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Some definable types that cannot be amalgamated

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We exhibit a theory where definable types lack the amalgamation property.

If $q_0(x, y)$ and $q_1(x, z)$ are types over a given set A, both extending the same type $p(x) \in S(A)$, it is an easy exercise to show that there is $r(x, y, z) \in S(A)$ extending $q_0(x, y)$ and $q_1(x, z)$ simultaneously. In other words, in every theory, types have the *amalgamation property*. Suppose now that p, q_0, q_1 all belong to some special class $\mathcal{K}$ of types, and consider the following question: among the amalgams r as above, is there always one which belongs to $\mathcal{K}$? In this short note, we prove that, for $\mathcal{K}$ the class of definable types, the answer is in general negative.

By a fundamental result of Shelah, a complete theory is stable if and only if all types over models are definable. Definable types, and the tightly related notion of *stable embeddedness*, recently attracted considerable attention in unstable contexts as well. For instance, Hrushovski isolated a criterion for elimination of imaginaries in terms of density of definable types, which yielded a simplified proof of the classification of imaginaries in algebraically closed valued fields [7], and similar classification results in other (enriched) henselian valued fields [4, 5, 11]. In o-minimal theories, stable embeddedness of elementary substructures corresponds to relative Dedekind completeness [9], and in benign theories of henselian valued fields stable embeddedness obeys an Ax–Kochen–Ershov principle [2, 12]. Definable types are also central in Hrushovski and Loeser's celebrated work on Berkovich analytifications [6]: their *stable completions* of algebraic varieties are certain spaces of definable types which, crucially, form *strict pro-definable sets*.

This brings us to the main motivation for the present paper. If T is stable, then definable types over M may be seen as a pro-definable set in M^{eq} (this is a special case of [6, Lemma 2.5.1]), albeit this pro-definability need not be strict: it follows from work of Poizat on *belles paires* [10] that, if T is stable, then definable types over models form strict pro-definable sets if and only if T is nfcp, if and only if all belles paires of models of T are ω-saturated (cf. [3, § 3.1]). In order to establish strict pro-definability for other spaces of definable types, Cubides, Ye and the first author [3] recently introduced *beautiful pairs* in an arbitrary L-theory T. Poizat's belles paires are beautiful, and his theory generalises smoothly to unstable T, provided the latter satisfies certain assumptions: an extension property called (EP), and the amalgamation property (AP) for definable types.

In stable theories, (AP) and (EP) always hold. Similarly in o-minimal theories, where (AP) follows from a result of Baisalov–Poizat [1] (cf. [3, § 3.2]). In benign valued fields, there is an Ax–Kochen–Ershov type reduction for (AP) [3, § 8]. In [3, Corollary 2.4.16] an example of a (dp-minimal) theory satisfying (AP) but not (EP) was found, namely the levelled binary tree with level set ω. Whether there is a theory where (AP) fails is left open in [3].

Building on the aforementioned tree, we construct such an example.

1 | THE THEORY

Models M of the theory in which our counterexample lives are four-sorted, and are roughly obtained as follows. We start with a binary tree $\mathrm{T}(M)$, with discrete level set $\mathrm{L}(M)$. We then introduce two levelled sets $\mathrm{A}(M)$ and $\mathrm{B}(M)$, both with the same level set as the tree, namely $\mathrm{L}(M)$, and cover each level x of $\mathrm{T}(M)$ with a generic surjection from the cartesian product of the xth levels of $\mathrm{A}(M)$ and $\mathrm{B}(M)$. In this section, we spell out this construction in detail.

Definition 1.1. Let L be the following language.

1. L has four sorts A, B, T, and L.
2. T has a binary relation $\leq_{\mathrm{T}}$, a binary function $\sqcap$, a unary function $\mathrm{pred}_{\mathrm{T}}$, and constants g_{T}, r.
3. L has a binary relation $\leq_{\mathrm{L}}$, a unary function $\mathrm{pred}_{\mathrm{L}}$, and constants g_{L}, 0.
4. There are functions $\ell_{\mathrm{T}} : \mathrm{T} \to \mathrm{L}$, $\ell_{\mathrm{A}} : \mathrm{A} \to \mathrm{L}$, and $\ell_{\mathrm{B}} : \mathrm{B} \to \mathrm{L}$.
5. There is a function $f : \mathrm{A} \times \mathrm{B} \to \mathrm{T}$.

Definition 1.2. Let T be the L-theory expressing the following properties.

