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GENERIC PROPERTIES OF EIGENVALUES OF THE FRACTIONAL LAPLACIAN

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Abstract. We consider the Dirichlet eigenvalues of the fractional Laplacian $(-\Delta)^s$, with $s \in (0,1)$, related to a smooth bounded domain Ω. We prove that there exists an arbitrarily small perturbation $\tilde{\Omega} = (I+\psi)(\Omega)$ of the original domain such that all Dirichlet eigenvalues of the fractional Laplacian associated to $\tilde{\Omega}$ are simple. As a consequence we obtain that all Dirichlet eigenvalues of the fractional Laplacian on an interval are simple. In addition, we prove that for a generic choice of parameters all the eigenvalues of some non-local operators are also simple.

1. Introduction and statement of the result

The present paper is concerned with the Dirichlet eigenvalue fractional problem

$$(-\Delta)^s\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega. \tag{1.1}$$

Here Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ with $n \geq 1$ and $(-\Delta)^s$ with $s \in (0,1)$ is the fractional Laplacian defined, for $u \in C^2_c(\mathbb{R}^n)$, as

$$(-\Delta)^s u = C_{n,s}\mathtt{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dx = C_{n,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dx,$$

where $C_{n,s} := s4^s\frac{\Gamma(s+n/2)}{\pi^{n/2}\Gamma(1-s)}$ is a renormalization constant and $B_\varepsilon(x)$ is the ball of radius ε centered in x.

To avoid a priori regularity assumptions, we consider the eigenvalue problem in a weak sense. We consider the space

$$\mathcal{H}^s_0(\Omega) := \{u \in H^s(\mathbb{R}^n) \ : \ u \equiv 0 \text{ on } \Omega^c\},$$

where

$$H^s(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) \ : \ \frac{u(x)-u(y)}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n\times\mathbb{R}^n)\right\}.$$

On $\mathcal{H}^s_0(\Omega)$ we consider the quadratic form

$$(u,v) \mapsto \mathcal{E}^\Omega_s(u,v) := \frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy.$$

Then, we call $\varphi_s \in \mathcal{H}^s_0(\Omega)$ an eigenfunction corresponding to the eigenvalue λ if

$$\mathcal{E}^\Omega_s(\varphi_s, v) = \lambda\int_{\mathbb{R}^n}\varphi_s v dx \ \ \forall v \in \mathcal{H}^s_0(\Omega).$$

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In the following, to simplify notation, we will omit the renormalization constant $C_{n,s}$.

It is well known (see e.g. [1] and the reference therein for an exhaustive introduction about these topics) that (1.2) admits an ordered sequence of eigenvalues

$$0 < \lambda_{1,s} < \lambda_{2,s} \leq \lambda_{3,s} \leq \cdots \leq \lambda_{1,s} \leq \cdots \to +\infty.$$

Since the first eigenvalue is strictly positive, we can endow $\mathcal{H}_0^s(\Omega)$ with the norm

$$\|u\|^2_{\mathcal{H}_0^s(\Omega)} = \mathcal{E}_s^\Omega(u,u).$$

In the local case, i.e. $s = 1$, it is well known (see [8, 9]) that all the eigenvalues are simple for *generic* domains Ω.

It is natural to ask if the same results hold true in the non-local case, i.e. $s \in (0,1)$. As far as we know, there are only two results dealing with the simplicity issue. Very recently, in [2] the authors prove the simplicity of radial eigenvalues in a ball or an annulus. In [5, 6], the authors prove that all the eigenvalues of the fractional Laplacian $(-\Delta)^s$ with $s \in [1/2, 1)$ in the interval $\Omega = (-1,1)$ are simple. However, to our knowledge, the simplicity eigenvalues on an interval for all $s \in (0,1)$ remains an open problem. The present paper solves this open question, as a consequence of our main result.

To study domain perturbations we will consider the space

$$C^1(\mathbb{R}^n,\mathbb{R}^n) := \{\psi : \mathbb{R}^n \to \mathbb{R}^n \ : \ \psi,\ D\psi \text{ continuous and bounded}\}$$

endowed with the norm

$$\|\psi\|_1 = \sup_{x\in\mathbb{R}^n} \left(|\psi(x)| + |D\psi(x)|\right).$$

The first question is: if $\bar\lambda$ is an eigenvalue of multiplicity $\nu > 1$ of the operator $(-\Delta)^s_\Omega$ associated with the domain Ω with Dirichlet boundary condition, and U is an interval such that the intersection of the spectrum of $(-\Delta)^s_\Omega$ with U consist of the only number $\bar\lambda$, there exists a perturbation $\Omega_\psi = (I+\psi)(\Omega)$ of the domain Ω such that the intersection of the spectrum of $(-\Delta)^s_{\Omega_\psi}$ with the interval U consists exactly of ν simple eigenvalues of $(-\Delta)^s_{\Omega_\psi}$? Consequently, a second question arises: do there exist any perturbed domains $\Omega_\psi = (I+\psi)(\Omega)$ such that *all* the eigenvalues of $(-\Delta)^s_{\Omega_\psi}$ are simple?

The answer is affirmative and our main result reads as follows.

Theorem 1. *Let $s \in (0,1)$. Let Ω be a smooth bounded domain with $C^{1,1}$ boundary. Then for any $\varepsilon > 0$ there exists $\psi \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, with $\|\psi\|_{C^1} < \varepsilon$, such that all the eigenvalues of the problem*

$$(-\Delta)^s\varphi = \lambda\varphi \text{ in } \Omega_\psi = (I+\psi)(\Omega), \qquad \varphi = 0 \text{ in } \mathbb{R}^n \smallsetminus \Omega_\psi$$

are simple.

In other words, it can be said that all the eigenvalues of the problem (1.1) are simple for *generic* domains Ω, where with generic we mean that, given a domain Ω, there exists at least an arbitrarily close domain $\tilde\Omega = (I+\psi)\Omega$ for which all eigenvalues of (1.1) are simple. As a consequence of Theorem 1, we obtain the simplicity of eigenvalues of the fractional laplacian on intervals.

Corollary 2. *Let $s \in (0,1)$. Then all eigenvalues of the eigenvalue problem*

$$(-\Delta)^s\varphi = \lambda\varphi \quad \text{in } (-1,1), \qquad \varphi = 0 \quad \text{in } \mathbb{R} \smallsetminus (-1,1)$$

are simple.

Corollary 2 follows from Theorem 1 which implies that there exists an open interval $\tilde{\Omega}$ (a perturbation of an open bounded interval Ω) such that all its Dirichlet eigenvalues are simple. Since the dimension of the eigenspaces are invariant under scaling and translation, Corollary 2 follows immediately.

In the spirit of Theorem 1, we obtain a similar result considering Dirichlet eigenvalue fractional problem with nonconstant coefficients of the type

$$(-\Delta)^s\varphi + a(x)\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega \tag{1.2}$$

and

$$(-\Delta)^s\varphi = \lambda\alpha(x)\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \tag{1.3}$$

where $a, \alpha \in C^0(\mathbb{R}^n)$. Again, if $(-\Delta)^s + a(x)I$ is a positive operator (e.g. $\min_{\overline{\Omega}} a > 0$ or $\|a\|_{C^0(\Omega)}$ is small enough) or $\min_{\overline{\Omega}} \alpha > 0$, from a (fractional analogue) of Rellich's compactness lemma it is quite standard to deduce that there is an unbounded ordered sequence of eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ (see [1,3] and the references therein) and that each eigenvalue has finite multiplicity and the first one is simple.

