Series expansions for Maass forms on the full modular group from the Farey transfer operators

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Abstract

We deepen the study of the relations previously established by Mayer, Lewis and Zagier, and the authors, among the eigenfunctions of the transfer operators of the Gauss and the Farey maps, the solutions of the Lewis-Zagier three-term functional equation and the Maass forms on the modular surface $PSL(2,\mathbb{Z})\backslash\mathcal{H}$. In particular we introduce an "inverse" of the integral transform studied by Lewis and Zagier, and use it to obtain new series expansions for the Maass cusp forms and the non-holomorphic Eisenstein series restricted to the imaginary axis. As corollaries we obtain further information on the Fourier coefficients of the forms, including a new series expansion for the divisor function.

1 Introduction

One of the most interesting objects in the mathematics literature are the Maass forms on the full modular group $\mathrm{PSL}(2,\mathbb{Z})$. Letting $\Delta := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ denote the hyperbolic Laplacian, Maass forms are smooth $\mathrm{PSL}(2,\mathbb{Z})$-invariant complex functions ϕ defined on the upper half-plane $\mathcal{H}=\{z=x+iy\,:\,y>0\}$, increasing less than exponentially as $y\to\infty$, and satisfying $\Delta\phi=\lambda\phi$ for some $\lambda\in\mathbb{C}$. Maass forms divide into cusp and non-cusp forms according to their behaviour at the cusp of the modular surface $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$, and into even and odd forms according to whether $\phi(-x+iy)=\pm\phi(x+iy)$.
Despite their importance, Maass cusp forms remain mysterious objects. No explicit construction exists and all basic information about their existence comes from the Selberg trace formula. Much more is known for the non-cusp forms, which are generated by the non-holomorphic Eisenstein series. The standard approach to Maass forms uses the methods of harmonic analysis on $\mathcal{H}$, which leads to the Fourier expansion of the forms in terms of Whittaker function (see e.g. [14]).
In recent years, a new approach to Maass forms has been developed using the relation between the Selberg zeta function $Z(q)$ and the Fredholm determinant of the transfer operators $\mathcal{L}_q$ of the Gauss map (see [18, 6]), also in connection with a functional approach introduced in [15, 16]. This connection becomes clearer if one considers the transfer operators $\mathcal{P}_q$ of the Farey map, a "slow" version of the Gauss map, as shown by the authors in [4], where we also studied the properties of the eigenfunctions of the operators $\mathcal{P}_q$. In this paper we continue the work set out in [4], by transferring the information on the eigenfunctions of $\mathcal{P}_q$ to Maass forms. In particular we use the integral transform studied in [16] to obtain series expansions for the Maass forms restricted to the imaginary axis that to our knowledge are entirely new. We first obtain expansions in terms of Legendre functions P^{μ}_{ν}. In Section 3 we prove the following

Theorem A. *If $u(x+iy)$ is an even Maass cusp form on $\mathrm{PSL}(2,\mathbb{Z})$ with eigenvalue $q(1-q)$, then there*

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exists a sequence $\{a_{n,q}\}$ *satisfying* $\limsup_n |a_{n,q}|^{\frac{1}{n}} \leq 1$*, such that*

$$u(iy) = \sum_{n=0}^{\infty} (-1)^n \, a_{n,q} \, \frac{y^{\frac{1}{2}} \, \mathrm{P}_{n+q-\frac{1}{2}}^{-q+\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right) + y^{n+q} \, \mathrm{P}_{n+q-\frac{1}{2}}^{-q+\frac{1}{2}}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}$$

uniformly in y *on any compact interval in* $(0, +\infty)$*.*

The non-holomorphic Eisenstein series $E(x+iy, q)$ *can be written for* $x = 0$ *as a meromorphic function on the half-plane* $\Re(q) > 0$ *as*

$$E(iy, q) = \zeta(2q) \left(y^q + y^{-q}\right) - 2\,\zeta(2q) \left(\frac{y}{1+y^2}\right)^q +$$

$$+ 2^{q+\frac{1}{2}} \, \Gamma\left(q + \frac{1}{2}\right) \sum_{n=0}^{\infty} (-1)^n \, b_{n,q} \, \frac{y^{\frac{1}{2}} \, \mathrm{P}_{n+q-\frac{1}{2}}^{-q+\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right) + y^{n+q} \, \mathrm{P}_{n+q-\frac{1}{2}}^{-q+\frac{1}{2}}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}$$

with

$$b_{n,q} := (-1)^n \, \frac{\Gamma(n+2q)}{(n+1)!\,\Gamma(2q)} \sum_{i=0}^{n} \binom{n+1}{i} B_i \, \zeta(2q-1+i)\,,$$

where $\{B_i\}$ *are the Bernoulli numbers and* $\zeta(s)$ *is the Riemann zeta function.*

An analogous result is given in Theorem 3.10 for odd Maass cusp forms. Moreover, in Appendix B we show a curious way of using the Legendre functions to expand y^q, and then to write the non-holomorphic Eisenstein series in terms of the Legendre functions.

Then in Section 4 we study the Fourier coefficients of Maass forms − which for the non-cusp case are related to the divisor function $\sigma_\ell(n)$ − and obtain some results which can be summarized in the following

Theorem B. *Let* $\{c_{n,q}\}$ *denote the coefficients of the Fourier expansion of an even Maass cusp form with eigenvalue* $q(1-q)$*. Then we have, up to a constant depending on* q*,*

$$c_{n,q} = 2\, n^{q-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k a_{2k,q}}{\Gamma(2k+2q)} (2\pi n)^{2k}$$

where $\{a_{n,q}\}$ *is the sequence introduced in Theorem A.*

In the case of non-cusp forms we prove that for $n \geq 1$ *and* $\Re(q) > 0$ *it holds*

$$\sigma_{2q-1}(n) = n^{2q-1} \sum_{k=1}^{\infty} \frac{(-1)^k \, \tilde{A}_{2k,q}}{(2k)!} \, (2\pi n)^{2k}$$

with

$$\tilde{A}_{k,q} := \frac{1}{k+1} \sum_{i=2}^{k} \binom{k+1}{i} B_i \, \zeta(2q-1+i)\,.$$

In Remark 4.2 we show that the series expansion for the divisor function can be considered an extension of the Ramanujan expansion.

Finally, in Section 5 we exploit the properties of the Legendre function to obtain new series expansions for the Maass forms. In the cusp case these are only formal since we don't have control on the coefficients, whereas in the non-cusp case we prove

Theorem C. *For q with $\Re(q) > 0$ it holds*

$$E(iy,q) = 2\left(\frac{y}{1+y^2}\right)^q \left[\zeta(2q-1)\,{}_2F_1\left(1,q\,;\,\frac{3}{2}\,;\,\frac{1}{1+y^2}\right) + \zeta(2q-1)\,{}_2F_1\left(1,q\,;\,\frac{3}{2}\,;\,\frac{y^2}{1+y^2}\right)\right] +$$

$$+\,4\left(\frac{y}{1+y^2}\right)^q \sum_{s=1}^{\infty} 2^s\,\frac{\Gamma(s+q)}{s!\,\Gamma(q)}\left(\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{2^k\,(s+k)}\,B_{s+k}\,\zeta(2q-1+s+k)\right)\frac{1+y^{2s}}{(1+y^2)^s}$$

uniformly in y on any compact interval in $(0,+\infty)$.

We believe that these new series expansions will turn useful in the study of Maass forms and their Fourier coefficients, as they involve the coefficients $\{a_{n,q}\}$ which come from the totally different approach described below. In particular we hope that this will stimulate new numerical investigations of the coefficients $\{a_{n,q}\}$ (see also Remark 3.6).

For the benefit of the reader we now briefly recall the main steps of the approach to the Maass forms as developed by Mayer, Lewis, Zagier and the authors in [18, 16, 4].
In [18] Mayer used the definition of the Selberg zeta function $Z(q)$ as a product over the length spectrum of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$ to prove a relation between $Z(q)$ and the Smale-Ruelle zeta function for the geodesic flow on the modular surface. We recall that the length spectrum of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$ is the set of lengths of the closed geodesics on $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$, and the closed geodesics appear in the definition of the Smale-Ruelle zeta function. The aforementioned relation together with results in [17] entails the main result of [18], the equality

$$Z(q) = \det(1-\mathcal{L}_q)\,\det(1+\mathcal{L}_q)\,, \qquad q\in\mathbb{C}\,. \tag{1.1}$$

Here "det" indicates the determinant in the sense of Fredholm, and $\mathcal{L}_q$ denotes the meromorphic extension to $q\in\mathbb{C}$ of the family of nuclear of order zero endomorphisms defined by

$$(\mathcal{L}_q h)\,(z) = \sum_{n=1}^{\infty}\frac{1}{(z+n)^{2q}}\,h\left(\frac{1}{z+n}\right)$$

for $\Re(q) > \frac{1}{2}$, on the space $H(D)$ of holomorphic functions in the disk $D = \left\{z\in\mathbb{C}\,:\,|z-1| < \frac{3}{2}\right\}$. The connection (1.1) comes from the arithmetic properties of the length spectrum of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$ and the fact that the endomorphisms $\mathcal{L}_q$ are the transfer operators of the Gauss map, which generates a dynamical system related to the continued fractions expansion of a real number. Combining Mayer's equality (1.1) sharpened by Efrat in [9] with the known positions of the zeroes of $Z(q)$ as implied by the Selberg trace formula, one can state the following

Theorem 1.1 ([9],[18])**.** *Let $q = \xi + i\eta$ be a complex number with $\xi > 0$ and $q \neq \frac{1}{2}$. Then:*

- *(i) there exists a nonzero $h\in H(D)$ such that $\mathcal{L}_q h = h$ if and only if q is either an even spectral parameter of Γ, that is there exists an even Maass cusp form u such that $\Delta u = q(1-q)u$, or $2q$ is a non-trivial zero of the Riemann zeta function, or $q = 1$;*
- *(ii) there exists a nonzero $h\in H(D)$ such that $\mathcal{L}_q h = -h$ if and only if q is an odd spectral parameter of Γ, that is there exists an odd Maass cusp form u such that $\Delta u = q(1-q)u$.*

In the papers [15, 16] Lewis and Zagier introduced a three-term functional equation whose solutions are in one-to-one correspondence with the Maass cusp and non-cusp forms. Using the results for the spectral parameters of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$ and for the Maass non-cusp forms, they proved the following result.

Theorem 1.2 ([16])**.** *There is an isomorphism between the Maass cusp forms with eigenvalue $q(1-q)$ and the space of real-analytic solutions of the three-term functional equation*

$$\psi(x) = \psi(x+1) + (x+1)^{-2q}\psi\left(\frac{x}{x+1}\right), \qquad x\in\mathbb{R}^+ \tag{1.2}$$

with the conditions

$$\psi(x) = O(1) \text{ as } x \to 0^+ , \quad \psi(x) = O(1/x) \text{ as } x \to +\infty \tag{1.3}$$

Moreover, the Maass non-cusp forms, which for any given q lie in a one-dimensional space generated by the non-holomorphic Eisenstein series $E(z,q)$, are in one-to-one correspondence with the functions

$$\psi_q^+(x) = \frac{\zeta(2q)}{2}\left(1 + x^{-2q}\right) + \sum_{m,n\geq 1} \frac{1}{(mx+n)^{2q}} , \qquad \Re(q) > 1 \tag{1.4}$$

which, when multiplied by $\frac{\Gamma(2q)}{\Gamma(q-1)}$, can be analytically continued to $q \in \mathbb{C}$ as solutions of (1.2).

In [16] the solutions of equation (1.2) are called *period functions* because of an analogy, explored in the paper, with the classical Eichler-Shimura-Manin period polynomials of the holomorphic cusp forms. Moreover, the period functions associated to a Maass forms are divided into even and odd functions.
Putting together Theorems 1.1 and 1.2 we have the following situation for the zeroes of the Selberg zeta function $Z(q)$:

- if q is an even spectral parameter with $\xi = \frac{1}{2}$, then there exist a nonzero $h \in H(D)$ such that $\mathcal{L}_q h = h$ and an even real-analytic function $\psi(x)$ which satisfies (1.2) with conditions (1.3);
- if q is an odd spectral parameter with $\xi = \frac{1}{2}$, then there exist a nonzero $h \in H(D)$ such that $\mathcal{L}_q h = -h$ and an odd real-analytic function $\psi(x)$ which satisfies (1.2) with conditions (1.3);
- if $2q$ is a non-trivial zero of the Riemann zeta function, then there exist a nonzero $h \in H(D)$ such that $\mathcal{L}_q h = h$ and (1.2) has solutions given by multiples of the analytic continuation of the function ψ_q^+;
- if $q = 1$ then there exist a nonzero $h \in H(D)$ such that $\mathcal{L}_q h = h$, in fact we have $h(x) = \frac{1}{x+1}$, and (1.2) has solutions given by multiples of the function $\psi_1^+(x) = \frac{1}{x}$.

Moreover there is an explicit relation between the eigenfunctions of the operator $\mathcal{L}_q$ and the period functions relative to the same q. Namely $h(x) = \psi(x+1)$, and the same holds on D, where $\psi(z+1)$ is the holomorphic extension of ψ to $\mathbb{C} \setminus (-\infty, 0]$.
The beauty of Mayer's result lies in the displaying of the power of the theory of transfer operators for dynamical systems, but the spectral properties of the operators $\mathcal{L}_q$ turned out to be difficult to study, see [6] and [1]. On the other side, Lewis and Zagier approach has the advantage of introducing a relation of Maass forms with solutions of an equation with a finite number of terms, which might be easier to handle.
These two aspects are combined in our paper [4], where we used a family of signed transfer operators $\mathcal{P}_q^\pm$ for the Farey map, a "slow" version of the Gauss map, defined for $\xi = \Re(q) > 0$ by

$$(\mathcal{P}_q^\pm f)(z) = (\mathcal{P}_{0,q} f)(z) \pm (\mathcal{P}_{1,q} f)(z) := \left(\frac{1}{z+1}\right)^{2q} f\left(\frac{z}{z+1}\right) \pm \left(\frac{1}{z+1}\right)^{2q} f\left(\frac{1}{z+1}\right)$$

where $z \in B = \left\{ z \in \mathbb{C} \ : \ |z - \frac{1}{2}| < \frac{1}{2} \right\}$ and $f \in H(B)$. We studied the problem of existence of eigenfunctions for $\mathcal{P}_q^\pm$ and proved the following

Theorem 1.3 ([4])**.** *(a) If $f \in H(B)$ satisfies $\mathcal{P}_q^+ f = \lambda f$ with $\lambda \neq 0$ then $f \in H(\{\Re(z) > 0\})$ and we call it even in the sense that $\mathcal{I}_q f = f$, where*

$$(\mathcal{I}_q f)(z) := \frac{1}{z^{2q}} f\left(\frac{1}{z}\right) \tag{1.5}$$

Moreover it satisfies

$$\lambda f(z) = f(z+1) + (z+1)^{-2q} f\left(\frac{z}{z+1}\right) , \qquad \Re(z) > 0 \, . \tag{1.6}$$

(b) If $f \in H(B)$ satisfies $\mathcal{P}_q^- f = \lambda f$ with $\lambda \neq 0$ then $f \in H(\{\Re(z) > 0\})$ and we call it odd in the sense that $\mathcal{I}_q f = -f$. Moreover it satisfies (1.6).

(c) If $f \in H(\{\Re(z) > 0\})$ satisfies (1.6) for $\lambda \neq 0$, then $\mathcal{P}_q^\pm (f \pm \mathcal{I}_q f) = \lambda (f \pm \mathcal{I}_q f)$.

(d) If $f \in H(B)$ satisfies $\mathcal{P}_q^+ f = \lambda f$ with $\lambda \notin [0,1)$ then there exists $\phi \in L^2((0,+\infty), t^{2\xi-1} e^{-t} dt)$ such that f can be written as

$$f(z) = c\, \frac{\lambda^{\frac{1}{z}}}{z^{2q}} + b\, \frac{\Gamma(2q-1)}{\Gamma(2q)}\, \frac{1}{z} + \frac{1}{z^{2q}} \int_0^\infty e^{-\frac{t}{z}}\, \phi(t)\, t^{2q-1}\, dt\,, \qquad \Re(z) > 0\,, \tag{1.7}$$

where $c, b \in \mathbb{C}$, $\phi(0)$ is finite and $\phi(t) - \phi(0) = O(t)$ as $t \to 0^+$, and the last term is bounded as $\Re(z) \to 0$. Moreover if $\lambda \neq 1$ then $b = 0$.

(e) If $f \in H(B)$ satisfies $\mathcal{P}_q^- f = \lambda f$ with $\lambda \notin [0,1)$ then there exists $\phi \in L^2((0,+\infty), t^{2\xi-1} e^{-t} dt)$ such that f can be written as

$$f(z) = c\, \frac{\lambda^{\frac{1}{z}}}{z^{2q}} + \frac{1}{z^{2q}} \int_0^\infty e^{-\frac{t}{z}}\, \phi(t)\, t^{2q-1}\, dt\,, \qquad \Re(z) > 0\,, \tag{1.8}$$

where $c \in \mathbb{C}$, $\phi(0)$ is finite and $\phi(t) - \phi(0) = O(t)$ as $t \to 0^+$, and the last term is bounded as $\Re(z) \to 0$.

