

## **Sense-making in algebraic mathematizing discourse: The profiles of Bea and Nico**

Elena Macchioni<sup>1</sup>, Giulia Lisarelli<sup>2</sup>, Elisa Miragliotta<sup>3</sup> and Anna Baccaglini-Frank<sup>4</sup>

<sup>1</sup>University of Pisa, Italy; [elena.macchioni@phd.unipi.it](mailto:elena.macchioni@phd.unipi.it)

<sup>2</sup>University of Pisa, Italy;

<sup>3</sup>University of Pavia, Italy;

<sup>4</sup>University of Pisa, Italy;

*This paper analyzes the discourse of two low-achieving high school students as they solve equations. Using a commognitive perspective, we show in fine-grained detail how attempts at sense-making provide an important dimension through which to explore aspects of their mathematizing discourse.*

*Keywords: Algebra, commognition, low achievement in mathematics, sense-making, task situation.*

### **Difficulties in learning algebra: Interviews in DynaMat**

Students' difficulties in solving equations and in giving meaning to algebraic symbols, unknowns and variables are well documented (e.g., Kieran, 2022). Common difficulties lie in the transition to the letter-symbolic form of equations, for which students need to interpret algebraic expressions as mathematical objects as well as computational processes (e.g., Arcavi et al., 2017). For many (usually low-achieving) students algebra remains a meaningless manipulation of symbols (Xin et al., 2022).

This study is part of a larger research project (DynaMat) aimed at describing characteristics (both affective and cognitive) of students with a history of persistent low achievement in mathematics, and specifically in algebra. In this study, we take a commognitive perspective (Sfard, 2008) to explore aspects of the mathematizing discourse (constituting the student's "profile") of two high school students, Bea and Nico, as they solve tasks involving equations in their initial interview. Specifically, we focus on aspects of their discursive profiles related to sense-making, such as whether and when it is elicited, to what extent it guides students' construction of narratives, and what is its interplay with students' carrying out procedures in response to the interview questions.

### **Commognitive constructs used and our research questions**

We adopt the commognitive lens (Sfard, 2008). Following Lavie and colleagues (2019), we define a *task situation* as a situation in which a person feels bound to act. In school, task situations are, for example, the assignments the teacher gives students. The student interprets the *given task situation* and produces a *routine* consisting of a *task* and a *procedure*. The *task* elicited by the task situation and the implemented procedure are personal, that is, they depend on the person acting. Students' performance in a task situation depends on *precedent identifiers* (PIs) that allow them to select a past task situation, called a *precedent*, which is similar enough to the present one to warrant repeating what was done then. A mathematical routine is a *ritual* if it is a simple implementation of memorized procedures for the sake of themselves, with the performer never attending to any product that could later be used independently of the procedure that produced it.

In DynaMat we are interested in identifying students' mathematical discursive profiles; the work presented in this paper focuses on the comparison of two students' profiles with respect to their sense-making. Indeed, on the one hand, Baccaglini-Frank (2021) shows how low-achieving students, unwilling or unable to act ritualistically, either refuse to take part in mathematical discourse or they attempt to initiate sense-making processes to tell meaningful stories. On the other hand, the constant interplay between the use of well-known rituals and the effort of producing meaningful endorsed narratives is a characteristic of experts' mathematizing discourse. However, while experts can also produce a discourse on rituals and compare different rituals, many students keep acting ritualistically.

Sense-making processes occur where the discourse is more *explorative*, that is, aimed at constructing a meaningful narrative in order to *make sense* of a particular task situation (Baccaglini-Frank, 2021). Other discursive features suggesting sense-making are *de-ritualization* processes, which include: *flexibility*, *bondedness*, *applicability*, *performer's agentivity*, *objectification of the discourse*, and *substantiability* (Lavie et al., 2008). Specific aspects of de-ritualization can be seen as crucial steps for making sense of a task situation, that is to construct a *consistent*, *comprehensive*, and *cohesive* story (Sfard, 2008, 2019). Baccaglini-Frank (2021) attempted to capture sense-making in students' discourse, analyzing indicators such as the tone of the voice, logical linguistic connectors, and the coherent use of mathematical words. Sense-making is also deeply intertwined with students' *agency*, which concerns the independent decisions of the solver who proposes new actions or outcomes.

In this study, we focus on *sense-making* in students' mathematizing discourse comparing the mathematizing discourse of two students. Specifically, we explore:

- (1) whether (and if so, how) the given task situations elicit instances of sense-making;
- (2) the interplay between the implementation of rituals and processes of sense-making (not necessarily as mutually exclusive).

