Blowup dynamics for Mass Critical Half-wave equation in 3D

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Abstract

We consider the half-wave equation $iu_t = Du - |u|^{\frac{2}{3}}u$ in three dimensions and in the mass critical. For initial data $u(t_0,x) = u_0(x) \in H^{1/2+\delta}_{rad}(\mathbb{R}^3)$ with radial symmetry, we construct a new class of the radial blowup solutions with the blow up rate $\|D^{\frac{1}{2}}u(t)\|_2 \sim \frac{C(u_0)}{|t|}$ as $t \to 0^-$.

Keywords: Half-wave equation; Mass critical; Blow-up; Ground state mass
Math. Subject Classification 35Q55, 35B44, 35B40

1 Introduction and Main Result

In this paper, we consider the half-wave equation in three dimensions

$$\begin{cases} i\partial_t u = Du - |u|^{\frac{2}{3}}u, \\ u(t_0,x) = u_0(x), \ u : I \times \mathbb{R}^3 \to \mathbb{C}. \end{cases} \tag{1.1}$$

Here, $I \subset \mathbb{R}$ is an interval containing the initial time $t_0 \in \mathbb{R}$, and

$$\widehat{(Df)}(\xi) = |\xi|\hat{f}(\xi)$$

denotes the first-order nonlocal fractional derivative.

Let us mention that nonlinear half-wave equation have recently attracted some attentions in the area of dispersive nonlinear PDE. Evolution problems like (1.1) arise in various physical settings, which include equations range from turbulence phenomena, wave propagation, continuum limits of lattice systems and models for gravitational collapse in astrophysics [6, 11, 13, 21, 22, 27, 33]. We also refer to [8, 12, 15, 23] and the references therein for the background of the fractional Schrödinger model in mathematics and physics.

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For equation (1.1), the quantities of charge $M(u)$ and energy $E(u)$ given by

$$\text{Mass} \quad M(u) = \int_{\mathbb{R}^3} |u(t,x)|^2 dx, \tag{1.2}$$

$$\text{Energy} \quad E(u) = \frac{1}{2}\int_{\mathbb{R}^3} \bar{u}(t,x) D u(t,x) dx - \frac{3}{8}\int_{\mathbb{R}^3} |u(t,x)|^{\frac{8}{3}} dx \tag{1.3}$$

are conserved. The equation (1.1) also has the following symmetry:

$$u(t,x) \to \lambda_0^{\frac{3}{2}} u(\lambda_0 t, \lambda_0 x) e^{i\theta},$$

for $\lambda_0 > 0$ and $\theta \in \mathbb{R}$.

The Cauchy problem (1.1) is L^2-critical since the L^2-norm is invariant under the scaling rule $u_\lambda(x) = \lambda^{3/2} u(\lambda x)$:

$$\|u_\lambda\|_2 = \|u\|_2, \text{ for all } \lambda > 0.$$

From [2] it is known that the Cauchy problem (1.1) is locally well-posed in the Sobolev space $H^1_{rad}(\mathbb{R}^3)$. Furthermore, Hidano and Wang [20] obtained the local well-posed in the Sobolev space $H^s_{rad}(\mathbb{R}^3)$, where $s \in (\frac{1}{2}, 1]$ or in the Sobolev space $H^s(\mathbb{R}^3)$, where $s \in (1, \frac{5}{3})$.

On the other hand, from [25], it is known that the mass critical half wave equation in one demensional is local well-posed in the Sobolev space $H^s(\mathbb{R})$, where $s \geq \frac{1}{2}$. It also has the blowup alternative property in $H^{\frac{1}{2}}(\mathbb{R})$: Either $T(u_0) = +\infty$ or $T(u_0) < +\infty$ and $\|u(t)\|_{H^{1/2}} \to +\infty$ as $t \to T^-$.

A classical criterion of global-in-time existence for $H^{1/2}(\mathbb{R})$ initial data is derived by using the Gagliardo-Nirenberg inequality with best constant

$$\|u\|_{L^4}^4 \leq C_{opt} \|D^{\frac{1}{2}} u\|_2^2 \|u\|_2^2, \text{ for } u \in H^{1/2}(\mathbb{R}),$$

where $C_{opt} = 2\|Q\|_2^{-2}$ and Q is the unique ground state solution to

$$DQ + Q = |Q|^2 Q, \ Q(x) > 0, \ Q(x) \in H^{1/2}(\mathbb{R}). \tag{1.4}$$

Note that the existence of this equation follows from standard variational techniques, but the uniqueness of Q, which was obtained by Frank, Lenzmann [13]. For the higher dimensional cases, one can see [14]. The mass and energy conservation and the blowup criterion implies that initial data $u_0 \in H^{1/2}(\mathbb{R})$ with $\|u_0\|_2 < \|Q\|_2$ generate global-in-time solution. But in the higher dimensional case, the well-posedness in the energy space $H^{\frac{1}{2}}$ of the half wave equation is still an open problem.

In addition, the study of general half-wave equation attracted a great quantity of attentions; the topics cover over well-posedness, ill-posedness, traveling solitary waves, soliton solution, see [2–4, 7, 10, 17–19, 26, 30] and the references therein.

In this paper, we focus on the existence of nondispersive dynamics, we will describe example of such dynamics:

Blowup solution. There is no general criterion for blowup solutions for L^2-critical or L^2-supercritical half-wave equation in $\mathbb{R}^N$. This is still an open problem, which we can see [5]. However, Krieger, Lenzmann and Raphaël [25] constructed a minimal mass blow-up solutions to the mass critical half-wave equation in one dimension. We also obtained the two-dimensional result, see [16]. Now we state our main result.

Theorem 1.1. ***(Existence the blowup elements)*** *Let $\delta > 0$ be small enough. For all $E_0 \in \mathbb{R}_+^*$, there exist $t^* < 0$, independent of E_0, and a radial blowup solution $u \in C^0([t^*, 0); H_{rad}^{1/2+\delta}(\mathbb{R}^3))$ of equation (1.1) with*

$$\|u(t)\|_2 = \|Q\|_2 \text{ and } E(u(t)) = E(u_0),$$

which blow up at time $T = 0$. More precisely, it holds that

$$u(t,x) - \frac{1}{\lambda(t)^{\frac{3}{2}}} Q\left(\frac{x}{\lambda(t)}\right) e^{i\gamma(t)} \to 0 \text{ in } L^2(\mathbb{R}^3) \text{ as } t \to 0^-,$$

where

$$\lambda(t) = \lambda^* t^2 + \mathcal{O}(t^3),\ \gamma(t) = \frac{1}{\lambda^* |t|} + \mathcal{O}(t),$$

with some constant $\lambda^* > 0$, and the blowup speed is given by:

$$\|D^{\frac{1}{2}} u(t)\|_2 \sim \frac{C(u_0)}{|t|},$$

where $C(u_0) > 0$ only depend on the initial data u_0.

Let us point out the new points in the proof of this result.

The construction of a blow up element is starting with the standard ansatz

$$u(t,x) = \frac{1}{\lambda^{\frac{3}{2}}(t)} [Q_{b(t)} + \epsilon]\left(s, \frac{x}{\lambda(t)}\right) e^{i\gamma(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}, \tag{1.5}$$

where $b(t), \lambda(t), \gamma(t)$ are appropriate remodulation parameters, $Q_{b(t)}$ is approximate blow up profile build as approximation un to order 5 of equation of type

$$-i\frac{b^2}{2}\partial_b Q_b - DQ_b - Q_b + ib\Lambda Q_b + |Q_b|^{\frac{2}{3}} Q_b = -\Phi_b.$$

The remainder term ϵ can be controlled in $H^{1/2+\delta}$ norm in one dimensional case, since the Nemytskii operator

$$u \to |u|^{p-1} u$$

is a bounded operator in $H^{1/2+\delta}(\mathbb{R})$. This fact is sufficient to deduce the strong convergence in $H^{1/2}$ in one dimensional case [25]. In our case of dimension three the Nemytskii operator

$$u \to |u|^{p-1} u$$

is not bounded $H^{1/2+\delta}(\mathbb{R}^2)$ for $\delta > 0$ small.

However, we are able to get $H_{rad}^{1/2+\theta}$ control (see estimate (6.7)) of the remainder in a small backward time interval with $\theta \in (0, 3/10)$. This estimate is crucial to obtain the strong convergence in $H_{rad}^{1/2+\delta}$ (see (6.9)) with $\delta \in (0, \theta)$.

Unlike one [25] or two dimensional [16] cases, we need to use some new estimates in order to control the nonlinear term. In particular, we use the following Kato type smoothing estimate (see Proposition 2.3 in [2])

$$\left\| [x]_\delta^{-\frac{1}{q_1}} \int_0^t e^{i(t-s)D} F(s) ds \right\|_{L^{q_1}(\mathbb{R});L^2(\mathbb{R}^N)} \leq C \left\| |[x]_\delta^{\frac{1}{q_2}} F \right\|_{L^{q_2}(\mathbb{R});L^2(\mathbb{R}^N)},$$

where $q_1 \in [2, +\infty]$ and $q_2 \in (2, +\infty]$.

Remark 1. *In our discussion, the radial assumption of u is necessary. We take initial data in $H^{1/2+\varepsilon}_{rad}(\mathbb{R}^3)$. For any initial data in $H^{1/2}(\mathbb{R}^3)$, we do not know how to deal with the nonlinear term in section 4. On the other hand, we do not know the well-posedness in the general Sobolev space H^s, where $s \geq \frac{1}{2}$.*

This paper is organized as follows: in section 2, we construct the approximate blowup profile; in section 3, we estimate the parameter is small enough; in section 4, we define the new energy functional and obtain the bound estimate; in section 5, we apply the energy estimate to establish a bootstrap argument; in section 6 we prove our main result; the finally section is Appendix.

Notations and definitions

- $(f, g) = \int \bar{f} g$ as the inner product on $L^2(\mathbb{R}^3)$.
- $\| \cdot \|_{L^p}$ denotes the $L^p(\mathbb{R}^3)$ norm for $p \geq 1$.
- $\widehat{f}$ denotes the Fourier transform of function f.
- We shall use $X \lesssim Y$ to denote that $X \leq CY$ holds, where the constant $C > 0$ may change from line to line, but C is allowed to depend on universally fixed quantities only.
- Likewise, we use $X \sim Y$ to denote that both $X \lesssim Y$ and $Y \lesssim X$ hold.

For a sufficiently regular function $f : \mathbb{R}^3 \to \mathbb{C}$, we define the generator of L^2 scaling given by

$$\Lambda f := \frac{3}{2} f + x \cdot \nabla f.$$

Note that the operator Λ is skew-adjoint on $L^2(\mathbb{R}^3)$, that is, we have

$$(\Lambda f, g) = -(f, \Lambda g).$$

We write $\Lambda^k f$, with $k \in \mathbb{N}$, for the iterates of Λ with the convention that $\Lambda^0 f \equiv f$.

In some parts of this paper, it will be convenient to identity any complex-valued function $f : \mathbb{R}^3 \to \mathbb{C}$ with the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ by setting

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \Re f \\ \Im f \end{bmatrix}.$$

Corresponding, we will identity the multiplication by i in $\mathbb{C}$ with the multiplication by the real 2×2-matrix defined as

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2 Approximate Blowup Profile

This section is devoted to the construction of the approximate blowup profile. We start with a general observation: If $u = u(t,x)$ solves (1.1), then we define the function $v = v(s,y)$ by setting

$$u(t,x) = \frac{1}{\lambda^{\frac{3}{2}}(t)} v\left(t, \frac{x}{\lambda(t)}\right) e^{i\gamma(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}.$$

It is easy to check that $v = v(s,y)$ with $y = \frac{x}{\lambda}$ satisfies

$$i\partial_s v - Dv - v + |v|^{\frac{2}{3}} v = i\frac{\lambda_s}{\lambda}\Lambda v + \tilde{\gamma}_s v,$$

where we set $\tilde{\gamma}_s = \gamma_s - 1$. Here the operators D is understood as $D = D_y$. Following the slow modulated ansatz strategy developed in [24, 25, 31]. We freeze the modulation

$$-\frac{\lambda_s}{\lambda} = b.$$

And we look for the approximate solution of the form

$$v(s,y) = Q_{b(s)}(y),$$

with an expansion

$$Q_b = Q(y) + \sum_{k\geq 1} b^k R_k(y),$$

where $b \in \mathbb{R}$. The terms $R_k(y)$ is decomposed in real and imaginary parts as follows

$$R_k(y) = T_k + iS_k.$$

We will define ODE for $b(s)$ of type

$$b_s = P_1(b),$$

where P_1 is the approximate polynomials in b. We adjust the modulation equation for b_s to ensure the solvability of the obtained system, and a specific algebra leads to the laws to leading order:

$$b_s = -\frac{1}{2}b^2.$$

This allows us to construct a high order approximation Q_b solution to

$$-i\frac{b^2}{2}\partial_b Q_b - DQ_b - Q_b + ib\Lambda Q_b + |Q_b|^{\frac{2}{3}} Q_b = -\Phi_b,$$

where Φ_b is some small and well-localized error term.

