HECKE OPERATORS AND DRINFELD CUSP FORMS OF LEVEL t

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Abstract. We use a linear algebra interpretation of the action of Hecke operators on Drinfeld cusp forms to prove that, when the dimension of the $\mathbb{C}_\infty$-vector space $S_{k,m}(\mathrm{GL}_2(\mathbb{F}_q[t]))$ is one, the operator $\mathbf{T}_t$ is injective on $S_{k,m}(\mathrm{GL}_2(\mathbb{F}_q[t]))$ and $S_{k,m}(\Gamma_0(t))$ is direct sum of oldforms and newforms.

1. Introduction

Let $S_k(\mathrm{SL}_2(\mathbb{Z}))$ be the space of weight k cusp forms of level 1. It is well-known that it admits a basis of *Hecke eigenforms*, i.e. normalized eigenfunctions for all the Hecke operators T_n's. The space $S_k(\Gamma_0(N))$ still admits a basis of Hecke eigenforms, but only for those T_n's such that $(n,N) = 1$. In order to find a basis of eigenforms for all the T_n's we have to focus on forms that are genuinely of level N and also to consider the operator U_p if $p|N$ as Atkin and Lehner realized in [1]. More precisely, we have to distinguish between *oldforms*, i.e. forms coming from lower level $M|N$, and *newforms*, which are defined as the orthogonal complement of oldforms with respect to the Petersson inner product (see [10, Chapter 5]).

The present paper mainly deals with a function field counterpart of the above results. It comes after a series of papers, see [3], [4], [6] and [14], in which we investigated the following problems for the *Drinfeld modular forms*:

1. diagonalizability of Hecke operators;
2. injectivity of Hecke operators;
3. newforms and oldforms.

Let $S_{k,m}(\mathrm{GL}_2(\mathbb{F}_q[t]))$, resp. $S_{k,m}(\Gamma_0(\mathfrak{p}))$, be the space of Drinfeld cusp forms of weight k, type m and level 1, resp. level $\mathfrak{p}$, where $\mathfrak{p} = (P)$ is a prime ideal of $\mathcal{O} := \mathbb{F}_q[t]$ and q a power of a fixed prime $p \in \mathbb{Z}$ (see Section 2 for details on definitions and notations appearing in this introduction). Moreover, denote by $\mathbf{T}_\mathfrak{p}$, resp. $\mathbf{U}_\mathfrak{p}$, the Hecke, resp. Atkin-Lehner, operator acting on $S_{k,m}(\mathrm{GL}_2(\mathcal{O}))$, resp. $S_{k,m}(\Gamma_0(\mathfrak{p}))$. One of the challenges in the positive characteristic setting comes from the lack of a suitable analogue of the Petersson inner product.
In [3] and [4] we got over this using a combination of a combinatorial argument, i.e. Teitelbaum interpretation of cusp forms as harmonic cocycles (see [13]), and of *twisted trace maps*, to describe what we identify as newforms. The combinatorial method allowed us to explicitly describe the matrix associated to the $\mathbf{U}_\mathfrak{p}$-operator acting on $S_{k,m}(\Gamma_0(\mathfrak{p}))$, when $\mathfrak{p}$ is prime generated by a degree one polynomial, and to formulate a series of conjectures, supported by numerical search, on the distribution of slopes, i.e $\mathfrak{p}$-adic valuations of eigenvalues of $\mathbf{U}_\mathfrak{p}$, as the weight varies. In the paper [4], among other things, we conjecture that

ConjMain **Conjecture 1.1.** [4, Conjecture 1.1]

1. $\mathbf{T}_\mathfrak{p}$ *is injective;*
2. $S_{k,m}(\Gamma_0(\mathfrak{p}))$ *is the direct sum of oldforms* $S^{old}_{k,m}(\Gamma_0(\mathfrak{p}))$ *and newforms* $S^{old}_{k,m}(\Gamma_0(\mathfrak{p}))$.

We provided some evidence in particular

(a) for the case $\deg(P) = 1$ and $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(\mathcal{O}))) = 0$, in [4, Section 5];
(b) for the case $\deg(P) = 1$ and $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(\mathcal{O}))) = 1$, in the (unpublished) Section 3 of [5].

Recently T. Dalal and N. Kumar [7, Theorem 4.6] provided a new proof for case (b): their method is based on the analysis of the Fourier coefficients of the image of a generator via the Hecke operator $\mathbf{T}_\mathfrak{p}$ and, hopefully, it is suitable for more generalizations. Since there seems to be quite some interest in this type of results, we decided to present here our original proof of this fact based on the linear algebra interpretation of the Hecke operators $\mathbf{T}_\mathfrak{p}$ and $\mathbf{U}_\mathfrak{p}$, and of the trace maps Tr and

Tr' ([3] and [4]). The proof is via direct computation, exploiting the symmetries of the matrices representing these operators. We believe that such symmetries are the key to improve the results but, to go further, we probably need a deeper understanding of how they reflect on the action on oldforms, i.e. find the oldforms counterpart of the antidiagonal action on newforms (see [4, Section 5.2]). We mention that the statement and an explicit example already appeared in [6, Example 2.19].

The paper is organized as follows. In Section 2 we recall the objects we shall work with: Drinfeld modular forms, Hecke operators, degeneracy and trace maps that will enable us to define oldforms and newforms despite the absence of an appropriate inner product.
In Section 3 we specialize at $\mathfrak{p}=(t)$ and, as in [3], we associate explicit matrices to all operators. In particular, we describe a matrix M, see (6), that is involved in the description of $\mathbf{U}_t$ and has lots of peculiar symmetries. After that, we briefly deal with the diagonalizability of M and then prove our main results.