### **Method: Design of the interview and analytic scheme**

The study involves twelve 10th-grade students (ages 16-17) in Italy with a history of low achievement or difficulties in mathematics. Such an identification is acknowledged as follows: by the students' high school teacher, who has known them for at least a full school year; and by the students themselves. First, they are interviewed individually for 45 minutes. The interview is designed to highlight the presence, or the absence, of sense-making instances in students' discourse. We chose to present task situations both in familiar and unfamiliar forms because we hypothesized that failing to recall a ritual, and attempting to transform an unfamiliar task into a more familiar one, could elicit sense-making processes. The participants were interviewed individually by a researcher, whom they had never met before, in a quiet room with non-invasive recording devices. The researcher stood behind them and read the tasks aloud as she also showed them on a screen, remaining out of the students' field of vision, in order to reduce her influence over their task interpretation. For the same reason, she did not provide feedback on the correctness of their responses.

This paper focuses on two students, fictionally named Bea and Nico, while they are dealing with a subset of task situations about equations, that are divided into three types. In Figure 1 we show an example of the three task situations that we will analyze in more detail. In all the cases, simply showing the symbols without the written question, the interviewer would preliminarily ask the student

“Have you ever seen it written like this? What does it remind you of?”. Then, four questions for each type were presented and assigned non-sequentially. The questions were designed to elicit the same (or similar) tasks for the expert, and to allow us to explore whether this was the case or not for the students, and to study possible processes of sense-making along with implementation of known rituals. The preliminary question is to investigate which precedents might be elicited simply by the symbols shown, and the student's familiarity with the task.

<p><b>(a)</b> <math>13 - a = 13 + 11</math></p> <p>Does it have any solutions? Can you find them or possibly explain why there cannot be any?</p>	<p><b>(b)</b> <math>13 - \square = 13 + 11</math></p> <p>Is there a value that, when placed inside the square, makes this writing true? Can you find it or possibly explain why there cannot be one?</p>	<p><b>(c)</b> <math>13 - x = 13 + 11</math></p> <p>Does it have any solutions? Can you find them or possibly explain why there cannot be any?</p>
---	--	---

**Figure 1: Task situation (a), task situation (b), task situation (c)**

To answer our research questions, our analyses will consider the following aspects:

- (1) Task (for the students):** Which features of the task situation are precedent identifiers for the student? Is the task situation familiar or unfamiliar for the student?  
Indicators: Answers to the researcher’s preliminary question. Reformulations of the task situation. Verbal descriptions of specific precedents. References to memorized procedures / to a precedent recognized as “the same” task situation.
- (2) De-ritualization:** What characteristics does the performed routine have?  
Indicators: Several procedures are used for the same task situation (*flexibility*). The procedure performed in a specific task situation constitutes the precedent for another task situation (*applicability*). At the end of the procedure the outcome is used to go back to the task / There is a shift from the process to the outcome (*substantiability*).
- (3) Agency:** Are independent decisions made? If so, which ones?  
Indicators: New proposals for how to proceed. Attempts to construct an endorsed narrative, not only to perform memorized procedures. How procedures are performed or mentioned (in first or in third person, with the use of modal verbs such as “can” or “have to”...).

## Analyses of two cases

Bea and Nico were chosen for this study because, although they both perform rituals, their profiles differ. In the following, the indicators described in the analytical scheme are highlighted in the transcripts with different underlining marks, as indicated in the comments under each excerpt.

### The case of Bea

When Bea sees the equation in **task situation (a)** (Figure 1a) she spontaneously describes a specific procedure. Therefore, the symbolic inscription is a PI and Bea seems to be familiar with it.

Bea: So, it should be an equation...if not, if...then mmm we have to kind of apply it seems to me the first principle [in Italy this is the name of the rule by which if you add or subtract the same quantity to both sides of the equation you get an equivalent one], so we should separate the part, that is the terms, those that have only the parts with letters from those that instead are known terms. So, kind of changing the sign, so

for example  $13$  [hand gesture to indicate “I move that way”] minus  $13$  [same gesture] minus  $11$  and on the other side [same gesture] plus  $a$ . And then I sum up the similar terms, and I divide it, applying the second principle, by the coefficient and here I get the value of  $a$ .

Her discourse shows a certain hesitation: “it should be”, “if not, if... then mmm...”. There is a clear reference to a memorized procedure, and we can see the effort to use specific mathematical terms. The task interpreted by Bea is not just *solving the equation through algebraic manipulations*, but it involves stating the steps she has decided that need to be applied. She carries out the procedure with determination, suggested by her confident tone of voice, but no instances of *substantiability of the routine* appear. Indeed, Bea focuses only on carrying out all the announced steps, independent of the form of the equation she was tackling.

