{1 + |b|^2}. \tag{5.24}$$

Rotating (conjugation by V_i) so to align V to the third direction and relabelling the coefficients we get

$$X \mapsto e_- + bV_3 + \gamma e_+. \tag{5.25}$$

Suppose now $\alpha = 0$, $b \neq 0$. Then the same procedure as before gives

$$X \mapsto b_i V_i + (c + b \times y)_i e_i + (d + b \times x)_i e_i^* + \tilde{\gamma} e_+.$$

In this case the equation

$$c + b \times y = 0$$

has solution if and only if $b \cdot c = 0$, in which case $y = \frac{b \times c}{|b|^2}$. Similarly $d + x \times b = 0$ has solution if and only if $b \cdot d = 0$, in which case $x = \frac{b \times d}{|b|^2}$. Therefore we can only kill the part of $c_i e_i$ or $d_i e_i^*$ which is normal to $b_i V_i$. Unless $c = d = 0$ we can still conjugate by $x_k e_k$ or $x_k e_k^*$ with x such that $x \times b = 0$ so to kill the e_+ part while not affecting the e_i, e_i^* part. Rescaling and rotating we thus get

$$X \mapsto V_3 + ce_3 + de_3^*.$$

We now conjugate by $\exp(x^- e_-)$ obtaining

$$X \mapsto V_3 + (c \cosh x^- + d \sinh x^-)e_3 + (c \sinh x^- + d \cosh x^-)e_3^*. \tag{5.26}$$

Since $\tanh : \mathbb{R} \to (-1, 1)$ is surjective, if $|c/d| < 1$ we can kill the coefficient of e_3 and if $|d/c| < 1$ we can kill that of e_3^*. Therefore if $\alpha = 0$, $b \neq 0$ then X can be brought to one of the following forms

$$X = V_3 + \gamma e_+, \quad X = V_3 + c e_3, \quad X = V_3 + d e_3^*, \quad X = V_3 + c(e_3 \pm e_3^*). \tag{5.27}$$

If $\alpha = 0 = b$ but c, d are not both zero we can eliminate the e_+ part. If $c = 0$ then we can rotate and rescale so to get

$$X = e_3^*. \tag{5.28}$$

If $c \neq 0$ rotating and rescaling we obtain

$$X = e_3 + d_3 e_3^* + d_1 e_1^*. \tag{5.29}$$

If in addition $d_1 = 0$, $d_3 \neq \pm 1$ then acting with e_- we can bring (5.29) to the form (5.28) or to

$$X = e_3. \tag{5.30}$$

Finally if α, b, c, d all vanish then we rescale to get

$$X = e_+. \tag{5.31}$$

Thus, up to conjugation and rescaling, the possible forms of X are

$$X_1 = e_- + bV_3 + \gamma e_+, \tag{5.32}$$

$$X_2 = V_3 + c e_3, \quad c \neq 0, \tag{5.33}$$

$$X_3 = V_3 + d e_3^*, \quad d \neq 0, \tag{5.34}$$

$$X_4^\pm = V_3 + c(e_3 \pm e_3^*), \quad c \neq 0, \tag{5.35}$$

$$X_5 = V_3 + \gamma e_+, \tag{5.36}$$

$$X_6 = e_3, \tag{5.37}$$

$$X_7 = e_3^*, \tag{5.38}$$

$$X_8^\pm = e_3 \pm e_3^*, \tag{5.39}$$

$$X_9 = e_3 + d_3 e_3^* + d_1 e_1^*, \quad d_1 \neq 0 \tag{5.40}$$

$$X_{10} = e_+, \tag{5.41}$$

where any parameter can vanish unless otherwise specified.

Table 5. Fraction $\nu > 0$ of SUSY preserved by the KVFs of proposition 5.

$X \in \mathfrak{g} \rtimes \mathfrak{so}(3)$	Condition	ν
X_9	—	$1/2$
X_6	—	$1/2$
$X_8^{\pm}$	—	$1/2$
X_1	$4b^2 = 9$ or $4b^2 = 1$	$1/4$

Equations (5.10)–(5.13) give the KVF associated to $X_i \in \mathfrak{g} \rtimes \mathfrak{so}(3)$. Using (A.39) to compute their norms we find

$$\begin{aligned}
|\xi_{X_1}|^2 &= (1+b^2)(x_1^2+x_2^2) + x_3^2 + 2\gamma \geqslant 2\gamma,\\
|\xi_{X_2}|^2 &= x_1^2 + x_2^2 + c^2\cosh^2 x^- > 0,\\
|\xi_{X_3}|^2 &= x_1^2 + x_2^2 + d^2\sinh^2 x^-,\\
|\xi_{X_4^{\pm}}|^2 &= x_1^2 + x_2^2 + c^2\,\mathrm{e}^{\mp 2x^-} > 0,\\
|\xi_{X_5}|^2 &= x_1^2 + x_2^2,\\
|\xi_{X_6}|^2 &= \cosh^2 x^- > 0,\\
|\xi_{X_7}|^2 &= \sinh^2 x^-,\\
|\xi_{X_8^{\pm}}|^2 &= \mathrm{e}^{\mp 2x^-} > 0,\\
|\xi_{X_9}|^2 &= (\cosh x^- - d_3\sinh x^-)^2 + d_1^2\sinh^2 x^- > 0,\\
|\xi_{X_{10}}|^2 &= 0.
\end{aligned}$$

The KVFs X_3, X_5, X_7, X_{10} have zeros and need to be excluded, while in the case of X_1 we need to impose $\gamma > 0$. □

Making use of the representation (5.9) it can be checked that the one-parameter groups associated to (5.14)–(5.19) are all non-compact and thus have the topology of a line.