Theorem 1.2. *Assume* $\dim_{\mathbb{C}_\infty} S_{k,m}(\mathrm{GL}_2(\mathcal{O}))=1$. *Then*
1. $\mathbf{T}_t$ *is injective (see Theorem 3.1);*
2. $S_{k,m}(\Gamma_0(t)) = S^{old}_{k,m}(\Gamma_0(t)) \oplus S^{new}_{k,m}(\Gamma_0(t))$ *(see Theorem 3.2).*

SecNot

2. Setting and notations

Let K be the global function field $\mathbb{F}_q(t)$, where q is a power of a fixed prime $p\in\mathbb{Z}$, fix the prime $\frac{1}{t}$ at ∞ and denote by $\mathcal{O}:=\mathbb{F}_q[t]$ its ring of integers (i.e., the ring of functions regular outside ∞). Let $K_\infty=\mathbb{F}_q((\frac{1}{t}))$ be the completion of K at $\frac{1}{t}$ and denote by $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞.

2.1. **Drinfeld modular forms.** The *Drinfeld upper half-plane* is the set $\Omega:=\mathbb{P}^1(\mathbb{C}_\infty)-\mathbb{P}^1(K_\infty)$ together with a structure of rigid analytic space (see [11]).
The group $\mathrm{GL}_2(K_\infty)$ acts on Ω via Möbius transformation $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)(z)=\frac{az+b}{cz+d}$. Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(\mathcal{O})$: Γ has finitely many cusps, represented by $\Gamma\backslash\mathbb{P}^1(K)$. For $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{GL}_2(K_\infty)$, $k,m\in\mathbb{Z}$ and $\varphi:\Omega\to\mathbb{C}_\infty$, we define

Mod0 (1)
$$(\varphi|_{k,m}\gamma)(z):=\varphi(\gamma z)(\det\gamma)^m(cz+d)^{-k}.$$

Definition 2.1. *A rigid analytic function* $\varphi:\Omega\to\mathbb{C}_\infty$ *is called a* Drinfeld modular function of weight k and type $m\in\mathbb{Z}/o(\Gamma)\mathbb{Z}$ for Γ *if*

Mod (2)
$$(\varphi|_{k,m}\gamma)(z)=\varphi(z)\quad\forall\gamma\in\Gamma,$$

where $o(\Gamma)$ *is the number of scalar matrices in* Γ.
A Drinfeld modular function φ *of weight* $k\geqslant 0$ *and type* m *for* Γ *is called a* Drinfeld modular form *if* φ *is holomorphic at all cusps, it is called a* cusp form *if it vanishes at all cusps.*
The space of Drinfeld modular forms of weight k *and type* m *for* Γ *will be denoted by* $M_{k,m}(\Gamma)$. *The subspace of cuspidal modular forms is denoted by* $S_{k,m}(\Gamma)$.

The above definition coincides with [8, Definition 5.1], other authors require the function to be meromorphic (in the sense of rigid analysis, see for example [9, Definition 1.4]) and would call our functions *weakly modular*.
Let $\mathfrak{p}=(P)\subset\mathcal{O}$ be a prime with P irreducible of degree 1. We shall work only with the arithmetic subgroup $\Gamma_0(\mathfrak{p})$ of upper triangular matrices modulo $\mathfrak{p}$, and the spaces $S_{k,m}(\mathrm{GL}_2(\mathcal{O}))$ and $S_{k,m}(\Gamma_0(\mathfrak{p}))$ which we call, respectively, cusp forms of level 1 and of level $\mathfrak{p}$. Note that in both cases $o(\Gamma)=q-1$, so, to have nontrivial forms, weight and type must verify $k\equiv 2m \pmod{q-1}$.

2.2. **Hecke operators.** We have the following Hecke operators
$$\mathbf{T}_\mathfrak{p}(\varphi)(z):=P^{k-m}(\varphi|_{k,m}\left(\begin{smallmatrix} P & 0\\ 0 & 1\end{smallmatrix}\right))(z)+P^{k-m}\sum_{Q\in\mathbb{F}_q}(\varphi|_{k,m}\left(\begin{smallmatrix} 1 & Q\\ 0 & P\end{smallmatrix}\right))(z),\quad \text{on } S_{k,m}(\mathrm{GL}_2(\mathcal{O}));$$
$$\mathbf{U}_\mathfrak{p}(\varphi)(z):=P^{k-m}\sum_{Q\in\mathbb{F}_q}(\varphi|_{k,m}\left(\begin{smallmatrix} 1 & Q\\ 0 & P\end{smallmatrix}\right))(z),\quad \text{on } S_{k,m}(\Gamma_0(\mathfrak{p})).$$

2.3. **Newforms and oldforms.** As already mentioned in the introduction, in the positive characteristic setting we do not have an analogue of the Petersson inner product, therefore we need a different approach. In [4, Section 3] we defined oldforms and newforms of level t. The definition has been generalized in [6] and [14], but T. Dalal and N. Kumar in [7, Section 4.3] pointed out the existence of a twisted Eisenstein form both new and old (for our definition) when the level is $\mathfrak{pq}$ ($\mathfrak{q}$ another prime different from $\mathfrak{p}$). Since we shall only work with levels 1 and $\mathfrak{p}$, we can still rely on our original definition, which we now recall.
We have an injective map:

$$\begin{aligned}&(\delta_1,\delta_{\mathfrak{p}}) : S_{k,m}(\mathrm{GL}_2(\mathcal{O}))^2 \longrightarrow S_{k,m}(\Gamma_0(\mathfrak{p}))\\ &(\delta_1,\delta_{\mathfrak{p}})(\varphi,\psi) = \varphi(z) + (\psi|_{k,m}\left(\begin{smallmatrix} P & 0\\ 0 & 1\end{smallmatrix}\right))(z) = \varphi(z) + P^m\psi(Pz).\end{aligned}$$

Definition 2.2. *The space of* oldforms of level $\mathfrak{p}$*, denoted by* $S^{old}_{k,m}(\Gamma_0(\mathfrak{p}))$*, is the subspace of* $S_{k,m}(\Gamma_0(\mathfrak{p}))$ *generated by* $\mathrm{Im}(\delta_1,\delta_{\mathfrak{p}})$.