We have the following result about an approximate blowup profile $\mathbf{Q}_b$, parameterized by b, around the ground state $\mathbf{Q} = [Q, 0]^\top$.

Lemma 2.1. *(**Approximate Blowup Profile**) Let $b \in \mathbb{R}$. Then there exists a smooth radial function $\mathbf{Q}_b = \mathbf{Q}_b(x)$ of the form*

$$\mathbf{Q}_b = \mathbf{Q} + b\mathbf{R}_1 + b^2\mathbf{R}_2 + b^3\mathbf{R}_3 + b^4\mathbf{R}_4$$

that satisfies the equation

$$-J\frac{1}{2}b^2\partial_b\mathbf{Q}_b - D\mathbf{Q}_b - \mathbf{Q}_b + Jb\Lambda\mathbf{Q}_b + |\mathbf{Q}_b|^{\frac{2}{3}}\mathbf{Q}_b = -\mathbf{\Phi}_b. \tag{2.1}$$

Here the functions $\{\mathbf{R}_k\}_{0\le k\le 4}$ satisfy the following regularity and decay bounds:

$$\|\mathbf{R}_k\|_{H^m} + \|\Lambda\mathbf{R}_k\|_{H^m} + \|\Lambda^2\mathbf{R}_k\|_{H^m} \lesssim 1, \text{ for } m \in \{0,1\}, \tag{2.2}$$

$$|\mathbf{R}_k| + |\Lambda\mathbf{R}_k| + |\Lambda^2\mathbf{R}_k| \sim \langle x\rangle^{-4}, \text{ for } x \in \mathbb{R}^3. \tag{2.3}$$

Moreover, the term on the right-hand side of (2.1) *satisfies*

$$\|\mathbf{\Phi}_b\|_{H^m} \lesssim \mathcal{O}(b^5),\ |\nabla\mathbf{\Phi}_b| \lesssim \mathcal{O}(b^5)\langle x\rangle^{-4}, \tag{2.4}$$

for $m \in \{0,1\}$ and $x \in \mathbb{R}^3$.

In addition, the mass and the energy of $\mathbf{Q}_b$ satisfy

$$\int |\mathbf{Q}_b|^2 = \int Q^2 + \mathcal{O}(b^4),$$
$$E(\mathbf{Q}_b) = e_1 b^2 + \mathcal{O}(b^4).$$

Here $e_1 > 0$ is the positive constant given by

$$e_1 = \frac{1}{2}(L_-S_1, S_1),$$

where S_1 satisfy $L_-S_1 = \Lambda Q$.

Proof. We recall that the definition of linear operator

$$L_+ = D + 1 - \frac{5}{3}Q^{2/3},\ L_- = D + 1 - Q^{2/3}.$$

From [14] we have the key property that the kernel of L_+ and L_- is given by

$$\ker L_+ = \{\partial_{x_1}Q, \partial_{x_3}Q, \partial_{x_3}Q\} \text{ and } \ker L_- = \{Q\}.$$

It follows from the above properties (see [1] for the properties of Helmholtz kernel and [9, Appendix A] or proof of [29, Lemma 3.2] for similar arguments) that

$$\forall g \in L^2,\ (g, \nabla Q) = 0,\ \exists f_+ \in L^2,\ L_+f_+ = g,$$
$$\forall g \in L^2,\ (g, Q) = 0,\ \exists f_- \in L^2,\ L_-f_- = g. \tag{2.5}$$

Using the above properties (2.5), we discuss our ansatz for Q_b to solve (4.1) order by order. Following the similar argument, see [16,25], we can prove this lemma, and here we omit the details. □

Remark 2. *1. We know that Q_b have the following form*

$$Q_b = Q + ibS_1 + b^2T_2 + ib^3S_3 + b^4T_4.$$

2. We have the following identity:

$$(S_1, S_1) = -2(T_2, Q). \tag{2.6}$$

3. Since $L_- \ge 0$ and $L_- > 0$ on $Q^\perp$, we have $S_1 \perp Q$.

3 Geometrical Decomposition and Modulation Equation

Let $u \in H^{1/2+\delta}_{rad}(\mathbb{R}^3)$ be a radial solution of (1.1) on some time interval $[t_0, t_1]$ with $t_1 < 0$. Assume that $u(t)$ admits a geometrical decomposition of the form

$$u(t,x) = \frac{1}{\lambda^{\frac{3}{2}}(t)}[Q_{b(t)} + \epsilon]\left(s, \frac{x}{\lambda(t)}\right)e^{i\gamma(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda(t)}, \tag{3.1}$$

and we impose the uniform smallness bound

$$b^2(t) + \|\epsilon\|^2_{H^{1/2}} \ll 1.$$

Furthermore, we assume that $u(t)$ has almost critical mass in the sense that

$$\left| \int |u(t)|^2 - \int Q^2 \right| \lesssim \lambda^2(t), \quad \forall t \in [t_0, t_1]. \tag{3.2}$$

To fix the modulation parameters $\{b(t), \lambda(t), \gamma(t)\}$ uniquely, we impose the following orthogonality conditions on $\epsilon = \epsilon_1 + i\epsilon_2$ as follows:

$$\begin{aligned} (\epsilon_2, \Lambda Q_{1b}) - (\epsilon_1, \Lambda Q_{2b}) &= 0, \\ (\epsilon_2, \partial_b Q_{1b}) - (\epsilon_1, \partial_b Q_{2b}) &= 0, \\ (\epsilon_2, \rho_1) - (\epsilon_1, \rho_2) &= 0, \end{aligned} \tag{3.3}$$

the function $\rho = \rho_1 + i\rho_2$ is defined by

$$\begin{cases} L_+\rho_1 = S_1, \\ L_-\rho_2 = b\frac{2}{3}Q^{-\frac{1}{3}}S_1\rho_1 + b\Lambda\rho_1 - 2bT_2, \end{cases} \tag{3.4}$$

where S_1, T_2 are the radial functions, see Remark 2. Note that L_+^{-1} exists on $L^2_{rad}(\mathbb{R}^3)$ and thus ρ_1 is well-defined. Moreover, it is easy to see that the right-hand side in the equation for ρ_2 is orthogonality to Q. Indeed

$$\begin{aligned} (Q, \frac{2}{3}Q^{-\frac{1}{3}}S_1\rho_1 + \Lambda\rho_1 - 2T_2) &= \frac{2}{3}(Q^{\frac{2}{3}}S_1, \rho_1) - (\Lambda Q, \rho_1) - 2(Q, T_2) \\ &= \frac{2}{3}(Q^{\frac{2}{3}}S_1, \rho_1) - (S_1, L_-\rho_1) + (S_1, S_1) \\ &= -(S_1, L_+\rho_1) + (S_1, S_1) = 0, \end{aligned}$$

using that $(S_1, S_1) = -2(T_2, Q)$, see (2.6), and the definition of ρ_1. Hence ρ_2 is well-defined.

In the orthogonality conditions (3.3), we use the notation

$$Q_b = Q_{1b} + iQ_{2b} = Q + b^2T_2 + b^4T_4 + i(bS_1 + b^3S_3),$$

which (in terms of the vector notation used in Section 3) means that

$$\mathbf{Q}_b = \begin{bmatrix} Q_{1b} \\ Q_{2b} \end{bmatrix}.$$

By the similar arguments as [16, 25, 28], we can obtain the modulation parameters

$$\{b(t),\ \lambda(t),\ \gamma(t)\}$$

are uniquely determined, provided that $\epsilon = \epsilon_1 + i\epsilon_2 \in H^{1/2}_{rad}(\mathbb{R}^3)$ is sufficiently small. Moreover, it follows from the standard arguments that $\{b(t), \lambda(t), \gamma(t)\}$ are C^1-functions.

If we insert the decomposition (3.1) into (1.1), we obtain the following system

$$\begin{aligned}&\left(b_s + \frac{1}{2}b^2\right)\partial_b Q_{1b} + \partial_s \epsilon_1 - M_-(\epsilon) + b\Lambda\epsilon_1\\ =&\left(\frac{\lambda_s}{\lambda} + b\right)(\Lambda Q_{1b} + \Lambda\epsilon_1) + \tilde{\gamma}_s(Q_{2b} + \epsilon_2) + \Im(\Phi_b) - R_2(\epsilon), \qquad (3.5)\\ &\left(b_s + \frac{1}{2}b^2\right)\partial_b Q_{2b} + \partial_s \epsilon_2 + M_+(\epsilon) + b\Lambda\epsilon_2\\ =&\left(\frac{\lambda_s}{\lambda} + b\right)(\Lambda Q_{2b} + \Lambda\epsilon_2) - \tilde{\gamma}_s(Q_{1b} - \epsilon_1) - \Re(\Phi_b) + R_1(\epsilon). \qquad (3.6)\end{aligned}$$

Here Φ_b denotes the error term from lemma 2.1, and $M = (M_+, M_-)$ are the small deformations of the linearized operator $L = (L_+, L_-)$ given by

$$\begin{aligned}M_+(\epsilon) =& D\epsilon_1 + \epsilon_1 - \frac{4}{3}|Q_b|^{\frac{2}{3}}\epsilon_1 - \frac{1}{3}|Q_b|^{-\frac{4}{3}}(Q_{1b}^2 - Q_{2b}^2)\epsilon_1\\ &- \frac{2}{3}|Q_b|^{-\frac{4}{3}}Q_{1b}Q_{2b}\epsilon_2, \qquad (3.7)\\ M_-(\epsilon) =& D\epsilon_2 + \epsilon_2 - \frac{4}{3}|Q_b|^{\frac{2}{3}}\epsilon_2 - \frac{1}{3}|Q_b|^{-\frac{4}{3}}(Q_{1b}^2 - Q_{2b}^2)\epsilon_2\\ &- \frac{2}{3}|Q_b|^{-\frac{4}{3}}Q_{1b}Q_{2b}\epsilon_1. \qquad (3.8)\end{aligned}$$

And $R_1(\epsilon)$, $R_2(\epsilon)$ are the high order terms about ϵ.

$$\begin{aligned}R_1(\epsilon) =& \frac{2}{3}|Q_b|^{-\frac{4}{3}}(Q_{1b}\epsilon_1 + Q_{2b}\epsilon_2)\epsilon_1 + \frac{1}{3}|Q_b|^{-\frac{4}{3}}|\epsilon|^2 Q_{1b}\\ &- \frac{4}{9}|Q_b|^{-\frac{10}{3}}(Q_{1b}\epsilon_1 + Q_{2b}\epsilon_2)Q_{1b} + \mathcal{O}(\epsilon^3),\\ R_2(\epsilon) =& \frac{2}{3}|Q_b|^{-\frac{4}{3}}(Q_{1b}\epsilon_1 + Q_{2b}\epsilon_2)\epsilon_2 + \frac{1}{3}|Q_b|^{-\frac{4}{3}}|\epsilon|^2 Q_{2b}\\ &- \frac{4}{9}|Q_b|^{-\frac{10}{3}}(Q_{1b}\epsilon_1 + Q_{2b}\epsilon_2)Q_{2b} + \mathcal{O}(\epsilon^3).\end{aligned}$$

We have the following energy type bound.

Lemma 3.1. *For $t \in [t_0, t_1]$, it holds that*

$$b^2 + \|\epsilon\|^2_{H^{1/2}} \lesssim \lambda|E_0| + \mathcal{O}(\lambda^2 + b^4).$$

Here $E_0 = E(u_0)$ denote the conserved energy of $u = u(t, x)$.

Proof. By the similar arguments as [16], we can obtain this estimate. Here we omit the details. $\square$

We continue with estimating the modulation parameters. To this end, we define the vector-valued function

$$\mathbf{Mod}(t) := \left(b_s + \frac{1}{2}b^2, \tilde{\gamma}_s, \frac{\lambda_s}{\lambda} + b \right). \tag{3.9}$$

We have the following result.

Lemma 3.2. *For $t \in [t_0, t_1]$, we have the bound*

$$|\mathbf{Mod}(t)| \le \lambda^2 + b^4 + b^2\|\epsilon\|_2 + \|\epsilon\|_2^2 + \|\epsilon\|_{H^{1/2}}^3. \tag{3.10}$$

Furthermore, we have the improved bound

$$\left| \frac{\lambda_s}{\lambda} + b \right| \le b^5 + b^2\|\epsilon\|_2 + \|\epsilon\|_2^2 + \|\epsilon\|_{H^{1/2}}^3.$$

Proof. We shall give only the sketch of the proof since the details can be found in [16, Lemma 4.2].