Using the operators $\mathcal{P}_q^\pm$ we introduced a generalization of the transfer operators $\mathcal{L}_q$, namely the two variable operator-valued function $\mathcal{L}_{q,w}$ formally defined as

$$\mathcal{L}_{q,w} = w\, \mathcal{P}_{1,q} (1 - w\, \mathcal{P}_{0,q})^{-1}$$

We proved that as operators acting on the Banach space $H_\infty(D_\varepsilon)$ of functions holomorphic on $D_\varepsilon = \left\{ z \in \mathbb{C} \,:\, |z-1| < \frac{3}{2} - \varepsilon \right\}$ and bounded on $\overline{D_\varepsilon}$, they are nuclear of order zero for $\Re(q) > 0$ and $w \in \mathbb{C} \setminus (1, \infty)$. Moreover the function $q \mapsto \mathcal{L}_{q,w}$ is analytic in $\Re(q) > 0$ for any $w \in \mathbb{C} \setminus [1, \infty)$ and is meromorphic in $\Re(q) > 0$ for $w = 1$ with a simple pole at $q = \frac{1}{2}$. Analogously the function $w \mapsto \mathcal{L}_{q,w}$ is analytic in $w \in \mathbb{C} \setminus [1, \infty)$ for any q with $\Re(q) > 0$. Hence we can compute the Fredholm determinants of the operators $(1 \pm \mathcal{L}_{q,w})$ and define the *two-variable Selberg zeta function*

$$Z(q,w) := \det(1 - \mathcal{L}_{q,w})\, \det(1 + \mathcal{L}_{q,w})$$

for $\Re(q) > 0$ and $w \in \mathbb{C} \setminus (1, \infty)$. For $w = 1$ the function $Z(q,1)$ is meromorphic in $\Re(q) > 0$ with a simple pole at $q = \frac{1}{2}$ and coincides with the Selberg zeta function $Z(q)$.
Finally we obtained a relation between the eigenfunctions of $\mathcal{L}_{q,w}$ and those of $\mathcal{P}_q^\pm$, and therefore, thanks to Theorem 1.3-(a,b), a relation between the solutions of the generalized three-term functional equation (1.6) and the zeroes of the function $Z(q,w)$. More precisely, using [4, Theorem 3.6] and [4, Corollary 3.7], and the definition of $Z(q,w)$, together with the spectral characterisation of the zeroes of the Selberg zeta function given in Theorem 1.1, it follows

Theorem 1.4 ([4]). *(a) Let $w = 1$. Then:*

- *q is an even spectral parameter with $\xi = \frac{1}{2}$ if and only if there exists an even $f \in H(B)$ such that $\mathcal{P}_q^+ f = f$, f satisfies (1.6) with $\lambda = 1$ (or (1.2)) and it can be written as in (1.7) with $c = b = 0$;*

- *q is an odd spectral parameter with $\xi = \frac{1}{2}$ if and only if there exists an odd $f \in H(B)$ such that $\mathcal{P}_q^- f = f$, f satisfies (1.6) with $\lambda = 1$ (or (1.2)) and it can be written as in (1.8) with $c = 0$;*

- *$2q$ is a non-trivial zero of the Riemann zeta function if and only if there exists an even $f \in H(B)$ such that $\mathcal{P}_q^+ f = f$, f satisfies (1.6) with $\lambda = 1$ (or (1.2)) and it can be written as in (1.7) with $c = 0$ and $b \neq 0$;*

- *$q = 1$ is a zero of $Z(q,1)$ since $f(z) = \frac{1}{z}$ satisfies $\mathcal{P}_1^+ f = f$.*

(b) Let $w \in \mathbb{C} \setminus [1,\infty)$. Then:

- *q is an "even" zero of $Z(q,w)$ if and only if there exists an even $f \in H(B)$ such that $\mathcal{P}_q^+ f = \frac{1}{w} f$, f satisfies* (1.6) *with $\lambda = \frac{1}{w}$ and it can be written as in* (1.7) *with $c = b = 0$;*
- *q is an "odd" zero of $Z(q,w)$ if and only if there exists an odd $f \in H(B)$ such that $\mathcal{P}_q^- f = \frac{1}{w} f$, f satisfies* (1.6) *with $\lambda = \frac{1}{w}$ and it can be written as in* (1.8) *with $c = 0$.*

Since by Theorem 1.3-(a,b), eigenfunctions of $\mathcal{P}_q^{\pm}$ satisfy a three-term equation which is a generalization of the Lewis-Zagier equation (1.2), we call the functions f of Theorem 1.4 *generalized period functions (gpf)* associated to the zeroes of the zeta function $Z(q,w)$, *even* and *odd* according to whether they correspond to even or odd zeroes. In addition we distinguish the two classes of gpf with $b = 0$, which we call 0-gpf, and $b \neq 0$, which we call b-gpf. In the $w = 1$ case the 0-gpf correspond to the Maass cusp forms and the b-gpf to the non-cusp forms. On the contrary in the $w \neq 1$ case the set of b-gpf is empty.
In Section 3 we consider the general case $w \in \mathbb{C} \setminus (1,\infty)$, so we find a series expansion as in Theorem A also for functions u_w corresponding to 0-gpf with $w \neq 1$. However it is not clear if these functions play a role in the spectral theory of hyperbolic surfaces. The original aim of this research was exactly to find a characterization for general u_w, and even if we don't achieve this result in this paper, we believe this is an interesting problem to be studied.

Finally we would like to point out that this paper includes one possible extension of the works [16, 18]. Other directions can be found in [8, 7, 19], where the authors study the relation between period functions and Maass wave forms for subgroups of $\mathrm{PSL}(2,\mathbb{Z})$, and in [20, 22], where the role of the "slow" dynamics and its advantage of introducing a transfer operator with finitely many terms is studied in relation to the cohomological approach of [5].

2 Notations for special functions and integral transforms

We use standard notations: ${}_2F_1(a,b;c;x)$ for the hypergeometric function; $J_\nu(z)$ for the Bessel functions of first kind; $K_\nu(z)$ for the modified Bessel functions of the third kind; $L_n^\nu(t)$ for the generalized Laguerre polynomials; $\Gamma(\nu)$ for the Gamma function; $\zeta(q)$ for the Riemann zeta function; P_ν^μ for the Legendre functions in the real interval $(-1,1)$.
In the following we use the following integral transforms:

- *Laplace transform*
$$\mathscr{L}[\varphi](z) := \int_0^\infty e^{-zt}\, \varphi(t)\, dt$$
- *Symmetric Hankel transform*
$$\mathscr{H}_\nu[\varphi](z) := \int_0^\infty J_\nu(tz)\, \sqrt{tz}\, \varphi(t)\, dt$$
- *Asymmetric Hankel transform*
$$\mathscr{J}_\nu[\varphi](z) := \int_0^\infty J_{2\nu}(2\sqrt{tz}) \left(\frac{t}{z}\right)^\nu \varphi(t)\, dt$$
- *Borel generalized transform*
$$\mathscr{B}_\nu[\varphi](z) := \frac{1}{z^{2\nu}} \int_0^\infty e^{-\frac{t}{z}}\, t^{2\nu-1} \varphi(t)\, dt$$
- *Mellin transform*
$$\mathscr{M}[\varphi](\rho) := \int_0^\infty \varphi(t)\, t^{\rho-1}\, dt$$

The asymmetric Hankel transform has been introduced in [15], and the Borel generalized transform in [13]. For the other transforms see [11] and [12]. For the convergence of the Hankel transforms, we recall that the Bessel function $J_\nu(t)$ satisfies the estimates $J_\nu(t) = O(t^\nu)$ as $t \to 0^+$, and $J_\nu(t) = O(t^{-\frac{1}{2}})$ as $t \to \infty$ (see [10, vol. II]).
We also use the notation

$$\chi_\alpha(t) := t^\alpha \quad \text{and} \quad \exp_\alpha(t) := e^{\alpha t}\,, \quad \alpha \in \mathbb{C}$$

and write $q = \xi + i\eta$, with $\xi > 0$ and $\eta \in \mathbb{R}$. Moreover we write $f(z) \doteq g(z)$ for two functions f, g which coincides up to a non-vanishing multiplication constant possibly depending only on q.

3 From gpf to Maass forms on the imaginary axis

To study the set of gpf, we used in [4] the integral transform $\mathscr{B}_q$ on the spaces of functions $L^p(m_q)$ in $\mathbb{R}^+$ with $m_q(dt) = t^{2\xi-1}\, e^{-t}\, dt$. Letting

$$L^p(m_q) := \left\{ \phi : \mathbb{R}^+ \to \mathbb{C} \ : \ \int_0^\infty |\phi(t)|^p\, t^{2\xi-1} e^{-t}\, dt < \infty \right\}$$

with the norm

$$\|\phi\|_p := \left(\int_0^\infty |\phi(t)|^p\, t^{2\xi-1} e^{-t}\, dt \right)^{\frac{1}{p}}\,,$$

it is immediate to check that

$$L^1(m_q) \ni \phi \mapsto \mathscr{B}_q[\phi] \in H(B)$$

and that $\mathscr{B}_q$ is continuous on $L^1(m_q)$ with values on $H(B)$ with the standard topology induced by the family of supremum norms on compact subsets of B. Moreover, since $m_q(0,\infty) = \Gamma(2\xi)$, one has $L^p(m_q) \subset L^1(m_q)$ for all $p \in [1, \infty]$.
We also need to introduce the linear operators M and N_q defined by

$$M(\phi)(t) := e^{-t}\, \phi(t)$$

$$N_q(\phi)(t) := \mathscr{J}_{q-\frac{1}{2}}[\exp_{-1} \phi](t) = \int_0^\infty J_{2q-1}(2\sqrt{st}) \left(\frac{s}{t}\right)^{q-\frac{1}{2}} e^{-s}\, \phi(s)\, ds\,.$$

In [4] it is proved that

$$L^2(m_q) \ni \phi \mapsto N_q[\phi] \in L^2(m_q)\,.$$

The same is clearly true also for M. Moreover in [4, Proposition 2.5], it is proved that the transfer operators of the Farey map $\mathcal{P}_q^\pm$ has a particularly nice behaviour with respect to the Borel generalized transform. In particular for all $\phi \in L^2(m_q)$ it holds

$$\mathcal{P}_q^\pm \Big(\mathscr{B}_q\, [\chi_{-1} + \phi] \Big)(z) = \mathscr{B}_q \Big[(M \pm N_q)\, (\chi_{-1} + \phi) \Big](z)\,. \tag{3.1}$$

Finally, putting together Theorem 2.8 and Corollary 2.10 in [4], we have

Proposition 3.1 ([4]). *If f is a generalized period function associated to a zero q and to the eigenvalue $\lambda = \frac{1}{w}$, then there exist $b \in \mathbb{C}$ and a function $\varphi \in L^2(m_q)$ such that*

$$f(z) = \mathscr{B}_q \left[\frac{b}{\Gamma(2q)}\, \chi_{-1} + \varphi \right](z) \tag{3.2}$$

and

$$(M \pm N_q) \left(\frac{b}{\Gamma(2q)}\, \chi_{-1} + \varphi \right) = \lambda \left(\frac{b}{\Gamma(2q)}\, \chi_{-1} + \varphi \right)\,,$$

where the signs "+" and "-" correspond to the case of even or odd gpf respectively. Moreover, there exists a sequence $\{a_{n,q}\}_{n\geq 0}$ *with* $\limsup |a_{n,q}|^{\frac{1}{n}} \leq 1$ *such that: in the even case, for* $w = 1$

$$\varphi(t) = \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(-1)^n a_{n,q}\, t^n}{\Gamma(n+2q)} + \frac{a_{0,q}}{\Gamma(2q)} \left(\frac{e^{-t}}{1-e^{-t}} - \frac{1}{t} \right) \tag{3.3}$$

with $a_{0,q} = b$*, and for* $w \in \mathbb{C} \setminus [1, \infty)$*,*

$$\varphi(t) = \frac{we^{-t}}{1-we^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q}\, t^n}{\Gamma(n+2q)}\,; \tag{3.4}$$

in the odd case, for all $w \in \mathbb{C} \setminus (1, \infty)$*, the constant* b *in* (3.2) *vanishes and the function* φ *can be written as in* (3.4)*.*
Finally, the invariance under the involution $\mathcal{I}_q$ *defined in* (1.5)*, implies that if* $b = 0$

$$\mathscr{B}_q[\varphi] = \pm \mathscr{L}[\chi_{2q-1}\varphi] \tag{3.5}$$

where again the signs "+" and "-" correspond to the case of even or odd gpf respectively.

3.1 The even case for 0-gpf

In [15] and [16] it is proved that the set of even period functions, that is even 0-gpf with $w = 1$, is in one-to-one correspondence with the set of even Maass cusp forms. This correspondence is proved using the Fourier series expansions of the even cusp forms given by

$$u(x+iy) = y^{\frac{1}{2}} \sum_{n\geq 1} c_{n,q}\, K_{q-\frac{1}{2}}(2\pi n y)\, \cos(2\pi n x) \tag{3.6}$$

where the coefficients $c_{n,q}$ have at most polynomial growth. In particular the correspondence is given in [15] in terms of the Laplace and Hankel transforms as

$$\psi(z) = \mathscr{L}\left[\chi_q \mathscr{H}_{q-\frac{1}{2}}[u(iy)]\right](z)\,. \tag{3.7}$$

Since the gpf $f(z)$ of Proposition 3.1 coincides with $\psi(z)$ up to a multiplication constant, using (3.2) with $b = 0$ and (3.5), we obtain

$$\mathscr{L}\left[\chi_q \mathscr{H}_{q-\frac{1}{2}}[u(iy)]\right](z) \doteq \mathscr{L}[\chi_{2q-1}\varphi](z)$$

from which we obtain an integral correspondence between cusp forms and the eigenfunctions φ of Proposition 3.1, namely

$$\varphi(t) \doteq t^{1-q}\, \mathscr{H}_{q-\frac{1}{2}}[u(iy)](t) \tag{3.8}$$

see [16, equation (2.27)].

Remark 3.2. We have used the notation $\doteq$ here and in the following to denote an equality up to a multiplicative constant between cusp forms and the eigenfunctions φ. However, once this constant has been fixed it remains the same in all the equations where $\doteq$ appears. The known constants have been written explicitly. In particular, if one chooses the right constant so that (3.8) is an equality, then all the other equations where $\doteq$ appears become equalities by using the same constant.

We would like to use the involution property of the Hankel transform to introduce the inverse relation of (3.8). Unfortunately we are outside the standard functional spaces where the involution property is valid, since for example an eigenfunction φ of Proposition 3.1 satisfies $\varphi(t) = O(e^{\varepsilon t})$ as $t \to \infty$ for all $\varepsilon > 0$. Hence we first explicitly construct the Hankel transform of the term $\chi_{q-1}\varphi$.

Definition 3.3. *For any q with $\Re(q) > 0$ and $w \in \mathbb{C} \setminus (1, \infty)$, define the one-parameter family of functions*

$$u_\beta(iy) := \mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1}\, \varphi](y)\,, \qquad \Re(\beta) > 0 \tag{3.9}$$

for functions $\varphi : (0, +\infty) \to \mathbb{C}$ which make the integral converge.

Thanks to the properties of the Bessel function recalled in Section 2, the integral in (3.9) is absolutely convergent if φ is as in Proposition 3.1, that is φ is in $L^2(m_q)$, satisfies $(M + N_q)\varphi = \frac{1}{w}\varphi$ and can be written as in (3.3) with $a_{0,q} = b = 0$ for $w = 1$, and as in (3.4) for $w \in \mathbb{C} \setminus [1, \infty)$. In fact by definition φ satisfies $\varphi(t) = O(1)$ as $t \to 0^+$ and $\varphi(t) = O(e^{\varepsilon t})$ as $t \to \infty$ for all $\varepsilon > 0$.

Theorem 3.4. *For any q with $\Re(q) > 0$ and any $w \in \mathbb{C} \setminus (1, \infty)$, and for φ as in Proposition 3.1 with $b = 0$, the function $u_\beta(iy)$ can be extended for all $y > 0$ as an analytic function of β to a small domain containing the origin. Moreover $u_0(iy)$ satisfies*

$$u_0(iy) = w\left[g(y) + g\left(\frac{1}{y}\right)\right], \qquad \forall\, y > 0 \tag{3.10}$$

where

$$g(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\, \chi_{q-1}\, \varphi](y) = \sum_{n=0}^{\infty} (-1)^n\, a_{n,q}\, \frac{y^{n+q}}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}\, \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right),$$

and $\{a_{n,q}\}$ is given in (3.3) with $a_{0,q} = 0$ for $w = 1$, and in (3.4) for $w \in \mathbb{C} \setminus [1, \infty)$.