In the local case, simplicity of the eigenvalues with respect to a perturbation of the coefficients where proved in [11] and we are able to show the nonlocal counterpart of this result. In particular, we prove that all the eigenvalues of (1.2) and (1.3) are simple for *generic* functions a and α, respectively, in this two results.

Theorem 3. *Let $a \in C^0(\mathbb{R}^n)$ such that $\min_{\overline{\Omega}} a > 0$ or $\|a\|_{C^0(\Omega)}$ is small enough. For any $\varepsilon > 0$ there exists $b \in C^0(\mathbb{R}^n)$, with $\|b\|_{C^0} < \varepsilon$, such that all the eigenvalues of the problem*

$$(-\Delta)^s\varphi + (a(x) + b(x))\,\varphi = \lambda\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

are simple.

Theorem 4. *Let $\alpha \in C^0(\mathbb{R}^n)$ such that $\min_{\overline{\Omega}} \alpha > 0$. For any $\varepsilon > 0$ there exists $\beta \in C^0(\mathbb{R}^n)$, with $\|\beta\|_{C^0} < \varepsilon$, such that all the eigenvalues of the problem*

$$(-\Delta)^s\varphi = \lambda\,(\alpha(x) + \beta(x))\,\varphi \text{ in } \Omega, \qquad \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega$$

are simple.

Remark 1.1. *It would be interesting to study eigenvalues problems associated with higher order fractional laplacians (i.e. $s > 1$). However, a lot of work should be done starting from the choice of the spaces and the boundary conditions. For example the bilaplacian operator (i.e. $s = 2$) can be considered with both Navier (i.e. $u = \Delta u = 0$ on $\partial\Omega$) or Dirichlet (i.e. $u = \partial_\nu u = 0$ on $\partial\Omega$) boundary conditions. Moreover, a suitable version of Lemma 15 which is a key point in our proof would be needed and this is far from being avalaible and understood.*

The strategy of the proofs of the above theorems relies on an abstract result which is presented in Section 3. In particular, Theorem 13 provides us a so called *splitting condition*, which is crucial to find the perturbation term ψ (or b,β) for which all eigenvalues are simple as claimed in Theorem 1 (Th. 3 and Th. 4, respectively). We will give a detailed proof of Theorem 1 from Section 2 to Section 5. In this part Lemma 15 plays a crucial role. In Section 6 and in Section 7 we will only describe the main steps to get Theorem 3 and Theorem 4.

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2. Domain perturbations

In this section we study how a perturbation of the domain affects the multiplicity of eigenvalues. The main point is, given a smooth perturbation of the domain of the form $I+\psi$, to introduce, by a suitable change of variables, the bilinear form $\mathcal{B}_s^\psi$ in (2.1) to which we apply the splitting condition of Theorem 13. The problem of the splitting of the eigenvalues with respect to domain perturbation was studied for the standard Laplacian in [4, 7–9], from which we derive this strategy and which we refer to for a bibliography on the subject.

For a function $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, we define

$$\Omega_\psi := (I+\psi)\Omega.$$

If $\|\psi\|_{C^1} \le L$ for some $L < 1$ then $(I+\psi)$ is invertible on Ω_ψ with inverse mapping $(I+\psi)^{-1} = I + \chi$. In the following we always consider $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\psi\|_{C^1} \le L$. Also, we denote $J_{I+\psi}$ as the Jacobian determinant of the mapping $I+\psi$. Whenever no ambiguity is possible, we use also the short notation $J_\psi := J_{I+\psi}$.

Remark 5. It is well known that, if ψ is sufficiently regular, the following expansion holds for ε small

$$J_{I+\varepsilon\psi} = 1 + \varepsilon \mathrm{div}\psi + \varepsilon^2 a_2 + \cdots + \varepsilon^n a_n$$

for suitable a_i.

By the change of variables given by the mapping $(I+\psi)$, and denoted $\tilde{u}(\xi) := u(\xi+\psi(\xi))$, we obtain the bilinear form $\mathcal{B}_s^\psi$ on $\mathcal{H}_0^s(\Omega)$

$$\begin{aligned}\mathcal{E}_s^{\Omega_\psi}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\tilde{u}(\xi)-\tilde{u}(\eta))(\tilde{v}(\xi)-\tilde{v}(\eta))}{|\xi-\eta+\psi(\xi)-\psi(\eta)|^{n+2s}} J_\psi(\xi) J_\psi(\eta) d\xi d\eta \\ &=: \mathcal{B}_s^\psi(\tilde{u},\tilde{v}), \quad (2.1)\end{aligned}$$

for $\tilde{u},\tilde{v} \in \mathcal{H}_0^s(\Omega)$. Notice that $\mathcal{B}_s^0(\tilde{u},\tilde{v}) = \mathcal{E}_s^\Omega(\tilde{u},\tilde{v})$.

At this point, one can prove by direct computation the following result.

Lemma 6. *Let $\psi \in C^1$, and take $\tilde{u} \in \mathcal{H}_0^s(\Omega)$. Then*

$$\mathcal{B}_s^\psi(\tilde{u},\tilde{u}) = \mathcal{E}_s^{\Omega_\psi}(u,u) \le C_1\left[\mathcal{E}_s^\Omega(\tilde{u},\tilde{u}) + \|\tilde{u}\|_{L^2(\Omega)}\right] \le C_2 \mathcal{E}_s^\Omega(\tilde{u},\tilde{u})$$

for some positive contants C_1, C_2.

Remark 7. Let us define the map

$$\begin{aligned}\gamma_\psi &: \mathcal{H}_0^s(\Omega_\psi) \to \mathcal{H}_0^s(\Omega); \\ \gamma_\psi(u) &:= \tilde{u}(\xi) = u(\xi+\psi(\xi)).\end{aligned}$$

By the previous lemma we have that, if $\|\psi\|_{C^1}$ is sufficiently small the following maps are continuous isomorphisms

$$\begin{aligned}\gamma_\psi &: \mathcal{H}_0^s(\Omega_\psi) \to \mathcal{H}_0^s(\Omega) \\ \gamma_\psi^{-1} = \gamma_\chi &: \mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^s(\Omega_\psi).\end{aligned}$$

In addition $\mathcal{B}_s^\psi(\tilde{u},\tilde{v})$ is a scalar product on $\mathcal{H}_0^s(\Omega)$, and the norm induced by $\mathcal{B}_s^\psi(\cdot,\cdot)$ is equivalent to the one induced by $\mathcal{E}_s^\Omega(\cdot,\cdot)$.

It is well known that the embedding $i : \mathcal{H}_0^s(\Omega) \to L^2(\Omega)$ is compact, so we consider the adjoint operator, with respect to $\mathcal{E}_s^\Omega$,

$$i^* : L^2(\Omega) \to \mathcal{H}_0^s(\Omega).$$

The composition $E_\Omega := (i^* \circ i)_\Omega : \mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^s(\Omega)$ is selfadjoint, compact, injective with dense image in $\mathcal{H}_0^s(\Omega)$ and it holds

$$\mathcal{E}_s^\Omega \left((i^* \circ i)_\Omega v, u \right) = \int_\Omega uv. \tag{2.2}$$

Remark 8. If $\varphi_k \in \mathcal{H}_0^s(\Omega)$ is an eigenfunction of the fractional Laplacian with eigenvalue λ_k, then φ_k is an eigenfunction of $(i^* \circ i)_\Omega$ with eigenvalue $\mu_k^\Omega := 1/\lambda_k$. In fact, it holds

$$\mathcal{E}_s^\Omega(\varphi_k, v) = \lambda_k \int_{\mathbb{R}^n} \varphi_k v dx = \int_{\mathbb{R}^n} \lambda_k \varphi_k v dx = \mathcal{E}_s^\Omega \left(\lambda_k (i^* \circ i)_\Omega \varphi_k, v \right),$$

thus $\lambda_k (i^* \circ i)_\Omega \varphi_k = \varphi_k$.