- Bea: These are similar so we eliminate them. So it is [Figure 2a] Mmm... Ah it's already solved basically... right? [embarrassed tone] Yes! I think it's already solved, however, I don't know...
- Bea: Because... I mean I would do minus 11 divided by minus 11. Which could be that... yes! Maybe that's how it should be done [laughs and writes]. So... Mmm... So 1... Then plus 1 minus 11 makes... minus  $10a$ ? No [...] Yes, okay. So... [writes  $-10a$ ] Mmm... And then from here... [Figure 2b]. I think it is
- Int: Okay. So about the question: does it have any solutions?
- Bea: Eh yes... 1 [giggles embarrassed]

In the last part of the excerpt, Bea's voice suggests insecurity when she comments on the outcome of the performed procedure. The acceptability of the outcome as a solution for her is determined by the procedure itself, and it suggests a ritualistic nature of the routine. Bea's giggling and hesitations (see “could be”, “maybe”) show her struggle in producing a meaningful narrative.

**Figure 2: (a) First part and (b) second part of the procedure carried out by Bea**

Bea declares the **task situation (b)** to be unfamiliar to her, and the symbolic writing with the empty square (Figure 1b) does not seem to elicit PIs related to equations. Once the researcher shows the written question, Bea focuses on the numerical values:

- Bea: So... 13 plus 11 makes 24. Right? Yes. Mmm... so 13... I don't think there is because if I take away I can't have the same result as when I add something. So I don't think there is any solution.

In a previous task situation with the empty square, she said “I think it's an equation”, even if she did not manipulate it algebraically. However, in task situation (b) she does not evoke the algebraic routine implemented in (a), perhaps because of its low applicability for her. Now she seems to get stuck: she appears to be more concerned with the substantiability of her routine, and more willing to construct a meaningful narrative: indeed, the narrative above is meaningful (and even endorsable if solutions are searched only within the natural numbers).

Interestingly, later, Bea asks the researcher to go back to this task situation:

Bea: If I applied the rule I would have to do the product of the extremes, so plus 13 plus 11, which precisely makes plus 24. But if I then divide it by 13 I wouldn't get a whole number, so because plus 24 divided by 13 I do [type on calculator 24/13] and it makes a decimal number, so no. Yes, then I was right, it is impossible [...] I applied the rule... the fundamental property of equations which is that the product of the extremes is equal to the product of the means. So I have both extremes, but I miss a mean. But knowing that all these elements are related to each other, if I do the product of the extremes, and to that then divide the mean, I get the other mean.

Now the PI includes the structure of an equation in the form (typical in Italian schools) of proportions in which there is precisely one unknown extreme or mean. The precedent elicited, related to proportions, makes the task familiar to Bea. The procedure is correctly memorized and recalled, but the task performed by Bea is not consistent with any of the possible task interpretations of an expert. Bea seems to recognize a specific situation (involving proportions) by selecting only some dominant PIs. She does not seem to focus on the other elements, and she gets lost in numerical manipulations (“product of extremes 13 plus 11, makes precisely 24”). The application of the memorized routine seems to strengthen the conclusion of her narrative: “Yes, then I was right, it is impossible”, and it leads her to not being concerned anymore about the substantiability of her routine. Overall, in task situations (a) and (b) there is no agency, making no independent proposals about how to proceed once she has chosen the ritual to use.

Finally, we report on a short excerpt of Bea's discourse when answering **task situation (c)**:

Bea: Then we have to do 13 plus 11 which makes precisely 24. So it means that even 13 minus the value I assign to  $x$  must give 24. So... But this value must be negative because there is the minus. But I don't think there are any solutions because if I do plus 13 minus 24 it should come out... 23, 22... like minus 11. And so it would not come with the same result as 13 plus 11. And so it doesn't have... it doesn't have solutions.

Bea does not recognize the task situation (c) as completely familiar. She confidently talks about  $x$  as unknown (this seems to be her PI), and she initially restates the task situation, but she does not mention or use the previous rituals. We can also recognize seeds of agency as she starts to plan how to solve it. Though hesitantly (“it should”, “like”), she performs the calculation (+13-24=-11) and attempts to analyze the suitability of her outcome (“like minus 11”) as a solution. This can be seen as a seed of substantiability, even though she ends up rejecting the correct solution.