Proof. Let us fix $y > 0$. Using the functional equation $(M + N_q)\varphi = \frac{1}{w}\varphi$, we can write

$$u_\beta(iy) = w\, \mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1} M\varphi](y) + w\, \mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1} N_q\varphi](y) \tag{3.11}$$

since the first integral on the right hand side is absolutely convergent. Moreover we can change the order of integration in the second integral, that is

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1} N_q\varphi](y) = \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{ty}\, e^{-\beta t} t^{q-1} \int_0^\infty J_{2q-1}(2\sqrt{st}) \left(\frac{s}{t}\right)^{q-\frac{1}{2}} e^{-s}\varphi(s)\, ds\, dt =$$

$$= \int_0^\infty e^{-s}\, s^{q-1}\sqrt{s}\, \varphi(s) \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{ty}\, J_{2q-1}(2\sqrt{st})\, e^{-\beta t}\, t^{-\frac{1}{2}} dt\, ds$$

since again the two-variable integral is absolutely convergent under the assumption $\Re(\beta) > 0$. Hence, applying [11, vol. II, eq. 8.12.(17), p. 58], we get

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1} N_q\varphi](y) = \int_0^\infty J_{q-\frac{1}{2}}\left(\frac{sy}{y^2+\beta^2}\right)\sqrt{sy}\; e^{-s-\frac{s\beta}{y^2+\beta^2}}\; \frac{s^{q-1}}{\sqrt{y^2+\beta^2}}\, \varphi(s)\, ds\,.$$

The integral on the right hand side is absolutely convergent if

$$\Re\left(1 + \frac{\beta}{y^2+\beta^2}\right) > 0$$

hence the left hand side can be extended as an analytic function of β to a small domain containing the origin. In particular we find that

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1} N_q\varphi]\Big|_{\beta=0}(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\, \chi_{q-1}\, \varphi]\left(\frac{1}{y}\right). \tag{3.12}$$

Coming back to (3.11), the first term

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_{q-1} M\varphi](y) = \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{ty}\, e^{-\beta t} t^{q-1}\, e^{-t}\, \varphi(t)\, dt$$

is absolutely convergent for $\Re(\beta) > -1$, hence again can be extended as an analytic function of β to $\Re(\beta) > -1$, satisfying

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_{q-1} M\varphi]\Big|_{\beta=0}(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\varphi](y)\,. \tag{3.13}$$

Hence, putting together (3.13) and (3.12), we have proved that $u_\beta(iy)$ can be extended, as an analytic function of β, to a small domain containing the origin for all $y > 0$, and

$$u_0(iy) = w\,\mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\varphi](y) + w\,\mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\varphi]\left(\frac{1}{y}\right)\,. \tag{3.14}$$

This establishes (3.10) with $g = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\varphi]$. We now use the power series expansion for φ to obtain the series representations for g.

First we write $g(y) = G(y,\beta)\Big|_{\beta=0}$ where

$$G(y,\beta) := \mathscr{H}_{q-\frac{1}{2}}[\exp_{-(1+\beta)}\ \chi_{q-1}\,\varphi](y)$$

for $y > 0$ and $\Re(\beta) > -1$, the integral on the right hand side being absolutely convergent by the estimates used to justify the convergence in (3.9). Then we use the identity

$$\frac{w\,e^{-t}}{1-w\,e^{-t}} = \frac{1}{1-w\,e^{-t}} - 1$$

in the definition of $G(y,\beta)$ to obtain

$$G(y,\beta) = \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{y}\,e^{-(1+\beta)t}\,\frac{w\,e^{-t}}{1-w\,e^{-t}}\sum_{n=0}^\infty \frac{(-1)^n\,a_{n,q}\,t^{n+q-\frac{1}{2}}}{\Gamma(n+2q)}\ dt =$$

$$= \frac{1}{w}\int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{y}\,\frac{w\,e^{-(1+\beta)t}}{1-w\,e^{-t}}\sum_{n=0}^\infty \frac{(-1)^n\,a_{n,q}\,t^{n+q-\frac{1}{2}}}{\Gamma(n+2q)}\ dt - \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{y}\,e^{-(1+\beta)t}\sum_{n=0}^\infty \frac{(-1)^n\,a_{n,q}\,t^{n+q-\frac{1}{2}}}{\Gamma(n+2q)}\ dt =$$

$$= \frac{1}{w}\,u_\beta(iy) - \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{y}\,e^{-(1+\beta)t}\sum_{n=0}^\infty \frac{(-1)^n\,a_{n,q}\,t^{n+q-\frac{1}{2}}}{\Gamma(n+2q)}\ dt\,.$$

Using [10, vol II, p. 14], it holds

$$|J_{q-\frac{1}{2}}(ty)| \le \frac{\sqrt{\pi}}{|\Gamma(q)|}\left(\frac{1}{2}ty\right)^{\Re(q-\frac{1}{2})}\ , \quad \forall\, t,y > 0$$

hence

$$\sup_{t\in\mathbb{R}^+}\left|J_{q-\frac{1}{2}}(ty)\,e^{-(1+\beta)t}\,t^{n+q-\frac{1}{2}}\right| \le \text{const.}(w,q,y)\sup_{t\in\mathbb{R}^+}\left|e^{-(1+\beta)t}\,t^{n+2q-1}\right|$$

where const.(w,q,y) denotes a constant only depending on w,q and y. Since

$$\sup_{t\in\mathbb{R}^+}\left|e^{-(1+\beta)t}\,t^{n+2q-1}\right| \le e^{-n-2\Re(q)+1}\left|\frac{n+2q-1}{\Re(1+\beta)}\right|^{n+2\Re(q)-1}$$

we find

$$\limsup_{n\to\infty}\left(\sup_{t\in\mathbb{R}^+}\left|\frac{a_{n,q}}{\Gamma(n+2q)}\,e^{-(1+\beta)t}\,t^{n+2q-1}\right|\right)^{\frac{1}{n}} \le \frac{1}{\Re(1+\beta)} < 1\,, \qquad \text{for } \Re(\beta) > 0$$

where we also used that $\limsup_n |a_{n,q}|^{\frac{1}{n}} \le 1$. Hence we can write

$$G(y,\beta) = \frac{1}{w}\,u_\beta(iy) - \sum_{n=0}^\infty \frac{(-1)^n\,a_{n,q}}{\Gamma(n+2q)}\ \mathscr{H}_{q-\frac{1}{2}}\left[\exp_{-(1+\beta)}\ \chi_{n+q-1}\right](y)\,, \qquad \text{for } \Re(\beta) > 0\,.$$

Using now the proved analytic extension for u_β, we can write for the second term on the right hand side

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, a_{n,q}}{\Gamma(n+2q)}\ \mathscr{H}_{q-\frac{1}{2}}\left[\exp_{-1}\ \chi_{n+q-1}\right](y) = \frac{1}{w}\, u(iy) - G(y,0) = g\left(\frac{1}{y}\right). \tag{3.15}$$

Moreover, using [11, vol. II, eq. 8.6.(6), p. 29], we obtain

$$g\left(\frac{1}{y}\right) = \sum_{n=0}^{\infty} (-1)^n\, a_{n,q}\, \frac{y^{\frac{1}{2}}}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}\ \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right)$$

and the proof is complete. □

We have thus proved the validity of the following expansion for $y \in (0,\infty)$

$$u_0(iy) = w \sum_{n=0}^{\infty} (-1)^n\, a_{n,q}\, \frac{y^{\frac{1}{2}}\, \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right) + y^{n+q}\, \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}. \tag{3.16}$$

Moreover, letting $y = \tan\vartheta$ with $\vartheta \in (0, \frac{\pi}{2})$ in (3.16), we get

$$u_0(iy) = w\,(\sin\vartheta\,\cos\vartheta)^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n\, a_{n,q}\left[(\cos\vartheta)^{n+q-\frac{1}{2}}\, \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}(\cos\vartheta) + (\sin\vartheta)^{n+q-\frac{1}{2}}\, \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}(\sin\vartheta)\right], \tag{3.17}$$

and from the integral representation valid for $\xi > 0$ (see [10, vol. I, eq. (27), p. 159])

$$\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}(\cos\vartheta) = \frac{\sqrt{2}\,(\sin\vartheta)^{-q+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(q)} \int_0^{\vartheta} \frac{\cos((n+q)t)}{(\cos t - \cos\vartheta)^{1-q}}\, dt$$

and the similar one for $\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}(\sin\vartheta)$, we see that the convergence in (3.17) is uniform on any compact interval contained in $(0, \frac{\pi}{2})$. Hence the convergence in (3.16) is uniform on any compact interval contained in $(0,\infty)$.

Corollary 3.5. *Letting $w = 1$ and $\{a_{n,q}\}$ as in (3.3) with $a_{0,q} = 0$, the function $u_0(iy)$ in (3.16) is the restriction to the imaginary axis of an even Maass cusp form.*

Proof. It follows from the fundamental theorem of Maass (see [24, Theorem 2, p. 234] and [15, Proposition 2.1]) that even Maass cusp forms are uniquely determined as functions with restriction on the imaginary axis of the form (3.6) for $x = 0$ and coefficients $\{c_{n,q}\}$ which make the series (3.6) satisfy $u(iy) = u(i\frac{1}{y})$.
By definition we have that the function $u_0(iy)$ in (3.10) corresponds to an eigenfunction φ of the operator $M + N_q$ as explained in Proposition 3.1, and by (3.8) and the involution property of the Hankel transform, it admits a Fourier expansion as in (3.6). Moreover by the properties of $u_0(iy)$ found in Theorem 3.4 it also satisfies $u_0(iy) = u_0(i\frac{1}{y})$. Hence the proof is finished. □

Remark 3.6. A consequence of this result is that Maass Theorem can be reformulated by saying that even Maass cusp forms are uniquely determined as functions with restriction on the imaginary axis of the form (3.16) and coefficients $\{a_{n,q}\}$ which satify $\limsup_n |a_{n,q}|^{\frac{1}{n}} \leq 1$ and the identity

$$\sum_{n=1}^{\infty} a_{n,q}\, z^n = \sum_{n=1}^{\infty} a_{n,q}\left((z-1)^n - \frac{(-1)^n}{(z+1)^{2q+n}}\right) \tag{3.18}$$

for all z where the series converge. This identity follows from Proposition 3.1 and [4].
As far as numerical computations are concerned, we also remark that (3.18) has been reduced in [2, 3] to a linear algebra identity for infinite matrices in the case q real. The same can be done for general complex values of q with positive real part (unpublished notes).

3.2 The even case for b-gpf

We now extend Theorem 3.4 to the case of even b-gpf, which do exist only for $w = 1$. We recall that non-cuspidal Maass forms of eigenvalue λ form a one-dimensional subspace which is spanned by the non-holomorphic Eisenstein series defined for $\xi > 1$ as

$$E(z,q) = \zeta(2q)\, y^q \left(1 + \frac{1}{|z|^{2q}}\right) + 2 \sum_{c,d\geq 1} \left(\frac{y}{|cz+d|^2}\right)^q , \qquad z = x+iy\,, \tag{3.19}$$

and extended to $\mathbb{C}$ as a meromorphic function with a simple pole at $q = 1$ with residue the constant function $\frac{\pi}{2}$, by the Fourier series expansions

$$E(x+iy,q) = \zeta(2q)\, y^q + \frac{\pi^{\frac{1}{2}}\, \Gamma(q-\frac{1}{2})}{\Gamma(q)}\, \zeta(2q-1)\, y^{1-q} + y^{\frac{1}{2}} \sum_{n\geq 1} \tilde{c}_{n,q}\, K_{q-\frac{1}{2}}(2\pi n y)\, \cos(2\pi n x) \tag{3.20}$$

where

$$\tilde{c}_{n,q} = \frac{4\pi^q}{\Gamma(q)}\, n^{\frac{1}{2}-q} \sum_{d|n} d^{2q-1}\,.$$

Notice that the extension of $E(x+iy,q)$ to $\mathbb{C}$ has no pole at $q = \frac{1}{2}$ since the contribution from the term $\zeta(2q)$ is cancelled by the contribution of the term containing $\Gamma(q-\frac{1}{2})$. Moreover it is proved in [6] and [16] (see the proof of equation (2.30) and page 243) that the function ψ_q^+ defined in (1.4) for $\xi > 1$, which is an eigenfunction of $\mathcal{P}_q^+$ with eigenvalue $\lambda = 1$, satisfies

$$\psi_q^+(z) = \frac{\zeta(2q)}{2}\left(1 + z^{-2q}\right) + \frac{2^{-q-\frac{1}{2}}}{\Gamma(q+\frac{1}{2})}\, \mathscr{L}\left[\chi_q\, \mathscr{H}_{q-\frac{1}{2}}[\tilde{E}(iy,q)]\right](z) \tag{3.21}$$

where

$$\tilde{E}(iy,q) = 2 \sum_{c,d\geq 1} \left(\frac{y}{c^2y^2+d^2}\right)^q . \tag{3.22}$$

It is shown in [16] that the function $\frac{\Gamma(2q)}{\Gamma(q-1)}\, \psi_q^+$ can be analytically continued to $\mathbb{C}$, we give here a proof of this fact for $\{\xi > 0\}$ using the $\mathscr{B}_q$ transform.

Theorem 3.7. *The equation*

$$\psi_q^+(z) = \mathscr{B}_q \left[\frac{\zeta(2q)}{2}\, \frac{\delta_0(t)}{t^{2q-1}} + \frac{e^{-t}}{1-e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q}\, t^n}{\Gamma(n+2q)}\right](z) \tag{3.23}$$

where[1]

$$\begin{cases} a_{0,q} = \zeta(2q-1) \\ a_{n,q} = (-1)^n\, \frac{\Gamma(n+2q)}{n!\,\Gamma(2q)} \left(\frac{\zeta(2q)}{2} + \frac{1}{n+1} \sum_{i=0}^n \binom{n+1}{i} B_i\, \zeta(2q-1+i)\right), \qquad n \geq 1 \end{cases}$$

defines a meromorphic extension of $\psi_q^+(z)$ to $\{\xi > 0\}$ with simple pole at $q = 1$ and residue the function $\frac{1}{2z}$, which is the density of the invariant measure of the Farey map, up to a multiplicative constant.

Proof. We first show that expression (3.23) coincides with the definition (1.4) of the function ψ_q^+ for $\xi > 1$. Then we show the meromorphic extension of (3.23) to the half-plane $\{\xi > 0\}$.
We first use [4, Remark 2.6] and in particular

$$\mathscr{B}_q \left[\frac{\zeta(2q)}{2}\, \frac{\delta_0(t)}{t^{2q-1}}\right](z) = \frac{\zeta(2q)}{2} z^{-2q} \tag{3.24}$$

[1] Here by δ_0 we denote the Dirac delta function at 0, and we use its definition when it is used as argument of an integral transform.

to obtain the second term on the right hand side of (1.4). The first term is obtained by

$$\frac{\zeta(2q)}{2} = \mathscr{B}_q\left[\frac{\zeta(2q)}{2\,\Gamma(2q)}\right](z) = \mathscr{B}_q\left[\frac{\zeta(2q)}{2\,\Gamma(2q)}\,\frac{e^{-t}}{1-e^{-t}}\,\sum_{n=1}^{\infty}\frac{t^n}{n!}\right](z)\,. \tag{3.25}$$

For the other terms we argue as follows

$$\sum_{m,n\geq 1}\frac{1}{(mz+n)^{2q}} = \frac{1}{z^{2q}}\sum_{m,n\geq 1}\frac{1}{n^{2q}\left(\frac{m}{n}+\frac{1}{z}\right)^{2q}} = \frac{1}{\Gamma(2q)\,z^{2q}}\sum_{m,n\geq 1}\frac{1}{n^{2q}}\,\mathscr{L}\left[t^{2q-1}e^{-\frac{m}{n}\,t}\right]\left(\frac{1}{z}\right) =$$

$$= \frac{1}{\Gamma(2q)}\sum_{m,n\geq 1}\frac{1}{n^{2q}}\,\mathscr{B}_q\left[e^{-\frac{m}{n}\,t}\right](z) = \mathscr{B}_q\left[\frac{1}{\Gamma(2q)}\sum_{m,n\geq 1}\frac{e^{-\frac{m}{n}\,t}}{n^{2q}}\right](z) = \mathscr{B}_q\left[\frac{1}{\Gamma(2q)}\sum_{n\geq 1}\frac{1}{n^{2q}}\,\frac{e^{-\frac{t}{n}}}{1-e^{-\frac{t}{n}}}\right](z)\,.$$