We recall two min-max characterizations of eigenvalues μ_k^Ω. We have that

$$\mu_1^\Omega := \sup_{u \in \mathcal{H}_\Omega^s \setminus \{0\}} \frac{\int_\Omega u^2 dx}{\mathcal{E}_s^\Omega(u,u)}; \qquad \mu_\nu^\Omega := \sup_{\substack{u \in \mathcal{H}_\Omega^s \setminus \{0\} \\ \mathcal{E}_s^\Omega(u, e_t) = 0 \\ t = 1, \dots \nu - 1}} \frac{\int_\Omega u^2 dx}{\mathcal{E}_s^\Omega(u,u)};$$

where $(i^* \circ i)_\Omega e_t = \mu_t^\Omega e_t$; equivalently,

$$\mu_\nu^\Omega := \inf_{V = \{v_1, \dots, v_{\nu-1}\}} \sup_{\substack{u \in \mathcal{H}_\Omega^s \setminus \{0\} \\ \mathcal{E}_s^\Omega(u, v_t) = 0 \\ t = 1, \dots \nu - 1}} \frac{\int_\Omega u^2 dx}{\mathcal{E}_s^\Omega(u,u)}.$$

By this characterization, and by (2.1), it is easy to prove the following result

Lemma 9. *Every eigenvalue μ_k of the operator $E_\psi := E_{\Omega_\psi}$ is continuous at 0 with respect to $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.*

Finally, since in Remark 8 we proved that if φ_k is an eigenfuntion of $(-\Delta)^s$ with Dirichlet boundary conditions on Ω_ψ with eigenvalue λ_k, then φ_k is an eigenfunction of E_ψ with eigenvalue $\mu_k := 1/\lambda_k$, to obtain the main result of this paper, we study the multiplicity of the eigenvalues μ_k of the operator E_ψ. For this purpose, in the next section we collect an abstract result which we will apply to the operator E_ψ.

3. An abstract result

We recall a series of abstract results which holds in general in a Hilbert space X endowed with scalar product $< \cdot, \cdot >_X$. Later, in the paper, we will apply these abstract results to derive a splitting condition for multiple eigenvalues. The proof of these results, are contained in [9, Section 2]. However, to make this paper self contained, we recall them in the following.

Let

$$F_{ij} := \{ A \in L(X,X) \ : \ \text{codim Im} A = i \text{ and } \dim \ker A = j \}$$

be the set of Fredholm operator with indices i and j in the Banach space $L(X,X) := \{ A : X \to X \ : \ A \text{ linear and continuous} \}$.

We show first that F_{ij} is a smooth submanifold of codimension ij in $L(X,X)$. It is well known that if $A \in F_{ij}$, there exist closed subspaces $V, W \subset X$ such that

$$X = \ker A \oplus V \text{ and } X = W \oplus \mathrm{Im}A.$$

Let us call $P, Q, \bar{P}$ and $\bar{Q}$ the projector on $\ker A, V, W, \mathrm{Im}A$, respectively. It holds

Lemma 10. *We have*

$$L(X,X) = L \oplus \mathcal{V},$$

where

$$\begin{aligned} \mathcal{V} &:= \{T \in L(X,X) \ : \ T(\ker A) \subset \mathrm{Im}A\} \\ L &:= \left\{\bar{P}HP \in L(X,X) \text{ with } H \in L(X,X)\right\}. \end{aligned}$$

Proof. The claim can be showed immediately noticing that $T = \bar{P}TP + \bar{Q}TQ + \bar{P}TQ + \bar{Q}TP$ and that $\bar{Q}TQ + \bar{P}TQ + \bar{Q}TP \in \mathcal{V}$. □

Lemma 11. *We have that F_{ij} is an analytic submanifold of $L(X,X)$. In addition, for any $A \in L(X,X)$, the tangent space in A to F_{ij}, $T_A F_{ij} = \mathcal{V}$.*

The proof of this result is postponed to appendix. Here we limit ourselves to give the main idea. Given $A_0 \in F_{ij}$, and given H such that $A_0 + H$ still belongs to F_{ij}, it is possible to write $H = \bar{P}HP + f(V)$ where $V \in \mathcal{V}$ and f is an analytic function. Then F_{ij} near A_0 is a smooth graph on $\mathcal{V}$.

Lemma 12. *Let $A \in F_{ij}$ such that $\ker A \not\subset \mathrm{Im}A$. Then*

$$M = \{A + H + \lambda I \in L(X,X) \ : \ \lambda \in \mathbb{R}, A + H \in F_{ij} \text{ and } H \text{ suff. small}\}$$

is an analytic manifold at $A + \lambda I$, and $T_{A+\lambda I}M = \mathcal{V} \oplus \mathrm{Span} < I >$ where $T_{A+\lambda I}M$ is the tangent space in $A + \lambda I$ to M.

Proof. By definition of $\mathcal{V}$, we have that $I \in \mathcal{V}$ if and only if $\ker A \subset \mathrm{Im}A$, which is not possible by our hypothesis on A. Thus, by Lemma 11 we have that M is a ruled manifold and the thesis follows immediately. □

We can recast the previous result considering $T : X \to X$ a selfadjoint compact operator with an eigenvalue $\bar{\lambda}$ with multiplicity ν. By Riesz theorem we have that $T - \bar{\lambda}I \in F_{\nu\nu}$ and that $\ker(T - \bar{\lambda}I) \cap \mathrm{Im}(T - \bar{\lambda}I) = \{0\}$. Moreover by Lemma 12 if U is a suitable neighborhood of $T - \bar{\lambda}I$ we have that

$$\tilde{M} = \left\{\tilde{T} + \lambda I \in L(X,X) \ : \ \lambda \in \mathbb{R} \text{ and } \tilde{T} \in F_{\nu\nu} \cap U\right\}$$

is a smooth manifold and $T_{T-\bar{\lambda}I}\tilde{M} = \tilde{\mathcal{V}} \oplus \mathrm{Span} < I >$ where

$$\tilde{\mathcal{V}} = \left\{H \in L(X,X) \ : \ H(\ker(T - \bar{\lambda}I)) \subset \mathrm{Im}(T - \bar{\lambda}I)\right\}. \tag{3.1}$$

At this point we are in position to enunciate the main result of this section.

Theorem 13. *Let $T_b : X \to X$ be a selfadjoint compact operator which depends smoothly on a parameter b belonging to a real Banach space B. Let $T_0 = T$ and let T_b be Frechet differentiable in $b = 0$. Let $x_1^0, \dots, x_\nu^0$ be an orthonormal basis for the eigenspace relative to the eigenvalue $\bar{\lambda}$ of T. If $T_b \in \tilde{M}$ for all b with $\|b\|_{C^0}$ small, then for all b there exist a $\rho = \rho(b) \in \mathbb{R}$ such that*

$$\left\langle T'(0)[b]x_j^0, x_i^0 \right\rangle_X = \rho\delta_{ij} \text{ for } i,j = 1,\dots,\nu. \tag{3.2}$$

Proof. By Lemma 12 we have that, if $T_b \in \tilde{M}$ for all b, then

$$T'(0)[b] \in \tilde{\mathcal{V}} \oplus \text{Span} < I > .$$

So, by (3.1), for all b, there exists $\bar{\lambda}(b) \in \mathbb{R}$, such that

$$\left[T'(0)[b] - \bar{\lambda}(b)I\right] (\ker(T - \bar{\lambda}I)) \subset \text{Im}(T - \bar{\lambda}I),$$

that is

$$\left\langle \left[T'(0)[b] - \bar{\lambda}(b)I\right] x_j^0, x_i^0 \right\rangle_X = 0$$

for all $i, j = 1, \ldots, \nu$, which implies (3.2). □

This theorem says that if condition (3.2) is fulfilled, then the eigenvalue $\bar{\lambda}(b)$ has still multiplicity ν in a neighborhood of $b = 0$.