Law for b. Here the treatment is quite close to the corresponding proof in [16, Lemma 4.2] so we can write

$$\begin{aligned} -\left(b_s + \frac{1}{2}b^2 \right) [2e_1 + \mathcal{O}(b^2)] = & - \|\epsilon\|_{L^2}^2 + (R_2(\epsilon), \Lambda Q_{2b}) + (R_1(\epsilon), \Lambda Q_{1b}) \\ & + \mathcal{O}\left((b^2 + |\mathbf{mod}(t)|)(\|\epsilon\|_2 + b^2) + |\|u\|_2^2 - \|Q\|_2^2| + b^4 \right). \end{aligned}$$

By the similar arguments, we can obtain the following estimates:

Law for λ.

$$\begin{aligned} \left(\frac{\lambda_s}{\lambda} + b \right) [2e_1 + \mathcal{O}(b^2)] =& (R_2(\epsilon), \partial_b Q_{2b}) + (R_1(\epsilon), \partial_b Q_{1b}) \\ & + \mathcal{O}\left((b^2 + |\mathbf{mod}(t)|)(\|\epsilon\|_2 + b^2) + b^5 \right). \end{aligned}$$

Furthermore, by this estimate, we deduce the improved bound for $\left| \frac{\lambda_s}{\lambda} + b \right|$.

Law for $\tilde{\gamma}_s$.

$$\begin{aligned} & \tilde{\gamma}_s((Q, \rho_1) + \mathcal{O}(b^2)) \\ = & - \left(b_s + \frac{1}{2}b^2 \right) \left((S_1, \rho_1) + \mathcal{O}(b^2) \right) + \left(\frac{\lambda_s}{\lambda} + b \right) \mathcal{O}(b) \\ & + (R_2(\epsilon), \rho_2) + (R_1(\epsilon), \rho_1) + \mathcal{O}\left((b^2 + |\mathbf{mod}(t)|)(\|\epsilon\|_2 + b^2) + b^5 \right). \end{aligned}$$

Conclusion. We collect the previous equation and estimate the nonlinear terms in ϵ by Sobolev inequalities. This gives us

$$\begin{aligned} (A + B)\mathbf{Mod}(t) =& \mathcal{O}\big((b^2 + |\mathbf{Mod}(t)|)\|\epsilon_2\| + \|\epsilon\|_2^2 + \|\epsilon\|_{H^{1/2}}^3 \\ & + |\|u\|_2^2 - \|Q\|_2^2| + b^4 \big). \end{aligned}$$

Here $A = O(1)$ is invertible 3×3-matrix, and $B = \mathcal{O}(b)$ is some 3×3-matrix that is polynomial in b. For $|b| \ll 1$, we can thus invert $A + B$ by Taylor expansion and derive the estimate for $\mathbf{Mod}(t)$ stated in this lemma. $\square$

4 Refined Energy bounds

In this section, we establish a refined energy estimate, which will be a key ingredient in the compactness argument to construct the blowup solutions. Let $u = u(t,x)$ be a radial solution (1.1) on the time interval $[t_0, 0)$ and suppose that w is a radial approximate solution to (1.1) such that

$$iw_t - Dw + |w|^{\frac{2}{3}}w = \psi, \tag{4.1}$$

with the priori bounds

$$\|w\|_2 \lesssim 1,\ \|D^{\frac{1}{2}}w\|_2 \lesssim \lambda^{-\frac{1}{2}},\ \|\nabla w\|_2 \lesssim \lambda^{-1}. \tag{4.2}$$

We decompose $u = w + \tilde{u}$, and hence $\tilde{u}$ is radial and satisfies

$$i\tilde{u}_t - D\tilde{u} + (|u|^{\frac{2}{3}}u - |w|^{\frac{2}{3}}w) = -\psi, \tag{4.3}$$

where we assume the priori estimate

$$\|D^{\frac{1}{2}}\tilde{u}\|_2 \lesssim \lambda^{\frac{1}{2}},\ \|\tilde{u}\|_2 \lesssim \lambda, \tag{4.4}$$

as well as

$$|\lambda_t + b| \lesssim \lambda^2,\ b \lesssim \lambda^{\frac{1}{2}},\ |b_t| \lesssim 1. \tag{4.5}$$

Next, Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth and radial function with the following properties

$$\phi'(x) = \begin{cases} x & \text{for } \ 0 \le x \le 1, \\ 3 - e^{-|x|} & \text{for } x \ge 2, \end{cases} \tag{4.6}$$

and the convexity condition

$$\phi''(x) \ge 0 \ \text{ for } \ x \ge 0. \tag{4.7}$$

Furthermore, we denote

$$F(u) = \frac{3}{8}|u|^{\frac{8}{3}},\ f(u) = |u|^{\frac{2}{3}}u,\ F'(u)\cdot h = \Re(f(u)\bar{h}).$$

Let $A > 0$ be a large constant and define the quantity

$$\begin{aligned} J_A(u) :=& \frac{1}{2}\int |D^{\frac{1}{2}}\tilde{u}|^2 + \frac{1}{2}\int \frac{|\tilde{u}|^2}{\lambda} - \int [F(u) - F(w) - F'(w)\cdot\tilde{u}] \\ &+ \frac{b}{2}\Im\left(\int A\nabla\phi\left(\frac{x}{A\lambda}\right)\cdot\nabla\tilde{u}\bar{\tilde{u}}\right). \end{aligned} \tag{4.8}$$

Our strategy will be to use the preceding functional to bootstrap control over $\|\tilde{u}\|_{H^{\frac{1}{2}}}$.

Lemma 4.1. *(Localized energy estimate) Let J_A be as above. Then we have*

$$\begin{aligned}\frac{dJ_A}{dt} =& - \Im\left(\psi, D\tilde{u} + \frac{1}{\lambda}\tilde{u} - f'(w)\tilde{u}\right) - \frac{1}{\lambda}(\tilde{u}, f'(w)\tilde{u}) \\ &+ \frac{b}{2\lambda}\int\frac{|\tilde{u}|^2}{\lambda} - \frac{2b}{\lambda}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta\phi(\frac{x}{A\lambda})|\nabla\tilde{u}_s|^2 dxds \\ &+ \frac{b}{2A^2\lambda^3}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta^2\phi(\frac{x}{A\lambda})|\tilde{u}_s|^2 dxds \\ &+ \Im\left(\int\left[ibA\nabla\phi(\frac{x}{A\lambda})\cdot\nabla\psi + i\frac{b}{2\lambda}\Delta\psi(\frac{x}{A\lambda})\psi\right]\bar{\tilde{u}}\right) \\ &- b\Re\left(\int A\nabla\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\overline{\nabla\tilde{u}}\right) \\ &- \frac{1}{2}\frac{b}{\lambda}\Re\left(\int\Delta\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\bar{\tilde{u}}\right) \\ &+ \mathcal{O}\left(\|\tilde{u}\|^2_{H^{1/2}} + \|\tilde{u}\|^{\frac{1}{3}}_{H^{1/2}} + \lambda\|\psi\|_2\right), \end{aligned} \tag{4.9}$$

where $\tilde{u}_s := \sqrt{\frac{2}{\pi}}\frac{1}{-\Delta+s}\tilde{u}$ *with* $s > 0$.

Proof. **Step 1** : (Estimating the energy part). Using (4.3), a computation

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}\int|D^{\frac{1}{2}}\tilde{u}|^2 + \frac{1}{2}\int\frac{|\tilde{u}|^2}{\lambda} - \int[F(u) - F(w) - F'(w)\cdot\tilde{u}]\right\} \\ =&\Re\left(\partial_t\tilde{u}, D\tilde{u} + \frac{1}{\lambda}\tilde{u} - (f(u) - f(w))\right) - \frac{\lambda_t}{2\lambda^2}\int|\tilde{u}|^2 \\ &- \Re\left(\partial_t w, (f(u) - f(w) - f'(w)\cdot\tilde{u})\right) \\ =& - \Im\left(\psi, D\tilde{u} + \frac{1}{\lambda}\tilde{u} - (f(u) - f(w))\right) - \frac{\lambda_t}{2\lambda^2}\int|\tilde{u}|^2 \\ &- \Re\left(\partial_t w, (f(u) - f(w) - f'(w)\cdot\tilde{u})\right) \\ &- \Im\left(D\tilde{u} - (f(u) - f(w)), D\tilde{u} + \frac{1}{\lambda}\tilde{u} - (f(u) - f(w))\right) \\ =& - \Im\left(\psi, D\tilde{u} + \frac{1}{\lambda}\tilde{u} - (f(u) - f(w))\right) - \frac{\lambda_t}{2\lambda^2}\int|\tilde{u}|^2 + \Im\left(f(u) - f(w), \frac{1}{\lambda}\tilde{u}\right) \\ &- \Re\left(\partial_t w, (f(u) - f(w) - f'(w)\cdot\tilde{u})\right) \\ =& - \Im\left(\psi, D\tilde{u} + \frac{1}{\lambda}\tilde{u} - f'(w)\tilde{u}\right) - \frac{1}{\lambda}(\tilde{u}, f'(w)\tilde{u}) - \frac{\lambda_t}{2\lambda^2}\int|\tilde{u}|^2 \\ &+ \Im\left(\psi - \frac{1}{\lambda}\tilde{u}, f(u) - f(w) - f'(w)\cdot\tilde{u}\right) - \Re\left(\partial_t w, (f(u) - f(w) - f'(w)\cdot\tilde{u})\right), \end{aligned} \tag{4.10}$$

where we denote

$$f'(w)\tilde{u} = \frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}}.$$

From (4.5) we obtain that

$$
\begin{aligned}
-\frac{\lambda_t}{2\lambda^2}\int |\tilde{u}|^2 &= \frac{b}{2\lambda}\int \frac{|\tilde{u}|^2}{\lambda} - \frac{1}{2\lambda^2}(\lambda_t + b)\|\tilde{u}\|_2^2 \\
&= \frac{b}{2\lambda}\int \frac{|\tilde{u}|^2}{\lambda} + \mathcal{O}(\|\tilde{u}\|^2_{H^{1/2}}). \qquad (4.11)
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
&\left|\Im\left(\psi - \frac{1}{\lambda}\tilde{u}, f(u) - f(w) - f'(w)\cdot\tilde{u}\right)\right| \\
\lesssim &\left(\|\psi\|_2 + \frac{1}{\lambda}\|\tilde{u}\|_2\right)\|f(u) - f(w) - f'(w)\cdot\tilde{u}\|_2 \\
\lesssim &\left(\|\psi\|_2 + \frac{1}{\lambda}\|\tilde{u}\|_2\right)\|\tilde{u}^{\frac{2}{3}+1}\|_2 \\
\lesssim &\left(\|\psi\|_2 + \frac{1}{\lambda}\|\tilde{u}\|_2\right)\||x|^{\frac{1}{q}}\tilde{u}^{\frac{2}{3}+1}\|_2. \qquad (4.12)
\end{aligned}
$$

Here in the last step we use the inequality

$$
\left\||x|^{-\frac{1}{q_1}}\int_0^t e^{i(t-s)D}F(s)ds\right\|_{L^{q_1}(\mathbb{R});L^2(\mathbb{R}^N)} \le C\left\||x|^{\frac{1}{q_2}}F\right\|_{L^{q_2}(\mathbb{R});L^2(\mathbb{R}^N)},
$$

where $q_1 \in [2, +\infty]$ and $q_2 \in (2, +\infty]$. This inequality can be find [2, Proposition 2.3] and we can choose $q \in (2, \infty]$. From Sobolev embedding and the Strauss inequality [2], we know that

$$
\|u\|_{L^{\frac{6}{3-2s_1}}} \le \|u\|_{\dot{H}^{s_1}} \text{ and } \||x|u\|_{L^\infty} \le \|u\|_{\dot{H}^{\frac{1}{2}+\delta}}, \text{ where } \delta > 0.
$$

Using above two inequalities and Stein-interpolation theorem, we have

$$
\||x|^{\frac{1}{q}}|u|^{\frac{5}{3}}\|_{L^2} = \||x|^{\frac{3}{5q}}u\|^{\frac{5}{3}}_{\frac{10}{3}} \le \|u\|^{\frac{5}{3}}_{\dot{H}^s(\mathbb{R}^3)} \qquad (4.13)
$$

where

$$
\frac{3}{10} = \frac{1-\theta}{\frac{6}{3-2s_1}} + \frac{\theta}{\infty},\ \frac{3}{5q} = 1\cdot\theta + 0\cdot(1-\theta) = \theta \text{ and } s = \left(\frac{1}{2}+\delta\right)\theta + (1-\theta)s_1.
$$