5.2. Preserved SUSY

Proposition 6. *The KVFs of proposition 5 preserving a fraction $\nu > 0$ of SUSY are given by table 5.*

Proof. Let us work at the point with coordinates, $x^i = x^{\pm} = 0$, where

$$\begin{aligned}
\xi_{e_\pm} &= \partial_\pm, & \mathrm{d}\xi^\flat_{e_\pm} &= 0,\\
\xi_{e_i} &= \partial_i, & \mathrm{d}\xi^\flat_{e_i} &= 0,\\
\xi_{e_i^*} &= 0, & \mathrm{d}\xi^\flat_{e_i^*} &= 2\,\mathrm{d}x^i \wedge \mathrm{d}x^-,\\
\xi_{V_i} &= 0, & \mathrm{d}\xi^\flat_{V_i} &= \epsilon_{ijk}\,\mathrm{d}x_j \wedge \mathrm{d}x_k.
\end{aligned} \tag{5.42}$$

Since $\varphi = \partial_+$, the β-term contribution to $L_\xi\epsilon$ is

$$\beta_\xi \epsilon = \frac{1}{4}\xi^i(\gamma_i\gamma_+ + 3g_{i+})r\epsilon, \tag{5.43}$$

which is non-zero only for ξ_{e_i} and ξ_-,

$$\beta_{\xi_{e_i}}\epsilon = \frac{1}{4}\gamma_i\gamma_+ r\epsilon, \quad \beta_{\xi_-}\epsilon = \frac{1}{4}(3+\gamma_-\gamma_+)r\epsilon. \tag{5.44}$$

The other contribution comes from $\frac{1}{4}(\mathrm{d}\xi^\flat)\cdot\epsilon$, which gives

$$\begin{aligned} \frac{1}{4}\,\mathrm{d}\xi^\flat_{e_i^*}\cdot\epsilon &= \frac{1}{2}\gamma^i\gamma^-\,\epsilon, \\ \frac{1}{4}\,\mathrm{d}\xi^\flat_{V_i}\cdot\epsilon &= \frac{\epsilon_{ijk}}{4}\gamma^j\gamma^k\epsilon. \end{aligned} \tag{5.45}$$

Putting all together

$$L_{\xi_{e_+}}\epsilon = 0, \tag{5.46}$$

$$L_{\xi_{e_-}}\epsilon = \frac{1}{4}(3+\gamma_-\gamma_+)r\epsilon, \tag{5.47}$$

$$L_{\xi_{e_i}}\epsilon = \frac{1}{4}\gamma_i\gamma_+ r\epsilon, \tag{5.48}$$

$$L_{\xi_{e_i^*}}\epsilon = \frac{1}{2}\gamma_i\gamma_+\epsilon, \tag{5.49}$$

$$L_{\xi_{V_i}}\epsilon = \frac{1}{4}\epsilon_{ijk}\gamma_j\gamma_k\epsilon. \tag{5.50}$$

For (5.14) we have

$$L_{\xi_{X_1}}\epsilon = 0 \Leftrightarrow \left[(3+\gamma_-\gamma_+)r + 2b\gamma_1\gamma_2\right]\epsilon = 0. \tag{5.51}$$

Left multiplying by γ_- gives the necessary condition

$$(3r + 2b\gamma_1\gamma_2)\gamma_-\epsilon = 0. \tag{5.52}$$

Taking $\epsilon_1, \epsilon_2 \in \operatorname{Ker}\gamma_-$ and substituting in (5.51) gives

$$(r + 2b\gamma_1\gamma_2)\epsilon = 0 \tag{5.53}$$

which has non-trivial solutions if and only if $b^2 = 1/4$, in which case we need

$$\epsilon_1 \in \operatorname{Ker}\gamma_- \cap \operatorname{Ker}(i + 2b\gamma_1\gamma_2), \quad \epsilon_2 \in \operatorname{Ker}\gamma_- \cap \operatorname{Ker}(-i + 2b\gamma_1\gamma_2), \tag{5.54}$$

and $\operatorname{Ker}\gamma_- \cap \operatorname{Ker}(\pm i + 2b\gamma_1\gamma_2)$ is one-dimensional, so $\nu = \frac{1}{4}$. Taking instead $(3r + 2b\gamma_1\gamma_2)\epsilon = 0$, which has non-trivial solutions if and only if $b^2 = 9/4$, and substituting in (5.51) gives $\gamma_-\gamma_+\epsilon_i = 0$ which has non-trivial solutions. So if $b^2 = 9/4$ we need

$$\epsilon_1 \in \operatorname{Ker}(3i + 2b\gamma_1\gamma_2) \cap \operatorname{Ker}(\gamma_-\gamma_+), \quad \epsilon_2 \in \operatorname{Ker}(-3i + 2b\gamma_1\gamma_2) \cap \operatorname{Ker}(\gamma_-\gamma_+), \tag{5.55}$$

and $\operatorname{Ker}(\pm 3i + 2b\gamma_1\gamma_2) \cap \operatorname{Ker}(\gamma_-\gamma_+)$ is one-dimensional, so again $\nu = \frac{1}{4}$.

A similar reasoning shows that (5.15) and (5.16) preserve no SUSY and (5.17) and (5.18) require $\epsilon_1, \epsilon_2 \in \operatorname{Ker}\gamma_+$ which is two-dimensional, so $\nu = \frac{1}{2}$. Finally (5.19) gives

$$(\gamma_3 r + 2d_3\gamma_3 + 2d_1\gamma_1)\gamma_+\epsilon = 0 \tag{5.56}$$

Table 6. Hereditary isometry algebra $\mathfrak{l}$ of the KK reductions of CW_5 by the KVFs of proposition 5.