We recall that a system of coset representative for $\Gamma_0(\mathfrak{p})\backslash\mathrm{GL}_2(\mathcal{O})$ is $R = \left\{Id, \left(\begin{smallmatrix} 0 & -1\\ 1 & Q\end{smallmatrix}\right) : Q \in \mathbb{F}_q\right\}$.

DefFrTrTr' **Definition 2.3.** *We have the following maps defined on* $S_{k,m}(\Gamma_0(\mathfrak{p}))$*:*

- *the* Fricke involution*, which preserves the space* $S_{k,m}(\Gamma_0(\mathfrak{p}))$*, represented by the matrix* $\gamma_{\mathfrak{p}} := \left(\begin{smallmatrix} 0 & -1\\ P & 0\end{smallmatrix}\right)$ *and defined by* $\varphi^{Fr} = (\varphi|_{k,m}\gamma_{\mathfrak{p}})$*;*
- *the* trace map*, defined by*

$$\begin{aligned}Tr : S_{k,m}(\Gamma_0(\mathfrak{p})) &\to S_{k,m}(\mathrm{GL}_2(\mathcal{O}))\\ \varphi &\mapsto \sum_{\gamma\in R}(\varphi|_{k,m}\gamma)(z);\end{aligned}$$

- *the* twisted trace map*, defined by*

$$\begin{aligned}Tr' : S_{k,m}(\Gamma_0(\mathfrak{p})) &\to S_{k,m}(\mathrm{GL}_2(\mathcal{O}))\\ \varphi &\mapsto Tr(\varphi^{Fr}).\end{aligned}$$

DefNew **Definition 2.4.** *The space of* newforms of level $\mathfrak{p}$*, denoted by* $S^{new}_{k,m}(\Gamma_0(\mathfrak{p}))$*, is* $\mathrm{Ker}(Tr)\cap\mathrm{Ker}(Tr')$.

The following important criterion is [6, Theorem 2.8 and Corollary 2.10].

ThmDirSum1 **Theorem 2.5.** *We have a direct sum decomposition* $S_{k,m}(\Gamma_0(\mathfrak{p})) = S^{old}_{k,m}(\Gamma_0(\mathfrak{p})) \oplus S^{new}_{k,m}(\Gamma_0(\mathfrak{p}))$ *if and only if the map* $\mathcal{D} := Id - P^{k-2m}(Tr')^2$ *is bijective. Moreover*

$$\mathrm{Ker}(\mathcal{D}) = \{\delta_1\varphi : \varphi\in S_{k,m}(\mathrm{GL}_2(\mathcal{O})),\ \mathbf{T}_{\mathfrak{p}}\varphi = \pm P^{k/2}\varphi\}.$$

SecDim1

3. Main results

For the level t (i.e. actually for any prime of degree 1) we computed the matrix associated to the operator $\mathbf{U}_t$ acting on $S_{k,m}(\Gamma_0(t))$ (using Teitelbaum's interpretation with harmonic cocycles, see [3, Section 4] and [4, Sections 3 and 4]): for the convenience of the reader we shortly recall the matrices involved in our computations.
In order to have $S_{k,m}(\Gamma_0(t)) \neq 0$ we need $k \equiv 2m \pmod{q-1}$, hence there exists a unique $n\in\mathbb{N}$ such that $k = 2m + (n-1)(q-1)$. For notational reasons we put $j+1 \equiv m \pmod{q-1}$ with $0 \leqslant j \leqslant q-2$: the letters j and n provide the type m and the dimension of the matrix U associated to the action of $\mathbf{U}_t$ on $S_{k,m}(\Gamma_0(t))$. The crucial ingredient is the following matrix: for even n we put

EqTMat

$$T := \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,\frac{n}{2}} & (-1)^{j+1}m_{1,\frac{n}{2}} & \cdots & (-1)^{j+1}m_{1,2} & (-1)^{j+1}m_{1,1}\\ m_{2,1} & m_{2,2} & \cdots & m_{2,\frac{n}{2}} & (-1)^{j+1}m_{2,\frac{n}{2}} & \cdots & (-1)^{j+1}m_{2,2} & (-1)^{j+1}m_{2,1}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ m_{\frac{n}{2},1} & m_{\frac{n}{2},2} & \cdots & m_{\frac{n}{2},\frac{n}{2}} & (-1)^{j+1}m_{\frac{n}{2},\frac{n}{2}} & \cdots & (-1)^{j+1}m_{\frac{n}{2},2} & (-1)^{j+1}m_{\frac{n}{2},1}\\ m_{\frac{n}{2}+1,1} & m_{\frac{n}{2}+1,2} & \cdots & 0 & 0 & \cdots & (-1)^{j+1}m_{\frac{n}{2}+1,2} & (-1)^{j+1}m_{\frac{n}{2}+1,1}\\ \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ m_{n-1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 & (-1)^{j+1}m_{n-1,1}\\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{3}$$

(the reason to denote it by T will become apparent shortly). For odd n one just needs to modify the indices a bit and add the central $\frac{n+1}{2}$-th column

$$(m_{1,\frac{n+1}{2}},\cdots,m_{\frac{n-1}{2},\frac{n+1}{2}},0,\cdots,0).$$

The entries of M are the binomial coefficients in $\mathbb{F}_p$

spam

$$m_{a,b}=\begin{cases} -\left[\binom{j+(n-a)(q-1)}{j+(n-b)(q-1)}+(-1)^{j+1}\binom{j+(n-a)(q-1)}{j+(b-1)(q-1)}\right] & \text{if } a\neq b \\ (-1)^j\binom{j+(n-a)(q-1)}{j+(a-1)(q-1)} & \text{if } a=b \end{cases}. \tag{4}$$