(i) $0 \neq g_{\mathrm{L}}$, and $(\mathrm{L} \setminus \{g_{\mathrm{L}}\}, \leq_{\mathrm{L}})$ is a discrete linear order with smallest element 0 and no largest element, with predecessor function $\mathrm{pred}_{\mathrm{L}}$, with the convention that $\mathrm{pred}_{\mathrm{L}}(0) = 0$. The "garbage" point g_{L} is not $\leq_{\mathrm{L}}$-related to anything, and $\mathrm{pred}_{\mathrm{L}}(g_{\mathrm{L}}) = g_{\mathrm{L}}$.

(ii) $(\mathrm{T} \setminus \{g_{\mathrm{T}}\}, \leq_{\mathrm{T}}, \sqcap)$ is a meet-tree, viewed as a semilinear order $\leq_{\mathrm{T}}$ with associated meet function $\sqcap$, root r, binary ramification,[1] and, for every fixed element, its set of predecessors is a discrete linear order, with predecessor function $\mathrm{pred}_{\mathrm{T}}$, with the convention that $\mathrm{pred}_{\mathrm{L}}(r) = r$. The "garbage" point g_{T} behaves similarly to the garbage point g_{L}.

(iii) ℓ_{T} is a a surjective level function $T \setminus \{g_{\mathrm{T}}\} \to L \setminus \{g_{\mathrm{L}}\}$, extended by $\ell_{\mathrm{T}}(g_{\mathrm{T}}) = g_{\mathrm{L}}$. For every fixed element $t \in \mathrm{T} \setminus \{g_{\mathrm{T}}\}$, the restriction of ℓ_{T} to the set of predecessors of t defines an order isomorphism onto an initial segment of $\mathrm{L} \setminus \{g_{\mathrm{T}}\}$ (in particular, $\ell_{\mathrm{T}} \circ \mathrm{pred}_{\mathrm{T}} = \mathrm{pred}_{\mathrm{L}} \circ \ell_{\mathrm{T}}$). Moreover, for any t from $\mathrm{T} \setminus \{g_{\mathrm{T}}\}$ and any y in L with $\ell_{\mathrm{T}}(t) \leq_{\mathrm{L}} y$ there is t' in T with $t \leq_{\mathrm{T}} t'$ such that $\ell_{\mathrm{T}}(t') = y$.

(iv) g_{L} is not in the image of $\ell_{\mathrm{A}} : \mathrm{A} \to \mathrm{L}$, nor in that of $\ell_{\mathrm{B}} : \mathrm{B} \to \mathrm{L}$.

(v) For every $c \in \mathrm{L} \setminus \{g_{\mathrm{L}}\}$ the fibers $\ell_{\mathrm{A}}^{-1}(c)$ and $\ell_{\mathrm{B}}^{-1}(c)$ are infinite.

(vi) $f(a, b) = g_{\mathrm{T}}$ if and only if $\ell_{\mathrm{A}}(a) \neq \ell_{\mathrm{B}}(b)$.

(vii) If $\ell_{\mathrm{A}}(a) = \ell_{\mathrm{B}}(b)$, then $\ell_{\mathrm{T}}(f(a, b)) = \ell_{\mathrm{A}}(a)$.

(viii) At any level, f defines a *generic surjection*: for any $c \in \mathrm{L} \setminus \{g_{\mathrm{L}}\}$, any $t_1, \ldots, t_n$ from $\mathrm{T} \setminus \{g_{\mathrm{T}}\}$ and any pairwise distinct $a_1, \ldots, a_n$ from A such that $\ell_{\mathrm{T}}(t_i) = c = \ell_{\mathrm{A}}(a_i)$ for all i, there are infinitely many b from B such that, for all i, we have $f(a_i, b) = t_i$; similarly with the roles of A and B interchanged.

Recall that a meet-tree together with a linear order and a map satisfying (iii) above is called a *levelled tree*.

Proposition 1.3. The following properties hold.

1. T is complete and admits quantifier elimination.
2. The union of the definable sets $\mathrm{T} \setminus \{g_{\mathrm{T}}\}$ and $\mathrm{L} \setminus \{g_{\mathrm{L}}\}$ is stably embedded, with induced structure a pure levelled (binary) meet-tree. In particular, $\mathrm{L} \setminus \{g_{\mathrm{L}}\}$ is stably embedded with induced structure a pure ordered set.

[1] We fix binary ramification for simplicity, but this is not important: any fixed finite ramification will work.

Proof. It is easy to see that T is consistent. It is enough to prove quantifier elimination: (2) is a direct consequence of the latter; as for completeness, it follows (cf., e.g., [8, Proposition 18.4]) from quantifier elimination and the fact that, given an arbitrary model of T, the L-substructure with underlying set the interpretations of the closed L-terms over $\varnothing$ embeds in every other model.