Hence

$$
\theta = 1 - \frac{9}{5(3-2s_1)} \quad \text{and } s = -\frac{2}{5} + \frac{9}{5(3-2s_1)} + \theta\delta.
$$

On the other hand, $q \in (2, \infty]$, from above we know that $q = \frac{3}{5\theta} > 2$, hence we have

$$
s_1 > \frac{3}{14}.
$$

Let $\delta > 0$ is small enough and $s_1 > \frac{3}{14}$ and close to $\frac{3}{14}$, we have $s \leq \frac{8}{25} < \frac{1}{2}$ and close to $\frac{8}{25}$. Combining (4.12) and (4.13), we deduce that

$$\begin{aligned}
&\left|\Im\left(\psi - \frac{1}{\lambda}\tilde{u}, f(u) - f(w) - f'(w)\cdot\tilde{u}\right)\right| \\
\lesssim&\left(\|\psi\|_2 + \frac{1}{\lambda}\|\tilde{u}\|_2\right)\||x|^{\frac{1}{q}}\tilde{u}^{\frac{2}{3}+1}\|_2 \\
\lesssim&\left(\|\psi\|_2 + \frac{1}{\lambda}\|\tilde{u}\|_2\right)\|u\|^{\frac{5}{3}}_{\dot{H}^s(\mathbb{R}^3)} \\
\lesssim&\lambda\|\psi\|_2 + \|\tilde{u}\|^{\frac{5}{3}}_{H^{1/2}}.
\end{aligned} \tag{4.14}$$

For the term that contain $\partial_t w$, we use the equation for w and the bounds (4.2) and (4.4). This leads us to

$$\begin{aligned}
&|\Re\left(\partial_t w, (f(u) - f(w) - f'(w)\cdot\tilde{u})\right)| \lesssim \left|\int \partial_t w|\tilde{u}|^{\frac{5}{3}}\right| \\
\lesssim&\int (Dw - |w|^{\frac{2}{3}}w + \psi)|\tilde{u}|^{\frac{5}{3}} \\
\lesssim&\left(\|Dw\|_2 + \|Dw\|_2\|w\|_2^{\frac{2}{3}} + \|\psi\|_2\right)\|\tilde{u}^{\frac{5}{3}}\|_2 \\
\lesssim&(\|Dw\|_2 + \|\psi\|_2)\|\tilde{u}\|^{\frac{5}{3}}_{\dot{H}^{8/25}} \\
\lesssim&\|\tilde{u}\|^{\frac{1}{3}}_{H^{1/2}} + \lambda\|\psi\|_2.
\end{aligned} \tag{4.15}$$

We now insert (4.11), (4.14) and (4.15) into (4.10). Combined with the assumed a priori bounds on $\tilde{u}$, we conclude

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\int |D^{\frac{1}{2}}\tilde{u}|^2 + \frac{1}{2}\int\frac{|\tilde{u}|^2}{\lambda} - \int[F(u) - F(w) - F'(w)\cdot\tilde{u}]\right\} \\
=&-\Im\left(\psi, D\tilde{u} + \frac{1}{\lambda}\tilde{u} - f'(w)\tilde{u}\right) - \frac{1}{\lambda}(\tilde{u}, f'(w)\tilde{u}) + \frac{b}{2\lambda}\int\frac{|\tilde{u}|^2}{\lambda} \\
&+\mathcal{O}\left(\|\tilde{u}\|^2_{H^{1/2}} + \|\tilde{u}\|^{\frac{1}{3}}_{H^{1/2}} + \lambda\|\psi\|_2\right).
\end{aligned} \tag{4.16}$$

Step 2 : Estimating the localized virial part. Using (4.3), we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(b\Im\left(\int A\nabla\Big(\frac{x}{A\lambda}\Big)\cdot\nabla\tilde{u}\bar{\tilde{u}}\right)\right) \\
=&\frac{1}{2}\Im\left(\int(\partial_t\nabla\tilde{\phi})\cdot\nabla u\bar{\tilde{u}}\right) + \frac{1}{2}\Im\left(\int\nabla\tilde{\phi}\cdot(\nabla\partial_t\tilde{u}\bar{\tilde{u}} + \nabla\tilde{u}\partial_t\bar{\tilde{u}})\right) \\
=&\frac{1}{2}\Im\left(\int(\partial_t\nabla\tilde{\phi})\cdot\nabla u\bar{\tilde{u}}\right) - \frac{1}{4}\Re\left(\int\bar{\tilde{u}}\left[-iD, \nabla\tilde{\phi}\cdot(-i\nabla) + (-i\nabla)\cdot\nabla\tilde{\phi}\right]\right) \\
&-b\Re(\int(|u|^{\frac{2}{3}}u - |w|^{\frac{2}{3}}w)A\nabla\phi\left(\frac{x}{A\lambda}\right)\cdot\overline{\nabla\tilde{u}}) \\
&-\frac{1}{2}\frac{b}{\lambda}\Re\left(\int(|u|^{\frac{2}{3}}u - |w|^{\frac{2}{3}}w)A\Delta\phi\left(\frac{x}{A\lambda}\right)\cdot\bar{\tilde{u}}\right)
\end{aligned}$$

$$- b\Re\left(\int \psi\nabla\phi\left(\frac{x}{A\lambda}\right)\cdot\overline{\nabla\tilde{u}}\right) - \frac{1}{2}\frac{b}{\lambda}\Re\left(\int \psi\Delta\left(\frac{x}{A\lambda}\right)\bar{\tilde{u}}\right), \tag{4.17}$$

where $\nabla\tilde{\phi}(t,x) = bA\nabla\phi\left(t, \frac{x}{A\lambda}\right)$. Using the bounds (4.5), we estimate

$$|\partial_t\nabla\tilde{\phi}| \leq |b_t| + b\frac{\lambda_t}{\lambda} \leq 1 \text{ and } |\partial_t\Delta\tilde{\phi}| \leq \lambda^{-1}.$$

Hence, by [25, Lemma F.1], we deduce that

$$\left|\Im\left((\partial_t\nabla\tilde{\phi})\cdot\nabla u\bar{\tilde{u}}\right)\right| \leq \|\tilde{u}\|^2_{\dot{H}^{1/2}} + \lambda^{-1}\|\tilde{u}\|_2^2. \tag{4.18}$$

Recalling that $\nabla\tilde{\phi}(t,x) = bA\nabla\phi(\frac{x}{A\lambda})$ and using that $(-\Delta+s)^{-1}$ is self-adjoint and the definition of $\tilde{u}_s$, we conclude that

$$\begin{aligned}
&\Re\left(\int \bar{\tilde{u}}\left[-iD, \nabla\tilde{\phi}\cdot(-i\nabla) + (-i\nabla)\cdot\nabla\tilde{\phi}\right]\tilde{u}\right)\\
=&-\frac{2b}{\lambda}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta\phi\left(\frac{x}{A\lambda}\right)|\nabla\tilde{u}_s|^2dxds\\
&+\frac{b}{2A^2\lambda^3}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta^2\phi\left(\frac{x}{A\lambda}\right)|\tilde{u}_s|^2dxds.
\end{aligned} \tag{4.19}$$

where $\tilde{u}_s(t,x) := \sqrt{\frac{2}{\pi}}\frac{1}{-\Delta+s}\tilde{u}(t,x)$, for $s > 0$. Next, we estimate the other term in (4.17). Integrating by part as well as the bound (4.2), (4.4) and (4.5), we find that

$$\begin{aligned}
&\Big| - b\Re\left(\int A\nabla\phi(\frac{x}{A\lambda})(f(u) - f(w) - f'(w)\tilde{u})\cdot\overline{\nabla\tilde{u}}\right)\\
&\quad - \frac{1}{2}\frac{b}{\lambda}\Re\left(\int \Delta\phi(\frac{x}{A\lambda})(f(u) - f(w) - f'(w)\tilde{u})\cdot\bar{\tilde{u}}\right)\Big|\\
\lesssim& b\Re\int A\nabla\phi\left(\frac{x}{A\lambda}\right)|\tilde{u}|^{\frac{2}{3}+1}\nabla\tilde{u}dx + \frac{b}{\lambda}\Re\left(\int\Delta\phi(\frac{x}{A\lambda})(\tilde{u}^{\frac{2}{3}+1}\cdot\bar{\tilde{u}}\right)\\
\lesssim& b\Re\int A\nabla\phi\left(\frac{x}{A\lambda}\right)\nabla(|\tilde{u}|^{\frac{8}{3}})dx + \frac{b}{\lambda}\Re\left(\int\Delta\phi(\frac{x}{A\lambda})(\tilde{u}^{\frac{2}{3}+1}\cdot\bar{\tilde{u}}\right)\\
\lesssim& \frac{b}{\lambda}\Re\int\Delta\phi\left(\frac{x}{A\lambda}\right)|\tilde{u}|^{\frac{8}{3}}dx\\
\lesssim& \lambda^{-\frac{1}{2}}\|\tilde{u}\|^2_{\dot{H}^{1/2}}\|\tilde{u}\|_2^{\frac{2}{3}} \lesssim \|\tilde{u}\|^2_{H^{1/2}}.
\end{aligned} \tag{4.20}$$

We consider the other term in (4.17). Integrating by parts, we obtain

$$\begin{aligned}
&- b\Re\left(\int\psi\nabla\phi(\frac{x}{A\lambda})\cdot\overline{\nabla\tilde{u}}\right) - \frac{1}{2}\frac{b}{\lambda}\Re\left(\int\psi\Delta\phi(\frac{x}{A\lambda})\bar{\tilde{u}}\right)\\
&= \Im\left(\int\left[ibA\nabla\phi(\frac{x}{A\lambda})\cdot\nabla\psi + i\frac{b}{2\lambda}\Delta\phi(\frac{x}{A\lambda})\psi\right]\bar{\tilde{u}}\right).
\end{aligned} \tag{4.21}$$

Finally, we insert (4.19), (4.20) and (4.21) into (4.17). This yield that

$$
\begin{aligned}
&\frac{1}{2}\Im\left(\nabla\tilde{\phi}\cdot(\nabla\partial_t\tilde{u}\bar{\tilde{u}}+\nabla\tilde{u}\partial_t\bar{\tilde{u}})\right)\\
=&-\frac{2b}{\lambda}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta\phi(\frac{x}{A\lambda})|\nabla\tilde{u}_s|^2dxds+\frac{b}{2A^2\lambda^3}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta^2\phi(\frac{x}{A\lambda})|\tilde{u}_s|^2dxds\\
&+\Im\left(\int\left[ibA\nabla\phi(\frac{x}{A\lambda})\cdot\nabla\psi+i\frac{b}{2\lambda}\Delta\psi(\frac{x}{A\lambda})\psi\right]\bar{\tilde{u}}\right)\\
&-b\Re\left(\int A\nabla\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u}+\frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\overline{\nabla\tilde{u}}\right)\\
&-\frac{1}{2}\frac{b}{\lambda}\Re\left(\int\Delta\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u}+\frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\bar{\tilde{u}}\right)\\
&+\mathcal{O}\left(\|\tilde{u}\|_{H^{1/2}}^2\right).
\end{aligned}
\tag{4.22}
$$

This completes the proof of lemma. □

5 Backwards Propagation in small time interval

We now apply the energy estimate of the previous section in order to establish a bootstrap argument that will be needed in the construction of ground state mass blowup solution. Let $u = u(t,x)$ be a radial solution to (1.1) defined in $[t_0, 0)$. Assume that $t_0 < t_1 < 0$ and suppose that u admits on $[t_0, t_1]$ a geometrical decomposition of the form

$$
u(t,x)=\frac{1}{\lambda^{\frac{3}{2}}(t)}\left(Q_{b(t)}+\epsilon\right)\left(s,\frac{x}{\lambda(t)}\right)e^{i\gamma(t)},
\tag{5.1}
$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ satisfies the orthogonality condition (3.3) and $b^2 + \|\epsilon\|_{H^{1/2}}^2 \ll 1$ holds. We set

$$
\tilde{u}(t,x)=\frac{1}{\lambda^{\frac{3}{2}}(t)}\epsilon\left(s,\frac{x}{\lambda(t)}\right)e^{i\gamma(t)}.
\tag{5.2}
$$

Define the constant

$$
A_0=\sqrt{\frac{e_1}{E_0}},
\tag{5.3}
$$

with the constants $e_1 = \frac{1}{2}(L_-S_1, S_1) > 0$ and $E_0 = E(u_0) > 0$.

Now we claim that the following backwards propagation estimate holds.