4. Splitting of a single eigenvalue

We recall that $E_\psi = (i^* \circ i)_{\Omega_\psi}$. Also, by (2.2), and by the definition of $\tilde{u}$ we have

$$\mathcal{E}_s^{\Omega_\psi}(E_\psi u, v) =< u, v >_{L^2(\Omega_\psi)}= \int_\Omega \tilde{u}\tilde{v}J_\psi.$$

By the definition of $\mathcal{B}_s^\psi$, we can rewrite the previous formula as

$$\mathcal{B}_s^\psi(\gamma_\psi E_\psi u, \tilde{v}) = \mathcal{E}_s^{\Omega_\psi}(E_\psi v, u) = \int_\Omega \tilde{u}\tilde{v}J_\psi.$$

Set

$$T_\psi \tilde{u} := \gamma_\psi E_\psi \gamma_\psi^{-1} \tilde{u}, \tag{4.1}$$

we get that $T_\psi : \mathcal{H}_0^s(\Omega) \to \mathcal{H}_0^s(\Omega)$ is a compact selfadjoint operator such that

$$\mathcal{B}_s^\psi(T_\psi \tilde{u}, \tilde{v}) = \int_\Omega \tilde{u}\tilde{v}J_\psi$$

for all ψ.

Remark 14. One can prove that T_ψ and $\mathcal{B}_s^\psi$ are differentiable in the ψ variable at 0. Then it holds

$$\left(\mathcal{B}_s^\psi\right)'(0)[\psi](T_0\tilde{u}, \tilde{v}) + \mathcal{B}_s^0(T_\psi'(0)[\psi]\tilde{u}, \tilde{v}) = \int_\Omega \tilde{u}\tilde{v}\,\text{div}\psi. \tag{4.2}$$

Lemma 15. *Let $\tilde{u}, \tilde{v} \in \mathcal{H}_0^s(\Omega)$ such that $(-\Delta)^s\tilde{u}, (-\Delta)^s\tilde{v} \in C_{loc}^\alpha(\Omega) \cap L^\infty(\Omega)$ with $\alpha > (1-2s)_+$. Then*

$$\left(\mathcal{B}_s^\psi\right)'(0)[\psi](\tilde{u}, \tilde{v}) = -\Gamma^2(1+s)\int_{\partial\Omega} \frac{\tilde{u}}{\delta^s}\frac{\tilde{v}}{\delta^s}\psi\cdot N d\sigma - \int_\Omega [\nabla\tilde{u}\cdot\psi(-\Delta)^s\tilde{v} + \nabla\tilde{v}\cdot\psi(-\Delta)^s\tilde{u}]dx \tag{4.3}$$

where $\delta(x) = \text{dist}(x, \mathbb{R}^n \setminus \Omega)$ and N is the exterior normal of Ω.

Proof. If $\|\psi\|_{C^1}$ is small, by direct computation we have that

$$\left(\mathcal{B}_s^\psi\right)'(0)[\psi](\tilde{u}, \tilde{v}) =$$

$$\frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\tilde{u}(\eta) - \tilde{u}(\xi))(\tilde{v}(\eta) - \tilde{v}(\xi))}{|\xi - \eta|^{n+2s}} \left\{\text{div}\psi(\xi) + \text{div}\psi(\eta) - \frac{(n+2s)(\xi - \eta)\cdot(\psi(\xi) - \psi(\eta))}{|\xi - \eta|^2}\right\} d\xi d\eta$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (\tilde{u}(\eta) - \tilde{u}(\xi))(\tilde{v}(\eta) - \tilde{v}(\xi))K(\xi, \eta)d\xi d\eta, \tag{4.4}$$

where

$$K(\xi,\eta) := \frac{1}{2}\left\{\mathrm{div}\psi(\xi) + \mathrm{div}\psi(\eta) - \frac{(n+2s)(\xi-\eta)\cdot(\psi(\xi)-\psi(\eta))}{|\xi-\eta|^2}\right\}\frac{1}{|\xi-\eta|^{n+2s}}.$$

At this point we use the result of Theorem 1.3 of [2] which allows to compute integrals of the form of (4.4) and we obtain the conclusion. □

We want to apply the previous result to eigenfunctions of $(-\Delta)^s$ on Ω with Dirichlet boundary conditions. We recall that, by Remark 8, this is equivalent to consider eigenfunctions of the operator T_0.

Corollary 16. *Let $u, v \in \mathcal{H}^s_0(\Omega)$ satisfy $T_0 u = \frac{1}{\lambda_0}u$, and $T_0 v = \frac{1}{\lambda_0}v$. Then we have*

$$\left(\mathcal{B}^\psi_s\right)'(0)[\psi](T_0 u, v) = -\frac{\Gamma^2(1+s)}{\lambda_0}\int_{\partial\Omega}\frac{u}{\delta^s}\frac{v}{\delta^s}\,\psi\cdot N\,d\sigma + \int_\Omega uv\mathrm{div}(\psi)dx.$$

Proof. By elliptic regularity the eigenfunctions belongs to $C^\alpha_{\mathrm{loc}}(\Omega)\cap L^\infty(\Omega)$ with $\alpha > (1-2s)_+$. Then, by Lemma 15 we have

$$\begin{aligned}\left(\mathcal{B}^\psi_s\right)'(0)[\psi](T_0 u, v) = &-\frac{\Gamma^2(1+s)}{\lambda_0}\int_{\partial\Omega}\frac{u}{\delta^s}\frac{v}{\delta^s}\,\psi\cdot N\,d\sigma\\ &-\frac{1}{\lambda_0}\int_\Omega \nabla u\cdot\psi(-\Delta)^s v\,dx - \frac{1}{\lambda_0}\int_\Omega \nabla v\cdot\psi(-\Delta)^s u\,dx\end{aligned}$$

Combining this with Remark 8 and integration by parts, we obtain

$$\begin{aligned}\left(\mathcal{B}^\psi_s\right)'(0)[\psi](T_0 u, v) &= -\frac{\Gamma^2(1+s)}{\lambda_0}\int_{\partial\Omega}\frac{u}{\delta^s}\frac{v}{\delta^s}\,\psi\cdot N\,d\sigma - \int_\Omega \nabla u\cdot\psi v\,dx - \int_\Omega \nabla v\cdot\psi u\,dx\\ &= -\frac{\Gamma^2(1+s)}{\lambda_0}\int_{\partial\Omega}\frac{u}{\delta^s}\frac{v}{\delta^s}\,\psi\cdot N\,d\sigma + \int_\Omega uv\mathrm{div}(\psi)dx,\end{aligned}$$

as desired. □

Now we apply Theorem 13 to the operator T_ψ defined in (4.1). This is the fundamental block to prove Theorem 1.