Since $\xi > 1$, we can write

$$\sum_{n\geq 1}\frac{1}{n^{2q}}\,\frac{e^{-\frac{t}{n}}}{1-e^{-\frac{t}{n}}} = \frac{e^{-t}}{1-e^{-t}}\sum_{n\geq 1}\frac{1}{n^{2q}}\,\frac{e^{t}-1}{e^{\frac{t}{n}}-1} = \frac{e^{-t}}{1-e^{-t}}\sum_{n\geq 1}\frac{1}{n^{2q}}\sum_{j=0}^{n-1}(e^{\frac{t}{n}})^j =$$

$$= \frac{e^{-t}}{1-e^{-t}}\left[\sum_{n\geq 1}\frac{1}{n^{2q}} + \sum_{k\geq 0}\left(\sum_{n\geq 2}\sum_{j=1}^{n-1}\frac{j^k}{n^{2q+k}}\right)\frac{t^k}{k!}\right] = \frac{e^{-t}}{1-e^{-t}}\sum_{k\geq 0}A_{k,q}\,\frac{t^k}{k!}$$

with

$$A_{0,q} = \zeta(2q) + \sum_{n\geq 2}\frac{n-1}{n^{2q}} = \zeta(2q-1)$$

and in general

$$A_{k,q} = \sum_{n\geq 2}\frac{S_k(n-1)}{n^{2q+k}}\,,\quad k\geq 1$$

where $S_k(n-1) = \sum_{j=1}^{n-1} j^k$. Notice that $S_k(n-1) \leq n^{k+1}$, thus for $\xi > 1$ the sum defining $A_{k,q}$ is convergent and $|A_{k,q}| \leq \zeta(2\xi-1)$ for all $k\geq 1$. Hence the series $\sum_{k\geq 0} A_{k,q}\,\frac{t^k}{k!}$ converges for $t\in\mathbb{R}$ and

$$\sum_{m,n\geq 1}\frac{1}{(mz+n)^{2q}} = \mathscr{B}_q\left[\frac{1}{\Gamma(2q)}\,\frac{e^{-t}}{1-e^{-t}}\sum_{k\geq 0}A_{k,q}\,\frac{t^k}{k!}\right](z)\,. \tag{3.26}$$

Moreover, we recall that

$$S_k(n) = \frac{1}{k+1}\,n^{k+1} + \frac{1}{2}\,n^k + \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_i\,n^{k+1-i}$$

where B_i are the Bernoulli numbers. Hence for $\xi > 1$ and $k\geq 1$

$$A_{k,q} = \sum_{n\geq 2}\frac{S_k(n)-n^k}{n^{2q+k}} = \sum_{n\geq 2}\frac{S_k(n)-\frac{1}{k+1}\,n^{k+1}-\frac{1}{2}\,n^k}{n^{2q+k}} + \frac{1}{k+1}\Big(\zeta(2q-1)-1\Big) - \frac{1}{2}\Big(\zeta(2q)-1\Big) =$$

$$= \frac{1}{k+1}\Big(\zeta(2q-1)-1\Big) - \frac{1}{2}\Big(\zeta(2q)-1\Big) + \frac{1}{k+1}\sum_{i=2}^{k}\binom{k+1}{i}B_i\Big(\zeta(2q-1+i)-1\Big) =$$

$$= \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i\Big(\zeta(2q-1+i)-1\Big) = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i\,\zeta(2q-1+i)\,,$$

using the identity $\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0$. These expressions for $A_{k,q}$ are holomorphic in $\{\xi > 0\}$ for all $k \geq 0$ except for simple poles at $q = \frac{1}{2}$ and $q = 1$. Moreover, using

$$\left| S_k(n) - \frac{1}{k+1}\, n^{k+1} - \frac{1}{2}\, n^k \right| \leq const\, k^2\, n^{k-1}\,, \qquad k \geq 1 \tag{3.27}$$

which is proved in the Appendix A, we have that

$$\left| \sum_{n\geq 2} \frac{S_k(n) - \frac{1}{k+1}\, n^{k+1} - \frac{1}{2}\, n^k}{n^{2q+k}} \right| \leq const\, k^2 \sum_{n\geq 2} \frac{n^{k-1}}{n^{2\xi+k}} = const\, k^2\, \zeta(2\xi+1)$$

for all q in $\{\xi > 0\}$, hence $|A_{k,q}| = O(k^2)$ for all q in $\{\xi > 0\}$. This implies that (3.26) is valid for $\xi > 0$, and putting together (3.24), (3.25) and (3.26), we get for $\xi > 0$

$$\psi_q^+(z) = \mathscr{B}_q \left[\frac{\zeta(2q)}{2}\, \frac{\delta_0(t)}{t^{2q-1}} + \frac{e^{-t}}{1-e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q}\, t^n}{\Gamma(n+2q)} \right] (z)$$

where

$$\begin{cases} a_{0,q} = \zeta(2q-1) \\ a_{n,q} = (-1)^n\, \frac{\Gamma(n+2q)}{n!\,\Gamma(2q)} \left(\frac{\zeta(2q)}{2} + A_{n,q} \right)\,, \qquad n \geq 1 \end{cases}$$

which are holomorphic except for a simple pole at $q = 1$.
There is also a pole at $q = \frac{1}{2}$ in the coefficient $\frac{\zeta(2q)}{2}$ of the first term in the argument of the $\mathscr{B}_q$ transform. However, when applying the $\mathscr{B}_q$, we obtain that ψ_q^+ can be written as in (1.7) with $c = \frac{\zeta(2q)}{2}$ and $b = \zeta(2q-1)$, so the first two terms are given by

$$\frac{\zeta(2q)}{2}\, \frac{1}{z^{2q}} + \frac{\zeta(2q-1)\,\Gamma(2q-1)}{\Gamma(2q)}\, \frac{1}{z}$$

so that there is no pole at $q = \frac{1}{2}$, as it happens for the Eisenstein series in (3.20).
Finally we can compute the residue for ψ_q^+ at $q = 1$ using (3.23). The only contributing terms are those containing $\zeta(2q-1)$, which has residue $\frac{1}{2}$. Hence $\mathrm{Res}_{q=1}(a_{n,q}) = \frac{(-1)^n}{2}$ and

$$\mathrm{Res}_{q=1} \left[\frac{\zeta(2q)}{2}\, \frac{\delta_0(t)}{t^{2q-1}} + \frac{e^{-t}}{1-e^{-t}} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,q}\, t^n}{\Gamma(n+2q)} \right] = \frac{e^{-t}}{1-e^{-t}} \sum_{n=0}^{\infty} \frac{t^n}{2\,\Gamma(n+2)}$$

which gives

$$\mathrm{Res}_{q=1}(\psi_q^+) = \mathscr{B}_1 \left[\frac{e^{-t}}{1-e^{-t}} \sum_{n=0}^{\infty} \frac{t^n}{2\,\Gamma(n+2)} \right] (z) = \mathscr{B}_1 \left[\frac{1}{2t} \right] (z) = \frac{1}{2z}\,.$$

This concludes the proof. □

By Theorem C-(a), the function ψ_q^+ satisfies the equation $\mathcal{I}_q \psi_q^+ = \psi_q^+$, as is easily verified using the definition (1.4). Then, using (3.26) and (3.21) it follows that the function

$$\tilde{\varphi}(t) := \frac{1}{\Gamma(2q)}\, \frac{e^{-t}}{1-e^{-t}} \sum_{k\geq 0} A_{k,q}\, \frac{t^k}{k!} \tag{3.28}$$

satisfies

$$\mathscr{B}_q[\tilde{\varphi}] = \mathscr{L}[\chi_{2q-1}\, \tilde{\varphi}] = \frac{2^{-q-\frac{1}{2}}}{\Gamma(q+\frac{1}{2})}\, \mathscr{L} \left[\chi_q\, \mathscr{H}_{q-\frac{1}{2}}[\tilde{E}(iy,q)] \right]\,,$$

from which we get the analogue of (3.8)

$$\tilde{\varphi}(t) = \frac{2^{-q-\frac{1}{2}}}{\Gamma(q+\frac{1}{2})}\, t^{1-q}\, \mathscr{H}_{q-\frac{1}{2}}[\tilde{E}(iy,q)](t)\,, \tag{3.29}$$

for $\tilde{E}(iy,q)$ defined in (3.22). From this we get an analytic continuation of $E(iy,q)$ different from the Fourier series expansion (3.20).

Theorem 3.8. *The function $U(iy)$ defined by*

$$\begin{aligned} U(iy) :=& \zeta(2q)\left(y^q + y^{-q}\right) - 2\,\zeta(2q)\left(\frac{y}{1+y^2}\right)^q + \\ &+ 2^{q+\frac{1}{2}}\,\Gamma\left(q+\frac{1}{2}\right)\sum_{n=0}^{\infty}(-1)^n\, b_{n,q}\, \frac{y^{\frac{1}{2}}\,\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right) + y^{n+q}\,\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}} \end{aligned} \tag{3.30}$$

with

$$b_{n,q} = (-1)^n\,\frac{\Gamma(n+2q)}{(n+1)!\,\Gamma(2q)}\sum_{i=0}^{n}\binom{n+1}{i}\, B_i\,\zeta(2q-1+i)\,,$$

gives an analytic continuation of the Eisenstein series $E(iy,q)$ in (3.19) to $q\in\mathbb{C}$ with a simple pole at $q=1$ with residue the constant function $\frac{\pi}{2}$.

Proof. Writing the Eisenstein series $E(iy,q)$ as in (3.19)

$$E(iy,q) = \zeta(2q)\left(y^q + y^{-q}\right) + \tilde{E}(iy,q)\,,$$

we proceed as in Theorem 3.4 to invert the relation (3.29).
The proof follows the same lines as that of Theorem 3.4 with some modifications. The first is that the function $\tilde{\varphi}$ satisfies the functional equation

$$((M+N_q)\tilde{\varphi})(t) = \tilde{\varphi}(t) - \frac{\zeta(2q)}{\Gamma(2q)}\, e^{-t}\,. \tag{3.31}$$

This follows by applying $\mathcal{P}^+_q$ to $\psi^+_q(z) = \frac{\zeta(2q)}{2}\,(1+z^{-2q}) + \mathscr{B}_q[\tilde{\varphi}](z)$. Indeed $\tilde{\varphi}$ is of the right form to apply (3.1), hence

$$\begin{aligned} \psi^+_q(z) &= (\mathcal{P}^+_q\psi^+_q)(z) = \mathcal{P}^+_q\left(\frac{\zeta(2q)}{2}\,(1+z^{-2q})\right) + (\mathcal{P}^+_q\mathscr{B}_q[\tilde{\varphi}])(z) = \\ &= \frac{\zeta(2q)}{2}\,(1+z^{-2q}) + \zeta(2q)\,(1+z)^{-2q} + \mathscr{B}_q[(M+N_q)\tilde{\varphi})](z)\,. \end{aligned}$$

Using

$$(1+z)^{-2q} = \frac{1}{\Gamma(2q)}\, z^{-2q}\int_0^\infty e^{-t(1+\frac{1}{z})}\, t^{2q-1}\, dt = \frac{1}{\Gamma(2q)}\,\mathscr{B}_q[e^{-t}](z)$$

we obtain (3.31).
Letting now

$$\tilde{U}_\beta(iy) := 2^{q+\frac{1}{2}}\,\Gamma\left(q+\frac{1}{2}\right)\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\,\chi_{q-1}\,\tilde{\varphi}](y)\,, \qquad \Re(\beta)>0$$

we get from (3.31)

$$\tilde{U}_\beta(iy) = 2^{q+\frac{1}{2}}\,\Gamma\left(q+\frac{1}{2}\right)\left(\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\,\chi_{q-1}\,(M+N_q)\tilde{\varphi}](y) + \frac{\zeta(2q)}{\Gamma(2q)}\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta-1}\,\chi_{q-1}](y)\right).$$

For the first term on the right hand side, for $\xi > \frac{1}{2}$ we can repeat the arguments of the proof of Theorem 3.4 leading to (3.14), to get

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_{q-1}\,(M+N_q)\tilde{\varphi}]\Big|_{\beta=0}(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\tilde{\varphi}](y) + \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\tilde{\varphi}]\left(\frac{1}{y}\right),$$

whereas the second term is absolutely convergent for $\beta = 0$, thus we simply have

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta-1}\ \chi_{q-1}]\Big|_{\beta=0}(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}](y)\,.$$

Hence we obtain the continuation of $\tilde{U}_\beta$ to a neighborhood of $\beta = 0$, and define $\tilde{U}(iy) := \tilde{U}_0(iy)$ by

$$\tilde{U}(iy) = 2^{q+\frac{1}{2}}\,\Gamma\left(q+\frac{1}{2}\right)\left(\mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\tilde{\varphi}](y) + \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}\,\tilde{\varphi}]\left(\frac{1}{y}\right) + \frac{\zeta(2q)}{\Gamma(2q)}\mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}](y)\right). \tag{3.32}$$

To finish the proof, we use (3.28) to write $\tilde{\varphi}$ as

$$\tilde{\varphi}(t) = \frac{e^{-t}}{1-e^{-t}}\sum_{n=0}^{\infty}\frac{(-1)^n b_{n,q}\,t^n}{\Gamma(n+2q)}$$

with $b_{n,q} = (-1)^n\,\frac{\Gamma(n+2q)}{n!\,\Gamma(2q)}\,A_{n,q}$, hence

$$\begin{cases} b_{0,q} = \zeta(2q-1) \\ b_{n,q} = (-1)^n\,\frac{\Gamma(n+2q)}{(n+1)!\,\Gamma(2q)}\,\sum_{i=0}^{n}\binom{n+1}{i}\,B_i\,\zeta(2q-1+i)\,, \quad n \geq 1 \end{cases}$$

Then we define as in the proof of Theorem 3.4 the function $\tilde{g}(y) = \tilde{G}(y,\beta)\Big|_{\beta=0}$ with

$$\tilde{G}(y,\beta) := \mathscr{H}_{q-\frac{1}{2}}[\exp_{-(1+\beta)}\ \chi_{q-1}\,\tilde{\varphi}](y)$$

and repeat the same argument used in the proof of Theorem 3.4 to prove (3.15), to show that

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,b_{n,q}}{\Gamma(n+2q)}\ \mathscr{H}_{q-\frac{1}{2}}\left[\exp_{-1}\ \chi_{n+q-1}\right](y) = \frac{2^{-q-\frac{1}{2}}}{\Gamma(q+\frac{1}{2})}\,\tilde{U}(iy) - \tilde{G}(y,0) = \tilde{g}\left(\frac{1}{y}\right) + R(y)\,,$$

with $R(y) := \frac{\zeta(2q)}{\Gamma(2q)}\mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\ \chi_{q-1}](y)$. The above equation can be used to obtain an expression for $\tilde{g}\left(\frac{1}{y}\right)$ and the analogous for $\tilde{g}(y)$, that when substituted in (3.32) finally give

$$\frac{2^{-q-\frac{1}{2}}}{\Gamma(q+\frac{1}{2})}\,\tilde{U}(iy) = \sum_{n=0}^{\infty}\frac{(-1)^n\,b_{n,q}}{\Gamma(n+2q)}\ \mathscr{H}_{q-\frac{1}{2}}\left[\exp_{-1}\ \chi_{n+q-1}\right](y) + \sum_{n=0}^{\infty}\frac{(-1)^n\,b_{n,q}}{\Gamma(n+2q)}\ \mathscr{H}_{q-\frac{1}{2}}\left[\exp_{-1}\ \chi_{n+q-1}\right]\left(\frac{1}{y}\right) - R\left(\frac{1}{y}\right).$$

The last step of the proof consists of the calculations of the Hankel transforms. The first one is the same as in Theorem 3.4, that is