$X \in \mathfrak{g} \rtimes \mathfrak{so}(3)$	$\mathfrak{l}$	Generators
$X_1, b = 0$	$\mathbb{R} \oplus \mathfrak{so}(3)$	$e_+; V_1, V_2, V_3$
$X_1, b \neq 0$	$\mathbb{R} \oplus \mathfrak{so}(2)$	$e_+; V_3$
X_2	$\mathbb{R} \oplus \mathfrak{so}(2)$	$e_+; V_3$
$X_4^{\pm}$	$\mathbb{R} \oplus \mathfrak{so}(2)$	$e_+; V_3$
X_6	$\mathfrak{h} \rtimes \mathfrak{so}(2)$	$e_+, e_2, e_2^*, e_1, e_1^*; V_3$
$X_8^{\pm}$	$\mathfrak{h} \rtimes (\mathbb{R} \oplus \mathfrak{so}(2))$	$e_+, e_2, e_2^*, e_1, e_1^*; e_-, V_3$
X_9	$\mathfrak{h}$	$e_+, e_2, e_2^*, e_1 + d_1 e_3^*, e_1^*$

which has two-dimensional kernel, so $\nu = \frac{1}{2}$. □

5.3. Geometry of the quotient

All the KVFs of proposition 5 are spacelike with orbit homeomorphic to a line, so the quotient M/Γ is in all cases a Lorentzian four-manifold with the topology of $\mathbb{R}^4$.

In order to determine the quotient metric we proceed as follows. Let ξ be the KVF generating Γ, and pick a basis $(\chi_1, \chi_2, \chi_3, \chi_4)$ for $\xi^\perp$. Then for a KK geometry we have

$$g = \frac{\xi^\flat \otimes \xi^\flat}{g(\xi, \xi)} + \sum_{i=1}^{4} C_{ij} \chi_i^\flat \odot \chi_j^\flat, \tag{5.57}$$

where the coefficients $C_{ij} = C_{ji}$ need to be determined, and the quotient metric h is

$$h = \sum_{i=1}^{4} C_{ij} \chi_i^\flat \odot \chi_j^\flat \tag{5.58}$$

provided that we re-express C_{ij}, $\chi_i^\flat$ in terms of coordinates $\tilde{x}_i$ well-defined on the quotient, i.e. such that $\xi(\tilde{x}_i) = 0$. We carry out this procedure explicitly for the cases X_1, $b = 0$, X_6, $X_8^{\pm}, X_9$ which have a larger isometry group, see table 6 below.

If $b = 0$, $X_1 = e_- + \gamma e_+$,

$$\xi_{X_1} = \partial_- + \gamma \partial_+, \tag{5.59}$$

and $\xi^\perp = \mathrm{Span}(\partial_1, \partial_2, \partial_3, \chi)$, with

$$\chi = \partial_- - (\gamma + |x|^2)\partial_+. \tag{5.60}$$

We find

$$h = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 - \frac{\mathrm{d}u^2}{2\gamma + |x|^2}, \tag{5.61}$$

with $\mathrm{d}u = \chi^\flat$,

$$u = x^+ - \gamma x^-. \tag{5.62}$$

Note that $\xi(u) = \xi(x^1) = \xi(x^2) = \xi(x^3) = 0$ so (u, x^i) are well-defined coordinates on the quotient. The isometry group of (5.61) is $O(3) \times \mathbb{R}$.

For $X_6 = e_3$,

$$\xi_{X_6} = \cosh(x^-)\partial_3 - x_3 \sinh(x^-)\partial_+, \tag{5.63}$$

and $\xi^\perp = \mathrm{Span}(\partial_+, \partial_1, \partial_2, \chi)$ with

$$\chi = x_3 \sinh(x^-)\partial_3 + \cosh(x^-)\partial_-. \tag{5.64}$$

We find

$$h = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x^- \left(C_1\, \mathrm{d}x^- + C_2\chi^\flat\right), \tag{5.65}$$

$$C_1 = -|x|^2 - x_3^2 \tanh^2(x^-), \quad C_2 = \frac{2}{\cosh(x^-)}. \tag{5.66}$$

We can rewrite h in terms of coordinates $(x_1, x_2, x^-, \tilde{x}^+)$ well-defined on the quotient as

$$h = 2\mathrm{d}x^-\mathrm{d}\tilde{x}^+ + (x_1^2 + x_2^2)(\mathrm{d}x^-)^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2, \tag{5.67}$$

where

$$\tilde{x}^+ = x^+ + \frac{x_3^2}{2} \tanh(x^-). \tag{5.68}$$

For $X_8^\pm = e_3 \pm e_3^*$,

$$\xi_{X_8^\pm} = \mathrm{e}^{\mp x^-}(\partial_3 \pm x_3\partial_+), \tag{5.69}$$

and $\xi^\perp = \mathrm{Span}(\partial_+, \partial_1, \partial_2, \chi)$ with

$$\chi = x_3\partial_3 \mp \partial_-. \tag{5.70}$$

We find

$$h = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 \mp 2\,\mathrm{d}x^- \odot \chi^\flat - (x_1^2 + x_2^2 + 2x_3^2)(\mathrm{d}x^-)^2, \tag{5.71}$$

which in terms of coordinates $(x^1, x^2, x^-, \tilde{x}^+)$ well-defined on the quotient becomes

$$h = 2\mathrm{d}\tilde{x}^+\,\mathrm{d}x^- + (x_1^2 + x_2^2)(\mathrm{d}x^-)^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2, \tag{5.72}$$

where

$$\tilde{x}^+ = x^+ \mp \frac{x_3^2}{2}. \tag{5.73}$$

Note that quotienting along X_8^+ and along X_8^- results in the same metric (5.72). This is a four-dimensional CW space with quadratic form $A = \eta^{0,2}$. Note also that reduction along X_6 and along $X_8^\pm$ results in the same quotient manifold, which makes intuitive sense since X_6 can be obtained from $X_8^\pm$ in the limit $x^- \to \pm\infty$ where x^- is the parameter appearing in (5.26).