Let A be the antidiagonal matrix

EqA

$$A=\begin{pmatrix} 0 & & (-1)^{j+1} \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{j+1} & & 0 \end{pmatrix}, \tag{5}$$

then $A^2=I$ (the identity matrix of dimension n) and the main simmetry of T can be expressed as $TA=T$. This is clear for even n. For odd n and even j note that the central column is identically 0 because of formula (4), while for odd j one is simply multiplying the central column by 1. Finally let

EqDefM

$$M:=T-A. \tag{6}$$

We list here the matrices associated to the maps involved in our computations.

- The action of $\mathbf{U}_t$ on $S_{k,m}(\Gamma_0(t))$ is described by

EqAt

$$U=MD:=M\begin{pmatrix} t^{s_1} & & 0 \\ & \ddots & \\ 0 & & t^{s_n} \end{pmatrix}, \tag{7}$$

  where for $1\leqslant i\leqslant n$ we put $s_i=j+1+(i-1)(q-1)$ (so that $s_i+s_{n+1-i}=k$ for $1\leqslant i\leqslant \frac{n}{2}$ or $1\leqslant i\leqslant\frac{n+1}{2}$ according to n being even or odd).
- The matrix for the Fricke involution $Fr(t)$ is

MatrixFricke

$$t^{m-k}F=t^{m-k}\begin{pmatrix} 0 & & (-t)^{s_n} \\ & \cdot^{\cdot^{\cdot}} & \\ (-t)^{s_1} & & 0 \end{pmatrix}=t^{m-k}\begin{pmatrix} 0 & & (-1)^{j+1}t^{s_n} \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{j+1}t^{s_1} & & 0 \end{pmatrix}. \tag{8}$$

  Note that $F^2=t^kI$ and $AF=D$.
- Direct computation (see [3, Section 3.3]) provides the equation

EqTr1

$$Tr(\varphi)=\varphi+t^{-m}\mathbf{U}_t(\varphi^{Fr}). \tag{9}$$

  Its immediate translation in matrix form is

eqTr

$$\begin{aligned} I+t^{-m}MD(t^{m-k}F) &= I+t^{-k}MAF^2=I+MA=A^2+(T-A)A \\ &=(A+T-A)A=TA=T. \end{aligned} \tag{10}$$

- The matrix for the twisted trace follows directly

eqTr'

$$T'=t^{m-k}TF=\begin{cases} t^{m-k}(M+A)F=t^{m-k}(MF+D) \\ t^{m-k}TAF=t^{m-k}TD \\ t^{m-k}TAF=t^{m-k}(I+MA)F=t^{m-k}(F+MD) \end{cases}. \tag{11}$$

- Finally, since the trace acts trivially on $\mathrm{Im}(\delta_1)$, it is easy to see that $\mathrm{Ker}(Tr-Id)=\mathrm{Im}(\delta_1)$, i.e. in terms of matrices $\mathrm{Im}(\delta_1)=\mathrm{Ker}(T-I)=\mathrm{Ker}(MA)$.

3.1. **Diagonalizability of M.** As seen above the matrices M and T verify a number of equations. We mention a few more, leading to the diagonalizability of M (unfortunately not equivalent to the diagonalizability of $U = MD$ which is included in [4, Conjecture 1.1] and is related to the conjectures treated in this paper), but we shall not pursue this topic further here.

(i) Like all trace maps $T^2 = T$ and T is diagonalizable. This obviously leads to n^2 equations in the entries $m_{i,j}$ which, anyway, are still difficult to handle for a generic n.

(ii) Let $\underline{v} \in (M)$, i.e. $\underline{v} = M\underline{w}$. Then

$$T\underline{v} = TM\underline{w} = T(T-A)\underline{w} = (T^2 - TA)\underline{w} = 0,$$

i.e. $\mathrm{Im}(M) \subseteq \mathrm{Ker}(T)$.
Viceversa, let $\underline{v} \in \mathrm{Ker}(T)$, then, writing $\underline{v} = -A\underline{w}$, we get $0 = T\underline{v} = -TA\underline{w} = -T\underline{w}$, i.e. $\underline{w} \in \mathrm{Ker}(T)$ as well. Therefore $M\underline{w} = (T-A)\underline{w} = -A\underline{w} = \underline{v} \in \mathrm{Im}(M)$. Hence $\mathrm{Im}(M) = \mathrm{Ker}(T)$.

(iii) Finally

$$\begin{aligned} M^3 &= (T-A)^3 = (T^2 - AT - TA + I)(T-A) \\ &= (-AT + I)(T-A) = -AT^2 + T + ATA - A \\ &= T - A = M. \end{aligned}$$

Therefore, for any $p \neq 2$, the matrix M is diagonalizable and we can write vectors $\underline{v}$ as

$$\underline{v} = \frac{M^2\underline{v} + \underline{v}}{2} + \frac{M^2\underline{v} - \underline{v}}{2} + (\underline{v} - M^2\underline{v}) := \underline{v}_1 + \underline{v}_{-1} + \underline{v}_0,$$

where each $\underline{v}_\alpha$ is in the M-eigenspace of eigenvalue $\alpha \in \{0, 1, -1\}$.
This somehow reflects the results of [2], where we found examples of non diagonalizability of $\mathbf{U}_t$ in characteristic 2, due to the presence of inseparable eigenvalues associated to newforms.