Let μ_0 be an eigenvalue of $T_0 = E_\Omega = (i^*\circ i)_\Omega$ which has multiplicity $\nu > 1$. If for all ψ with $\|\psi\|_{C^1}$ small, the operator T_ψ has an eigenvalue $\mu(\psi)$ with multiplicity ν for all ψ and such that $\mu(\psi)\to\mu_0$ while $\psi\to 0$, then Theorem 13 yields

$$\mathcal{B}^0_s(T'_\psi(0)[\psi]\varphi_i, \varphi_j) = \rho\delta_{ij}$$

for some $\rho = \rho(\psi)\in\mathbb{R}$. Here $\{\varphi_i\}_{i=1,\dots,\nu}$ is an orthonormal basis for the eigenspace $\mu(0)$. This, in light of (4.2) and Corollary 16 can be recast as

$$\begin{aligned}\rho\delta_{ij} &= -\left(\mathcal{B}^\psi_s\right)'(0)[\psi](T_0\varphi_i, \varphi_j) + \int_\Omega \varphi_i\varphi_j\mathrm{div}\psi dx\\ &= \Gamma^2(1+s)\mu_0\int_{\partial\Omega}\frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\,\psi\cdot N\,d\sigma.\end{aligned} \tag{4.5}$$

So, for all ψ with $\|\psi\|_{C^1}$ small,

$$\int_{\partial\Omega}\frac{\varphi_i}{\delta^s}\frac{\varphi_j}{\delta^s}\,\psi\cdot N\,d\sigma = 0 \text{ for } i\neq j; \qquad \int_{\partial\Omega}\left(\frac{\varphi_1}{\delta^s}\right)^2\psi\cdot N\,d\sigma = \dots = \int_{\partial\Omega}\left(\frac{\varphi_\nu}{\delta^s}\right)^2\psi\cdot N\,d\sigma.$$

This implies that $(\frac{\varphi_i}{\delta^s})^2 \equiv 0$ on $\partial\Omega$ for $i = 1, \ldots, \nu$. On the other hand, by the fractional Pohozaev identity (see [10] and [2, formula (1.6)]),

$$\Gamma^2(1+s) \int_{\partial\Omega} \left(\frac{\varphi_i}{\delta^s}\right)^2 x \cdot N \, d\sigma = \frac{2s}{\mu_0} \int_\Omega \varphi_i^2 dx = \frac{2s}{\mu_0} \neq 0.$$

This leads to a contradiction and thus $\mu(\psi)$ cannot have multiplicity ν for all ψ with $\|\psi\|_{C^1}$ small. This fact can be summarized in the next proposition, which is the main tool to prove Theorem 1.

Proposition 17. *Let $\bar\lambda$ an eigenvalue of the operator $(-\Delta)^s_\Omega$ with Dirichlet boundary condition which has multiplicity $\nu > 1$. Let U and open bounded interval such that*

$$\bar U \cap \sigma\left((-\Delta)^s_\Omega\right) = \left\{\bar\lambda\right\},$$

where $\sigma\left((-\Delta)^s_\Omega\right)$ is the spectrum of $(-\Delta)^s_\Omega$.

Then, there exists $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\psi\|_1$ small enough such that for $\Omega_\psi = (I+\psi)\Omega$ it holds

$$\bar U \cap \sigma\left((-\Delta)^s_{\Omega_\psi}\right) = \left\{\lambda_1^{\Omega_\psi}, \ldots, \lambda_k^{\Omega_\psi}\right\},$$

where $\lambda_i^{\Omega_\psi}$ is an eigenvalue of the operator $(-\Delta)^s_{\Omega_\psi}$ associated to the set Ω_ψ with Dirichlet boundary condition. Here $k > 1$ and the multiplicity of $\lambda_i^{\Omega_\psi}$ is $\nu_i < \nu$ with $\sum_{i=1}^k \nu_i = \nu$.

We recall that if $\|\psi\|_{C^1}$ is small, the multiplicity of an eigenvalue λ^{Ω_ψ} near $\bar\lambda$ can only be equal or smaller than the multiplicity of $\bar\lambda$. Here, in Proposition 17, we proved the existence of perturbations for which the multiplicity is strictly smaller.

The next corollary follows from Proposition 17, composing a finite number of perturbations.

Corollary 18. *There exists $\psi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\psi\|_1$ small enough such that for $\Omega_\psi = (I+\psi)\Omega$ it holds*

$$\bar U \cap \sigma\left((-\Delta)^s_{\Omega_\psi}\right) = \left\{\lambda_1^{\Omega_\psi}, \ldots, \lambda_\nu^{\Omega_\psi}\right\},$$

where $\lambda_i^{\Omega_\psi}$ is a simple eigenvalue of the operator $(-\Delta)^s_{\Omega_\psi}$ associated to the set Ω_ψ with Dirichlet boundary condition.

At this point we are in position to prove the main result of this paper.

5. Proof of Theorem 1

We start proving the following splitting property for a finite number of multiple eigenvalues.

Lemma 19. *Given a sequence $\{\sigma_l\}$ of positive real numbers there exists*

- *a sequence of bijective map $\{F_l\} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $F_l = (I + \psi_l)$ with $\|\psi_l\|_{C^1} \le \sigma_l$*
- *a sequence of open bounded C^1 sets with $\Omega_0 = \Omega$ and $\Omega_l = F_l(\Omega_{l-1})$*
- *a sequence of increasing integer numbers $\{q_l\}$ with $q_l \nearrow +\infty$*
- *a sequence of open bounded intervals $\{U_t\}_{t=1,\ldots,q_l}$ with $\bar U_i \cap \bar U_j = \emptyset$ for $i \neq j$*

such that the eigenvalues $\lambda_i^{\Omega_l}$ of the operator $(-\Delta)^s_{\Omega_l}$ are simple for $i = 1, \ldots, q_l$ and $\lambda_i^{\Omega_l} \in U_i$ for all $i = 1, \ldots, q_l$.

Proof. Take $q \in \mathbb{N}$ such that that $\lambda_1, \ldots, \lambda_q$ are simple eigenvalues for $(-\Delta)^s_\Omega$ and that λ_{q+1} is the first eigenvalue with multiplicity $\nu_{q+1} > 1$. For $t = 1, \ldots, q$ let U_t be open intervals such that $\bar U_i \cap \bar U_j = \emptyset$ for $i \neq j$ and $\lambda_t \in U_t$. Let us take W an open interval such that $\bar W \cap \bar U_t = \emptyset$ for all $t = 1, \ldots, q$ and $\bar W \cap \sigma((-\Delta)^s_\Omega) = \{\lambda_{q+1}\}$. At this point, by Corollary 18 we can choose $\bar\psi$ such that $\bar W \cap \sigma((-\Delta)^s_{\Omega_{\bar\psi}})$ contains exactly ν_{q+1} simple eigenvalues. Also, we can choose a number σ_{q+1} sufficiently small, with $\|\bar\psi\|_{C^1} \le \sigma_{q+1}$ so that $\lambda_t^{\bar\psi} \in U_t$ for all $t = 1, \dots, q$, since the eigenvalues depends continuously on ψ. At this point, by iterating this procedure a finite number of times we get the proof. □

Proof of Theorem 1. Let us take a sequence $\{\sigma_l\}$ with $0 < \sigma_l < \frac{1}{4^l}$, and a sequence $F_l = (1 + \psi_l)$ associated to σ_l as in the previous theorem. We set

$$\mathcal{F}_l = F_l \circ F_{l-1} \circ \cdots \circ F_1.$$

We can prove that, by the choice of σ_l, the sequence $\{\mathcal{F}_l - I\}_l$ converges to some function $\bar\psi$ in $C^1(\mathbb{R}^n, \mathbb{R}^n)$. In fact, by the previous lemma we have

$$\|\mathcal{F}_{i+1} - \mathcal{F}_i\|_\infty \le \|\psi_{i+1}\|_{C^1} < \left(\frac{1}{4}\right)^{i+1} \tag{5.1}$$

$$\|\mathcal{F}'_{i+1} - \mathcal{F}'_i\|_\infty \le \|\psi_{i+1}\|_{C^1} \|\mathcal{F}'_i\|_\infty \le \left(\frac{1}{4}\right)^{i+1} \|\mathcal{F}'_i\|_\infty. \tag{5.2}$$