Lemma 5.1. ***(Backwards propagation of smallness)*** *Assume that, for some $t_1 < 0$ sufficiently close to 0, we have the bounds*

$$
\begin{aligned}
&\left|\|u\|_2^2-\|Q\|_2^2\right|\lesssim\lambda^2(t_1),\\
&\|D^{\frac{1}{2}}\tilde{u}(t_1)\|_2^2+\frac{\|\tilde{u}\|_2^2}{\lambda(t_1)}\lesssim\lambda(t_1),
\end{aligned}
$$

$$\left|\lambda(t_1) - \frac{t_1^2}{4A_0^2}\right| \lesssim \lambda^{\frac{3}{2}}(t_1), \ \left|\frac{b(t_1)}{\lambda^{\frac{1}{2}}(t_1)}\right| \lesssim \lambda(t_1),$$

where A_0 is defined in (5.3). *Then there exists a time $t_0 < t_1$ depending on A_0 such that for all $t \in [t_0, t_1]$, it holds that*

$$\|D^{\frac{1}{2}}\tilde{u}(t)\|_2^2 + \frac{\|\tilde{u}\|_2^2}{\lambda(t)} \lesssim \lambda(t),$$
$$\left|\lambda(t) - \frac{t^2}{4A_0^2}\right| \lesssim \lambda^{\frac{3}{2}}(t), \ \left|\frac{b(t)}{\lambda^{\frac{1}{2}}(t)} - \frac{1}{A_0}\right| \lesssim \lambda(t).$$

Proof. By assumption, we have $u \in C^0([t_0, t_1]; H_{rad}^{1/2+\delta}(\mathbb{R}^3))$. Hence, by this continuity and the continuity of the functions $\{\lambda(t), b(t)\}$, there exists a time t_0 such that for all $t \in [t_0, t_1]$ we have the bounds

$$\|\tilde{u}\|_2^2 \leq K\lambda^2(t), \ \|\tilde{u}(t)\|_{H^{1/2}} \leq K\lambda(t), \tag{5.4}$$

$$\left|\lambda(t) - \frac{t^2}{4A_0^2}\right| \leq K\lambda^{\frac{3}{2}}(t), \ \left|\frac{b(t)}{\lambda^{\frac{1}{2}}(t)} - \frac{1}{A_0}\right| \leq K\lambda(t), \tag{5.5}$$

with some constant $K > 0$. We now claim that the bounds stated in this lemma hold on $[t_0, t_1]$, hence improving (5.4) and (5.5) on $[t_0, t_1]$ for $t_0 = t_0(A_0) < t_1$ small enough but independent of t_1. We divide the proof into the following steps.

Step 1 Bounds on energy and $\mathbf{L^2}$ – norm. We set

$$w(x,t) = \tilde{Q}(t,x) = \frac{1}{\lambda^{\frac{3}{2}}(t)} Q_{b(t)}\left(\frac{x}{\lambda(t)}\right) e^{i\gamma(t)}. \tag{5.6}$$

Let J_A be given by above section. Applying lemma 4.1, we claim that we obtain the following coercivity estimate:

$$\frac{dJ_A}{dt} \geq \frac{b}{\lambda^2}\int |\tilde{u}|^2 + \mathcal{O}\left(\|\tilde{u}\|_{H^{1/2}}^2 + \|\tilde{u}\|_{H^{1/2}}^{\frac{1}{3}} + K^4\lambda^{\frac{5}{2}}\right). \tag{5.7}$$

Assume (5.7) holds. By the Sobolev embedding and small of ϵ, we deduce the upper bound

$$|J_A| \leq \|D^{\frac{1}{2}}\tilde{u}\|_2^2 + \frac{1}{\lambda}\|\tilde{u}\|_2^2 \tag{5.8}$$

Here we use the following inequality

$$\left|\Im\left(\int A\nabla\phi\left(\frac{x}{A\lambda}\right)\cdot\nabla\tilde{u}\bar{\tilde{u}}\right)\right| \leq \|D^{\frac{1}{2}}\tilde{u}\|_2^2 + \frac{1}{\lambda}\|\tilde{u}\|_2^2,$$

where we can see [25, Lemma F.1]. Furthermore, we also need to prove the lower bound about J_A. We compute

$$F''(w)\cdot\tilde{u}\cdot\tilde{u} = \frac{1}{6}\overline{w^{2/3}}\tilde{u}^2 + \frac{4}{3}|w|^{2/3}|\tilde{u}|^2 + \frac{1}{6}|w|^{-3/4}w^2\bar{\tilde{u}}^2,$$

and by the elementary calculation, we deduce

$$\left|F(w+\tilde{u})-F(w)-F'(w)\cdot\tilde{u}-\frac{1}{2}F''(w)\cdot\tilde{u}\cdot\tilde{u}]\right|\lesssim|u|^{\frac{8}{3}}.$$

Then, we have

$$\begin{aligned}&\int[F(w+\tilde{u})-F(w)-F'(w)\cdot\tilde{u}]\\=&\int\left[\frac{1}{6}\overline{w^{2/3}}\tilde{u}^2+\frac{4}{3}|w|^{2/3}|\tilde{u}|^2+\frac{1}{6}|w|^{-3/4}w^2\bar{\tilde{u}}^2\right]+\mathcal{O}\left(\int|\tilde{u}|^{\frac{8}{3}}\right)\\=&\frac{5}{3}\int Q^{2/3}\tilde{u}_1^2+\int Q^{2/3}\tilde{u}_2^2+\mathcal{O}\left(\|\tilde{u}\|_{L^2}^{\frac{2}{3}}\|D^{1/2}\tilde{u}\|_{L^2}^2\right).\end{aligned}$$

Hence, from above we can obtain that

$$\begin{aligned}J_A(\tilde{u})=&\frac{1}{2}\int|D^{\frac{1}{2}}\tilde{u}|^2+\frac{1}{2}\int\frac{|\tilde{u}|^2}{\lambda}-\frac{5}{3}\int Q^{2/3}\tilde{u}_1^2-\int Q^{2/3}\tilde{u}_2^2+\mathcal{O}\left(\|\tilde{u}\|_{H^{1/2}}^2\right)\\&+\frac{b}{2}\Im\left(\int A\nabla\phi\left(\frac{x-\alpha}{A\lambda}\right)\cdot\nabla\tilde{u}\bar{\tilde{u}}\right)\\\geq&c_0\left(\int|D^{\frac{1}{2}}\tilde{u}|^2+\int\frac{|\tilde{u}|^2}{\lambda}\right)-\frac{c_0}{\lambda}(\epsilon_1,Q)^2.\end{aligned}\tag{5.9}$$

Here we using the orthogonality conditions (3.3) satisfied by ϵ and the coercivity estimate for the linearized operator $L=(L_+,L_-)$. From (3.2), we can obtain

$$(\epsilon_1,Q)^2\lesssim o(\|\epsilon\|_{L^2}^2)+K^4\lambda^4(t).\tag{5.10}$$

Next, we define

$$H(t):=\|D^{\frac{1}{2}}\tilde{u}(t)\|_2^2+\frac{1}{\lambda(t)}\|\tilde{u}(t)\|_2^2.\tag{5.11}$$

By integrating (5.7) in time and using (5.8), (5.9) and (5.10), we find

$$\begin{aligned}J_A(t)\leq&J_A(t_1)-\int_t^{t_1}\left(\frac{b}{\lambda^2}\|\tilde{u}\|_{L^2}^2+\mathcal{O}(K^4\lambda^{\frac{5}{2}})\right)\\\lesssim&H(t_1)+\lambda^3.\end{aligned}$$

Therefore, we deduce that

$$H(t)\lesssim H(t_1)+\lambda^3(t),\tag{5.12}$$

which closes the bootstrap for (5.4).

Step 2 Controlling the law for the parameters. Here the treatment is quite close to the corresponding proof in [16, 25, Lemma 4.2], so we can obtain

$$\left|\lambda^{\frac{1}{2}}(t)-\frac{t}{2A_0}\right|\lesssim\left|\lambda^{\frac{1}{2}}(t_1)-\frac{t_1}{2A_0}\right|+\mathcal{O}(t^3)\lesssim t^2,\ \left|\frac{b(t)}{\lambda^{\frac{1}{2}}(t)}-\frac{1}{A_0}\right|\lesssim\lambda.$$

This completes the proof of Step 2.

Step 3 Proof of the coercivity estimate (5.7). Recalling that $w = \tilde{Q}$. Let $\mathcal{K}_A(\tilde{u})$ denote the terms in $\tilde{u}$ on the righthand side in lemma 4.1, that is, we have

$$\begin{aligned}\mathcal{K}_A(\tilde{u}) = &\frac{b}{2\lambda}\int\frac{|\tilde{u}|^2}{\lambda} - \frac{2b}{\lambda}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta\phi(\frac{x}{A\lambda})|\nabla\tilde{u}_s|^2dxds - \frac{1}{\lambda}(\tilde{u}, f'(w)\tilde{u})\\ &+ \frac{b}{2A^2\lambda^3}\int_0^{+\infty}\sqrt{s}\int_{\mathbb{R}^3}\Delta^2\phi(\frac{x}{A\lambda})|\tilde{u}_s|^2dxds\\ &- b\Re\left(\int A\nabla\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\overline{\nabla\tilde{u}}\right)\\ &- \frac{1}{2}\frac{b}{\lambda}\Re\left(\int\Delta\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\bar{\tilde{u}}\right).\end{aligned} \tag{5.13}$$

Recalling that the function $\tilde{u}_s = \tilde{u}_s(t,x)$ with the parameter $s > 0$ was defined in lemma 4.1 to be $\tilde{u}_s = \sqrt{\frac{2}{\pi}}\frac{1}{-\Delta+s}\tilde{u}$ and $\tilde{u} = \frac{1}{\lambda^{\frac{3}{2}}}\epsilon(t, \frac{x}{\lambda})$, we now claim that the following estimate holds:

$$\mathcal{K}_A(\tilde{u}) \geq \frac{C}{\lambda^{\frac{3}{2}}}\int|\epsilon|^2 + \mathcal{O}(\|\tilde{u}\|^2_{H^{1/2}} + K^4\lambda^{\frac{5}{2}}), \tag{5.14}$$

where $C > 0$ is some positive constant. Indeed, from the lemma 3.2 and the estimate (5.4) we obtain that

$$|\mathbf{Mod}(t)| \lesssim K^2\lambda^2(t). \tag{5.15}$$

First, using $\epsilon(t,x) = \lambda^{\frac{3}{2}}\tilde{u}(t, \lambda x + \alpha)$, we estimate the term

$$\begin{aligned}&\left|b\Re\left(\int A\nabla\phi(\frac{x}{A\lambda})(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\overline{\nabla\tilde{u}}\right)\right|\\ \lesssim&\|\nabla\tilde{\phi}\|_{L^\infty}\Re\left(\int(\frac{4}{3}|w|^{\frac{2}{3}}\tilde{u} + \frac{1}{3}|w|^{-\frac{4}{3}}w^2\bar{\tilde{u}})\cdot\overline{\nabla\tilde{u}}\right)\\ \lesssim&b\lambda^{-2}\Re\left(\int(\frac{4}{3}|Q_b|^{\frac{2}{3}}\epsilon + \frac{1}{3}|Q_b|^{-\frac{4}{3}}Q_b^2\bar{\epsilon})\cdot\overline{\nabla\epsilon}\right)\\ \lesssim&\lambda^{-\frac{3}{2}}\|Q_b\|^{\frac{2}{3}}_{L^\infty}\int\epsilon\nabla\epsilon\\ \lesssim&\lambda^{-\frac{1}{2}}\|\tilde{u}\|^2_{H^{1/2}} \lesssim \|\tilde{u}\|_{H^{1/2}},\end{aligned} \tag{5.16}$$

where in the last step we use the uniform decay estimate $|Q_b| \lesssim \langle x\rangle^{-4}$. By the definition of $\mathcal{K}_A(\tilde{u})$ and expressing everything in terms, we have

$$\begin{aligned}\mathcal{K}_A(\tilde{u}) \gtrsim&\frac{b}{2\lambda^2}\Big\{\int_0^\infty\sqrt{s}\int\Delta\left(\frac{x}{A}\right)|\nabla\epsilon_s|^2dxds + \int|\epsilon|^2\\ &- \frac{1}{2A^2}\int_0^\infty\sqrt{s}\int\Delta^2\phi\left(\frac{x}{A}\right)|\epsilon_s|^2dxds\\ &- \int\frac{4}{3}|Q_b|^{\frac{2}{3}}\epsilon^2 + \frac{1}{3}|Q_b|^{-\frac{4}{3}}Q_b^2|\epsilon|^2\Big\} + \mathcal{O}(\|\tilde{u}\|_{H^{1/2}}).\end{aligned}$$

Furthermore, thanks to lemma A.3, we have

$$\left|\frac{1}{A^2}\int_{s=0}^{+\infty}\sqrt{s}\int\Delta^2\phi_A|\epsilon_s|^2dxds\right|\leq\frac{1}{A}\|\epsilon\|_2^2.$$