For X_9, taking for simplicity $d_3 = 0$, we have

$$\xi_{X_9} = \cosh x^-\partial_3 - d_1 \sinh x^-\partial_1 + (d_1x_1 \cosh x^- - x_3 \sinh x^-)\partial_+, \tag{5.74}$$

and $\xi^{\perp} = \mathrm{Span}(\partial_+, \partial_2, \chi_1, \chi_2)$ with

$$\begin{aligned}\chi_1 &= d_1 \sinh x^- \partial_3 + \cosh x^- \partial_1,\\ \chi_2 &= d_1 \sinh x^- \partial_- + (d_1 x_1 \cosh x^- - x_3 \sinh x^-)\partial_1.\end{aligned} \tag{5.75}$$

In terms of coordinates $(\tilde{x}^+, x^-, u, x^2)$ well-defined on the quotient we have

$$h = 2\mathrm{d}\tilde{x}^+ \mathrm{d}x^- + x_2^2(\mathrm{d}x^-)^2 + \frac{(\mathrm{d}u - 2u\coth(2x^-)\mathrm{d}x^-)^2}{\cosh^2 x^- + d_1^2 \sinh^2 x^-} + \mathrm{d}x_2^2, \tag{5.76}$$

where

$$\begin{aligned}u &= d_1 x^3 \sinh x^- + x^1 \cosh x^-,\\ \tilde{x}^+ &= x^+ + \frac{x_1^2}{2}\coth x^- + \frac{x_3^2}{2}\tanh x^-.\end{aligned} \tag{5.77}$$

The hereditary isometry algebra is

$$\mathfrak{l} = \frac{N_{X_\xi}(\mathfrak{g} \rtimes \mathfrak{so}(3))}{\mathbb{R}X_\xi}, \tag{5.78}$$

where $N_{X_\xi}(\mathfrak{g} \rtimes \mathfrak{so}(3))$ is the normaliser of X_ξ in $\mathfrak{g} \rtimes \mathfrak{so}(3)$. For the KVFs of proposition 5 $\mathfrak{l}$ is given in table 6. The Lie algebra $\mathfrak{h}$ is the five-dimensional Heisenberg algebra generated by $\{e_1, e_1^*, e_2, e_2^*, e_+\}$ in the case of X_6, $X_8^{\pm}$, and by $\{e_1 + d_1 e_3^*, e_1^*, e_2, e_2^*, e_+\}$ in the case of X_9. Note that while the quotient by X_6 is a CW space, cf equation (5.67), the hereditary isometry algebra associated to X_6 is only a proper subalgebra of the CW isometry algebra. Therefore this is an example where the quotient has additional accidental symmetry.

6. Conclusions and summary

In this paper we have classified four-dimensional KK reductions of certain supersymmetric five-dimensional Lorentzian geometries found in [21]; namely,

(a) $-\mathbb{R} \times S^4$;
(b) $\mathbb{R} \times \mathrm{AdS}_4$; and
(c) A conformally flat CW symmetric space.

We have concentrated on reductions leading to four-dimensional Lorentzian or Riemannian manifolds. Although we do not consider them in this paper, the question of null reductions is interesting in the context of non-relativistic supersymmetry and studying the null reductions of the above backgrounds might give four-dimensional supersymmetric Newton–Cartan geometries different from those in [14] in a similar way to how three-dimensional supersymmetric Newton–Cartan geometries can be obtained via null reduction of four-dimensional supersymmetric geometries [26].

For the three backgrounds listed above, we list the possible one-parameter subgroups of isometries resulting in a Lorentzian or Riemannian quotient, identify the hereditary isometries of the four-dimensional quotient, the fraction of the supersymmetry which is preserved and, in most cases, the form of the metric in the quotient. The possible generators of one-parameter subgroups are given in proposition 1 for $-\mathbb{R} \times S^4$, proposition 3 for $\mathbb{R} \times \mathrm{AdS}_4$ and proposition 5 for the CW spacetime. The hereditary isometries of the quotients are listed in table 1 for $-\mathbb{R} \times S^4$, table 4 for $\mathbb{R} \times \mathrm{AdS}_4$ and table 6 for the CW space. The conditions for preservation of supersymmetry and the fraction of supersymmetry which is preserved upon reduction are described in proposition 2 for $-\mathbb{R} \times S^4$ and listed in table 2 for $\mathbb{R} \times \mathrm{AdS}_4$ and table 5 for the CW space.

It is worth highlighting some of the half-BPS reductions; that is, those which preserve half of the supersymmetry. Firstly, there are four half-BPS Lorentzian reductions: one (labelled λ_4 with parameter $\beta = 2\|\varphi\|$) of $\mathbb{R} \times \mathrm{AdS}_4$ and three (labelled $X_6, X_8^{\pm}, X_9$) of the CW space. These give four-dimensional Lorentzian geometries admitting an $N = 1$ supersymmetry algebra. It is then a natural question to ask whether they are contained in the classification of [18]. The geometries in that paper have supersymmetry algebras which are filtered deformations of graded maximally supersymmetric subalgebras of the $N = 1$ Poincaré superalgebra and consist of Minkowski spacetime, AdS_4, $-\mathbb{R} \times S^3$, $\mathbb{R} \times \mathrm{AdS}_3$ and the Nappi–Witten group NW_4. They all share the property that the metric is conformally flat. The reductions labelled X_6 and $X_8^{\pm}$ of the CW space are isometric to the Nappi–Witten group, as shown by the explicit form of the quotient metric in equation (5.67) and (5.72). The other reduction (X_9) of the CW space depends on two parameters $d_1 \neq 0$ and d_3. We have calculated the Weyl curvature tensor in the case $d_3 = 0$ and found it to be non-vanishing, so that this reduction is not conformally flat, in contrast to the backgrounds in [18, theorem 14]. We have not calculated the curvature tensor for any $d_3 \neq 0$, but it seems likely that they are not conformally flat reductions either. Finally, the half-BPS Lorentzian reduction of $\mathbb{R} \times \mathrm{AdS}_4$ is novel to the best of our knowledge. Indeed, calculating the Weyl curvature tensor of the metric in equation (4.35) one sees that it too is non-vanishing.

Acknowledgments

GF would like to thank the Simons Foundation for its support under the Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics (Grant 488631).

Data availability statement

No new data were created or analysed in this study.

Appendix A. Cahen–Wallach spaces

CW spaces are locally symmetric Lorentzian manifolds which exist for any dimension $n \geqslant 3$. We first construct them as symmetric spaces, following [27] for the most part, and then derive a coordinate expression for their metric.