3.2. **Injectivity of $\mathbf{T}_t$.** In [4] we proved some special cases of Conjecture 1.1 building on the analog of Theorem 3.2 and on the above matrices/formulas (which are not avaliable for $\deg P \geqslant 2$). In particular, in [4, Theorem 5.5] we proved that when $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(\mathcal{O})) = 0$, i.e. there are no oldforms, the matrix M is antidiagonal and the conjectures hold. We shall now approach the case $\dim_{\mathbb{C}_\infty}(S_{k,m}(\mathrm{GL}_2(\mathcal{O})) = 1$, this will include many more cases since, for example, $\dim_{\mathbb{C}_\infty}(S_{k,0}(\mathrm{GL}_2(\mathcal{O})) = 1$ if and only if $q \leqslant n < 2q - 1$, by [9, Proposition 4.3] (compare with the bounds of [4, Theorems 5.8, 5.9, 5.12, 5.14]).

ThmInj **Theorem 3.1.** *Assume* $\dim_{\mathbb{C}_\infty} \mathrm{Im}(\delta_1) = 1$. *Then* $\mathbf{T}_t$ *is injective.*

Proof. We first observe that, by [6, Proposition 2.5], $\mathrm{Ker}(\mathbf{T}_t) = \mathrm{Ker}(MA) \cap \mathrm{Ker}(MDMD)$. Thanks to our assumption on the dimension of $\mathrm{Im}(\delta_1) = \mathrm{Ker}(MA)$ and to the fact that the entries of MA are in $\mathbb{F}_p$, we have $\dim_{\mathbb{C}_\infty}(\mathrm{Ker}(MA) \cap \mathrm{Ker}(MDMD)) \leqslant 1$ and we can fix a generator $\underline{a} = (a_1, \dots, a_n) \in \mathbb{F}_p^n$. Our goal is to prove $\underline{a} = 0$.
We prove the even dimension case, for odd n the argument is exactly the same: the vector $\underline{a}$ satisfies the following equations coming from $MA\underline{a} = 0$

v1 (12)
$$\begin{cases} (m_{1,1}-1)a_1 + m_{1,2}a_2 + \cdots + m_{1,\frac{n}{2}}a_{\frac{n}{2}} + (-1)^{j+1}m_{1,\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{1,1}a_n = 0 \\ m_{2,1}a_1 + (m_{2,2}-1)a_2 + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}} + (-1)^{j+1}m_{2,\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{2,1}a_n = 0 \\ \vdots \\ m_{\frac{n}{2},1}a_1 + m_{\frac{n}{2},2}a_2 + \cdots + (m_{\frac{n}{2},\frac{n}{2}} - 1)a_{\frac{n}{2}} + (-1)^{j+1}m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{\frac{n}{2},1}a_n = 0 \\ m_{\frac{n}{2}+1,1}a_1 + m_{\frac{n}{2}+1,2}a_2 + \cdots + m_{\frac{n}{2}+1,\frac{n}{2}-1}a_{\frac{n}{2}-1} - a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,1}a_n = 0 \\ \vdots \\ m_{n-1,1}a_1 - a_{n-1} + (-1)^{j+1}m_{n-1,1}a_n = 0 \\ -a_n = 0 \end{cases}.$$

Now put $\underline{p(t)} := MD\underline{a} \in \mathbb{F}_p[t]^n$, then (with $a_n = 0$)

p1 (13)

$$\underline{p(t)} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_{\frac{n}{2}}(t) \\ p_{\frac{n}{2}+1}(t) \\ \vdots \\ p_{n-1}(t) \\ p_n(t) \end{pmatrix} = \begin{pmatrix} m_{1,1}a_1t^{s_1} + \cdots + m_{1,\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}m_{1,\frac{n}{2}}a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{1,2}a_{n-1}t^{s_{n-1}} \\ m_{2,1}a_1t^{s_1} + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}m_{2,\frac{n}{2}}a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}(m_{2,2}-1)a_{n-1}t^{s_{n-1}} \\ \vdots \\ m_{\frac{n}{2},1}a_1t^{s_1} + \cdots + m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}(m_{\frac{n}{2},\frac{n}{2}}-1)a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{\frac{n}{2},2}a_{n-1}t^{s_{n-1}} \\ m_{\frac{n}{2}+1,1}a_1t^{s_1} + \cdots + (-1)^j a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + m_{\frac{n}{2}+1,\frac{n}{2}-1}a_{\frac{n}{2}+2}t^{s_{\frac{n}{2}+2}} + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,2}a_{n-1}t^{s_{n-1}} \\ \vdots \\ m_{n-1,1}a_1t^{s_1} + (-1)^j a_2t^{s_2} \\ (-1)^j a_1t^{s_1} \end{pmatrix}.$$

Since $MD\underline{p(t)} = 0$, we also have equations:

w1 (14)

$$\begin{cases} m_{1,1}t^{s_1}p_1(t) + \cdots + m_{1,\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t) + (-1)^{j+1}m_{1,\frac{n}{2}}t^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t) + \cdots + (-1)^{j+1}(m_{1,1}-1)t^{s_n}p_n(t) = 0 \\ m_{2,1}t^{s_1}p_1(t) + \cdots + m_{2,\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t) + (-1)^{j+1}m_{2,\frac{n}{2}}t^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t) + \cdots + (-1)^{j+1}m_{2,1}t^{s_n}p_n(t) = 0 \\ \vdots \\ m_{\frac{n}{2},1}t^{s_1}p_1(t) + \cdots + m_{\frac{n}{2},\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t) + (-1)^{j+1}(m_{\frac{n}{2},\frac{n}{2}}-1)t^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t) + \cdots + (-1)^{j+1}m_{\frac{n}{2},1}t^{s_n}p_n(t) = 0 \\ m_{\frac{n}{2}+1,1}t^{s_1}p_1(t) + \cdots + (-1)^j t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t) + (-1)^{j+1}m_{\frac{n}{2}+1,\frac{n}{2}-1}t^{s_{\frac{n}{2}+2}}p_{\frac{n}{2}+2}(t) + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,1}t^{s_n}p_n(t) = 0 \\ \vdots \\ m_{n-1,1}t^{s_1}p_1(t) + (-1)^j t^{s_2}p_2(t) + (-1)^{j+1}m_{n-1,1}t^{s_n}p_n(t) = 0 \\ (-1)^j t^{s_1}p_1(t) = 0 \end{cases}.$$