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,b_{n,q}}{\Gamma(n+2q)}\ \mathscr{H}_{q-\frac{1}{2}}\left[\exp_{-1}\ \chi_{n+q-1}\right](y) = \sum_{n=0}^{\infty}(-1)^n\,b_{n,q}\,\frac{y^{\frac{1}{2}}}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}\,\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right),$$

and the second one is

$$R(y) = \frac{\zeta(2q)}{\Gamma(2q)}\,\frac{2^{q-\frac{1}{2}}\,\Gamma(q)}{\pi^{\frac{1}{2}}}\left(\frac{y}{1+y^2}\right)^q = \zeta(2q)\,\frac{2^{-q+\frac{1}{2}}}{\Gamma(q+\frac{1}{2})}\ \left(\frac{y}{1+y^2}\right)^q$$

where we have used [11, vol. II, eq. 8.6.(5), p. 29] and $\Gamma(2q) = \pi^{-\frac{1}{2}}\, 2^{2q-1}\, \Gamma(q)\, \Gamma(q+\frac{1}{2})$.
In the proof of Theorem 3.7 we proved that $|A_{n,q}| = O(n^2)$, hence arguing as in (3.17), we obtain that the expansion (3.30) is well defined for all $q \in \mathbb{C}$, except $q = \frac{1}{2}$ and $q = 1$, and for all $y > 0$. Moreover it is uniformly convergent in y on any compact interval contained in $(0, \infty)$.
We now first show that the expression (3.30) has no pole at $q = \frac{1}{2}$. It is enough to show that the term multiplying $\zeta(2q)$ vanishes at $q = \frac{1}{2}$, indeed

$$\lim_{q\to\frac{1}{2}} (2q-1)\, U(iy) = y^{\frac{1}{2}} + y^{-\frac{1}{2}} - 2\left(\frac{y}{1+y^2}\right)^{\frac{1}{2}} - \sum_{n=1}^{\infty} \frac{y^{\frac{1}{2}}\, P_n\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right) + y^{n+\frac{1}{2}}\, P_n\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{1}{2}}}$$

where P_n are the Legendre polynomials. Using equation (see [21, eq. 18.12.11, p. 449])

$$\sum_{n=0}^{\infty} P_n(\alpha)\, \beta^n = (1 - 2\alpha\beta + \beta^2)^{-\frac{1}{2}}$$

for $\alpha \in (0, 1)$ and $|\beta| < 1$, we obtain

$$\sum_{n=1}^{\infty} \frac{y^{\frac{1}{2}}\, P_n\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right) + y^{n+\frac{1}{2}}\, P_n\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{1}{2}}} = \left(\frac{y}{1+y^2}\right)^{\frac{1}{2}} \left(\frac{(1+y^2)^{\frac{1}{2}}}{y} + (1+y^2)^{\frac{1}{2}} - 2\right),$$

hence

$$\lim_{q\to\frac{1}{2}} (2q-1)\, U(iy) = 0\,.$$

At $q = 1$, the expression (3.30) has instead a pole with a residue that can be computed using $\mathrm{Res}_{q=1}(b_{n,q}) = \frac{(-1)^n}{2}$. Letting $y = \tan\vartheta$ as above we find

$$\mathrm{Res}_{q=1}(U)(iy) = 2^{\frac{1}{2}}\, \Gamma\left(\frac{3}{2}\right) (\sin\vartheta\, \cos\vartheta)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left[(\cos\vartheta)^{n+\frac{1}{2}}\, \mathrm{P}^{-\frac{1}{2}}_{n+\frac{1}{2}}(\cos\vartheta) + (\sin\vartheta)^{n+\frac{1}{2}}\, \mathrm{P}^{-\frac{1}{2}}_{n+\frac{1}{2}}(\sin\vartheta)\right]$$

Using [21, eq. 14.5.12, p. 359] we get

$$(\sin\vartheta)^{\frac{1}{2}} (\cos\vartheta)^{n+1}\, \mathrm{P}^{-\frac{1}{2}}_{n+\frac{1}{2}}(\cos\vartheta) = \frac{1}{n+1} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (\cos\vartheta)^{n+1} \sin((n+1)\vartheta)$$

Hence

$$\sum_{n=0}^{\infty} (\sin\vartheta)^{\frac{1}{2}} (\cos\vartheta)^{n+1}\, \mathrm{P}^{-\frac{1}{2}}_{n+\frac{1}{2}}(\cos\vartheta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Im\left(\sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{1+\exp(2i\vartheta)}{2}\right)^{n+1}\right) =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Im\left(-\log\left(\frac{1-\exp(2i\vartheta)}{2}\right)\right) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \arctan\left(-\frac{\sin(2\vartheta)}{1-\cos(2\vartheta)}\right) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \arctan\frac{1}{y}$$

recalling that $y = \tan\vartheta$. Hence finally

$$\mathrm{Res}_{q=1}(U)(iy) = 2^{\frac{1}{2}}\, \Gamma\left(\frac{3}{2}\right) \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\arctan y + \arctan\frac{1}{y}\right) = \frac{\pi}{2}$$

recalling $\Gamma(\frac{3}{2}) = \frac{1}{2}\, \pi^{\frac{1}{2}}$. $\square$

3.3 The odd case for 0-gpf

Here we are going to repeat the approach of Section 3.1 for odd period functions, which in the case $w = 1$ are in one-to-one correspondence with the set of odd Maass cusp forms, as shown in [16]. Also in this case it is fundamental to use the Fourier series expansion for the odd cusp forms given by

$$u(x+iy) = y^{\frac{1}{2}} \sum_{n\geq 1} c_{n,q}\, K_{q-\frac{1}{2}}(2\pi n y)\, \sin(2\pi n x)\,, \tag{3.33}$$

with $c_{n,q}$ having at most a polynomial growth. The integral correspondence between even cusp forms and even period functions proved in [15] is extended to the odd case in [16, Section II.1]. We can formally proceed as for (3.7) by applying [15, Proposition 4.3] to get[2]

$$\psi(z) = -\frac{1}{z}\, \mathscr{L}\left[\chi_{q-1} \mathscr{H}_{q-\frac{3}{2}}[y\, u_x(iy)]\right](z)\,,$$

where $u_x = \frac{\partial}{\partial x} u$, and find by (3.5) that

$$\frac{1}{z}\, \mathscr{L}\left[\chi_{q-1} \mathscr{H}_{q-\frac{3}{2}}[y\, u_x(iy)]\right](z) \doteq \mathscr{L}\left[\chi_{2q-1}\varphi\right](z)$$

for a function $\varphi \in L^2(m_q)$ satisfying $(M - N_q)\varphi = \varphi$, with expansion as in (3.3) with $a_{0,q} = 0$. From this we obtain the analogous of (3.8) and [16, equation (2.27)] for odd Maass cusp forms. By [11, vol. I, eq. 4.1.(9), p. 130, and vol. II, eq. 8.1.(6), p. 5] we finally have[3]

$$\varphi(t) \doteq t^{1-2q} \int_0^t \tau^{q-1}\, \mathscr{H}_{q-\frac{3}{2}}[y\, u_x(iy)](\tau)\, d\tau = t^{-q}\, \mathscr{H}_{q-\frac{1}{2}}[u_x(iy)](t)\,. \tag{3.34}$$

Notice that the Hankel transform in (3.34) is absolutely convergent for $\xi > 0$ thanks to the rapid decay properties of Maass forms.
We now make use of the involution property of the Hankel transform as in Section 3.1 and repeat the proof of Proposition 3.4. We first define the "modified" inverse of (3.34)

Definition 3.9. *For any q with $\Re(q) > 0$ and $w \in \mathbb{C} \setminus (1, \infty)$, define the one-parameter family of functions*

$$v_\beta(iy) := \mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\, \chi_q\, \varphi](y)\,, \qquad \Re(\beta) > 0 \tag{3.35}$$

for functions $\varphi : (0, +\infty) \to \mathbb{C}$ which make the integral converge.

Then we show that if φ is an eigenfunction of $(M - N_q)$ then we can put $\beta = 0$ in (3.35).

Theorem 3.10. *For any q with $\Re(q) > 0$, any $w \in \mathbb{C}\setminus(1, \infty)$, and any φ as in Proposition 3.1 with $a_{0,q} = 0$, the function $v_\beta(iy)$ can be extended for all $y > 0$ as an analytic function of β to a small domain containing the origin. Moreover $v_0(iy)$ satisfies*

$$v_0(iy) = w\left[g(y) - \frac{1}{y^2}\, g\left(\frac{1}{y}\right)\right]\,, \qquad \forall\, y > 0 \tag{3.36}$$

where

$$g(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\, \chi_q\, \varphi](y) = \sum_{n=1}^{\infty} (-1)^n\, (n + 2q - 1)\, a_{n-1,q}\, \frac{y^{n+q-2}}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}\, \mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)\,,$$

and $\{a_{n,q}\}$ is given in (3.4) with $a_{0,q} = 0$.

[2] Proposition 4.3 in [15] can be applied only for $\xi > \frac{3}{2}$.
[3] See Remark 3.2.

Proof. Let us fix $y > 0$. Using the functional equation $(M - N_q)\varphi = \frac{1}{w}\varphi$, we can write

$$v_\beta(iy) = w\,\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_q M\varphi](y) - w\,\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_q N_q\varphi](y) \tag{3.37}$$

since the first integral on the right hand side is absolutely convergent. Moreover we can change the order of integration in the second integral, that is

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_q N_q\varphi](y) = \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{ty}\,e^{-\beta t}t^q \int_0^\infty J_{2q-1}(2\sqrt{st})\left(\frac{s}{t}\right)^{q-\frac{1}{2}} e^{-s}\varphi(s)\,ds\ dt =$$

$$= \int_0^\infty e^{-s}\,s^{q-1}\sqrt{s}\,\varphi(s) \int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{ty}\,J_{2q-1}(2\sqrt{st})\,e^{-\beta t}\,t^{\frac{1}{2}}dt\ ds$$

since again the two-variable integral is absolutely convergent. Now we use [11, vol. I, eq. 4.1.(6), p. 129, and eq. 4.14.(38), p. 186] to write

$$\int_0^\infty J_{q-\frac{1}{2}}(ty)\sqrt{ty}\,J_{2q-1}(2\sqrt{st})\,e^{-\beta t}\,t^{\frac{1}{2}}dt = -\sqrt{y}\,\frac{d}{d\beta}\int_0^\infty J_{q-\frac{1}{2}}(ty)\,J_{2q-1}(2\sqrt{st})\,e^{-\beta t}\,dt =$$

$$= -\sqrt{y}\,\frac{d}{d\beta}\left[e^{-\frac{s\beta}{y^2+\beta^2}}\,(y^2+\beta^2)^{-\frac{1}{2}}\,J_{q-\frac{1}{2}}\left(\frac{sy}{y^2+\beta^2}\right)\right],$$

and consequently

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_q N_q\varphi](y) = -\int_0^\infty e^{-s}\,s^{q-1}\sqrt{s}\,\varphi(s)\,\sqrt{y}\,\frac{d}{d\beta}\left[e^{-\frac{s\beta}{y^2+\beta^2}}\,(y^2+\beta^2)^{-\frac{1}{2}}\,J_{q-\frac{1}{2}}\left(\frac{sy}{y^2+\beta^2}\right)\right]ds\,.$$

Computing all the terms in the previous derivative, we see as in the proof of Theorem 3.4 that all the addends of the integral are absolutely convergent for

$$\Re\Big(1 + \frac{\beta}{y^2+\beta^2}\Big) > 0\,.$$

Hence we can again set $\beta = 0$ and it turns out that there is only one non-vanishing term, so

$$\mathscr{H}_{q-\frac{1}{2}}[\exp_{-\beta}\ \chi_q N_q\varphi]\Big|_{\beta=0}(y) = \frac{1}{y^2}\,\mathscr{H}_{q-\frac{1}{2}}[\chi_q M\varphi]\left(\frac{1}{y}\right).$$

So we argue as in Theorem 3.4 and from (3.37) we get (3.36) with

$$g(y) = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\chi_q\varphi](y)\,.$$

The proof is finished as in Theorem 3.4 since we can write $g = \mathscr{H}_{q-\frac{1}{2}}[\exp_{-1}\chi_{q-1}\tilde\varphi]$ with $\tilde\varphi(t) = t\varphi(t)$. Hence

$$\tilde\varphi(t) = \frac{we^{-t}}{1-we^{-t}}\sum_{n=1}^\infty \frac{(-1)^{n-1}a_{n-1,q}\,t^n}{\Gamma(n+2q-1)}$$

and at the end we get

$$-\frac{1}{y^2}\,g\left(\frac{1}{y}\right) = \sum_{n=1}^\infty (-1)^{n-1}\,(n+2q-1)\,a_{n-1,q}\,\frac{y^{\frac{1}{2}}}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}\,\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}\left(\frac{1}{(1+y^2)^{\frac{1}{2}}}\right).$$

This finishes the proof. □

As in the even case, one can show that the expansion for $v_0(iy)$ obtained in Theorem 3.10 is uniformly convergent on any compact interval of $(0,\infty)$. Moreover we have the following

Corollary 3.11. *Letting $w = 1$, the function $v_0(iy)$ in* (3.36) *is the restriction to the imaginary axis of the x-derivative of an odd Maass cusp form.*

Proof. It follows from the fundamental theorem of Maass (see [24, Theorem 2 and Exercise 6, p. 234]) that odd Maass cusp forms are uniquely determined by their restriction on the imaginary axis, and correspond to coefficients $\{c_{n,q}\}$ which make the series (3.33) satisfy $u_x(iy) = -y^2\, u_x(i\frac{1}{y})$.
By definition we have that the function $v_0(iy)$ in (3.36) satisfies $v_0(iy) = -y^2\, v_0(i\frac{1}{y})$. Then the proof is finished by using (4.8) and [16, Chap. II, Section 3], to show that (3.34) is a bijection between odd Maass cusp forms and the eigenfunctions of $M - N_q$ as in Proposition 3.1. □

4 From gpf to Fourier coefficients of Maass forms

We now use equations (3.8), (3.29) and (3.34), to obtain relations between the coefficients of the power series expansions of the eigenfunctions φ introduced in Proposition 3.1, and the Fourier coefficients of the Maass forms. In the case of the non-holomorphic Eisenstein series, this approach brings interesting results for the divisor function $\sigma_\alpha(n)$. The results are summarized in Theorem B in the Introduction.
The main equality we need is the symmetric Hankel transform of the Bessel functions K_ν, which is given in [11, vol. II, eq. 8.13.(2), p. 63], namely

$$\mathscr{H}_\nu\left[y^{\frac{1}{2}}\, K_\nu(ay)\right](t) = a^{-\nu}\,\frac{t^{\nu+\frac{1}{2}}}{t^2+a^2}\,, \qquad \Re(a) > 0\,,\, \Re(\nu) > -1\,. \tag{4.1}$$

4.1 Maass cusp forms

Let $u(x + iy)$ be an even Maass cusp form, we can then use its Fourier series expansion

$$u(x+iy) = y^{\frac{1}{2}} \sum_{n\geq 1} c_{n,q}\, K_{q-\frac{1}{2}}(2\pi n y)\, \cos(2\pi n x)$$

in

$$\varphi(t) \doteq t^{1-q}\, \mathscr{H}_{q-\frac{1}{2}}[u(iy)](t)\,,$$

from which using term-by-term (4.1) with $a = 2\pi n$ and $\nu = q - \frac{1}{2}$, one obtains (see [16, equation (2.28)])

$$\varphi(t) \doteq \sum_{n\geq 1} n^{\frac{1}{2}-q}\, c_{n,q}\, \frac{t}{t^2+(2\pi n)^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(4\,\pi^2)^{k+1}}\, L_u\Big(q+2k+\frac{3}{2}\Big)\, t^{2k+1} \tag{4.2}$$

where $L_u(\rho)$ is the Dirichlet L-series associated to u, namely

$$L_u(\rho) := \sum_{n\geq 1} c_{n,q}\, n^{-\rho}\,. \tag{4.3}$$

This shows that φ is an odd function, and gives a relation between the coefficients $\{a_{k,q}\}$ in

$$\varphi(t) = \frac{e^{-t}}{1-e^{-t}} \sum_{k=1}^{\infty} \frac{(-1)^k a_{k,q}\, t^k}{\Gamma(k+2q)}$$

and $\{c_{n,q}\}$. Indeed, recalling that

$$\frac{e^{-t}}{1-e^{-t}} = \sum_{i\geq 0} B_i\, \frac{t^{i-1}}{i!}$$

where $\{B_i\}$ are the Bernoulli numbers, we find[4]

$$\sum_{i\geq -1,\, j\geq 1,\, i+j=n} (-1)^j \,\frac{B_{i+1}\, a_{j,q}}{(i+1)!\,\Gamma(j+2q)} \doteq \begin{cases} \frac{(-1)^{\frac{n-1}{2}}}{(2\pi)^{n+1}}\, L_u\Big(q+n+\frac{1}{2}\Big)\,, & \text{if } n \text{ is odd} \\ 0\,, & \text{if } n \text{ is even} \end{cases} \tag{4.4}$$

The equality (4.2) can be further used to find an even more direct expression for the $c_{n,q}$ in terms of the $\{a_{n,q}\}$. Following a suggestion first given in [15] (see also [16]), let us introduce the *interpolating function* g_q, defined as

$$g_q(t) = \sum_{k=1}^{\infty} \frac{a_{k,q} t^k}{\Gamma(k+2q)} \equiv \sum_{k=1}^{\infty} \frac{\beta_{k,q} t^k}{k!} \tag{4.5}$$

where we have set

$$\beta_{k,q} = \frac{k!}{\Gamma(k+2q)}\, a_{k,q}$$

One readily sees that g_q is entire of exponential type and, moreover, we find

$$g_q(-t) = \sum_{k=1}^{\infty} \frac{(-1)^k \beta_{k,q} t^k}{k!} = (e^t-1)\varphi(t)$$