Let V be an $(n-2)$-dimensional vector space with basis (e_i), V^* its dual with basis (e_i^*), $Z = \mathrm{Span}(e_+)$, $Z^* = \mathrm{Span}(e_-)$. Let A be a symmetric bilinear form on V. Then the Lie algebra

$$\mathfrak{g} = V \oplus V^* \oplus Z \oplus Z^* \tag{A.1}$$

with non-zero Lie brackets

$$[e_-, e_i] = e_i^*, \quad [e_-, e_i^*] = A_{ij} e_j, \quad [e_i^*, e_j] = A_{ij} e_+, \tag{A.2}$$

is solvable since $\mathfrak{g}' = \mathrm{Span}\big(e_i^*, e_i, e_+\big)$, $\mathfrak{g}^{\mathfrak{sp}\prime\prime} = \mathrm{Span}(e_+)$, $\mathfrak{g}''' = 0$. Set

$$\mathfrak{h} = \mathrm{Span}(e_i^*), \quad \mathfrak{k} = \mathrm{Span}(e_i, e_+, e_-). \tag{A.3}$$

Then $\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{k}] = \mathfrak{h}$. Thus $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ is a symmetric splitting of $\mathfrak{g}$. Note that $\{e_i, e_i^*, e_+\}$ generates a $(2n+1)$-dimensional Heisenberg algebra. Let G (respectively H) be the unique simply connected Lie group with Lie algebra $\mathfrak{g}$ ($\mathfrak{h}$). Then $M = G/H$ is a locally symmetric space. Denote by o the identity coset, $o = eH$.

Let B be the symmetric bilinear form on $V \oplus Z \oplus Z^* \simeq \mathfrak{g}/\mathfrak{h}$ with non-zero components

$$B(e_i, e_j) = \delta_{ij}, \quad B(e_+, e_-) = 1. \tag{A.4}$$

Note that B is invariant under the action of H by conjugation. Since H is Abelian we can simply check that $h = \exp(ce_i^*)$ acts by isometries. Using (A.44) and (A.45) we have e.g.

$$\begin{aligned} B(he_-h^{-1}, he_-h^{-1}) &= B\left(e_- - cA_{ij}e_j - \frac{c^2}{2}A_{ii}^2 e_+, e_- - cA_{ik}e_k - \frac{c^2}{2}A_{ii}^2 e_+\right) \\ &= -c^2 A_{ii}^2 + c^2 A_{ij}A_{ik}\delta_{jk} = 0 = B(e_-, e_-). \end{aligned} \tag{A.5}$$

The other components can be checked similarly.

Define a Lorentzian G-invariant metric β on G by setting, for any $g \in G$,

$$\beta_g(U, V) = B(g^{-1}U, g^{-1}V). \tag{A.6}$$

The left action of H on TM corresponds to the action of Ad_h on $\mathfrak{g}/\mathfrak{h}$. Since B is H-invariant, β descends to a well defined metric on M, which we still denote by β,

$$\beta_{g\cdot o}(U, V) = B(g^{-1}U, g^{-1}V), \tag{A.7}$$

where we have identified T_oM with $\mathfrak{g}$, and more generally we identify $T_{g\cdot o}M$ with T_gG.

We now derive a coordinate expression for β. On $\mathbb{R}^n$ take coordinates $x^1, \ldots, x^{n-2}, x^+, x^-$ and define

$$\sigma : \mathbb{R}^n \to G, \quad (x^i, x^+, x^-) \mapsto \exp(x^+e_+ + x^-e_-)\exp(x^ie_i). \tag{A.8}$$

The map σ provides *modified exponential coordinates* on G and, acting on o, on M. The curve $(x^1, \ldots, x^{n-1}, x^+, x^- + t)$ on $\mathbb{R}^n$ has tangent $\partial/\partial x^-$, thus

$$\sigma_*(\partial_-) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \sigma(x^i, x^+, x^- + t) = e_-\sigma(x). \tag{A.9}$$

Note that

$$\begin{aligned} \sigma(x)^{-1}e_-\sigma(x) &= \exp(-x^je_j)e_- \ \exp(x^ke_k) = \exp(-x^k\mathrm{ad}_{e_k})(e_-) \\ &= e_- + x^ie_i^* + \frac{1}{2}x^ix^jA_{ij}e_+. \end{aligned} \tag{A.10}$$

Similarly

$$\sigma_*(\partial_+) = e_+\sigma(x) = \sigma(x)e_+, \tag{A.11}$$

$$\sigma_*(\partial_i) = \sigma(x)e_i. \tag{A.12}$$

Pulling back (A.7) by σ we have

$$(\sigma^*\beta)_x(U, V) = \beta_{\sigma(x)}(\sigma_*U, \sigma_*V) = B(\sigma(x)^{-1}(\sigma_*U), \sigma(x)^{-1}(\sigma_*V)). \tag{A.13}$$

Writing $g = \sigma^*\beta$ and taking into account that terms in e_i^* are zero on the quotient space we find

$$\begin{aligned} g_x(\partial_i, \partial_j) &= \delta_{ij}, \\ g_x(\partial_+, \partial_-) &= B(e_+, \sigma(x)^{-1}e_-\sigma(x)) = B(e_+, e_-) = 1, \\ g_x(\partial_-, \partial_-) &= B(\sigma(x)^{-1}e_-\sigma(x), \sigma(x)^{-1}e_-\sigma(x)) = x^i x^j A_{ij}, \\ g_x(\partial_i, \partial_\pm) &= g_x(\partial_+, \partial_+) = 0. \end{aligned} \tag{A.14}$$

Therefore, with respect to the modified exponential coordinates (A.8), the G-invariant metric (A.7) on M is

$$g = 2\,\mathrm{d}x^+\,\mathrm{d}x^- + A_{ij}x^i x^j(\mathrm{d}x^-)^2 + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{A.15}$$

The vector field $\varphi = \partial_+$ is null and parallel. All the coordinates $x^i, x^\pm$ range in $\mathbb{R}$ and the resulting space is complete. It is known that a CW space is indecomposable if and only if A is non-degenerate. The only non-zero components of Riemann and Ricci tensors Ricci tensor in the coordinates (A.15) are

$$R_{-i-j} = -A_{ij}, \quad \mathrm{Ric}_{--} = -\,\mathrm{Tr}\,A, \tag{A.16}$$

hence CW spaces are scalar-flat. The non-vanishing components of the Weyl tensor are

$$W_{-i-j} = -A_{ij} + \frac{1}{n-2}\,\mathrm{Tr}\,A\eta^{0,n-2}_{ij}, \tag{A.17}$$

so a CW space is conformally flat if and only if $A = a\eta^{0,n-2}$ for some constant a.