Note that in (14) we have polynomials in $\mathbb{F}_p[t]$, from now on we shall basically use the identity principle for polynomials to solve the equations in the a_i. From the last row in (14) we get $p_1(t) = 0$, i.e. comparing with (13) and recalling the s_i are distinct

$$m_{1,1}a_1 = m_{1,2}a_2 = \cdots = m_{1,\frac{n}{2}}a_{\frac{n}{2}} = m_{1,\frac{n}{2}}a_{\frac{n}{2}+1} = \cdots = m_{1,2}a_{n-1} = 0\,.$$

Substituting in the first and second-last equations in (12) we obtain

$$a_1 = a_{n-1} = 0,$$

which also means that $p_n(t) = 0$.
We can rewrite (12), (13) and (14) as

v2

$$\begin{cases} (m_{2,2}-1)a_2 + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}} + (-1)^{j+1}m_{2,\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{2,3}a_{n-2} = 0 \\ \vdots \\ m_{\frac{n}{2},2}a_2 + \cdots + (m_{\frac{n}{2},\frac{n}{2}}-1)a_{\frac{n}{2}} + (-1)^{j+1}m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{\frac{n}{2},3}a_{n-2} = 0 \\ m_{\frac{n}{2}+1,2}a_2 + \cdots + m_{\frac{n}{2}+1,\frac{n}{2}-1}a_{\frac{n}{2}-1} - a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,3}a_{n-2} = 0 \\ \vdots \\ m_{n-2,2}a_2 - a_{n-2} = 0 \\ a_1 = a_{n-1} = a_n = 0 \end{cases}, \tag{15}$$

p2 (16)

$$\begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_{\frac{n}{2}}(t) \\ p_{\frac{n}{2}+1}(t) \\ \vdots \\ p_{n-1}(t) \\ p_n(t) \end{pmatrix} = \begin{pmatrix} 0 \\ m_{2,2}a_2t^{s_2} + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}m_{2,\frac{n}{2}}a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{2,3}a_{n-2}t^{s_{n-2}} \\ \vdots \\ m_{\frac{n}{2},2}a_2t^{s_2} + \cdots + m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}(m_{\frac{n}{2},\frac{n}{2}}-1)a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{\frac{n}{2},3}a_{n-2}t^{s_{n-2}} \\ m_{\frac{n}{2}+1,2}a_2t^{s_2} + \cdots + (-1)^j a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + m_{\frac{n}{2}+1,\frac{n}{2}-1}a_{\frac{n}{2}+2}t^{s_{\frac{n}{2}+2}} + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,3}a_{n-2}t^{s_{n-2}} \\ \vdots \\ (-1)^j a_2t^{s_2} \\ 0 \end{pmatrix}$$

and

w2 (17)

$$\begin{cases} m_{1,2}t^{s_2}p_2(t)+\cdots+m_{1,\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+(-1)^{j+1}m_{1,\frac{n}{2}}t^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)+\cdots(-1)^{j+1}m_{1,2}t^{s_{n-1}}p_{n-1}(t)=0\\ m_{2,2}t^{s_2}p_2(t)+\cdots+m_{2,\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+(-1)^{j+1}m_{2,\frac{n}{2}}t^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)+\cdots+(-1)^{j+1}(m_{2,2}-1)t^{s_{n-1}}p_{n-1}(t)=0\\ \vdots\\ m_{\frac{n}{2},2}t^{s_2}p_2(t)+\cdots+m_{\frac{n}{2},\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+(-1)^{j+1}(m_{\frac{n}{2},\frac{n}{2}}-1)t^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)+\cdots+(-1)^{j+1}m_{\frac{n}{2},2}t^{s_{n-1}}p_{n-1}(t)=0\\ m_{\frac{n}{2}+1,2}t^{s_2}p_2(t)+\cdots+(-1)^{j}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+(-1)^{j+1}m_{\frac{n}{2}+1,\frac{n}{2}-1}t^{s_{\frac{n}{2}+2}}p_{\frac{n}{2}+2}(t)+\cdots+(-1)^{j+1}m_{\frac{n}{2}+1,2}t^{s_{n-1}}p_{n-1}(t)=0\\ \vdots\\ (-1)^j t^{s_2}p_2(t)=0\\ p_1(t)=p_n(t)=0 \end{cases}.$$

We repeat the same argument starting now from the second-last equation in (17), which yields $p_2(t)=0$. This means

$$m_{2,2}a_2=\cdots=m_{2,\frac{n}{2}}a_{\frac{n}{2}}=m_{2,\frac{n}{2}}a_{\frac{n}{2}+1}=\cdots=m_{2,3}a_{n-2}=0,$$

which, substituted in the first equation of (15), gives $a_2=0$. Thus (second-last equations in (15) and (16)) $a_{n-2}=0$ and $p_{n-1}(t)=0$ as well.
Iterating the process we see that the specular symmetries between MD ($(-1)^j$ on the antidiagonal) and MA (-1 on the diagonal), and the positions of the $m_{i,i}-1$ lead to $\underline{a}=0$. □