By induction, using 5.2, we can prove that

$$\|\mathcal{F}'_i\|_\infty \le \left(1 + \frac{1}{4}\right)^i \le \left(\frac{5}{4}\right)^i \tag{5.3}$$

and, combining all these equation, that

$$\|\mathcal{F}_{i+1} - \mathcal{F}_i\|_{C^1} \le \|\psi_{i+1}\|_{C^1} \le \left(\frac{1}{4}\right)^{i+1} \left(\frac{5}{4}\right)^i \tag{5.4}$$

and, by iterating, that, for all $p \in \mathbb{N}$

$$\begin{aligned} \|\mathcal{F}_{i+p} - \mathcal{F}_i\|_{C^1} &\le \|\psi_i\|_{C^1} \le \sum_{t=0}^{p} \left(\frac{1}{4}\right)^{i+t+1} \left(\frac{5}{4}\right)^{i+t} \\ &\le \frac{1}{4} \left(\frac{5}{16}\right)^i \sum_{t=0}^{p} \left(\frac{5}{16}\right)^t \to 0 \text{ as } i \to \infty. \end{aligned} \tag{5.5}$$

Thus the sequence $\{\mathcal{F}_i - I\}$ converges in C^1 to some $\bar\psi = \bar{\mathcal{F}} - I$ and, by (5.5), $\|\bar\psi\|_{C^1} \le 1/2$, so $\bar{\mathcal{F}}$ is invertible.

We claim that all the eigenvalues $(-\Delta)^s_{\Omega_{\bar\psi}}$ are simple. By contradiction, suppose that there exists a $\bar q$ such that $\lambda_{\bar q}^{\bar\psi}$ is the first multiple eigenvalue. Let us call $\Omega_l = \mathcal{F}_l(\Omega)$ and $\{\lambda_i^{\Omega_l}\}_i$ the eigenvalues of $(-\Delta)^s_{\Omega_l}$ on Ω_l with Dirichlet boundary conditions. By Theorem 19 we have that there exists an $l \in \mathrm{N}$ such that $(-\Delta)^s_{\Omega_l}$ has the first $\bar q + 1$ eigenvalues simple, and that there exists $U_1, \dots, U_{\bar q+1}$ open intervals, with disjoint closure, such that $\lambda_t^{\Omega_l} \in U_t$ for $t = 1, \dots, \bar q + 1$. On the one hand, $\lambda_{\bar q}^{\Omega_N} \to \lambda_{\bar q}^{\bar\psi}$ as well as $\lambda_{\bar q+1}^{\Omega_N} \to \lambda_{\bar q}^{\bar\psi}$ when $N \to \infty$ by continuity of the eigenvalues. On the other hand, $\lambda_{\bar q}^{\Omega_N} \in U_{\bar q}$ and $\lambda_{\bar q+1}^{\Omega_N} \in U_{\bar q+1}$ for all N, by Theorem 19. So $\lambda_{\bar q}^{\bar\psi} = \lambda_{\bar q+1}^{\bar\psi} \in \bar U_{\bar q} \cap \bar U_{\bar q+1}$ which leads us to a contradiction, and the theorem is proved. □

6. Proof of Theorem 3

In this case we call

$$\mathcal{B}^a(u,v) = \mathcal{E}(u,v) + \int_{\mathbb{R}^n} au^2 dx.$$

and, since $\min_\Omega a > 0$ or $\|a\|_{C^0(\Omega)}$ is small enough, we can endow $\mathcal{H}_0^s(\Omega)$ with the norm

$$\|u\|^2_{\mathcal{H}_0^s(\Omega)} = \mathcal{B}^a(u,u) = \mathcal{E}(u,u) + \int_{\mathbb{R}^n} au^2 dx.$$

We call $\varphi^a \in \mathcal{H}_0^s(\Omega)$ an eigenfunction of $((-\Delta)^s + a)$ corresponding to the eigenvalue λ^a. Given the embedding $i : \mathcal{H}_0^s(\Omega) \to L^2(\Omega)$ we consider its adjoint operator, with respect to the scalar product $\mathcal{B}^a$,

$$i^* : L^2(\Omega) \to \mathcal{H}_0^s(\Omega).$$

It holds

$$\mathcal{B}^a\left((i^* \circ i)_a u, v\right) = \mathcal{E}\left((i^* \circ i)_a u, v\right) + \int_\Omega au(i^* \circ i)_a v = \int_\Omega uv, \tag{6.1}$$

and, as before, if $\varphi_k^a \in \mathcal{H}_0^s(\Omega)$ is an eigenfunction of the fractional Laplacian with eigenvalue λ_k^a, then φ_k^a is an eigenfunction of $(i^* \circ i)_a$ with eigenvalue $\mu_k^a := 1/\lambda_k^a$.

In addiction (1.2) admits an ordered sequence of eigenvalues

$$0 < \lambda_1^a < \lambda_2^a \le \lambda_3^a \le \cdots \le \lambda_k^a \le \cdots \to +\infty$$

and all the eigenvalues λ_k^a depends continuously on a.

In the following, for $b \in C^0(\Omega)$ with $\|b\|_{L^\infty}$ small enough we consider $\mathcal{B}^{a+b}$ and $(i^* \circ i)_{a+b}$ and we put

$$B_b := \mathcal{B}^{a+b} \text{ and } E_b := (i^* \circ i)_{a+b}. \tag{6.2}$$

Similarly to what we proved in Section 4 we have the following lemma.

Lemma 20. *The maps $b \mapsto B_b$ and $b \mapsto E_b$ are differentiable at 0 and it holds*

$$(B'(0)[b]u, v) = \int_\Omega buv,$$

$$0 = \big(B'(0)[b]E_0 u, v\big) + B_0\left(E'(0)[b]u, v\right). \tag{6.3}$$

for all $u, v \in \mathcal{H}_0^s(\Omega)$.

Remark 21. Notice that, by Lemma 20 and by (6.3), it holds

$$-B_0\left(E'(0)[b]u, v\right) = \big(B'(0)[b]E_0 u, v\big) = \int_\Omega b(E_0 u)v = \int_\Omega b\left[(i^* \circ i)_a u\right] v.$$

Remark 22. If $\mu^a = \mu$ is an eigenvalue of the map $E_0 = (i^* \circ i)_a$ with multiplicity $\nu > 1$, and $\varphi_1^a, \dots, \varphi_\nu^a$ are orthonormal eigenvectors associated to μ, then, by the previous remark we have

$$\big(B'(0)[b]E_0\varphi_i^a, \varphi_j^a\big) = \int_\Omega bE_0(\varphi_i^a)\varphi_j^a = -\mu \int_\Omega b\varphi_i^a\varphi_j^a,$$

for all $i, j = 1, \dots, \nu$.

Now we apply the condition (3.2) to prove the splitting property for a chosen multiple eigenvalue.