### The case of Nico

Similar to Bea, just looking at the symbolic form of the equation in **task situation (a)** (Figure 1a) Nico invokes the following algebraic procedure for solving it:

Nico: So this is an equation, so to... I rewrite the equation... So 13 minus  $a$  equals 13 plus 11. So to solve it. I move the unknown which is indicated here by the letter  $a$  and instead on the other side the other numbers so 13 plus 11 minus 13. So then, 13 goes away with 13 and it comes out minus  $a$  equals plus 11. And that's the result.

In the task interpretation, PIs are the letter  $a$  and the equal sign. Nico's discourse reveals his reformulation of the task situation as the resolution of an equation where the unknown is  $a$ . A seed of agency appears in the use of the first person and in his effort to clarify his actions. Without any explicit request, Nico stresses that what he finished with “is the result”, so he focuses on the outcome.

Unlike Bea, he seems to be quite confident and, indeed, he selects a suitable precedent. However, the output of his procedure does not lead him to state the correct solution (-11).

**Task situation (b)** with the empty square is unfamiliar to Nico, who answers the preliminary question by explicitly saying he has never seen it before. Then, he also adds “it seems to me similar to the equation that was there before, but instead of the...square there was  $a$ , so I think it is... that is, the result is the same in the end”. For Nico the two task situations are not the same, but they are “similar” enough to share the outcome. The structure of the two equations acts as a PI for him. Nico recognizes (b) as being close to a task situation he has already faced (task situation (a)), so he plans to reinvest the previously found results in the current task situation (Figure 1b), which can be seen as an instance of algebraic routine applicability. The following excerpt shows Nico’s attempt at sense-making.

Nico: There is one, there is only one value that makes the equation true. If 13 plus plus 11 makes 24. So it equals 13 minus [...] I added 13 plus 11. It equals 24. So there is 13 minus, we have to put a number in there that makes the equation true, which can be for example mmm I had thought of putting minus 37 in brackets because then two minus make become, no... minus 37 [...] because 13 plus 11 makes 24 there is one 13 to be subtracted 37 minus 13 makes 24 and 13 plus 11 makes 24. I had thought about that. Mmm... But by putting minus 37, no then it would come minus 24. So... Mmm don’t... I don’t, I don’t know. I can't think of anything.

Nico changes the words present in the task situation: the “writing” becomes “equation”, and the “empty square” becomes “the brackets”. His reformulation allows us to highlight the elements he focuses on. The PI seems to be the question present in the task situation that resonates with “the value that makes the equation true”, a realization (of his) of solution. The procedure performed is a numerical substitution by trial and error, but it suggests substantiability in his concern about the consistency of the chosen number. Nico’s agency emerges when he makes independent proposals about how to proceed; he suggests considering a negative number, and then attempts to produce a narrative about whole numbers. Even though he does not find his mistake in managing negative numbers, he is aware of the fact that he has not reached an endorsable narrative (“I don’t know”).

Nico identifies **task situation (c)** (Figure 1c) as “the same as before”. Based on what he did before, the term “before” seems to refer to the equation in  $a$ . So Nico refers to the task situations involving equations in  $a$  as “the same as” the task situations involving equations in  $x$ . He also relates the task situations involving equations with an empty square to the task situations involving equations in  $x$ ; however these do not seem to be “the same” to him.

Nico: This is as before... Wait. I’ll try. 13 minus  $x$  equals 13 plus 11. Minus  $x$  equals... this can be taken off... 11. So,  $x$  equals...  $x$  equals minus 11 [Figure 3a]. If I try to do 13 minus minus 11 equals 13 plus 11. So minus and minus become, mmm a plus. So 13 plus 11 equals 13 plus 11. So 24 equals 24 [Figure 3b]. The  $x$  has a solution and it is minus 11. So then it also works for the one where there was... the square. The solution is the same.

(a)  $13 - x = 13 + 11$   
 $-x = 11$   
 $x = -11$

(b)  $13 - (-11) = 13 + 11$   
 $13 + 11 = 13 + 11$   
 $24 = 24$

Figure 3: (a) First part and (b) second part of the procedure carried out by Nico

The excerpt shows that in this task situation the letter realizing the unknown and the equal sign play the role of PIs, as in the case of task situation (a). Indeed, when different realizations (in this case equations in  $x$  and in  $a$ ) are involved in the same procedure, and they are linked in the discourse by verbal descriptions (in this case, “this is as before”), we can assume that they are elicited by the same PIs. Moreover, this analysis suggests that Nico had started to see the two given realizations of equation as “the same”. His discourse also shows instances of substantiability when he verifies whether the value found makes the equation true (Figure 3b), while he had never made a similar check in the other cases. It is reasonable to assume that this addition to his narrative is prompted by a connection to task situation (b), previously left undone, because right after finishing the calculation he states, “So then it also works for the one where there was... the square”. This utterance suggests flexibility and applicability.