Let N_0 and N_1 be models of T with N_0 countable and N_1 $\aleph_1$-saturated, and let M be a common L-substructure of N_0 and N_1. It is an easy exercise to M-embed N_0 into N_1, yielding quantifier elimination. $\square$

The theory induced on $\mathrm{T} \setminus \{g_{\mathrm{T}}\}$ and $\mathrm{L} \setminus \{g_{\mathrm{L}}\}$ is thus precisely the one used in [3, Fact 2.4.15].

Lemma 1.4. For all $M \vDash T$, all linearly ordered definable subsets of $\mathrm{T}(M)$ have a maximum.

Proof. This follows from quantifier elimination. Alternatively, one may use that no infinite branch is definable in the standard binary meet-tree $(2^{<\omega}, \omega)$, e.g., since for any $n \in \omega$ and branches $s, s' \in 2^{\omega}$ with $s\restriction_n = s'\restriction_n$ there is an automorphism σ over $2^{<n}$ with $\sigma(s) = s'$. $\square$

2 | THE TYPES

Let T be the theory defined in the previous section, and $\mathfrak{U} \vDash T$ a monster model. Failure of amalgamation of definable types boils down to the following phenomenon. All elements y of A with level larger than $\mathrm{L}(\mathfrak{U})$ have the same, definable, type, and similarly for z in B; nevertheless, if such y and z have the *same* infinite level, then $f(y, z)$ can be used to produce an externally definable subset of $\mathrm{T}(\mathfrak{U})$ which is not definable. More formally, we proceed as follows.

Definition 2.1. Define the following sets of $L(\mathfrak{U})$-formulas.

1. $p(x)$ is the global type of an element x of sort L such that $x > \mathrm{L}(\mathfrak{U})$.
2. $q_{\mathrm{A}}(x, y)$ restricts to p on x, and says that y is an element of sort A with $\ell_{\mathrm{A}}(y) = x$.
3. $q_{\mathrm{B}}(x, z)$ restricts to p on x, and says that z is an element of sort B with $\ell_{\mathrm{B}}(y) = x$.

Lemma 2.2. All of $p, q_{\mathrm{A}}, q_{\mathrm{B}}$ are complete types over $\mathfrak{U}$ which are $\varnothing$-definable.

Proof. Consistency and $\varnothing$-definability are clear. As for completeness, we argue as follows.

1. Completeness of $p(x)$ follows from Proposition 1.3(2).
2. As for $q_{\mathrm{A}}(x, y)$, note that, since y has a new level, it cannot be in $\mathrm{A}(\mathfrak{U})$. Again because $\ell_{\mathrm{A}}(y)$ is new, for all $b \in \mathrm{B}(\mathfrak{U})$ we must have $f(y, b) = g_{\mathrm{T}}$. By quantifier elimination, this is enough to determine a complete type.
3. The argument for $q_{\mathrm{B}}(x, z)$ is symmetrical. $\square$

Proposition 2.3. The types $q_{\mathrm{A}}(x, y)$ and $q_{\mathrm{B}}(x, z)$ cannot be amalgamated over $p(x)$ into a definable type. In other words, no completion of $q_{\mathrm{A}}(x, y) \cup q_{\mathrm{B}}(x, z)$ is definable.

Proof. Suppose $r(x, y, z)$ is a completion of $q_{\mathrm{A}}(x, y) \cup q_{\mathrm{B}}(x, z)$. Then $r(x, y, z) \vdash \ell_{\mathrm{A}}(y) = x = \ell_{\mathrm{B}}(z)$, thus $r(x, y, z) \vdash \ell_{\mathrm{T}}(f(y, z)) = x$. Since $p(x)$ is not realised, $f(y, z)$, having a new level, cannot be in $\mathrm{T}(\mathfrak{U})$. Consider the set $\{d \in \mathrm{T}(\mathfrak{U}) \mid r(x, y, z) \vdash f(y, z) > d\}$. If $r(x, y, z)$ is definable, then this set is definable. As a set of predecessors, it must be linearly ordered, hence have a maximum by Lemma 1.4. But then $f(y, z) \notin \mathfrak{U}$ contradicts binary ramification. $\square$

Corollary 2.4. In T, global definable types do not have the amalgamation property.

This partially answers [3, Question 9.3.1], asking whether there is such a theory which, additionally, has uniform definability of types; note that T does not. Moreover, T is easily shown to have IP.

Question 2.5. Is there a NIP theory where global definable types do not have the amalgamation property?

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CONFLICT OF INTEREST

The authors declare no conflicts of interest.

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