3.3. **Direct sum.** We shall use the criterion of Theorem 2.5 and show that $\mathrm{Ker}(\mathcal{D})=0$. Note that $\varphi\in\mathrm{Ker}(\mathcal{D})$ yields $\varphi-t^{k-2m}(Tr')^2(\varphi)=0$, i.e. $\varphi=t^{k-2m}(Tr')^2(\varphi)\in S_{k,m}(\mathrm{GL}_2(\mathcal{O}))$; hence φ is old and, in particular, belongs to $\mathrm{Im}(\delta_1)=\mathrm{Ker}(MA)$. So we write $\varphi=\delta_1\psi$ and, by [3, Equation (3.2)], $\mathbf{U}_t(\delta_1\psi)=\delta_1\mathbf{T}_t\psi-t^{k-m}\delta_t\psi$ is old as well. Moreover

$$\begin{aligned} t^{2m-k}\delta_1\psi &= (Tr')^2(\delta_1\psi)=(Tr')(Tr'(\delta_1\psi))\\ &= Tr'((\delta_1\psi)^{Fr}+t^{m-k}\mathbf{U}_t(\delta_1\psi)) \quad \text{(use the twisted version of (9))}\\ &= Tr(((\delta_1\psi)^{Fr})^{Fr})+t^{m-k}Tr'(\mathbf{U}_t(\delta_1\psi))\\ &= t^{2m-k}Tr(\delta_1\psi)+t^{m-k}Tr'(\mathbf{U}_t(\delta_1\psi))\\ &= t^{2m-k}\delta_1\psi+t^{m-k}Tr'(\mathbf{U}_t(\delta_1\psi)), \end{aligned}$$

i.e. $Tr'(\mathbf{U}_t(\delta_1\psi))=0$. We can similarly prove that $Tr(\mathbf{U}_t(\delta_1\psi))=0$ as well (i.e. $\mathbf{U}_t(\delta_1\psi)$ is old and new), but the equations coming from MA and $T'U$ will be enough for our purposes.

ThmDirSum **Theorem 3.2.** *Assume that* $\dim_{\mathbb{C}_\infty}\mathrm{Im}(\delta_1)=1$. *Then* $S_{k,m}(\Gamma_0(t))=S^{old}_{k,m}(\Gamma_0(t))\oplus S^{new}_{k,m}(\Gamma_0(t))$.

Proof. Take $\underline{a}\in\mathbb{F}_p^n$ which verifies $MA\underline{a}=0$ and represents an element $\eta=\delta_1\varphi\in\mathrm{Ker}(\mathcal{D})$, so that, as seen above, $TF(MD\underline{a})=0$. We prove that these two relations yield $\underline{a}=0$, so that $\mathrm{Ker}(\mathcal{D})=0$ and $\mathcal{D}$ is invertible. As before we only treat the case of even n.
The equation $MA\underline{a}=0$ gives again the system (12) (in particular $a_n=0$), then, writing $\underline{p(t)}=MD\underline{a}$ as in (13), from $TF(MD\underline{a})=0$ we get

d1

$$(18)\quad \begin{cases} m_{1,1}t^{s_1}p_1(t)+\cdots+m_{1,\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+m_{1,\frac{n}{2}}(-t)^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)+\cdots+m_{1,1}(-t)^{s_n}p_n(t)=0\\ m_{2,1}t^{s_1}p_1(t)+\cdots+m_{2,\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+m_{2,\frac{n}{2}}(-t)^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)+\cdots+m_{2,1}(-t)^{s_n}p_n(t)=0\\ \vdots\\ m_{\frac{n}{2},1}t^{s_1}p_1(t)+\cdots+m_{\frac{n}{2},\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t)+m_{\frac{n}{2},\frac{n}{2}}(-t)^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)+\cdots+m_{\frac{n}{2},1}(-t)^{s_n}p_n(t)=0\\ m_{\frac{n}{2}+1,1}t^{s_1}p_1(t)+\cdots+m_{\frac{n}{2}+1,1}(-t)^{s_n}p_n(t)=0\\ \vdots\\ m_{n-2,1}t^{s_1}p_1(t)+m_{n-2,2}t^{s_2}p_2(t)+m_{n-2,2}(-t)^{s_{n-1}}p_{n-1}(t)+m_{n-2,1}(-t)^{s_n}p_n(t)=0\\ m_{n-1,1}t^{s_1}p_1(t)+m_{n-1,1}(-t)^{s_n}p_n(t)=0 \end{cases}.$$

In the last equation of (18) the term with the highest degree in t is $m_{n-1,1}(-t)^{s_n}(-1)^j a_1 t^{s_1}=-m_{n-1,1}a_1t^k$ (note that $p_1(t)$ has degree at most s_{n-1} because $a_n=0$): therefore $m_{n-1,1}a_1=0$ and the second-last equation in (12) tells us that $a_{n-1}=0$. Now (12) and (13) turn into

s2 (19)
$$\begin{cases} (m_{1,1}-1)a_1 + m_{1,2}a_2 + \cdots + m_{1,\frac{n}{2}}a_{\frac{n}{2}} + (-1)^{j+1}m_{1,\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{1,3}a_{n-2} = 0 \\ m_{2,1}a_1 + (m_{2,2}-1)a_2 + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}} + (-1)^{j+1}m_{2,\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{2,3}a_{n-2} = 0 \\ \vdots \\ m_{\frac{n}{2},1}a_1 + m_{\frac{n}{2},2}a_2 + \cdots + (m_{\frac{n}{2},\frac{n}{2}}-1)a_{\frac{n}{2}} + (-1)^{j+1}m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{\frac{n}{2},3}a_{n-2} = 0 \\ m_{\frac{n}{2}+1,1}a_1 + m_{\frac{n}{2}+1,2}a_2 + \cdots + m_{\frac{n}{2}+1,\frac{n}{2}-1}a_{\frac{n}{2}-1} - a_{\frac{n}{2}+1} + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,3}a_{n-2} = 0 \\ \vdots \\ m_{n-2,1}a_1 + m_{n-2,2}a_2 - a_{n-2} = 0 \\ m_{n-1,1}a_1 = 0 \\ a_{n-1} = a_n = 0 \end{cases}$$