A.1. Killing vector fields

It is clear from the symmetric space description that a CW space has isometry algebra

$$\mathfrak{g} \rtimes \mathfrak{s}, \tag{A.18}$$

where

$$\mathfrak{g} = V \oplus V^* \oplus Z \oplus Z^* \tag{A.19}$$

as in (A.1) and

$$\mathfrak{s} = \{s \in \mathfrak{so}(V) : s^{\mathrm{T}}A + As = 0\} \tag{A.20}$$

is the subalgebra of $\mathfrak{so}(V)$ leaving A invariant. The action of $\mathfrak{s}$ on $\mathfrak{g}$ is the natural action of $\mathfrak{so}(V)$ on V, the adjoint action on V^*, and the trivial action on Z, Z^*. Such an action preserves brackets. In fact for $s \in \mathfrak{s}$ we have e.g.

$$\begin{aligned} s\cdot[e_-, e_i] &= [s\cdot e_-, e_i] + [e_-, s\cdot e_i] = 0 + [e_-, s^j{}_i e_j] = s^j{}_i e_j^* = -s^i{}_j e_j^* = s\cdot e_i^*, \\ s\cdot[e_-, e_i^*] &= [e_-, -s^i{}_j e_j^*] = -s^i{}_j A_{jk}e_k = -A_{ik}s^k{}_j e_k = s\cdot A_{ik}e_k, \end{aligned} \tag{A.21}$$

having used $s^{\mathrm{T}} = -s$, $s^{\mathrm{T}}A = -As$. We can therefore form the semidirect product $\mathfrak{g} \rtimes \mathfrak{s}$ with, for $s_i \in \mathfrak{s}$, $g_i \in \mathfrak{g}$, bracket

$$[(s_1, g_1), (s_2, g_2)] = ([s_1, s_2], [g_1, g_2] + s_1\cdot g_2 - s_2\cdot g_1). \tag{A.22}$$

We now want to write down the KVFs corresponding to the generators (e_i, e_i^*, e_+, e_-) of $\mathfrak{g}$ in terms of the modified exponential coordinates (x^i, x^+, x^-). We define an action $x \mapsto g \cdot x$ of G on the exponential coordinates so that (A.8) is G-equivariant,

$$g\sigma(x) \cdot o = \sigma(g \cdot x) \cdot o. \tag{A.23}$$

It follows $\sigma(g \cdot x)^{-1} g\sigma(x) \in H$ as it fixes o. Thus, taking $X \in \mathfrak{g}$, $g = \exp(tX)$ to be the corresponding one-parameter subgroup of G, we can write

$$\exp(tX)\sigma(x) = \sigma(\exp(tX) \cdot x)h(t, X) \tag{A.24}$$

for some $h \in H$ depending on t, X. Clearly $h(0, X) = e$ for any X. Differentiating and evaluating at $t = 0$ we get

$$X\sigma(x) = \sigma_*|_x(\xi_X) + \sigma(x)Y, \tag{A.25}$$

where ξ_X is the KVF corresponding to $X \in \mathfrak{g}$ with respect to modified exponential coordinates x and $Y \in \mathfrak{h}$.

As calculated before, see (A.11), (A.9) and (A.12),

$$\sigma_*(\partial_\pm|_x) = e_\pm\sigma(x), \tag{A.26}$$

$$\sigma_*(\partial_i|_x) = \sigma(x)e_i. \tag{A.27}$$

Comparing (A.26) with (A.25) we immediately see that

$$\xi_{e_\pm}|_x = \partial_\pm|_x. \tag{A.28}$$

For $\xi_{e_i}|_x$ we calculate, writing σ for $\sigma(x)$,

$$\begin{aligned}
e_i\sigma = \sigma\sigma^{-1}e_i\sigma &= \sigma(\exp(-x^k e_k)\exp(-x^- e_-)e_i \, \exp(x^- e_-)\exp(x^k e_k)) \\
&= \sigma(\exp(-x^k \mathrm{ad}_{e_k}))(\exp(-x^- \mathrm{ad}_{e_-})(e_i)) \\
&= \sigma(\exp(-x^k \mathrm{ad}_{e_k}))\left(Ce_i - \frac{S}{\sqrt{|\lambda_i|}}e_i^*\right) \\
&= \sigma\left(Ce_i - \frac{S}{\sqrt{|\lambda_i|}}(e_i^* + x^i\lambda_i e_+)\right) \\
&= \sigma_*\left(C\partial_i - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}Sx^i\partial_+\right) - \frac{S}{\sqrt{|\lambda_i|}}e_i^*,
\end{aligned} \tag{A.29}$$

having used the equations given in appendix A.2 and with C, S given by (A.51). It follows

$$\xi_{e_i}|_x = C\partial_i|_x - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}Sx^i\partial_+|_x \quad (\text{no sum over } i). \tag{A.30}$$