Recalling the definitions of $L_{+,A}$ and $L_{-,A}$ in (A.1) and (A.2), respectively. We deduce that

$$K_A(\tilde{u})=\frac{a}{2\lambda^2}\left\{(L_{+,A}\epsilon_1,\epsilon_1)+(L_{-,A}\epsilon_2,\epsilon_2)+\mathcal{O}\left(\frac{1}{A}\|\epsilon\|_2^2\right)\right\}+\mathcal{O}(\|\tilde{u}\|_{H^{1/2}}).$$

Next, we recall that $b\sim\lambda^{\frac{1}{2}}$ due to the above. Hence, by lemma A.2 and choosing the $A>0$ sufficiently large, we deduce from previous estimates that

$$K_A(\tilde{u})\gtrsim\frac{1}{\lambda^{\frac{3}{2}}}\left\{\int|\epsilon|^2-(\epsilon_1,Q)^2\right\}\gtrsim\frac{1}{\lambda^{\frac{3}{2}}}\int|\epsilon|^2+\mathcal{O}(\|\tilde{u}\|_{H^{1/2}}+K\lambda^{\frac{5}{2}}).\tag{5.17}$$

Step 4 Controlling the remainder terms in $\frac{d}{dt}J_A$. We now control the terms that appear in lemma 4.1 and contain ψ. Here we recall that $w=\tilde{Q}$ and (4.3), which yields

$$\psi=\frac{1}{\lambda^{\frac{3}{2}+1}}\left[i\left(b_s+\frac{1}{2}b^2\right)\partial_bQ_b-i\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q_b+\tilde{\gamma}_sQ_b+\Phi_b\right]\left(\frac{x}{\lambda}\right)e^{i\gamma}.$$

Here Φ_b is the error term given in lemma 2.1. In fact, by the estimate for Q_b and Φ_b from lemma 2.1 and recalling (5.15), we deduce the rough pointwise estimate

$$\left|\nabla^k\psi(x)\right|\lesssim\frac{1}{\lambda^{\frac{5}{2}+k}}\left\langle\frac{x}{\lambda}\right\rangle^{-4}K^2\lambda^2,\ \text{for}\ k=0,1.\tag{5.18}$$

Hence

$$\|\nabla^k\psi\|_2\lesssim K^2\lambda^{1-k},\ \text{ for } k=0,1.\tag{5.19}$$

In particular, we obtain the following bounds

$$\begin{aligned}&\lambda\|\psi\|_2^2\lesssim K^4\lambda^3,\\&\left|\Im\left(\int[ibA\nabla\phi\left(\frac{x}{A\lambda}\right)\cdot\nabla\psi+i\frac{b}{2\lambda}\Delta\phi\left(\frac{x}{A\lambda}\right)\psi]\bar{\tilde{u}}\right)\right|\\\lesssim&\lambda^{\frac{1}{2}}\|\nabla\psi\|_2\|\tilde{u}\|_2+\lambda^{-\frac{1}{2}}\|\psi\|_2\|\tilde{u}\|_2\\\lesssim&K^2\lambda^{\frac{1}{2}}\|\epsilon\|_2\lesssim o\left(\frac{\|\epsilon\|_2^2}{\lambda^{\frac{3}{2}}}\right)+K^4\lambda^{\frac{5}{2}}.\end{aligned}\tag{5.20}$$

Write $\psi=\psi_1+\psi_2$ with $\psi_2=\mathcal{O}(b|\mathbf{Mod}|+b^5)=\mathcal{O}(\lambda^{\frac{5}{2}})$, that is, we denote

$$\psi_1=\frac{1}{\lambda^{\frac{3}{2}+1}}\left[-\left(b_s+\frac{1}{2}b^2\right)S_1-i\left(\frac{\lambda_s}{\lambda}+b\right)\Lambda Q+\tilde{\gamma}_sQ\right]\left(\frac{x}{\lambda}\right)e^{i\gamma}.$$

Let us first deal with estimating the contributions coming from ψ_2. Indeed, since $b^2\sim\lambda$ we note that $\psi_2=\mathcal{O}(\lambda^{\frac{5}{2}})$ satisfies the pointwise bound

$$|\nabla^k \psi_2(x)| \leq \frac{1}{\lambda^{\frac{3}{2}+k+1}} \left\langle \frac{x}{\lambda} \right\rangle^{-4} K^2 \lambda^{\frac{5}{2}}, \text{ for } k = 0, 1.$$

Hence

$$\|\nabla^k \psi_2\|_2 \leq K^2 \lambda^{\frac{3}{2}-k}, \text{ for } k = 0, 1.$$

Therefore, we obtain that

$$\begin{aligned}
&\left|\Re\left(\int \left[-D\psi_2 - \frac{\psi_2}{\lambda} + (\frac{1}{3}+1)|w|^{\frac{2}{3}}\psi_2 + \frac{1}{3}|w|^{\frac{2}{3}-2}w^2\bar{\psi}_2\right]\bar{\tilde{u}}\right)\right| \\
\leq &\left(\|\nabla\psi_2\|_2 + \lambda^{-1}\|\psi_2\|_2 + \|w\|_2^{\frac{1}{3}}\|\psi_2\|_6\right)\|\epsilon\|_2 \\
\leq &\left(\|\nabla\psi_2\|_2 + \lambda^{-1}\|\psi_2\|_2 + \|w\|_2^{\frac{2}{3}}\|\nabla\psi_2\|_2\right)\|\epsilon\|_2 \\
\leq &K^2\lambda^{\frac{1}{2}}\|\epsilon\|_2 \leq o\left(\frac{\|\epsilon\|_2^2}{\lambda^{\frac{3}{2}}}\right) + K^4\lambda^{\frac{5}{2}}, \qquad (5.21)
\end{aligned}$$

which is acceptable. Here we used the Hölder inequality and Sobolev inequality.

Finally, we estimate the terms that contain ψ_1 have the same bounded. Indeed, using (5.15) once again and $|b| \leq \lambda^{\frac{1}{2}}$, as well as $(\epsilon_2, L_- S_1) = (\epsilon_2, \Lambda Q) = \mathcal{O}(b\|\epsilon\|_2)$, thanks to the orthogonality conditions for ϵ, we find the following bound

$$\begin{aligned}
&\left|\Re\left(\int \left[-D\psi_1 - \frac{\psi_1}{\lambda} + (\frac{1}{3}+1)|w|^{\frac{2}{3}}\psi_1 + \frac{1}{3}|w|^{\frac{2}{3}-2}w^2\bar{\psi}_1\right]\bar{\tilde{u}}\right)\right| \\
\leq &\frac{|\mathbf{Mod}(t)|}{\lambda^2}\left[|(\epsilon_2, L_- S_1)| + |(\epsilon_2, L_- Q)| + \mathcal{O}(b\|\epsilon\|_2)\right] \\
&+ \frac{1}{\lambda^2}\left|\frac{\lambda_s}{\lambda} + b\right| |(\epsilon_1, L_+\Lambda Q)| \\
\leq &K^2\lambda^{\frac{1}{2}}\|\epsilon\|_2 + \frac{K^2\lambda\|\epsilon\|_2 + \lambda^{\frac{5}{2}}}{\lambda^2}(K\lambda^{\frac{1}{2}}\|\epsilon\|_2 + K^2\lambda^2) \\
\leq &o\left(\frac{\|\epsilon\|_2^2}{\lambda^{\frac{3}{2}}}\right) + K^4\lambda^{\frac{5}{2}}, \qquad (5.22)
\end{aligned}$$

Here we used that $L_+\Lambda Q = -Q$ together with the improved bound in lemma 3.2, combined with the fact that $|(\epsilon_1, Q)| \leq \lambda^{\frac{1}{2}}\|\epsilon\|_2 + K^2\lambda^2$, which follows from $\|\epsilon\|_2 \leq \lambda$ and the conservation of L^2-norm. And the proof of this lemma is complete. □

6 Proof of the Theorem 1.1

In this section, we prove the following result.

Theorem 6.1. *Let $\gamma_0 \in \mathbb{R}$, $\delta > 0$ be small enough and $E_0 > 0$ be given. Then there exist a time $t_0 < 0$ and a radial solution $u \in C^0([t_0, 0); H^{\frac{1}{2}+\delta}(\mathbb{R}^3))$ of* (1.1) *such that u blowup at time $T = 0$ with*

$$E(u(t)) = E_0 \text{ and } \|u(t)\|_2^2 = \|Q\|_2^2.$$

Furthermore, we have $\|D^{\frac{1}{2}}u(t)\|_2 \sim t^{-1}$ *as* $t \to 0^-$, *and* u *is of the form*

$$u(t,x) = \frac{1}{\lambda^{\frac{3}{2}}(t)}[Q_{b(t)} + \epsilon]\left(t, \frac{x}{\lambda(t)}\right)e^{i\gamma(t)} = \tilde{Q} + \tilde{\epsilon},$$

and ϵ *satisfies the orthogonality condition* (3.3). *Finally, the following estimates hold:*

$$\begin{aligned}
&\|\tilde{\epsilon}\|_2 \lesssim \lambda, \ \|\tilde{\epsilon}\|_{H^{1/2}} \lesssim \lambda^{\frac{1}{2}}, \\
&\lambda(t) - \frac{t^2}{4A_0^2} = \mathcal{O}(\lambda^2), \ \frac{b}{\lambda^{\frac{1}{2}}} - \frac{1}{A_0} = \mathcal{O}(\lambda), \\
&\gamma(t) = -\frac{4A_0^2}{t} + \gamma_0 + \mathcal{O}(\lambda^{\frac{1}{2}}).
\end{aligned}$$

Here $A_0 > 0$ *is the constant defined in* (5.3).

Proof. Motivated by [16, 25], we are now ready to complete our result by the following two steps.

Step 1. Backwards uniform bounds.

Let $t_n \to 0^-$ be a sequence of negative times and let u_n be the radial solution to (1.1) with initial data at $t = t_n$ given by

$$u_n(t_n, x) = \frac{1}{\lambda_n(t_n)}Q_{b_n(t_n)}\left(\frac{x}{\lambda_n(t_n)}\right)e^{i\gamma_n(t_n)}, \tag{6.1}$$

where the sequence $b_n(t_n)$ and $\lambda_n(t_n)$ are given by

$$b_n(t_n) = -\frac{t_n}{2A_0}, \ \lambda_n(t_n) = \frac{t_n^2}{4A_0^2}, \ \gamma_n(t_n) = \gamma_0 - \frac{4A_0^2}{t_n}. \tag{6.2}$$

By lemma 2.1, we have

$$\int |u_n(t_n)|^2 = \int |Q|^2 + \mathcal{O}(t_n^4), \tag{6.3}$$

and $\tilde{\epsilon}_n(t_n) = 0$ by construction. Thus u_n satisfies the assumptions of lemma 5.1. Hence we can find a backwards time t_0 independent of n such that for all $t \in [t_0, t_n)$ we have the geometric decomposition

$$u_n(t,x) = \frac{1}{\lambda_n(t)}Q_{b_n(t)}\left(\frac{x}{\lambda_n(t)}\right) + \tilde{\epsilon}_n(t,x), \tag{6.4}$$

with the uniform bounds (see (5.12) and lemma 5.1) given by

$$\|D^{\frac{1}{2}}\tilde{\epsilon}_n(t)\|_2^2 + \frac{\|\tilde{\epsilon}_n(t)\|_2^2}{\lambda_n(t)} \lesssim \lambda_n^3(t), \tag{6.5}$$

$$\left|\lambda_n(t) - \frac{t^2}{4A_0^2}\right| \lesssim K\lambda_n^2(t), \ \left|\frac{b_n(t)}{\lambda_n^{\frac{1}{2}}(t)} - \frac{1}{A_0}\right| \lesssim K\lambda_n(t). \tag{6.6}$$

and

$$\|\tilde{\epsilon}_n(t)\|_{H^{\frac{1}{2}+\theta}} \lesssim \lambda_n^{\frac{3}{4}-\frac{5\theta}{2}}, \text{ where } \theta \in (0, \frac{3}{10}). \tag{6.7}$$

which we prove in step 2.