Proposition 23. *Let $a \in C^0(\mathbb{R}^n)$ be positive on Ω or with $\|a\|_{C^0(\Omega)}$ sufficiently small. Let $\bar{\lambda}$ an eigenvalue of the operator $(-\Delta)^s_\Omega + aI$ on $\mathcal{H}^s_0$ with Dirichlet boundary condition with multiplicity $\nu > 1$. Let U be an open bounded interval such that*

$$\bar{U} \cap \sigma\left((-\Delta)^s_\Omega + aI\right) = \left\{\bar{\lambda}\right\},$$

where $\sigma\left((-\Delta)^s_\Omega + aI\right)$ is the spectrum of $(-\Delta)^s_\Omega + aI$.

Then, there exists $b \in C^0(\mathbb{R}^n)$ such that for

$$\bar{U} \cap \sigma\left((-\Delta)^s_\Omega + (a+b)I\right) = \left\{\lambda^b_1, \dots, \lambda^b_k\right\},$$

where λ^b_i is an eigenvalue of the operator $(-\Delta)^s_\Omega + (a+b)I$. Here $k > 1$ and the multiplicity of λ^b_i is ν_i with $\sum_{i=1}^k \nu_i = \nu$.

The next corollary follows from the previous proposition, after composing a finite number of perturbations.

Corollary 24. *There exists $b \in C^0(\mathbb{R}^n)$ with $\|b\|_{C^0}$ small enough such that*

$$\bar{U} \cap \sigma\left((-\Delta)^s_\Omega + (a+b)I\right) = \left\{\lambda^b_1, \dots, \lambda^b_\nu\right\},$$

where λ^b_i is a simple eigenvalue of the operator $(-\Delta)^s_\Omega + (a+b)I$ with Dirichlet boundary condition.

Proof of Proposition 23. We apply Theorem 13 to the operator $E_b = (i^* \circ i)_{a+b}$ introduced in (6.2).

If μ^{a+b} is an eigenvalue of E_b which has multiplicity ν at $b = 0$ and at any b with $\|b\|_{C^0}$ small, then by condition (3.2) of Theorem 13 we have

$$B_0(E'(0)[b]\varphi_i, \varphi_j) = \rho\delta_{ij} \text{ for some } \rho \in \mathbb{R},$$

where $\{\varphi_i\}_{i=1,\dots,\nu}$ is an L^2-orthonormal basis for the eigenspace relative to μ^a. Then, in light of Remark 22, we should have that for any $b \in C^0$ small, there exists $\rho = \rho(b)$ such that

$$\mu^a \int_\Omega b\varphi_i\varphi_j = \rho(b)\delta_{ij}.$$

Then, in particular, we deduce that

$$\int_\Omega b\varphi_1\varphi_2 = 0 \text{ and } \int_\Omega b\varphi_1^2 = \int_\Omega b\varphi_2^2 \text{ for all } b \in C^0.$$

Thus $\varphi_1\varphi_2 \equiv 0$ and $\varphi_1^2 \equiv \varphi_2^2$ almost everywhere in Ω. Thus $\varphi_1 \equiv \varphi_2 \equiv 0$ a.e. in Ω, which leads us to a contradiction. Then there exists $b \in C^0$ small such that the multiplicity of μ^{a+b} is smaller that ν. Since the eigenvalue μ^{a+b} depends continuosly on b, given a neighborhood U of μ^a, for $\|b\|_{C^0}$ small we have that $\bar{U} \cap \sigma(E_b) = \left\{\mu^{a+b}_1, \dots, \mu^{a+b}_k\right\}$ with ν_i the multiplicity of μ^{a+b}_i, and where $\sum_{i=1}^k \nu_i = \nu$, and $k > 1$. Remebering the definition of E_b and that $\mu^{a+b} = 1/\lambda^{a+b}$ we have the claim. □

We proceed similarly as the proof of Theorem 1 to obtain Theorem 3

Lemma 25. *Given $a \in C^0(\mathbb{R}^n)$ as in the hypotesis of Theorem 3, and a sequence $\{\sigma_l\}$ of positive real numbers there exists*

- *a sequence of functions $\{b_l\} \in C^0(\mathbb{R}^n)$ with $\|b_l\|_{C^0} \le \sigma_l$*
- *a sequence of increasing integer numbers $\{q_l\}$ with $q_l \nearrow +\infty$*
- *a sequence of open bounded intervals $\{U_t\}_{t=1,\dots,q_l}$ with $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$*

such that the eigenvalues $\lambda_i^{a+\sum_{j=i}^l b_j}$ *of the operator* $(-\Delta)^s_\Omega + (a+\sum_{j=i}^l b_j)I$ *are simple for* $i = 1, \dots, q_l$ *and* $\lambda_i^{a+\sum_{j=i}^l b_j} \in U_i$ *for all* $i = 1, \dots, q_l$.

Proof. Take $q \in \mathbb{N}$ such that that $\lambda_1^a, \dots, \lambda_q^a$ are simple eigenvalues for $(-\Delta)^s_\Omega + aI$ and that λ_{q+1}^a is the first eigenvalue with multiplicity ν_{q+1}. For $t = 1, \dots, q$ let $\{U_t\}$ open intervals such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$ and $\lambda_t^a \in U_t$. Let us take W an open interval such that $\bar{W} \cap \bar{U}_t = \emptyset$ for all $t = 1, \dots, q$ and $\bar{W} \cap \sigma((-\Delta)^s_\Omega + aI) = \{\lambda_{q+1}^a\}$. At this point, by Corollary 24 we can choose $\bar{b}$ such that $\bar{W} \cap \sigma((-\Delta)^s_\Omega + (a+\bar{b})I$ contains exactly ν_{q+1} simple eigenvalues. Also, we can choose a number σ_{q+1} sufficiently small, with $\|b_{q+1}\|_{C^0} \leq \sigma_{q+1}$ so that $\lambda_t^{a+\bar{b}} \in U_t$ for all $t = 1, \dots, q$, since the eigenvalues depends continuosly on b. At this point, by iterating this procedure a finite number of times we get the proof. □

At this point we can conclude.

Proof of Theorem 3. Let us take a sequence $\{\sigma_l\}$ with $0 < \sigma_l < \frac{1}{2^l}$, and a sequence b_l associated to σ_l as in the previous theorem. By the choice of σ_l, we have that $\sum_l b_l$ converge to some function b in $C^0(\mathbb{R}^n)$. We claim that all the eigenvalues $(-\Delta)^s_\Omega + (a+b)I$ are simple. By contradiction, suppose that there exists a $\bar{q}$ such that $\lambda_{\bar{q}}^{a+b}$ is the first multiple eigenvalue. By Theorem 19 we have that $(-\Delta)^s_\Omega + (a + \sum_{l=1}^{\bar{q}+1} b_l)I$ has the first $\bar{q}+1$ eigenvalues simple, and that there exists $U_1, \dots, U_{\bar{q}+1}$ open intervals, with disjoint closure, such that $\lambda_t^{a+\sum_{l=1}^{\bar{q}+1} b_l} \in U_t$ for $t = 1, \dots, \bar{q}+1$. On the one hand, $\lambda_{\bar{q}}^{a+\sum_{l=1}^N b_l} \to \lambda_{\bar{q}}^{a+b}$ as well as $\lambda_{\bar{q}+1}^{a+\sum_{l=1}^N b_l} \to \lambda_{\bar{q}}^{a+b}$ when $N \to \infty$ by continuity of the eigenvalues. On the other and, $\lambda_{\bar{q}}^{a+\sum_{l=1}^N b_l} \in U_{\bar{q}}$ and $\lambda_{\bar{q}+1}^{a+\sum_{l=1}^N b_l} \in U_{\bar{q}+1}$ for all N, by Theorem 19. So $\lambda_{\bar{q}}^{a+b} = \lambda_{\bar{q}+1}^{a+b} \in \bar{U}_{\bar{q}} \cap \bar{U}_{\bar{q}+1}$ which lead as to a contradiction, and the theorem is proved. □

7. Sketch of the proof of Theorem 4.

In this section we adapt the abstract scheme to the last result of this paper. Since the proof is very similar to the one of Theorem 3, we provide only the main tools.