Now, since (the meromorphic continuation of) φ is odd, we have $g_q(t) = (1-e^{-t})\varphi(t)$. We thus see that g_q satisfies the functional equation $g_q(-t) = e^t g_q(t)$. The name of the function g_q comes from the following fact: taking the limit $t \to \pm 2\pi i n$ in (4.5), using the first identity of (4.2) and observing that

$$\lim_{t\to\pm 2\pi i n} \frac{t(e^t-1)}{t^2+(2\pi n)^2} = \lim_{t\to\pm 2\pi i n} \frac{t(e^t-1)}{(t+2\pi i n)(t-2\pi i n)} = \frac{1}{2}$$

we obtain the following formula for the Fourier coefficients of u:

$$c_{n,q} \doteq 2\, n^{q-\frac{1}{2}}\, g_q(\pm 2\pi i n) \quad , \quad n \geq 1 \tag{4.6}$$

In order to better understand the consequences of this formula we have to study the behavior of the entire function g_q on the imaginary axis. For the moment we just put the above formula in a more explicit form, using (4.5) and (4.6),

$$c_{n,q} \doteq 2\, n^{q-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{\beta_{k,q}}{k!} (\pm 2\pi i n)^k \quad , \quad n \geq 1$$

The symmetry with respect to the change of sign yields

$$\sum_{k\geq 0} \frac{\beta_{2k+1,q}}{(2k+1)!}\,(2\pi i n)^{2k+1} = 0 \quad , \quad \forall n \geq 1$$

so that we can finally write

$$c_{n,q} \doteq 2\, n^{q-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k \beta_{2k,q}}{(2k)!} (2\pi n)^{2k} = 2\, n^{q-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k a_{2k,q}}{\Gamma(2k+2q)} (2\pi n)^{2k} \quad , \quad n \geq 1 \tag{4.7}$$

and finish the proof of Theorem B for even Maass cusp forms.
Finally, we remark that the functional equation $g_q(-t) = e^t g_q(t)$ gives

$$\sum_{k=1}^{\infty} (-1)^k \frac{\beta_{k,q}}{k!} t^k = \sum_{k=1}^{\infty} \left(\sum_{\ell=1}^{k} \frac{\beta_{\ell,q}}{\ell!(k-\ell)!} \right) t^k$$

[4] See Remark 3.2.

and therefore

$$\sum_{\ell=1}^{k} \binom{k}{\ell} \beta_{\ell,q} = (-1)^k \beta_{k,q} \quad , \quad k \geq 1$$

which (for k even) is akin to the recursive property of the Bernoulli numbers.
In the odd case, everything works similarly. Using (3.33) and integrating term-by-term we get the analogous of (4.2), namely

$$\begin{aligned} \varphi(t) &\doteq t^{-q} \sum_{n\geq 1} n\, c_{n,q}\, \mathscr{H}_{q-\frac{1}{2}} \Big[\chi_{\frac{1}{2}}(y)\, K_{q-\frac{1}{2}}(2\pi n y) \Big](t) \\ &\doteq \sum_{n\geq 1} n^{-q-\frac{1}{2}}\, c_{n,q} \Big(\frac{1}{(\frac{t}{2\pi n})^2 + 1} - 1 \Big) + L_u\Big(q + \frac{1}{2}\Big) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\pi)^{2k}}\, L_u\Big(q + 2k + \frac{1}{2}\Big)\, t^{2k} \end{aligned} \tag{4.8}$$

where in the second line we have used [11, vol. II, eq. 8.13.(2), p. 63], the definition of the L-series L_u in (4.3) and their analytic extensions. It follows that φ is an even function and the analogous of (4.4) and (4.7) are immediate.

4.2 Non-holomorphic Eisenstein series

Analogous computations can be performed starting from (3.29) and applying (4.1) term-by-term. In this case however the coefficients are known, hence we obtain explicit equalities. Letting

$$\tilde{\varphi}(t) := \frac{1}{\Gamma(2q)}\, \frac{1}{e^t - 1} \sum_{k\geq 0} A_{k,q}\, \frac{t^k}{k!}$$

with

$$A_{k,q} = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i\, \zeta(2q - 1 + i)\,, \qquad k \geq 0\,,$$

where B_i are the Bernoulli numbers, we first rewrite it with

$$\tilde{A}_{k,q} := A_{k,q} - \frac{\zeta(2q-1)}{k+1} + \frac{\zeta(2q)}{2} = \frac{1}{k+1} \sum_{i=2}^{k} \binom{k+1}{i} B_i\, \zeta(2q - 1 + i) \qquad k \geq 2\,,$$

from which we obtain

$$\tilde{\varphi}(t) = \frac{\zeta(2q-1)}{\Gamma(2q)}\, \frac{1}{t} - \frac{\zeta(2q)}{2\,\Gamma(2q)} + \frac{1}{\Gamma(2q)}\, \frac{1}{e^t - 1} \sum_{k\geq 2} \tilde{A}_{k,q}\, \frac{t^k}{k!}\,.$$

Then writing

$$\tilde{E}(iy, q) = \frac{\pi^{\frac{1}{2}}\, \Gamma(q - \frac{1}{2})}{\Gamma(q)}\, \zeta(2q-1)\, y^{1-q} - \zeta(2q)\, y^{-q} + \frac{4\,\pi^q}{\Gamma(q)}\, y^{\frac{1}{2}} \sum_{n\geq 1} n^{\frac{1}{2}-q}\, \sigma_{2q-1}(n)\, K_{q-\frac{1}{2}}(2\pi n y)$$

where $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$ is the divisor function, equality (3.29) gives

$$\tilde{\varphi}(t) = \frac{\zeta(2q-1)}{\Gamma(2q)}\, \frac{1}{t} - \frac{\zeta(2q)}{2\,\Gamma(2q)} + \frac{2}{\Gamma(2q)} \sum_{n\geq 1} n^{1-2q}\, \sigma_{2q-1}(n)\, \frac{t}{t^2 + (2\pi n)^2}\,,$$

where we have used [11, vol. II, eq. 8.5.(7), p. 22], (4.1) and the equality $2^{1-2q}\, \pi^{\frac{1}{2}}\, \Gamma(2q) = \Gamma(q)\, \Gamma(q + \frac{1}{2})$, hence

$$\frac{1}{e^t - 1} \sum_{k\geq 2} \tilde{A}_{k,q}\, \frac{t^k}{k!} = 2 \sum_{n\geq 1} n^{1-2q}\, \sigma_{2q-1}(n)\, \frac{t}{t^2 + (2\pi n)^2} = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(4\,\pi^2)^{k+1}}\, L_\sigma\Big(2q + 2k + 1\Big)\, t^{2k+1} \tag{4.9}$$

where now $L_\sigma(\rho)$ is the Dirichlet L-series

$$L_\sigma(\rho) := \sum_{n\geq 1} \sigma_{2q-1}(n)\, n^{-\rho}\,.$$

Arguing as above, from (4.9) we obtain the analogous of (4.4) and (4.7)

$$\sum_{i\geq -1,\, j\geq 1,\, i+j=n} \frac{B_{i+1}\, \tilde{A}_{j,q}}{(i+1)!\, j!} = \begin{cases} \frac{(-1)^{\frac{n-1}{2}}}{2^n\, \pi^{n+1}}\, L_\sigma\Big(2q+n\Big)\,, & \text{if } n \text{ is odd} \\ 0\,, & \text{if } n \text{ is even} \end{cases} \tag{4.10}$$

and

$$\sigma_{2q-1}(n) = n^{2q-1} \sum_{k=1}^{\infty} \frac{(-1)^k\, \tilde{A}_{2k,q}}{(2k)!}\, (2\pi n)^{2k}\,, \qquad \forall\, n\geq 1\,. \tag{4.11}$$

for all q with $\xi > 0$. The convergence on the right hand side of (4.11) is absolutely since $|\tilde{A}_{k,q}| \leq$ const $k^2\,\zeta(2\xi+1)$, as we prove in Theorem 3.7. This finishes the proof of Theorem B.
From (4.11) we can obtain a new formulation also for the partial sums of $\frac{\sigma_{2q-1}(n)}{n^{2q-1}}$. Using

$$\sum_{n=1}^{N} n^{2k} = \frac{1}{2k+1}(B_{2k+1}(N) - B_{2k+1})$$

where $B_n(x)$ are the Bernoulli polynomials, we get

$$\sum_{n=1}^{N} \frac{\sigma_{2q-1}(n)}{n^{2q-1}} = \sum_{k=1}^{\infty} \frac{(-1)^k\,(2\pi)^{2k}\,\tilde{A}_{2k,q}}{(2k+1)!}\Big(B_{2k+1}(N) - B_{2k+1}\Big)\,.$$

Notice that the function

$$F(x) := \sum_{k=1}^{\infty} \frac{(-1)^k\,(2\pi)^{2k}\,\tilde{A}_{2k,q}}{(2k+1)!}\Big(B_{2k+1}(x) - B_{2k+1}\Big)$$

is uniformly convergent on any compact interval of the real line.

Remark 4.1. Equation (4.10) is equivalent to the formulas $\zeta(2k) = (-1)^{k+1}\,\frac{B_{2k}(2\pi)^{2k}}{2(2k)!}$ and $L_\sigma(2q+2k+1) = \zeta(2q+2k+1)\,\zeta(2k+2)$, with $k\geq 0$. In particular assuming one of the two formulas and (4.10), one obtains the other formula.

Remark 4.2. By using the expansion

$$\tilde{A}_{2k,q} = \sum_{\ell\geq 2} \frac{S_{2k}(\ell) - \frac{1}{2k+1}\,\ell^{2k+1} - \frac{1}{2}\,\ell^{2k}}{\ell^{2q+2k}} - \frac{1}{2k+1} + \frac{1}{2}$$

for $\Re(q) > 1$ one can change the order of summation in (4.11), and use the expression $c_r(n) = \sum_{(i,r)=1\,,\, i\leq r} \cos\big(2\pi \frac{i}{r} n\big)$ for the Ramanujan's sums to write

$$\frac{\sigma_{2q-1}}{n^{2q-1}} = \sum_{k=1}^{\infty} \frac{(-1)^k\,\tilde{A}_{2k,q}}{(2k)!}\,(2\pi n)^{2k} = T_1 + T_2 + T_3 + T_4 + T_5$$

where

$$T_2 = -\sum_{k=1}^{\infty}\left(\frac{1}{2k+1}\sum_{\ell\geq 2}\frac{\ell^{2k+1}}{\ell^{2q+2k}}\right)\frac{(-1)^k}{(2k)!}\,(2\pi n)^{2k} = -\Big(\zeta(2q-1)-1\Big)\,\frac{\sin t - t}{t}\Big|_{t=2\pi n} = \zeta(2q-1) - 1$$

$$T_3 = -\sum_{k=1}^{\infty} \left(\frac{1}{2} \sum_{\ell \geq 2} \frac{\ell^{2k}}{\ell^{2q+2k}} \right) \frac{(-1)^k}{(2k)!} (2\pi n)^{2k} = -\frac{1}{2}\Big(\zeta(2q) - 1\Big) (\cos t - 1)\Big|_{t=2\pi n} = 0$$

$$T_4 = -\sum_{k=1}^{\infty} \frac{1}{2k+1} \frac{(-1)^k}{(2k)!} (2\pi n)^{2k} = -\frac{\sin t - t}{t}\Big|_{t=2\pi n} = 1$$

$$T_5 = \sum_{k=1}^{\infty} \frac{1}{2} \frac{(-1)^k}{(2k)!} (2\pi n)^{2k} = (\cos t - 1)\Big|_{t=2\pi n} = 0$$

and for the first term one gets

$$T_1 = \sum_{k=1}^{\infty} \left(\sum_{\ell \geq 2} \sum_{j=1}^{\ell} \frac{j^{2k}}{\ell^{2q+2k}} \right) \frac{(-1)^k}{(2k)!} (2\pi n)^{2k} = \sum_{\ell \geq 2} \frac{1}{\ell^{2q}} \sum_{j=1}^{\ell} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \left(2\pi \frac{j}{\ell} n \right)^{2k} \right) =$$

$$= \sum_{\ell \geq 2} \frac{1}{\ell^{2q}} \sum_{j=1}^{\ell} \left(\cos \left(2\pi \frac{j}{\ell} n \right) - 1 \right) = -\zeta(2q-1) + 1 + \sum_{\ell \geq 2} \frac{1}{\ell^{2q}} \sum_{j=1}^{\ell} \cos \left(2\pi \frac{j}{\ell} n \right) =$$

$$= -\zeta(2q-1) + \sum_{\ell \geq 1} \frac{1}{\ell^{2q}} \sum_{j=1}^{\ell} \cos \left(2\pi \frac{j}{\ell} n \right) = -\zeta(2q-1) + \sum_{\ell \geq 1} \frac{1}{\ell^{2q}} \sum_{r|\ell} c_r(n) =$$

$$= -\zeta(2q-1) + \zeta(2q) \sum_{\ell \geq 1} \frac{c_\ell(n)}{\ell^{2q}}$$

where in the last equality we have used Dirichlet multiplication to write $\sum_{r|\ell} c_r(n) = (c(n) \star u)(\ell)$, where $u(k) = 1$ for all $k \geq 1$. We have thus obtained Ramanujan's expansion for the divisor function for $\Re(q) > 1$ (see [23]), namely

$$\frac{\sigma_{2q-1}}{n^{2q-1}} = \sum_{k=1}^{\infty} \frac{(-1)^k \tilde{A}_{2k,q}}{(2k)!} (2\pi n)^{2k} = \zeta(2q) \sum_{\ell \geq 1} \frac{c_\ell(n)}{\ell^{2q}} \, ,$$

We remark that Ramanujan expansion is known to hold only for $\Re(q) > \frac{1}{2}$, where the Dirichlet series is absolutely convergent, for $q = \frac{1}{2}$ where the convergence of the Dirichlet series is equivalent to the Prime Number Theorem, and can be extended to $\Re(q) \in \left(\frac{1}{4}, \frac{1}{2}\right]$ assuming Riemann Hypothesis. On the other hand, expansion (4.11) holds for $\Re(q) > 0$.

5 Power series expansions for Maass forms on the imaginary axis

Equations (3.16) and (3.30) provide series expansions for Maass forms in terms of the Legendre functions P^{μ}_{ν}. Moreover, in the case of non-cusp forms we have explicit expressions for the coefficients $b_{n,q}$ of the series. We now use properties of the Legendre functions to obtain different expansions in terms of rational functions. The functions involved are P^{μ}_{ν} with $\nu = n + q - \frac{1}{2}$ and $\mu = -q + \frac{1}{2}$, so that $\mu + \nu = n$ is an integer. Consequently, using [21, eq. 14.3.11, p. 354], we get

$$\mathrm{P}^{-q+\frac{1}{2}}_{n+q-\frac{1}{2}}(t) = \begin{cases} (-1)^{\frac{n}{2}} \frac{2^{-q+\frac{1}{2}}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+1}{2}+q)} (1-t^2)^{\frac{q}{2}-\frac{1}{4}} \, {}_2\mathbf{F}_1\left(-\frac{n}{2}, \, \frac{n}{2}+q; \, \frac{1}{2}; \, t^2\right), & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \frac{2^{-q+\frac{1}{2}}\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+q)} \, t \, (1-t^2)^{\frac{q}{2}-\frac{1}{4}} \, {}_2\mathbf{F}_1\left(-\frac{n-1}{2}, \, \frac{n+1}{2}+q; \, \frac{3}{2}; \, t^2\right), & \text{if } n \text{ is odd} \end{cases}$$

where ${}_2\mathbf{F}_1$ is the scaled hypergeometric function. Moreover, since the first variable of ${}_2\mathbf{F}_1$ is in both case a non-positive integer, then the hypergeometric function is a polynomial in t^2, more precisely for $k \in \mathbb{N}$ and

$c \neq 0, -1, -2, \ldots$, it holds

$$ {}_2\mathbf{F}_1(-k,\, b;\, c;\, t^2) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\Gamma(b+j)}{\Gamma(b)\,\Gamma(c+j)}\, t^{2j}\,. $$

It follows that for the terms in (3.16) and (3.30), we get

$$ \frac{y^{\frac12}\, \mathrm{P}^{-q+\frac12}_{n+q-\frac12}\left(\frac{1}{(1+y^2)^{\frac12}}\right) + y^{n+q}\, \mathrm{P}^{-q+\frac12}_{n+q-\frac12}\left(\frac{y}{(1+y^2)^{\frac12}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac14}} = $$

$$ \begin{cases} (-1)^{\frac{n}{2}}\, \frac{2^{-q+\frac12}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+1}{2}+q)\,\Gamma(\frac{n}{2}+q)} \left(\frac{y}{1+y^2}\right)^q \sum_{j=0}^{n/2} (-1)^j \binom{n/2}{j} \frac{\Gamma(\frac{n}{2}+q+j)}{\Gamma(\frac12+j)}\, \frac{1+y^{n+2j}}{(1+y^2)^{\frac{n}{2}+j}}\,, & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}\, \frac{2^{-q+\frac12}\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2}+q)\,\Gamma(\frac{n}{2}+q)} \left(\frac{y}{1+y^2}\right)^q \sum_{j=0}^{(n-1)/2} (-1)^j \binom{(n-1)/2}{j} \frac{\Gamma(\frac{n+1}{2}+q+j)}{\Gamma(\frac32+j)}\, \frac{1+y^{n+1+2j}}{(1+y^2)^{\frac{n+1}{2}+j}}\,, & \text{if } n \text{ is odd} \end{cases} \tag{5.1} $$