Next, we conclude that $\{u_n(t_0)\}_{n=1}^{\infty}$ converges strongly in $H^s_{rad}(\mathbb{R}^3)$ (after passing to a subsequence if necessary), where $s \in [0, \frac{1}{2}+\theta)$. Indeed, from the uniform bound $\|\tilde{\epsilon}_n(t_0)\|_{H^{\frac{1}{2}+\theta}_{rad}} \lesssim 1$ we can assume (after passing to a subsequence if necessary) that $u_n(t_0) \rightharpoonup u_0$ weakly in $H^s_{rad}(\mathbb{R}^3)$ for any $s \in [0, \frac{1}{2}+\theta]$. Moreover, we note the uniform bound

$$\begin{aligned}\left|\frac{d}{dt}\int \chi_R |u_n|^2\right| =& \left|\int \chi_R\left((u_n)_t \bar{u}_n + u_n(\bar{u}_n)_t\right)\right| \\ =& \left|\int u_n \left[\chi_R, iD\right] \bar{u}_n\right| \\ \lesssim& \|\nabla \chi_R\|_\infty \|u_n\|_2^2 \lesssim \frac{1}{R},\end{aligned} \tag{6.8}$$

with a smooth cutoff function $\chi_R(x) = \chi(\frac{x}{R})$ where $\chi(x) \equiv 0$ for $|x| \le 1$ and $\chi(x) \equiv 1$ for $|x| \ge 2$. Note that we used the commutator estimate (which we can see [32])

$$\|[\chi_R, D]\|_{L^2 \to L^2} \lesssim \|\nabla \chi_R\|_\infty.$$

By integrating the previous bound from t_1 to t_0 and using the previous estimate (6.1) and (6.2), we derive that for every $\tau > 0$ there is a radius $R > 0$ such that

$$\int_{|x| \ge R} |u_n((t_0)|^2 \le \tau \text{ for all } n \ge 1.$$

Combining this fact with the weak convergence of $\{u_n(t_0)\}_{n=1}^{\infty}$ in $H^s_{rad}(\mathbb{R}^3)$, we deduce that

$$u_n(t_0) \to u_0(t_0) \text{ strongly in } H^s_{rad}(\mathbb{R}^3) \text{ for every } s \in [0, \frac{1}{2}+\theta). \tag{6.9}$$

Thus, by local well-posedness in $H^{\frac{1}{2}+\delta}_{rad}(\mathbb{R}^3)$ (see [20]), we can solve the Cauchy problem (1.1) and find

$$u \in C([t_0, T); H^{\frac{1}{2}+\delta}_{rad}(\mathbb{R}^3)), \text{ for } 0 < \delta < \theta,$$

and obtain

$$u_n(t) \to u(t) \text{ strongly in } H^s_{rad}(\mathbb{R}^3)\ s \in [0, \frac{1}{2}+\delta) \text{ for } t \in [t_0, T), \tag{6.10}$$

where $0 > T > t_0$ is the lifetime of u on the right. Moreover, u admits, for $t < \min\{T, 0\}$, a geometrical decomposition of the form state in above with

$$b_n(t) \to b(t),\ \lambda_n(t) \to \lambda(t),\ \gamma_n(t) \to \gamma(t). \tag{6.11}$$

Furthermore, we deduce that

$$\|\tilde{\epsilon}(t)\|_2 \lesssim \lambda \text{ and } \|\tilde{\epsilon}(t)\|_{H^{1/2}} \lesssim \lambda^{\frac{1}{2}}.$$

for $t \in [t_0, T)$. In particular, this implies that $u(t)$ blows up at time $T = 0$ such that

$$\|D^{\frac{1}{2}}u\|_2^2 \sim \lambda^{-1}(t) \sim |t|^{-2} \text{ as } t \to 0^-.$$

In addition, we deduce from L^2-mass conversation and the strong convergence that

$$\|u\|_2 = \lim_{n\to\infty} \|u_n(t_n)\|_2 = \|Q\|_2.$$

As for the energy, we note that

$$E(u(t)) = \frac{b^2}{\lambda}e_1 + o(1) \to E_0 \text{ as } t \to 0^-,$$

by the choice of A_0, $b_n(t_n)$ and $\lambda_n(t_n)$. By energy conversation, this implies that

$$E(u) = E_0.$$

Next, we recall that rough bound

$$|\tilde{\gamma}_s| \lesssim \lambda_n.$$

Therefore, using that $\frac{ds}{dt} = \lambda^{-1}$ and the estimates for λ_n

$$\left|\frac{d}{dt}\left(\gamma_n + \frac{4A_0^2}{t}\right)\right| = \frac{1}{\lambda_n}\left|(\gamma_n)_s - \frac{4A_0^2\lambda_n}{t^2}\right| = \frac{1}{\lambda_n}\left|(\tilde{\gamma}_n)_s - \frac{4A_0^2\lambda_n}{t^2} + 1\right| \lesssim 1.$$

Integrating this bound and using (6.2) and $\lambda \sim t^2$, we find

$$\gamma_n(t) + \frac{4A_0^2}{t} = \gamma_0 + \mathcal{O}(\lambda^{\frac{1}{2}}),$$

whence the claim for γ follows, since we have $\lambda \sim t^2$.

Step 2. $H^{\frac{1}{2}+\theta}$ bound.

It remains to prove the $H^{1/2+\theta}$ bound (6.7). Our point of departure is again the identity

$$i\partial_t\tilde{\epsilon}_n = D\tilde{\epsilon}_n - \psi_n - F_n,$$

where

$$F_n = |\tilde{Q}_n + \tilde{\epsilon}_n|^{\frac{2}{3}}(\tilde{Q}_n + \tilde{\epsilon}_n) - |\tilde{Q}_n|^{\frac{2}{3}}\tilde{Q}_n.$$

We plan to obtain a $H^{1/2+\theta}$-bound on $\tilde{\epsilon}_n$, taking advantage of the bounds at time $t_n \sim \lambda_n^{1/2}$ and those assumed for $t \in [t_0, t_n]$. We make partition of the interval $[t_0, t_n]$ into

$$t_0 = s_0 < s_1 < \cdots < s_N = t_n,\ s_j - s_{j-1} = h,\ j = 1, \cdots, N,$$

where

$$h \sim \lambda_n^3, N \sim (t_n - t_0)/h \sim (\lambda_n)^{-3}.$$

We can obtain estimates in each interval $\Delta_j = [s_{j-1}, s_j]$. For the purpose we follow [2] and for any time interval $I = (a, b)$ we use estimate (see Proposition 1.3 in [2])

$$\left\| \int_a^t e^{i(t-s)\sqrt{-\Delta}} F(s) ds \right\|_{X_I} \leq C \| [x]_\delta^{\frac{1}{\bar{q}}} F \|_{L^{\bar{q}'}(I; L^2(\mathbb{R}^3))},$$

where the norm in the space X_I is given by

$$\|u\|_{X_I} = \|u(t,x)\|_{L_I^\infty H^1(\mathbb{R}^3)} + \left\| [x]_\delta^{-\frac{1}{q}} \nabla_x u(t,x) \right\|_{L_I^q L^2(\mathbb{R}^3)} + \left\| [x]_\delta^{-\frac{1}{q}} u(t,x) \right\|_{L_I^q L^2(\mathbb{R}^3)},$$

and $q, \bar{q} > 2$ and $\delta \in (0,1)$ satisfy

$$\frac{1-\delta}{\bar{q}} + \frac{1-\delta}{q} = \frac{2}{3} \text{ and } [x]_\delta = |x|^{1+\delta} + |x|^{1-\delta}.$$

Using the same argument as the one in [2, section 2.3], in the interval $\Delta_N = [s_{N-1}, s_N]$, we can obtain

$$\begin{aligned} \|\tilde{\epsilon}_n\|_{X_{\Delta_N}} \lesssim & h^{1-\frac{1}{\bar{q}}-\frac{1}{q}} \left(\|\tilde{Q}_n\|^{\frac{2}{3}}_{L^\infty_{\Delta_N} H^1(\mathbb{R}^3)} + \|\tilde{\epsilon}_n\|^{\frac{2}{3}}_{L^\infty_{\Delta_N} H^1(\mathbb{R}^3)} \right) \|\tilde{\epsilon}_n\|_{X_{\Delta_N}} \\ & + \left\| |x|^{-\frac{1}{2}} |x|^{\frac{1-\delta}{\bar{q}}} (|x|^{\frac{1}{2}} \tilde{\epsilon}_n) \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^{\bar{q}'}_{\Delta_N} L^2(|x|\leq 2)} + \left\| \tilde{\epsilon}_n \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^1_{\Delta_N} L^2(|x|\geq 2)} \\ & + h \|\psi_n\|_{X_{\Delta_N}} \\ \lesssim & h^{1-\frac{1}{\bar{q}}-\frac{1}{q}} \|\tilde{\epsilon}_n\|_{X_{\Delta_N}} \left(\lambda_n^{-2/3} + \|\tilde{\epsilon}_n\|^{\frac{2}{3}}_{L^\infty_{\Delta_N} H^1(\mathbb{R}^3)} \right) + \left\| |x|^{\frac{1}{6}-\frac{1-\delta}{q}} (|x|^{\frac{1}{2}} \tilde{\epsilon}_n) \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^{\bar{q}'}_{\Delta_N} L^2(|x|\leq 2)} \\ & + \left\| \tilde{\epsilon}_n \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^1_{\Delta_N} L^2(|x|\geq 2)} + h \|\psi_n\|_{X_{\Delta_N}}, \end{aligned} \tag{6.12}$$

due to the bounds (4.2), the choice (5.6) of w and $\tilde{\epsilon}_n(t_n) = 0$. Now taking $q < 6 - 6\delta$, such that $\bar{q} \geq 2$ and $\bar{q}$ close to 2, we estimate the second term

$$\begin{aligned} & \left\| |x|^{\frac{1}{6}-\frac{1-\delta}{q}} (|x|^{\frac{1}{2}} \tilde{\epsilon}_n) \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^{\bar{q}'}_{\Delta_N} L^2(|x|\leq 2)} \\ \lesssim & \left\| |x|^{\frac{1}{6}-\frac{1-\delta}{q}} \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^{\bar{q}'}_{\Delta_N} L^2(|x|\leq 2)} \|\tilde{\epsilon}_n\|_{L^\infty_{\Delta_N} H^1(\mathbb{R}^3)} \\ \lesssim & |\Delta_N|^{1/\bar{q}'} \lambda_n^{\frac{1}{2}} \left\| |x|^{\frac{1}{6}-\frac{1-\delta}{q}} \nabla \tilde{Q}_n \right\|_{L^\infty_{\Delta_N} L^2(|x|\leq 2)} \|\tilde{\epsilon}_n\|_{L^\infty_{\Delta_N} H^1(\mathbb{R}^3)} \\ \lesssim & |\Delta_N|^{1/\bar{q}'} \lambda_n^{\frac{1}{2}} \lambda_n^{-1+\frac{1}{6}-\frac{1-\delta}{q}} \|\tilde{\epsilon}_n\|_{L^\infty_{\Delta_N} H^1(\mathbb{R}^3)}, \end{aligned} \tag{6.13}$$

where we used the estimate $\left\| |x|^\alpha \nabla \tilde{Q}_n \right\|_{L^2} \lesssim \lambda_n^{-1+\alpha}$ and $\widetilde{Q}_n \gtrsim 1$ that come from (5.6) and we also used the estimate $\left\| |x|^{\frac{1}{2}} u \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \|u\|_{H^1(\mathbb{R}^3)}$.

For the third term, we have

$$\left\| \left\| \tilde{\epsilon}_n \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^2(|x|\geq 2)} \right\|_{L^1_{\Delta_N}} \lesssim \left\| \|\tilde{\epsilon}_n\|_{L^3(\mathbb{R}^3)} \left\| \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^6(|x|\geq 2)} \right\|_{L^1_{\Delta_N}}$$

$$
\begin{aligned}
&\lesssim \Big\| \|\tilde{\epsilon}_n\|_{H^{1/2}(\mathbb{R}^3)} \|\lambda_n^{-\frac{3}{2}} \tilde{Q}_n^{\frac{2}{3}}\|_{L^6(|x\geq 2|)} \Big\|_{L^1_{\Delta_N}} \\
&\lesssim \Big\| \lambda_n^{\frac{3}{2}} \lambda_n^{-\frac{3}{2}} \|\nabla \tilde{Q}_n\|_{L^2}^{\frac{1}{2}} \|\tilde{Q}_n\|_{L^2}^{\frac{1}{6}} \Big\|_{L^1_{\Delta_N}} \\
&\lesssim \Big\| \lambda_n^{-\frac{1}{2}} \Big\|_{L^1_{\Delta_N}} \lesssim \lambda_n^{-\frac{1}{2}} |\Delta_N|. \qquad (6.14)
\end{aligned}
$$

Here we used the estimate

$$
|\nabla Q_n| \lesssim |Q_n| \text{ for } |x| \geq 2. \qquad (6.15)
$$

To check this estimate we use Proposition 3.1 from [14] and therefore we have $Q \sim \langle x \rangle^{-4}$ and $Q \in H^2(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3)$. Hence, we deduce that $|\nabla Q| \lesssim Q$ for $|x| \geq 2$. Hence we only have to check the estimate

$$
|R_k| \lesssim Q \text{ for } |x| \geq 2 \text{ and } k = 1, 2, 3, 4. \qquad (6.16)
$$

Indeed, since $L_- R_1 = \Lambda Q$, (2.4) and R_1 is a radial function, we deduce that

$$
(D+1) R_1 = Q^{\frac{2}{3}} R_1 + \Lambda Q = f \lesssim |x|^{-\frac{11}{3}}, \text{ for } |x| \geq 2. \qquad (6.17)
$$

Using the [14, Lemma C.3], we have

$$
|R_1| \lesssim |x|^{-\frac{10}{3}}.
$$

Hence, the right hand side of (6.17) have the decay estimate $|f| \lesssim |x|^{-4}$ for $|x| \geq 2$. On the other hand, by the similar argument as [16, 25, Lemma A.1], we have

$$
\|\langle x \rangle^4 g\|_{L^\infty} = \|\langle x \rangle^4 L_\pm f\|_{L^\infty} \lesssim \|\langle x \rangle^4 f\|_{L^\infty}.
$$

From the above estimates, we have

$$
|R_1| \lesssim |x|^{-4} \lesssim Q \text{ for } |x| \geq 2.
$$

By the same argument as R_1, we can obtain that $|R_k| \lesssim Q$ for $|x| \geq 2$ and $k = 2, 3, 4$.