Since $\alpha > 0$ on $\bar{\Omega}$, we endow the space $L^2(\Omega)$ with scalar product and norm given, respectively, by

$$\langle u, v \rangle_{L^2} = \int_\Omega \alpha uv; \qquad \|u\|^2_{L^2} = \int_\Omega \alpha u^2,$$

while on $\mathcal{H}_0^s$ we consider the usual scalar product $\mathcal{E}(u,v)$. We consider the embedding $i : \mathcal{H}_0^s \to L^2$ and its adjoint operator $i^* : L^2 \to \mathcal{H}_0^s$. Then we have

$$\mathcal{E}((i^* \circ i)_\alpha v, u) = \int_\Omega \alpha uv \quad \forall u, v \in \mathcal{H}_0^s.$$

As before, the map $(i^* \circ i)_\alpha$ is selfadjoint, compact and injective form $\mathcal{H}_0^s$ in itself. In addition, is φ^α is an eigenfunction associated to the eigenvalue μ^α for $(i^* \circ i)_\alpha$, then

$$\mu^a (-\Delta)^s \varphi = \alpha(x) \varphi_s \text{ in } \Omega, \ \varphi = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

thus $\lambda^\alpha = 1/\mu^\alpha$ is an eigenvalue with φ^α as eigenvector for Problem (1.3).

We want to prove that there exists $\beta \in C^0(\Omega)$, with $\|\beta\|_{L^\infty}$ sufficiently small, such that $(i^* \circ i)_{\alpha+\beta}$ has all eigenvalues simple.

Set

$$E_\beta := (i^* \circ i)_{\alpha+\beta},$$

we have the following Lemma

Lemma 26. *The map $\beta \mapsto E_\beta$ from a neighborhood of 0 in $C^0(\Omega)$ to the space of linear maps from $\mathcal{H}_0^s(\Omega)$ to $\mathcal{H}_0^s(\Omega)$ is continuous and differentiable at 0 and it holds*

$$\mathcal{E}(E'(0)[\beta]u, v) = \int_\Omega \beta uv.$$

Proof. Since $\Lambda_1 \int_\Omega u^2 \le \mathcal{E}(u,u)$, and $\Lambda_1 > 0$, where Λ_1 is the first eigenvalue of $(-\Delta)^s$, we have $\|E_\beta u\|_{L^2} \le c\|u\|_{L^2}$. Indeed

$$\Lambda_1 \int_\Omega (E_\beta u)^2 \le \mathcal{E}(E_\beta u, E_\beta u) = \int_\Omega (\alpha+\beta)uE_b u \le c\|u\|_{L^2}\|E_\beta u\|_{L^2}.$$

We can show now that $\mathcal{E}\left((E_\beta - E_0)u, (E_\beta - E_0)u\right) \to 0$ as $\|\beta\|_{L^\infty} \to 0$, proving the continuity of $\beta \to E_\beta$ at $b = 0$, in fact $\mathcal{E}\left((E_\beta - E_0)u, w\right) = \int_\Omega \beta uw$, so

$$\mathcal{E}\left((E_\beta - E_0)u, (E_\beta - E_0)u\right) = \int_\Omega \beta u(E_\beta - E_0)u \le c\|\beta\|_{L^\infty}\|u\|_{L^2}\mathcal{E}\left((E_\beta - E_0)u, (E_\beta - E_0)u\right)^{\frac{1}{2}}$$

which proves the claim.

Finally, given $\beta \in C^0(\Omega)$ and $u \in \mathcal{H}_0^s$, there exists $L(\beta,u) \in \mathcal{H}_0^s$ such that

$$\int_\Omega \beta uw = \mathcal{E}\left(L(\beta,u), w\right).$$

Thus, for any $w \in \mathcal{H}_0^s$ it holds

$$\mathcal{E}\left((E_\beta u - E_0 u - L(\beta,u)), w\right) = \int_\Omega (\alpha+\beta)uw - \int_\Omega \alpha uw - \int_\Omega \beta uw \equiv 0.$$

Thus $L(\beta,u) = E'(0)[\beta]u$ and $\mathcal{E}(E'(0)[\beta]u, v) = \int_\Omega \beta uv$, as claimed. □

It remains to us to apply Theorem 13 to conclude the proof of Theorem 4.

Proof of Theorem 4. If μ^α is an eigenvalue of multiplicity $\nu > 1$ of the operator $(i^* \circ i)_\alpha = E_0$ and $\varphi_1^\alpha, \dots, \varphi_\nu^\alpha$ are orthonormal eigenfunctions associated to μ^α, the condition of non splitting is that for any b with $\|\beta\|_{C^0}$ small there exists $\rho = \rho(\beta) \in \mathbb{R}$ such that

$$\int_\Omega \beta\varphi_i\varphi_j = \rho\delta_{ij}, \text{ for all } i,j = 1,\dots,\nu.$$

At this point, the proof can be achieved as the proof of Theorem 3. □

8. Appendix

Proof of Lemma 11. It is known that the Fredholm operator of a given index is open in $L(X,X)$. So, if $A_0 \in F_{ij}$, then $A_0 + H \in F_{ij}$ (if H is small) if and only if dim $(\ker(A_0+H)) =$ dim $(\ker(A_0))$, that is, if there exists j linearly independent solutions of $(A_0+H)x = 0$. By means of the projections $P, Q, \bar{P}, \bar{Q}$, this is equivalent to solve

$$\begin{cases} \bar{P}Hx = 0 \\ \bar{Q}A_0 x + \bar{Q}Hx = 0 \end{cases}; \tag{8.1}$$

Furthermore by Lemma 10, we can decompose $H = Y + S + Z + T$ where $Y = \bar{P}HP$, $S = \bar{Q}HP$, $Z = \bar{P}HQ$ and $T = \bar{Q}HQ$. Set $x = u + v$ where $u \in \ker A_0$ and $v \in \mathcal{V}$, we can recast (8.1) as

$$\begin{cases} Yu + Zv = 0 \\ \bar{Q}A_0 v + Su + Tv = 0 \end{cases}. \tag{8.2}$$

Now, $\bar{Q}A_0 : \mathcal{V} \to \mathrm{Im}A$ is invertible, and let us call R its inverse. Then the second equation of (8.2) becomes

$$v = -RSu - RTv.$$

If H is sufficiently small, then the operator $w \mapsto -RSu - RTw$ is a contraction from $\mathcal{V}$ to $\mathcal{V}$. Then we can find v as

$$v = -RSu - \sum_{i=0}^{\infty} (-1)^i (RT)^i RSu.$$

Plugging this expression in (8.2) we obtain

$$\left[Y + Z \left(-RS - \sum_{i=0}^{\infty} (-1)^i (RT)^i RS \right) \right] u = 0.$$

Recalling that $u \in \ker A_0$, we have that this equation has j linearly independent solutions if and only if

$$Y = Z \left(RS + \sum_{i=0}^{\infty} (-1)^i (RT)^i RS \right).$$

Then, when H is small, the set $\{A_0 + H \in F_{ij}\}$ is a graph of an analytic function with domain $\mathcal{V}$, and the claim follows easily. □

Declarations: conflict of interest. The authors declare no competing interests. Manuscript has no associated data.

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