Notice that each term of the finite sums is invariant with respect to the transformation $y \mapsto \frac{1}{y}$.
We now substitute (5.1) into (3.16) and (3.30) and get, with $\alpha_{n,q}$ being equal to $a_{n,q}$ and $b_{n,q}$ respectively,

$$ \sum_{n=0}^{\infty} (-1)^n \alpha_{n,q}\, \frac{y^{\frac12}\, \mathrm{P}^{-q+\frac12}_{n+q-\frac12}\left(\frac{1}{(1+y^2)^{\frac12}}\right) + y^{n+q}\, \mathrm{P}^{-q+\frac12}_{n+q-\frac12}\left(\frac{y}{(1+y^2)^{\frac12}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac14}} = $$

$$ = \sum_{k=0}^{\infty} \alpha_{2k,q}\, (-1)^k\, \frac{2^{-q+\frac12}\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{2k+1}{2}+q)\,\Gamma(k+q)} \left(\frac{y}{1+y^2}\right)^q \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\Gamma(k+q+j)}{\Gamma(\frac12+j)}\, \frac{1+y^{2(k+j)}}{(1+y^2)^{k+j}} + $$

$$ + \sum_{h=0}^{\infty} (-1)\, \alpha_{2h+1,q}\, (-1)^h\, \frac{2^{-q+\frac12}\Gamma(\frac{2h+1}{2}+1)}{\Gamma(h+1+q)\,\Gamma(\frac{2h+1}{2}+q)} \left(\frac{y}{1+y^2}\right)^q \sum_{j=0}^{h} (-1)^j \binom{h}{j} \frac{\Gamma(h+q+j+1)}{\Gamma(\frac32+j)}\, \frac{1+y^{2(h+1+j)}}{(1+y^2)^{h+1+j}} $$

where we have split the sum in one with even indices and one with odd indices. At this point we can group together all the coefficients multiplying terms of the form $\frac{1+y^{2s}}{(1+y^2)^s}$ and get

$$ \sum_{n=0}^{\infty} (-1)^n \alpha_{n,q}\, \frac{y^{\frac12}\, \mathrm{P}^{-q+\frac12}_{n+q-\frac12}\left(\frac{1}{(1+y^2)^{\frac12}}\right) + y^{n+q}\, \mathrm{P}^{-q+\frac12}_{n+q-\frac12}\left(\frac{y}{(1+y^2)^{\frac12}}\right)}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac14}} = 2^{-q+\frac12} \left(\frac{y}{1+y^2}\right)^q \sum_{s=0}^{\infty} (-1)^s\, \eta_{s,q}\, \frac{1+y^{2s}}{(1+y^2)^s} $$

where the coefficients $\eta_{s,q}$ are a given by a finite sum. In particular

$$ \eta_{s,q} := \begin{cases} \sum_{i=0}^{s} \alpha_{s+i,q}\, \gamma_{s+i,q}\, \delta_{s+i,\frac{s}{2}-\lfloor\frac{i+1}{2}\rfloor,q}\,, & s \text{ even} \\ \sum_{i=0}^{s} \alpha_{s+i,q}\, \gamma_{s+i,q}\, \delta_{s+i,\frac{s-1}{2}-\lfloor\frac{i}{2}\rfloor,q}\,, & s \text{ odd} \end{cases} \tag{5.2} $$

with

$$ \gamma_{2k,q} = \frac{\Gamma(\frac{2k+1}{2})}{\Gamma(\frac{2k+1}{2}+q)\,\Gamma(k+q)} \quad \text{and} \quad \gamma_{2k+1,q} = \frac{\Gamma(\frac{2k+1}{2}+1)}{\Gamma(\frac{2k+1}{2}+q)\,\Gamma(k+q+1)} $$

$$ \delta_{2k,j,q} = \binom{k}{j} \frac{\Gamma(k+q+j)}{\Gamma(\frac12+j)} \quad \text{and} \quad \delta_{2k+1,j,q} = \binom{k}{j} \frac{\Gamma(k+q+j+1)}{\Gamma(\frac32+j)} $$

Introducing the notation $\beta_{n,q} := (-1)^n\, \frac{n!\,\Gamma(2q)}{\Gamma(n+2q)}\, \alpha_{n,q}$ in (5.2) we get

$$ \eta_{s,q} = (-1)^s\, 2^s\, \frac{\Gamma(s+q)}{s!\,\Gamma(q+\frac12)\,\Gamma(q)} \sum_{i=0}^{s} (-1)^i\, 2^{-i}\, \beta_{s+i,q} \binom{s}{i}\,. \tag{5.3} $$

We have thus proved

Proposition 5.1. *An even Maass cusp form with eigevalue $q(1-q)$, can be formally written when restricted to the imaginary axis as*

$$u(iy) = 2^{-q+\frac{1}{2}} \left(\frac{y}{1+y^2}\right)^q \sum_{s=0}^{\infty} (-1)^s \, \eta_{s,q} \, \frac{1+y^{2s}}{(1+y^2)^s}$$

with $\eta_{s,q}$ as in (5.3) *and $\beta_{n,q} = (-1)^n \frac{n!\,\Gamma(2q)}{\Gamma(n+2q)} a_{n,q}$, where $\{a_{n,q}\}$ is given in* (3.3).

When writing the same expansion for the non-cusp forms $U(iy)$ as in (3.30), we can use the explicit expression for the coefficient $\{b_{n,q}\}$, which are defined in terms of the $\{A_{n,q}\}$ of Theorem 3.7. We first get

$$U(iy) = \zeta(2q)\left(y^q + y^{-q}\right) + 2\left(\frac{y}{1+y^2}\right)^q \left[-\zeta(2q) + \sum_{s=0}^{\infty} 2^s \frac{\Gamma(s+q)}{s!\,\Gamma(q)} \left(\sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i A_{s+i,q}}{2^i}\right) \frac{1+y^{2s}}{(1+y^2)^s}\right] \tag{5.4}$$

then recalling

$$A_{n,q} = \frac{1}{n+1} \sum_{\ell=0}^{n} \binom{n+1}{\ell} B_\ell \, \zeta(2q-1+\ell)\,,$$

we obtain for $s \geq 1$

$$\sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i A_{s+i,q}}{2^i} = \sum_{i=0}^{s} \sum_{\ell=0}^{s+i} B_\ell \, \zeta(2q-1+\ell) \binom{s+i+1}{\ell} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)} =$$

$$= \sum_{\ell=0}^{s} \left(\sum_{i=0}^{s} \binom{s+i+1}{\ell} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)}\right) B_\ell \, \zeta(2q-1+\ell) +$$

$$+ \sum_{\ell=s+1}^{2s} \left(\sum_{i=\ell-s}^{s} \binom{s+i+1}{\ell} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)}\right) B_\ell \, \zeta(2q-1+\ell)$$

Moreover for $\ell \geq 2$

$$\sum_{i=0}^{s} \binom{s+i+1}{\ell} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)} = \frac{1}{\ell} \binom{s}{\ell-1} \sum_{i=0}^{s} \binom{s}{i} \frac{\Gamma(s+i+1)\,\Gamma(s+2-\ell)}{\Gamma(s+1)\,\Gamma(s+i+2-\ell)} \left(-\frac{1}{2}\right)^i =$$

$$= \frac{1}{\ell} \binom{s}{\ell-1} {}_2F_1\Big(-s, s+1; s+2-\ell; \frac{1}{2}\Big) = \frac{1}{\ell} \binom{s}{\ell-1} \frac{\pi^{\frac{1}{2}}\,\Gamma(s+2-\ell)}{2^{s+1-\ell}\,\Gamma(1-\frac{\ell}{2})\,\Gamma(s+\frac{3-\ell}{2})}$$

where in the last equality we have used [21, eq. 15.4.30, p. 387]. Notice that the last expression vanishes for ℓ even because of the poles of the term $\Gamma(1-\frac{\ell}{2})$, and since $B_\ell = 0$ for $\ell \geq 2$ and odd, we obtain

$$\sum_{\ell=0}^{s} \left(\sum_{i=0}^{s} \binom{s+i+1}{\ell} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)}\right) B_\ell \, \zeta(2q-1+\ell) =$$

$$= B_0 \, \zeta(2q-1) \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)} + B_1 \, \zeta(2q) \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i}{2^i} = \frac{\pi^{\frac{1}{2}}}{2^{s+1}} \frac{\Gamma(s+1)}{\Gamma(s+\frac{3}{2})} \zeta(2q-1) - \frac{1}{2^{s+1}} \zeta(2q)\,.$$

Similarly, letting $k = \ell - s \geq 1$, we have

$$\sum_{i=k}^{s} \binom{s+i+1}{s+k} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)} = \frac{(-1)^k}{2^{k-1}\,(s+k)} \binom{s}{k-1} - \frac{(-1)^k\,\pi^{\frac{1}{2}}\,(s+k+1)}{(s+k)\,(s-k+1)} \binom{s}{k} \frac{\Gamma(k+1)}{\Gamma(\frac{k-s}{2})\,\Gamma(\frac{s+k+3}{2})}$$

whose second term vanishes for $s-k$ even, that is for $\ell = s-k+2k$ even, and since $B_\ell = 0$ for $\ell \geq 2$ and odd, we obtain for $s \geq 1$

$$\sum_{\ell=s+1}^{2s} \left(\sum_{i=\ell-s}^{s} \binom{s+i+1}{\ell} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)} \right) B_\ell\, \zeta(2q-1+\ell) =$$

$$= \sum_{k=1}^{s} \left(\sum_{i=k}^{s} \binom{s+i+1}{s+k} \binom{s}{i} \frac{(-1)^i}{2^i\,(s+i+1)} \right) B_{s+k}\, \zeta(2q-1+s+k) =$$

$$= \sum_{k=1}^{s} \frac{(-1)^k}{2^{k-1}\,(s+k)} \binom{s}{k-1} B_{s+k}\, \zeta(2q-1+s+k)\,.$$

We have thus shown that for $s \geq 1$

$$\begin{aligned} \sum_{i=0}^{s} \binom{s}{i} \frac{(-1)^i A_{s+i,q}}{2^i} =& \frac{\pi^{\frac12}}{2^{s+1}} \frac{\Gamma(s+1)}{\Gamma(s+\frac32)} \zeta(2q-1) - \frac{1}{2^{s+1}} \zeta(2q) + \\ &+ \sum_{k=1}^{s} \frac{(-1)^k}{2^{k-1}\,(s+k)} \binom{s}{k-1} B_{s+k}\, \zeta(2q-1+s+k) \end{aligned} \tag{5.5}$$

Substituting (5.5) in (5.4), and using the identities

$$\sum_{s=0}^{\infty} \frac{\Gamma(s+q)}{s!\,\Gamma(q)} z^s = (1-z)^{-q} \quad \text{and} \quad \sum_{s=0}^{\infty} \frac{\Gamma(s+q)}{\Gamma(s+\frac32)\,\Gamma(q)} z^s = \frac{2}{\pi^{\frac12}}\, {}_2F_1\left(1, q\,; \frac32\,; z\right),$$

we obtain that the Eisenstein series $E(iy,q)$ defined in (3.19) can be formally written as

$$\begin{aligned} E(iy,q) =& 2 \left(\frac{y}{1+y^2}\right)^q \left[\zeta(2q-1)\, {}_2F_1\left(1, q\,; \frac32\,; \frac{1}{1+y^2}\right) + \zeta(2q-1)\, {}_2F_1\left(1, q\,; \frac32\,; \frac{y^2}{1+y^2}\right) \right] + \\ &+ 4 \left(\frac{y}{1+y^2}\right)^q \sum_{s=1}^{\infty} 2^s \frac{\Gamma(s+q)}{s!\,\Gamma(q)} \left(\sum_{k=1}^{s} \binom{s}{k-1} \frac{(-1)^k}{2^k\,(s+k)} B_{s+k}\, \zeta(2q-1+s+k) \right) \frac{1+y^{2s}}{(1+y^2)^s} \end{aligned} \tag{5.6}$$

We point out that in the previous expression there is no dependence on $\zeta(2q)$, hence there is no pole at $q = \frac12$, the only pole being at $q = 1$ due to the term $\zeta(2q-1)$.

Theorem 5.2. *For any $q = \xi + i\eta$ with $\xi > 0$, the series in* (5.6) *is uniformly convergent for y in any compact interval in* $(0, +\infty)$.

Proof. We show that it is enough to prove that for all $q = \xi + i\eta$ with $\xi > 0$ it holds

$$\sum_{k=1}^{s} \binom{s}{k-1} \frac{(-1)^k}{2^k\,(s+k)} B_{s+k}\, \zeta(2q-1+s+k) = O\left(\frac{s^6}{2^s}\right).$$

This estimate is proved in four steps. In the fifth and last step we conclude the proof of the theorem.

Step 1. We first show that for $s \geq 2$ and even

$$2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi\ell)^s} \sum_{\substack{k=1 \\ k \text{ even}}}^{s} \binom{s}{k-1} \frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi\ell)^k} = \frac{s!}{2\,(2\pi)^s\,\ell^{s+\frac12}} J_{s+\frac12}(2\pi\ell)\,, \quad \forall\, \ell \geq 1 \tag{5.7}$$

For k even, we can write $(-1)^{\frac{k}{2}}$ as i^k and let $k' = k-1$, hence

$$\sum_{\substack{k=1\\ k \text{ even}}}^{s} \binom{s}{k-1} \frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi\ell)^k} = \sum_{\substack{k=0\\ k \text{ odd}}}^{s-1} \binom{s}{k} (s+k)! \left(\frac{i}{4\pi\ell}\right)^{k+1} =$$

$$= \frac{1}{2}\,\frac{i}{4\pi\ell}\Big(\sum_{k=0}^{s}\binom{s}{k}(s+k)!\left(\frac{i}{4\pi\ell}\right)^k - \sum_{k=0}^{s}\binom{s}{k}(s+k)!\,(-1)^k\,\left(\frac{i}{4\pi\ell}\right)^k\Big) =$$

$$= \frac{1}{2}\,\frac{i}{4\pi\ell}\Big(\sum_{k=0}^{s}\binom{s}{k}(s+k)!\left(\frac{i}{4\pi\ell}\right)^k - \sum_{k=0}^{s}\binom{s}{k}(s+k)!\,\overline{\left(\frac{i}{4\pi\ell}\right)^k}\Big) =$$

$$= -\frac{1}{4\pi\ell}\,\Im\Big(\sum_{k=0}^{s}\binom{s}{k}(s+k)!\left(\frac{i}{4\pi\ell}\right)^k\Big)$$

We now recall that using [21, eq. 10.39.6, p. 255 and eq. 13.2.8, p. 322], one can write

$$\sum_{k=0}^{s}\binom{s}{k}(s+k)!\,z^k = \frac{s!\,e^{\frac{1}{2z}}}{\sqrt{\pi z}}\,K_{s+\frac{1}{2}}\left(\frac{1}{2z}\right)$$

hence

$$2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi\ell)^s}\sum_{\substack{k=1\\ k \text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi\ell)^k} = \frac{2\,(-1)^{\frac{s}{2}}\,s!}{(2\pi)^{s+1}\,\ell^{s+\frac{1}{2}}}\,\Im\Big(e^{-i\,\frac{\pi}{4}}\,K_{s+\frac{1}{2}}(-2\pi i\ell)\Big) \tag{5.8}$$

Finally, we use [21, eq. 10.27.8, p. 251], relating the modified Bessel function K_ν to the Hankel function $H^{(1)}_\nu$, to write

$$e^{-i\,\frac{\pi}{4}}\,K_{s+\frac{1}{2}}(-2\pi i\ell) = \frac{i\,(-1)^{\frac{s}{2}}\,\pi}{2}\,H^{(1)}_{s+\frac{1}{2}}(2\pi\ell)\,,$$

and the relation

$$H^{(1)}_\nu(z) = J_\nu(z) + i\,Y_\nu(z)\,,$$

where the Bessel functions of first and second kind J_ν and Y_ν are real for real orders ν and positive real argument z, to obtain

$$\Im\Big(e^{-i\,\frac{\pi}{4}}\,K_{s+\frac{1}{2}}(-2\pi i)\Big) = \frac{(-1)^{\frac{s}{2}}\,\pi}{2}\,J_{s+\frac{1}{2}}(2\pi\ell)\,.$$

Using it in (5.8), we obtain (5.7).