Now we insert (6.13) and (6.14) into (6.12). Combined with the $\|\tilde{\epsilon}_n\|_{H^1}$ is small and $h \sim \lambda_n^3$. We have

$$
\|\tilde{\epsilon}_n\|_{X_{\Delta_N}} \lesssim \lambda_n^{\frac{5}{2}}.
$$

The term $\|\psi_n\|_{X_{\Delta_N}}$ can be estimated by the aid of interpolation between (5.18) and (5.19) so that we have

$$
\|\nabla \psi_n\|_{L^m} \lesssim \lambda_n^{-\frac{3}{2} + \frac{3}{m}}, \ m \in [2, \infty]
$$

so that

$$
\|\psi_n\|_{X_{\Delta_N}} \lesssim 1. \qquad (6.18)
$$

In fact, taking $m = 3$ we can write

$$
\Big\| \|[x]_\delta^{-1/q} \nabla \psi_n\|_{L^2(|x|\leq 1)} \Big\|_{L^q(\Delta_N)} \lesssim h^{1/q} \|[x]_\delta^{-1/q}\|_{L^6(|x|\leq 1)} \|\nabla \psi_n\|_{L^3(|x|\leq 1)} \lesssim \lambda_n^{3/q - 1/2} \ll 1.
$$

In a similar way we estimate $\left\| \| [x]_\delta^{-1/q} \psi_n \|_{L^2(|x|\le 1)} \right\|_{L^q(\Delta_N)}$, while the far field parts of the norm satisfy

$$\left\| \| [x]_\delta^{-1/q} \nabla^k \psi_n \|_{L^2(|x|\ge 1)} \right\|_{L^q(\Delta_N)} \lesssim \|\nabla^k \psi_n\|_{L^2} h^{1/q} \lesssim 1, \ k = 0, 1.$$

Hence we arrive at (6.18).

For the other intervals Δ_j, $j = 1, \cdots, N-1$, we have

$$\begin{aligned} \|\tilde{\epsilon}_n\|_{X_{\Delta_j}} \lesssim & \|\tilde{\epsilon}_n(s_j)\|_{H^1} + h^{1-\frac{1}{\bar{q}}-\frac{1}{q}} \left(\|\tilde{Q}_n\|_{L^\infty_{\Delta_j} H^1(\mathbb{R}^3)} + \|\tilde{\epsilon}_n\|_{L^\infty_{\Delta_j} H^1(\mathbb{R}^3)} \right) \|\tilde{\epsilon}_n\|_{X_{\Delta_j}} \\ & + \left\| |x|^{-\frac{1}{2}} |x|^{\frac{1-\delta}{\bar{q}}} (|x|^{\frac{1}{2}} \tilde{\epsilon}_n) \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^{\bar{q}'}_{\Delta_N} L^2(|x|\le 2)} + \left\| \tilde{\epsilon}_n \tilde{Q}_n^{-\frac{1}{3}} \nabla \tilde{Q}_n \right\|_{L^1_{\Delta_N} L^2(|x|\ge 2)} \\ & + h \|\psi_n\|_{L^\infty_{\Delta_j} H^1(\mathbb{R}^3)}. \end{aligned}$$

By the similar argument as above, we deduce

$$\|\tilde{\epsilon}_n\|_{X_{\Delta_j}} \lesssim \|\tilde{\epsilon}_n(s_j)\|_{H^1} + \lambda_n^{\frac{5}{2}} \lesssim \|\tilde{\epsilon}_n\|_{X_{\Delta_{j+1}}} + \lambda_n^{\frac{5}{2}},$$

and inductively we find

$$\|\tilde{\epsilon}_n\|_{X_{\Delta_j}} \lesssim (N-j) \|\tilde{\epsilon}_n\|_{X_{\Delta_N}} + \lambda_n^{\frac{5}{2}} \lesssim (N-j+1) \lambda_n^{\frac{5}{2}}.$$

Therefore, we have

$$\|\tilde{\epsilon}_n\|_{X_{[t_0,t_n]}} \lesssim \sup_{1\le j\le N} \|\tilde{\epsilon}_n\|_{X_{\Delta_j}} \lesssim N \lambda_n^{5/2} \lesssim \lambda_n^{-\frac{1}{2}}.$$

This means

$$\|\nabla \tilde{\epsilon}_n\|_{L^2} \lesssim \lambda_n^{-\frac{1}{2}}.$$

By using the interpolation inequality and (6.5), we can obtain

$$\|D^{\frac{1}{2}+\theta} \tilde{\epsilon}_n\|_{L^2} \lesssim \|\nabla \tilde{\epsilon}_n\|_{L^2}^{\frac{1}{2}+\theta} \|\tilde{\epsilon}_n\|_{L^2}^{\frac{1}{2}-\theta} \lesssim \lambda_n^{\frac{3}{4}-\frac{5\theta}{2}}.$$

Hence, we arrive at (6.7).

Now the proof of this Theorem is complete. □

A Appendix

In this section, we will give some lemmas, which are very important but the proof is relatively simple. By standard argument as [16, 25], we can obtain the following results. Here we omit the details.

In the following, we assume that $A > 0$ is a sufficiently large constant. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the smooth cutoff function introduced in Section 4. For $\epsilon = \epsilon_1 + i\epsilon_2 \in H^{1/2}(\mathbb{R}^3)$, we consider the quadratic forms

$$L_{+,A}(\epsilon_1) := \int_{s=0}^{\infty} \sqrt{s} \int \Delta\phi_A |\nabla(\epsilon_1)_s|^2 dx ds + \int |\epsilon_1|^2 - \frac{3}{8} \int Q^{\frac{2}{3}} |\epsilon_1|^2 \tag{A.1}$$

$$L_{-,A}(\epsilon_2) := \int_{s=0}^{\infty} \sqrt{s} \int \Delta\phi_A |\nabla(\epsilon_2)_s|^2 dx ds + \int |\epsilon_2|^2 - \int Q^{\frac{2}{3}} |\epsilon_2|^2, \tag{A.2}$$

where $\Delta\phi_A = \Delta(\phi(\frac{x}{A}))$. As in lemma 4.1, we denote

$$u_s = \sqrt{\frac{2}{\pi}} \frac{1}{-\Delta + s} u, \quad \text{for } s > 0. \tag{A.3}$$

We start with the following simple identity.

For $u \in H^{1/2}(\mathbb{R}^3)$, we have

$$\int_0^{\infty} \sqrt{s} \int_{\mathbb{R}^3} |\nabla u_s|^2 dx ds = \|D^{1/2} u\|_2^2. \tag{A.4}$$

Indeed, by applying Fubini's theorem and using Fourier transform, we find that

$$\int_0^{\infty} \sqrt{s} \int_{\mathbb{R}^3} |\nabla u_s|^2 dx ds = \frac{2}{\pi} \int_{\mathbb{R}^3} \int_0^{\infty} \frac{\sqrt{s}}{(\xi^2 + s)^2} |\xi|^2 |\hat{u}(\xi)|^2 d\xi = \|D^{1/2} u\|_2^2.$$

In particular, we have

$$\frac{2}{\pi} \int_0^{\infty} \sqrt{s} \int_{\mathbb{R}^3} |D^{\alpha} u_s|^2 dx ds = \|D^{\alpha - \frac{1}{2}} u\|_2^2. \tag{A.5}$$

Next, we establish a technical result, which show that, when taking the limit $A \to +\infty$, the quadratic form $\int_0^{\infty} \sqrt{s} \int \Delta\phi_A |\nabla u_s|^2 dx ds + \|u\|_2^2$ defines a weak topology that serves as a useful substitute for weak convergence in $H^{1/2}(\mathbb{R}^3)$. The precise statement reads as follows.

Lemma A.1. *Let $A_n \to \infty$ and suppose that $\{u_n\}_{n=1}^{\infty}$ is a sequence in $H^{1/2}(\mathbb{R}^3)$ such that*

$$\int_0^{\infty} \sqrt{s} \int \Delta\phi_{A_n} |\nabla(u_n)_s|^2 dx ds + \|u_n\|_2^2 \leq C,$$

for some constant $C > 0$ independent of n. Then, after possibly passing to a subsequence of $\{u_n\}_{n=1}^{\infty}$, we have that

$$u_n \rightharpoonup u \text{ weakly in } L^2(\mathbb{R}^3) \text{ and } u_n \to u \text{ strongly in } L^2_{loc}(\mathbb{R}^3),$$

and $u \in H^{1/2}(\mathbb{R}^3)$. Moreover, we have the bound

$$\|D^{1/2} u\|_2^2 \leq \liminf_{n \to \infty} \int_0^{\infty} \sqrt{s} \int \Delta\phi_{A_n} |\nabla(u_n)_s|^2 dx ds.$$

Lemma A.2. *Let $L_{+,A}(\epsilon_1)$ and $L_{-,A}(\epsilon_2)$ be the quadratic forms defined in* (A.1) *and* (A.2)*, respectively. Then there exist constants $C_0 > 0$ and $A_0 > 0$ such that, for all $A \geq A_0$ and all $\epsilon = \epsilon_1 + i\epsilon_2 \in H^{1/2}_{rad}(\mathbb{R}^3)$, we have the coercivity estimate*

$$(L_{+,A}\epsilon_1, \epsilon_1) + (L_{-,A}\epsilon_2, \epsilon_2) \geq C_0 \int |\epsilon|^2 - \frac{1}{C_0} \Big\{ (\epsilon_1, Q)^2 + (\epsilon_1, S_1)^2 + |(\epsilon_2, \rho_1)|^2 \Big\}.$$

Here S_1 is the unique functions such that $L_- S_1 = \Lambda Q$ with $(S_1, Q) = 0$ and the function ρ_1 is defined in (3.4)*.*

Lemma A.3. *For any* $u \in L^2(\mathbb{R}^3)$*, we have the bound*

$$\left| \int_{s=0}^{+\infty} \sqrt{s} \int \Delta^2 \phi_A |u_s|^2 dx ds \right| \leq \frac{1}{A} \|u\|_2^2.$$

Lemma A.4. *Let* $L_+\epsilon_1$ *and* $L_-\epsilon_2$ *be the defined as section 2, respectively. Then there exist constants* $C_0 > 0$ *and* $A_0 > 0$ *such that, for all* $A \geq A_0$ *and all* $\epsilon = \epsilon_1 + i\epsilon_2 \in H^{1/2}_{rad}(\mathbb{R}^3)$*, we have the coercivity estimate*

$$(L_+\epsilon_1, \epsilon_1) + (L_-\epsilon_2, \epsilon_2) \geq C_0 \int |\epsilon|^2 - \frac{1}{C_0} \Big\{ (\epsilon_1, Q)^2 + (\epsilon_1, S_1)^2 + |(\epsilon_2, \rho_1)|^2 \Big\}.$$

Here S_1 *is the unique functions such that* $L_- S_1 = \Lambda Q$ *with* $(S_1, Q) = 0$ *and the function* ρ_1 *is defined in* (3.4).

Acknowledgements

VG was partially supported by Project 2017“Problemi stazionari e di evoluzione nelle equazioni di campo nonlineari” of INDAM, GNAMPA - Gruppo Nazionale per l’Analisi Matematica, la Probabilita e le loro Applicazioni, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, by Top Global University Project, Waseda University and the Project PRA 2018 49 of University of Pisa. YL was supported by the China Scholarship Council (201906180041) and the NSF of China (12075102).

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