and

md2 (20)
$$\underline{p(t)} = \begin{pmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_{\frac{n}{2}}(t) \\ p_{\frac{n}{2}+1}(t) \\ \vdots \\ p_{n-1}(t) \\ p_n(t) \end{pmatrix} = \begin{pmatrix} m_{1,1}a_1t^{s_1} + \cdots + m_{1,\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}m_{1,\frac{n}{2}}a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{1,3}a_{n-2}t^{s_{n-2}} \\ m_{2,1}a_1t^{s_1} + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}m_{2,\frac{n}{2}}a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{2,3}a_{n-2}t^{s_{n-2}} \\ \vdots \\ m_{\frac{n}{2},1}a_1t^{s_1} + \cdots + m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + (-1)^{j+1}(m_{\frac{n}{2},\frac{n}{2}}-1)a_{\frac{n}{2}+1}t^{s_{\frac{n}{2}+1}} + \cdots + (-1)^{j+1}m_{\frac{n}{2},3}a_{n-2}t^{s_{n-2}} \\ m_{\frac{n}{2}+1,1}a_1t^{s_1} + \cdots + (-1)^j a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} + m_{\frac{n}{2}+1,\frac{n}{2}-1}a_{\frac{n}{2}+2}t^{s_{\frac{n}{2}+2}} + \cdots + (-1)^{j+1}m_{\frac{n}{2}+1,3}a_{n-2}t^{s_{n-2}} \\ \vdots \\ (-1)^j a_2t^{s_2} \\ (-1)^j a_1t^{s_1} \end{pmatrix}.$$

Consider the second-last equation in (18)

$$m_{n-2,1}t^{s_1}p_1(t) + m_{n-2,2}t^{s_2}p_2(t) + m_{n-2,2}(-t)^{s_{n-1}}p_{n-1}(t) + m_{n-2,1}(-t)^{s_n}p_n(t) = 0.$$

The term with the highest possible degree $s_1+s_n = s_2+s_{n-1} = k$ is

$$m_{n-2,2}(-t)^{s_{n-1}}(-1)^j a_2t^{s_2} + m_{n-2,1}(-t)^{s_n}(-1)^j a_1t^{s_1} = -(m_{n-2,2}a_2 + m_{n-2,1}a_1)t^k\,,$$

hence $m_{n-2,1}a_1 + m_{n-2,2}a_2 = 0$. Looking at the system (19) we obtain $a_{n-2} = 0$: hence the degree of $p_i(t)$ is boundend by s_{n-3} for all i.

The proof goes on in the same way: it may be less evident than the one of Theorem 3.1 (where the a_i vanished in couples), but looking always at the terms of degree k of the $(n-i)$-th equation of (18) and substituting in (19) we are able to prove that $a_{n-i} = 0$ and, as an immediate consequence from (13), that all the $p_i(t)$ have degree at most s_{n-i-1}. For example midway through the proof we get

$$\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{\frac{n}{2}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{p(t)} = \begin{pmatrix} m_{1,1}a_1t^{s_1} + \cdots + m_{1,\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} \\ \vdots \\ m_{\frac{n}{2},1}a_1t^{s_1} + \cdots + m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} \\ m_{\frac{n}{2}+1,1}a_1t^{s_1} + \cdots + (-1)^j a_{\frac{n}{2}}t^{s_{\frac{n}{2}}} \\ m_{\frac{n}{2}+2,1}a_1t^{s_1} + \cdots + (-1)^j a_{\frac{n}{2}-1}t^{s_{\frac{n}{2}-1}} \\ \vdots \\ (-1)^j a_1t^{s_1} \end{pmatrix}.$$

Therefore, what remains of (12) is

s3 (21)
$$\begin{cases} (m_{1,1}-1)a_1 + m_{1,2}a_2 + \cdots + m_{1,\frac{n}{2}}a_{\frac{n}{2}} = 0 \\ m_{2,1}a_1 + (m_{2,2}-1)a_2 + \cdots + m_{2,\frac{n}{2}}a_{\frac{n}{2}} = 0 \\ \vdots \\ m_{\frac{n}{2},1}a_1 + m_{\frac{n}{2},2}a_2 + \cdots + (m_{\frac{n}{2},\frac{n}{2}}-1)a_{\frac{n}{2}} = 0 \\ a_{\frac{n}{2}+1} = \cdots = a_n = 0 \end{cases}.$$

Finally, we observe that the $\frac{n}{2}$-th equation of (18) is

$$m_{\frac{n}{2},1}t^{s_1}p_1(t) + \cdots + m_{\frac{n}{2},\frac{n}{2}}t^{s_{\frac{n}{2}}}p_{\frac{n}{2}}(t) + m_{\frac{n}{2},\frac{n}{2}}(-t)^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t) + \cdots + m_{\frac{n}{2},1}(-t)^{s_n}p_n(t) = 0\,.$$

As before, the term of degree k must have coefficient 0 and it appears only in the final terms starting from $m_{\frac{n}{2},\frac{n}{2}}(-t)^{s_{\frac{n}{2}+1}}p_{\frac{n}{2}+1}(t)$. So we get

$$m_{\frac{n}{2},\frac{n}{2}}a_{\frac{n}{2}} + m_{\frac{n}{2},\frac{n}{2}-1}a_{\frac{n}{2}-1} + \cdots + m_{\frac{n}{2},1}a_1 = 0$$

and, by (21), $a_{\frac{n}{2}} = 0$ as well.
Iterating we get $\underline{a} = 0$ and so our claim. □

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