Step 2. Following the same ideas as in Step 1, we show that for $s \geq 1$ and odd

$$2\,\frac{(-1)^{\frac{s+1}{2}-1}}{(2\pi\ell)^s}\sum_{\substack{k=1\\ k \text{ odd}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k+1}{2}}(s+k-1)!}{(4\pi\ell)^k} = \frac{s!}{2\,(2\pi)^s\,\ell^{s+\frac{1}{2}}}\,J_{s+\frac{1}{2}}(2\pi\ell)\,,\quad \forall\,\ell\geq 1 \tag{5.9}$$

Step 3. For all $s \geq 1$ we have

$$\Big|\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{(2m)^k\,(s+k)}\,B_{s+k}\Big| \leq \frac{4\,s^2\,m^s}{2^s}\,,\quad \forall\,m\geq 1 \tag{5.10}$$

We recall that Bernoulli numbers satisfy $B_n = 0$ for $n \geq 2$ and odd, and the identity

$$B_{2n} = \frac{(-1)^{n-1}\,2\,(2n)!}{(2\pi)^{2n}}\,\zeta(2n)\,,\quad \forall\,n\geq 1\,.$$

Hence, we write the left hand side of (5.10) for s even, for which the only non-vanishing terms correspond to k even, as

$$2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi m)^k}\,\zeta(s+k) \tag{5.11}$$

and for s odd, for which the only non-vanishing terms correspond to k odd, as

$$2\,\frac{(-1)^{\frac{s+1}{2}-1}}{(2\pi)^s}\sum_{\substack{k=1\\k\text{ odd}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k+1}{2}}(s+k-1)!}{(4\pi m)^k}\,\zeta(s+k) \tag{5.12}$$

In both cases, using the Euler-MacLaurin formula for the Riemann zeta function, we can write for all fixed $N\geq 1$

$$\zeta(s+k)=\sum_{n=1}^{N}\frac{1}{n^{s+k}}+R_{s+k}(N)$$

where $|R_{s+k}(N)|\leq 2\max\{1,\frac{N}{s+k-1}\}\,N^{-s-k}$.
Let us first consider the case s even. From (5.11) we are reduced to study the sum

$$\sum_{n=1}^{N}2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi n)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi mn)^k}+2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi m)^k}\,R_{s+k}(N) \tag{5.13}$$

For the first term we apply Step 1, and applying (5.7) with $\ell=mn$ we find

$$m^s\sum_{n=1}^{N}2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi mn)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi mn)^k}=\frac{m^s\,s!}{2\,(2\pi)^s}\sum_{n=1}^{N}\frac{1}{(mn)^{s+\frac12}}\,J_{s+\frac12}(2\pi mn)$$

Since

$$\left|J_{s+\frac12}(2\pi\ell)\right|\leq\frac{(\pi\ell)^{s+\frac12}}{\Gamma(s+1)}$$

we get

$$\left|m^s\sum_{n=1}^{N}2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi mn)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi mn)^k}\right|\leq\frac{\pi^{\frac12}\,N\,m^s}{2^{s+1}}\,,\qquad\forall\,N\geq 1\,. \tag{5.14}$$

For the second term of (5.13), we apply the crude estimate

$$\left|2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi m)^k}\,R_{s+k}(N)\right|\leq\frac{4\,(2s)!\,\max\{1,\frac{N}{s}\}}{(2\pi N)^s}\sum_{k=1}^{s}\binom{s}{k-1}\frac{1}{(4\pi mN)^k}=$$

$$=\frac{2\,(2s)!\,\max\{1,\frac{N}{s}\}}{(2\pi N)^{s+1}\,m}\left(1+\frac{1}{4\pi mN}\right)^s$$

From this and (5.14) it follows that for s even, we obtain the estimate

$$\left|2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi)^s}\sum_{\substack{k=1\\k\text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi m)^k}\,\zeta(s+k)\right|\leq\frac{\pi^{\frac12}\,N\,m^s}{2^{s+1}}+\frac{2\,(2s)!\,\max\{1,\frac{N}{s}\}}{(2\pi N)^{s+1}\,m}\left(1+\frac{1}{4\pi mN}\right)^s \tag{5.15}$$

which holds for all $N \geq 1$. Hence, choosing $N = 2\,s^2$ in (5.15) and applying standard estimates for the factorial term we obtain

$$\Big|2\,\frac{(-1)^{\frac{s}{2}-1}}{(2\pi)^s}\sum_{\substack{k=1\\ k \text{ even}}}^{s}\binom{s}{k-1}\frac{(-1)^{\frac{k}{2}}(s+k-1)!}{(4\pi)^k}\,\zeta(s+k)\Big| \leq \frac{4\,s^2\,m^s}{2^s}$$

The case s odd follows exactly by the same argument using (5.9) in (5.12).

<u>Step 4</u>. For all $q = \xi + i\eta$ with $\xi > 0$, we have

$$\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{2^k\,(s+k)}\,B_{s+k}\,\zeta(2q-1+s+k) = O\left(\frac{s^6}{2^s}\right). \tag{5.16}$$

In Step 3 we have an estimate for the left hand side with $\zeta(2q-1+s+k) = 1$. Using the formula

$$\zeta(2q-1+s+k) = \sum_{m=1}^{M}\frac{1}{m^{2q-1+s+k}} + R_{2q-1+s+k}(M)\,, \quad \forall\, M \geq 1$$

where $|R_{2q-1+s+k}(M)| \leq 2\max\{1, \frac{M}{|2q-2+s+k|}\}\,M^{-2\xi+1-s-k}$, we have to estimate the sum

$$\sum_{m=1}^{M}\frac{1}{m^{2q-1+s}}\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{(2m)^k\,(s+k)}\,B_{s+k} + \sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{2^k\,(s+k)}\,B_{s+k}\,R_{2q-1+s+k}(M)$$

and we can use Step 3. For the first term we apply (5.10) to write

$$\Big|\sum_{m=1}^{M}\frac{1}{m^{2q-1+s}}\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{(2m)^k\,(s+k)}\,B_{s+k}\Big| \leq \frac{4\,s^2}{2^s}\sum_{m=1}^{M}\frac{1}{m^{2\xi-1}}$$

For the second term, we use the classical estimate

$$|B_{2n}| \leq 5\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}$$

and write the crude estimate

$$\Big|\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{2^k\,(s+k)}\,B_{s+k}\,R_{2q-1+s+k}(M)\Big| \leq \frac{5\sqrt{2\pi}\,\max\{1,\frac{M}{s-2}\}}{M^{2\xi+s-1}}\sum_{k=1}^{s}\binom{s}{k-1}\frac{(s+k)^{s+k}}{(2M)^k\,(2\pi e)^{s+k}\sqrt{s+k}} \leq$$

$$\leq \frac{5\sqrt{2\pi}\,\max\{1,\frac{M}{s-2}\}}{M^{2\xi+s-1}}\,\frac{(2s)^{2s}}{(2\pi e)^s}\sum_{k=1}^{s}\binom{s}{k-1}\frac{1}{(4\pi e M)^k} = \frac{5\sqrt{2\pi}\,\max\{1,\frac{M}{s-2}\}\,(2s)^{2s}}{M^{2\xi+s-1}\,(2\pi e)^s}\left(1+\frac{1}{4\pi e M}\right)^s$$

We have thus obtained that for $s \geq 3$ and all $q = \xi + i\eta$ with $\xi > 0$, the estimate

$$\Big|\sum_{k=1}^{s}\binom{s}{k-1}\frac{(-1)^k}{2^k\,(s+k)}\,B_{s+k}\,\zeta(2q-1+s+k)\Big| \leq \frac{4\,s^2}{2^s}\sum_{m=1}^{M}\frac{1}{m^{2\xi-1}} + \frac{5\sqrt{2\pi}\,\max\{1,\frac{M}{s-2}\}\,s^{2s}}{M^{2\xi+s-1}}\left(1+\frac{1}{4\pi e M}\right)^s$$

holds for all $M \geq 1$. Choosing $M = 2s^2$ the estimate (5.16) follows.

<u>Step 5</u>. The proof is finished by using (5.16) and the crude estimate

$$\frac{\Gamma(s+q)}{s!} = O(|s+q|^{\xi})\,,$$

since for all $a, b > 0$

$$\sup_{y\in[a,b]}\frac{1+y^{2s}}{(1+y^2)^s} \leq \frac{1}{(1+a^2)^s} + \frac{b^{2s}}{(1+b^2)^s}\,.$$

□

A Proof of (3.27)

Using the notation $\lfloor x\rfloor$ and $\{x\}$ for the integer and fractional part of a real number, we have

$$S_k(n) - \frac{1}{k+1}\, n^{k+1} - \frac{1}{2}\, n^k = \int_0^n \left[(\lfloor x\rfloor + 1)^k - x^k - \frac{1}{2}\, k x^{k-1}\right] dx =$$

$$= \int_0^n \left[\sum_{j=0}^{k} \binom{k}{j} \lfloor x\rfloor^j - \sum_{j=0}^{k} \binom{k}{j} \lfloor x\rfloor^j \{x\}^{k-j} - \frac{1}{2}\, k \sum_{j=0}^{k-1} \binom{k-1}{j} \lfloor x\rfloor^j \{x\}^{k-j-1}\right] dx =$$

$$= \int_0^n \Big[\lfloor x\rfloor^k \left(1 - \{x\}^{k-k}\right) + \lfloor x\rfloor^{k-1} \left(k - k\{x\} - \frac{1}{2}\, k\right) +$$

$$+ \sum_{j=0}^{k-2} \lfloor x\rfloor^j \left(\binom{k}{j} \left(1 - \{x\}^{k-j}\right) - \frac{1}{2}\, k \binom{k-1}{j} \{x\}^{k-j-1}\right)\Big]\, dx =$$

$$= \sum_{h=0}^{n-1} \int_h^{h+1} k\, \lfloor x\rfloor^{k-1} \left(\frac{1}{2} - \{x\}\right) dx + \int_0^n \left[\sum_{j=0}^{k-2} \binom{k}{j} \lfloor x\rfloor^j \left(1 - \{x\}^{k-j} - \frac{k-j}{2}\, \{x\}^{k-j-1}\right)\right] dx =$$

$$= \sum_{h=0}^{n-1} \int_h^{h+1} k\, h^{k-1} \left(\frac{1}{2} - x + h\right) dx + \int_0^n \left[\sum_{j=0}^{k-2} \binom{k}{j} \lfloor x\rfloor^j \left(1 - \{x\}^{k-j} - \frac{k-j}{2}\, \{x\}^{k-j-1}\right)\right] dx =$$

$$= \int_0^n \left[\sum_{j=0}^{k-2} \binom{k}{j} \lfloor x\rfloor^j \left(1 - \{x\}^{k-j} - \frac{k-j}{2}\, \{x\}^{k-j-1}\right)\right] dx$$

We can then write

$$\left|S_k(n) - \frac{1}{k+1}\, n^{k+1} - \frac{1}{2}\, n^k\right| \le k \int_0^n \sum_{j=0}^{k-2} \binom{k}{j} \lfloor x\rfloor^j\, dx =$$

$$= \int_0^n k^2(k-1) \sum_{j=0}^{k-2} \frac{1}{(k-j)(k-j-1)} \binom{k-2}{j} \lfloor x\rfloor^j\, dx \le$$

$$\le k^3 \int_0^n \sum_{j=0}^{k-2} \binom{k-2}{j} \lfloor x\rfloor^j\, dx = k^3 \int_0^n (\lfloor x\rfloor + 1)^{k-2}\, dx \le \mathrm{const}\, k^2\, n^{k-1}$$

B Spectral properties of the terms with Legendre functions

We now give some properties of the functions used in the series (3.16) in terms of the hyperbolic Laplacian $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. Using recurrence relations and the formulas for derivatives of the Legendre functions which can be found in [10, vol. I], with the notation

$$F_q(n,y) := \frac{y^{n+q}}{(1+y^2)^{\frac{n}{2}+\frac{q}{2}+\frac{1}{4}}}\, \mathrm{P}_{n+q-\frac{1}{2}}^{-q+\frac{1}{2}} \left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)$$

we find that for all $n \ge 0$

$$-y^2\, \tfrac{\partial^2}{\partial y^2} \left[F_q(n,y)\right] =$$

$$= -(n+q)(n+q-1)\, F_q(n,y) + 2(n+2q)(n+q)\, F_q(n+1,y) - (n+2q)(n+2q+1)\, F_q(n+2,y)\,.$$

Hence for a series

$$\mathbf{F}_q(y) := \sum_{n=0}^{\infty} (-1)^n a_n F_q(n,y)\,, \quad y \in (0,\infty)$$

with $\limsup |a_n|^{\frac{1}{n}} \leq 1$, we find

$$-y^2 \frac{\partial^2}{\partial y^2}\mathbf{F}_q(y) = \sum_{n=0}^{\infty} (-1)^n b_n\, F_q(n,y) \tag{B.1}$$

where

$$\begin{cases} b_0 = q(1-q)\,a_0 \\ b_1 = -q(1+q)\,a_1 - 4q^2\,a_0 \\ b_n = -(n+q)(n+q-1)a_n - 2(n+2q-1)(n+q-1)a_{n-1} - (n+2q-2)(n+2q-1)a_{n-2}, \quad n \geq 2 \end{cases} \tag{B.2}$$

It follows that

Theorem B.1. *For any q the function*

$$\mathbf{E}_q(y) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2q)}{n!\,\Gamma(2q)}\, F_q(n,y)$$

satisfies $\Delta\mathbf{E}_q(y) = q(1-q)\mathbf{E}_q(y)$.

Proof. Using (B.1) and (B.2) we have to find a solution of the system

$$\begin{cases} a_0 \in \mathbb{C} \\ a_1 = -2q\,a_0 \\ a_n = -2\,\frac{n+q-1}{n}\,a_{n-1} - \frac{n+2q-2}{n}\,a_{n-2}, \qquad n \geq 2 \end{cases} \tag{B.3}$$

with $\limsup |a_n|^{\frac{1}{n}} \leq 1$.
Consider the generating function

$$f(z) = \sum_{n\geq 0} a_n z^n$$

with $a_0 = 1$, of the solution of (B.3). From the recurrence relation of (a_n), it turns out that f satisfies

$$\begin{cases} (1+z)^2\, f'(z) + 2q(z+1)\,f(z) = 0 \\ f(0) = 1 \end{cases} \tag{B.4}$$

We now want to show that the solution f of (B.4) is analytic for $|z| < 1$. Letting for any $\alpha \in \mathbb{C}$

$$(1+z)^\alpha := \exp\Big(\alpha \log|1+z| + i\alpha \arg(1+z)\Big)$$

with $\arg(1+z) \in (-\pi,\pi]$, we have that $(1+z)^\alpha$ is well defined as a single-valued analytic function on the cut plane $\mathbb{C} \setminus (-\infty,-1]$, hence in particular for $|z| < 1$. It follows that

$$f(z) = (1+z)^{-2q}$$

is the solution of (B.4) and is analytic for $|z| < 1$. Hence for all $n \geq 0$

$$a_n = (-1)^n\,\frac{\Gamma(n+2q)}{n!\,\Gamma(2q)}$$

is the solution of (B.3) with $a_0 = 1$ and $\limsup |a_n|^{\frac{1}{n}} \leq 1$. $\square$

It is interesting to notice that the functions $\mathbf{E}_q(y)$ have a much simpler formulation.

Theorem B.2. *For all q we have*

$$\mathbf{E}_q(y) = \frac{2^{-q+\frac{1}{2}}}{\Gamma\left(q+\frac{1}{2}\right)}\, y^q\,,$$

hence for $\Re(q) > 1$ it holds

$$\mathbf{E}_q(z) := \sum_{(c,d)=1} \mathbf{E}_q\left(\frac{y}{|cz+d|^2}\right) = \text{const.(q)}\ E(z,q)$$

where $E(z,q)$ is the non-holomorphic Einsenstein series.

Proof. We start from the expression for $\mathbf{E}_q$ found in Theorem B.1. Using the relation [21, eq. 14.3.21, p. 355]

$$\mathrm{P}^{\mu}_{\nu}(t) = \frac{2^{\mu}\,\Gamma(1-2\mu)\,\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)\,\Gamma(1-\mu)\,(1-t^2)^{\frac{\mu}{2}}}\, C^{(\frac{1}{2}-\mu)}_{\nu+\mu}(t)$$

in terms of the Gegenbauer functions $C^{(\beta)}_{\alpha}$, we get

$$\mathbf{E}_q(y) = \frac{2^{-q+\frac{1}{2}}}{\Gamma\left(q+\frac{1}{2}\right)} \left(\frac{y}{1+y^2}\right)^q \sum_{n=0}^{\infty} \left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)^n C^{(q)}_{n}\left(\frac{y}{(1+y^2)^{\frac{1}{2}}}\right)$$

Finally, the generating function [21, eq. 18.12.4, p. 449]

$$(1-2t\,u+u^2)^{-\beta} = \sum_{n=0}^{\infty} u^n\, C^{(\beta)}_{\alpha}(t)$$

valid for $|u| < 1$, we conclude that

$$\mathbf{E}_q(y) = \frac{2^{-q+\frac{1}{2}}}{\Gamma\left(q+\frac{1}{2}\right)}\, y^q\,.$$

The last statement is immediate from the definition of the non-holomorphic Eisenstein series. □

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