Finally for $\xi_{e_i^*}|_x$ we have, again writing σ for $\sigma(x)$,

$$\begin{aligned} e_i^*\sigma &= \sigma\sigma^{-1}e_i^*\sigma = \sigma\big(\exp(-x^k \mathrm{ad}_{e_k})\big(\exp(-x^- \mathrm{ad}_{e_-})(e_i^*)\big) \\ &= \sigma\Big(\exp(-x^k \mathrm{ad}_{e_k})\Big(Ce_i^* - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}Se_i\Big)\Big) \\ &= \sigma\Big(Ce_i^* + Cx^i\lambda_i e_+ - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}Se_i\Big) \\ &= \sigma\big(Ce_i^*\big) + \sigma_*\Big(Cx^i\lambda_i\partial_+ - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}S\partial_i\Big), \end{aligned} \tag{A.31}$$

thus

$$\xi_{e_i^*}|_x = Cx^i\lambda_i\partial_+|_x - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}S\partial_i|_x \quad (\text{no sum over } i). \tag{A.32}$$

In summary

$$\xi_{e_\pm} = \partial_\pm, \tag{A.33}$$

$$\xi_{e_i} = C\partial_i - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}Sx^i\partial_+ \quad (\text{no sum over } i), \tag{A.34}$$

$$\xi_{e_i^*} = Cx^i\lambda_i\partial_+ - \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|}S\partial_i \quad (\text{no sum over } i). \tag{A.35}$$

Note that $\xi_{e_i^*}$ vanishes for $x = 0$, as it must.

Let us consider in more detail the case relevant for the paper, $n = 5$, $A = a\eta^{0,3}$, $a > 0$. In this case $\mathfrak{s} = \mathfrak{so}(3)$ and we choose generators (V_1, V_2, V_3) satisfying

$$[V_i, V_j] = -\epsilon_{ijk}V_k \tag{A.36}$$

so that the associated KVFs ξ_{V_i} are the usual generators of rotations in $\mathbb{R}^{0,3}$,

$$\xi_{V_i} = \epsilon_{ijk}x_j\partial_k. \tag{A.37}$$

By (A.22) the non-trivial brackets of $\mathfrak{so}(3)$ with $\mathfrak{g}$ are

$$[V_i, e_j] = V_i e_j = -\epsilon_{ijk}e_k, \quad [V_i, e_j^*] = V_i e_j^* = -\epsilon_{ijk}e_k^*. \tag{A.38}$$

For use in the main text, we give here the inner products with respect to the metric (A.15) of the KVFs $(\xi_{e_i}, \xi_{e_i^*}, \xi_{e^\pm}, \xi_{V_i})$ for $n = 5$, $A = a\eta^{0,3}$, $a > 0$.

$$\begin{aligned}
|\xi_{e_+}|^2 = \langle \xi_{e_i}, \xi_{e_+}\rangle = \langle \xi_{e_i^*}, \xi_{e_+}\rangle &= \langle \xi_{V_i}, \xi_{e_\pm}\rangle = 0, \\
\langle \xi_{e_+}, \xi_{e_-}\rangle &= 1, \\
|\xi_{e_-}|^2 &= a|x|^2, \\
\langle \xi_{e_i}, \xi_{e_-}\rangle &= -\sqrt{a}x_i \sinh(\sqrt{a}x^-), \\
\langle \xi_{e_i^*}, \xi_{e_-}\rangle &= a x_i \cosh(\sqrt{a}x^-), \\
\langle \xi_{e_i}, \xi_{e_j}\rangle &= \cosh^2(\sqrt{a}x^-)\delta_{ij}, \\
\langle \xi_{e_i}, \xi_{e_j^*}\rangle &= -\sqrt{a} \sinh(\sqrt{a}x^-)\cosh(\sqrt{a}x^-)\delta_{ij}, \\
\langle \xi_{e_i^*}, \xi_{e_j^*}\rangle &= a \sinh^2(\sqrt{a}x^-)\delta_{ij}, \\
\langle \xi_{V_i}, \xi_{V_j}\rangle &= |x|^2\delta_{ij} - x_i x_j, \\
\langle \xi_{V_i}, \xi_{e_j}\rangle &= -\cosh(\sqrt{a}x^-)x_k\epsilon_{kij}, \\
\langle \xi_{V_i}, \xi_{e_j^*}\rangle &= \sqrt{a}\sinh(\sqrt{a}x^-)x_k\epsilon_{kij}.
\end{aligned} \tag{A.39}$$

In particular ξ_{e_i} is spacelike, ξ_{e_+} is null, and $\xi_{e_i^*}$, ξ_{V_i}, ξ_{e_-} have non-negative norm.

A.2. Inner automorphisms of $\mathfrak{g} \rtimes \mathfrak{s}$

For use in the main text and general reference we record the following expressions,

$$\mathrm{Ad}_{\exp(c_k e_k)}(e_i^*) = e_i^* - A_{ik}c_k e_+, \tag{A.40}$$

$$\mathrm{Ad}_{\exp(c_k e_k)}(e_-) = e_- - c_k e_k^* + \frac{1}{2}c_i c_j A_{ij} e_+, \tag{A.41}$$

$$\mathrm{Ad}_{\exp(c_k e_k)}(e_i) = e_i, \tag{A.42}$$

$$\mathrm{Ad}_{\exp(c_k e_k)}(e_+) = e_+, \tag{A.43}$$

$$\mathrm{Ad}_{\exp(c_k e_k^*)}(e_i) = e_i + c_k A_{ik} e_+, \tag{A.44}$$

$$\mathrm{Ad}_{\exp(c_k e_k^*)}(e_-) = e_- - c_k A_{kj} e_j - \frac{1}{2}c_i c_k A_{ij} A_{jk} e_+, \tag{A.45}$$

$$\mathrm{Ad}_{\exp(c_k e_k^*)}(e_i^*) = e_i^*, \tag{A.46}$$

$$\mathrm{Ad}_{\exp(c_k e_k^*)}(e_+) = e_+. \tag{A.47}$$

Furthermore, assuming that A has been diagonalised,

$$A_{ij} = \lambda_i \delta_{ij}, \tag{A.48}$$

for $\lambda_i \neq 0$ we have

$$\exp(x^- \mathrm{ad}_{e_-})(e_i) = C e_i + \frac{S}{\sqrt{|\lambda_i|}} e_i^*, \tag{A.49}$$

$$\exp(x^- \mathrm{ad}_{e_-})(e_i^*) = C e_i^* + \mathrm{sign}(\lambda_i)\sqrt{|\lambda_i|} S e_i, \tag{A.50}$$

where

$$C = \begin{cases} \cos(\sqrt{-\lambda_i} x^-) & \text{if } \lambda_i < 0, \\ \cosh(\sqrt{\lambda_i} x^-) & \text{if } \lambda_i > 0, \end{cases} \qquad S = \begin{cases} \sin(\sqrt{-\lambda_i} x^-) & \text{if } \lambda_i < 0, \\ \sinh(\sqrt{\lambda_i} x^-) & \text{if } \lambda_i > 0. \end{cases} \tag{A.51}$$