## Conclusions

The two cases presented contribute to the overarching aim of characterizing low-achieving students’ algebraic discourse by investigating their attempts at sense-making.

The analysis of the students’ discourse highlights how task situation (b), that was unfamiliar for Bea and Nico, indeed elicits instances of sense-making in terms of agency: both students initially attempt to produce a meaningful narrative about numbers (natural for Bea’s and whole for Nico). In Nico’s discourse, we also found instances of de-ritualization, suggesting that task situation (c) constitutes an opportunity that he exploits for de-ritualization. On the other hand, Bea tries mostly to recall memorized procedures, with little sense-making. In the familiar task situations (to both students), ((a) and (c)), we identified different features of the students’ discourse: Nico’s readiness to move away from rituals, and Bea’s willingness to cling to them. The two students are prompted by different PIs.

Main differences in the students' sense-making processes are the following. Bea seems to have verbally memorized the steps in her rituals, and she rarely engages in processes of deritualization. Her realizations of the *equation* remain very disconnected, strongly related to different PIs. Nico’s discourse seems to be more flexible: for example, *equation* is a discursive object that he uses in different task situations, and for different realizations. We can say that Nico still relies on rituals, but his discourse contains seeds of de-ritualization that suggest a sort of readiness to initiate sense-making processes. On the other hand, Bea so desperately wants to carry out her selected ritual that she forces the task situation to be suitable (task (b)). In absence of a precise ritual to be performed (which she does confidently), she becomes hesitant. The effort of selecting a ritual seems to inhibit sense-making attempts. Instead, Nico is more concerned with local and global sense-making: he recognizes task situations as “similar”, even if not “the same”, suggesting processes of de-ritualization including substantiability, applicability, and agency. When Bea starts dealing with task situation b) she is less at ease when her ritual turns out to be ineffective, suggesting she needs to undertake an explorative routine. We find seeds of substantiability and agency in her discourse; her narrative would even be endorsable if solutions were to be searched only within the natural numbers. It is a glimpse at the initiation of a sense-making process.

The analyses suggest that the process of identifying PIs is dynamic: both students identify a PI from a subset of prompts in the task situation, and different prompts in a same question may become

dominant at different times, which identify different tasks for the students. For Nico, the driving force behind such a process seems to be initiating substantiability. Whereas Bea seems to change PIs, and hence tasks, to be able to perform her selected rituals.

As for Bea and Nico's discursive profiles, (long term and short term) memory is not a discriminating factor, unlike for some other participants in the DynaMat project. Indeed, Bea and Nico introduce and carry out procedures learned at school. Finally, our results suggest an additional dimension of students' profiles, that is, their disposition for sense-making in a given task situation. This dimension concerns the extent to which a solver *is comfortable* in ritualistic/explorative routines. Concerning this dimension, Bea and Nico's discourses present different features. Bea seems to be more at ease when she deals with well-established rituals, while her discourse is hedged and muddled when she explores an unfamiliar task situation or tries to check the outcome of a procedure. Nico willingly lets go of rituals in favor of sense-making, which includes recognizing as "the same" or similar situations that previously for him were different. We conjecture that this different disposition strongly affects the students' sense-making attempts. Further research is needed to better explore this conjecture and widely characterize the multiple dimensions of sense-making as a crucial aspect of discursive profile.

### Acknowledgments

This study is part of the funded project DynaMat (PRIN 2020BKWEXR); experimental work was conducted at CARME ([www.carme.center](http://www.carme.center)), UNISER Pistoia Srl, Italy.

### References

- Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The learning and teaching of algebra: Ideas, insights, and activities*. Routledge.
- Baccaglini-Frank, A. (2021). To tell a story, you need a protagonist: How dynamic interactive mediators can fulfill this role and foster explorative participation to mathematical discourse. *Educational Studies in Mathematics*, 106(2), 291–312.
- Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: Background, overarching dimensions, and new directions. *ZDM Mathematics Education*, 54(6), 1131–1150.
- Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics*, 101, 153–176.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.
- Sfard, A. (2019). Making sense of identities as sense-making devices. *ZDM Mathematics Education*, 51(3), 555–564.
- Xin, Y.P., Thur, R., & Thouless, H. (2022). *Enabling mathematics learning of struggling students*. Springer.