For $\lambda_i = 0$ instead

$$\exp(x^- \mathrm{ad}_{e_-})(e_i) = e_i + x^- e_i^*, \quad \exp(x^- \mathrm{ad}_{e_-})(e_i^*) = e_i^*, \tag{A.52}$$

so (A.49) and (A.50) still hold provided that we extend the definition of C and S by setting

$$C = \begin{cases} 1 & \text{if } \lambda_i = 0, \\ \cos(\sqrt{-\lambda_i} x^-) & \text{if } \lambda_i < 0, \\ \cosh(\sqrt{\lambda_i} x^-) & \text{if } \lambda_i > 0, \end{cases} \qquad S = \begin{cases} 0 \text{ and } S/\sqrt{|\lambda_i|} \to x^- & \text{if } \lambda_i = 0, \\ \sin(\sqrt{-\lambda_i} x^-) & \text{if } \lambda_i < 0, \\ \sinh(\sqrt{\lambda_i} x^-) & \text{if } \lambda_i > 0. \end{cases} \tag{A.53}$$

Using (A.38) we also have

$$\mathrm{Ad}_{\exp(\sum_k c_k e_k)}(V_i) = V_i - c_k \epsilon_{kij} e_j, \tag{A.54}$$

$$\mathrm{Ad}_{\exp(\sum_k c_k e_k^*)}(V_i) = V_i - c_k \epsilon_{kij} e_j^*, \tag{A.55}$$

and, for (i, j, k) a permutation of $(1, 2, 3)$,

$$\mathrm{Ad}_{\exp(tV_i)}(V_j) = \cos\, tV_j \mp \sin\, tV_k, \tag{A.56}$$

$$\mathrm{Ad}_{\exp(tV_i)}(e_j) = \cos\, te_j \mp \sin\, te_k, \tag{A.57}$$

$$\mathrm{Ad}_{\exp(tV_i)}(e_j^*) = \cos\, te_j^* \mp \sin\, te_k^*, \tag{A.58}$$

with the upper (lower) sign for an even (odd) permutation.

Appendix B. Gamma matrices

It is useful to have an explicit representation of Cl(1, 4) to compute the fraction of preserved SUSY. Let $(\sigma_1, \sigma_2, \sigma_3)$ be the Pauli matrices, I_2 the two-dimensional identity matrix.

In section 2 we used

$$\begin{aligned} \gamma_0 &= \sigma_3 \otimes I_2 = \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & -I_2 \end{array}\right), \\ \gamma_1 &= (i\sigma_2) \otimes \sigma_1 = \left(\begin{array}{c|c} 0 & \sigma_1 \\ \hline -\sigma_1 & 0 \end{array}\right), \\ \gamma_2 &= (i\sigma_2) \otimes \sigma_2 = \left(\begin{array}{c|c} 0 & \sigma_2 \\ \hline -\sigma_2 & 0 \end{array}\right), \\ \gamma_3 &= (i\sigma_2) \otimes \sigma_3 = \left(\begin{array}{c|c} 0 & \sigma_3 \\ \hline -\sigma_3 & 0 \end{array}\right), \\ \gamma_4 &= (i\sigma_1) \otimes I_2 = \left(\begin{array}{c|c} 0 & iI_2 \\ \hline iI_2 & 0 \end{array}\right), \end{aligned} \tag{B.1}$$

with $\gamma_0 = \gamma_1\gamma_2\gamma_3\gamma_4$ corresponding to the timelike direction and $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ to the spacelike ones.

In section 4 for

$$\lambda_5 = R_{13} - R_{34} \tag{B.2}$$

we chose generators $(\gamma_1, \gamma_y, \gamma_3, \gamma_4, \gamma_5)$, with γ_1 corresponding to the timelike direction, and used

$$\begin{aligned} \gamma_1 &= (\sigma_3) \otimes I_2, \\ \gamma_y &= (i\sigma_2) \otimes \sigma_1, \\ \gamma_3 &= (i\sigma_2) \otimes \sigma_2, \\ \gamma_4 &= (i\sigma_2) \otimes \sigma_3, \\ \gamma_5 &= (i\sigma_1) \otimes I_2. \end{aligned} \tag{B.3}$$

For all the other KVFs we took generators $(\gamma_2, \gamma_y, \gamma_3, \gamma_4, \gamma_5)$, with γ_2 corresponding to the timelike direction, and used

$$\begin{aligned} \gamma_2 &= (\sigma_3) \otimes I_2, \\ \gamma_y &= (i\sigma_2) \otimes \sigma_1, \\ \gamma_3 &= (i\sigma_2) \otimes \sigma_2, \\ \gamma_4 &= (i\sigma_2) \otimes \sigma_3, \\ \gamma_5 &= (i\sigma_1) \otimes I_2. \end{aligned} \tag{B.4}$$

Note that (B.1), (B.3) and (B.4) only differ by a relabelling of the gamma matrices.

In section 6 we used the representation (B.1). Additionally we defined

$$\gamma_\pm = \frac{1}{\sqrt{2}}(\gamma_4 \pm \gamma_0) = \frac{1}{\sqrt{2}}\left(\begin{array}{c|c} \pm I_2 & iI_2 \\ iI_2 & \mp I_2 \end{array}\right). \tag{B.5}$$

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