Pacific Journal of Mathematics

REMARKS ON EIGENSPECTRA OF ISOLATED SINGULARITIES

BEN CASTOR, HAOHUA DENG, MATT KERR AND GREGORY PEARLSTEIN

Volume 327 No. 1 November 2023

PACIFIC JOURNAL OF MATHEMATICS
Vol. 327, No. 1, 2023
https://doi.org/10.2140/pjm.2023.327.29

REMARKS ON EIGENSPECTRA OF ISOLATED SINGULARITIES

BEN CASTOR, HAOHUA DENG, MATT KERR AND GREGORY PEARLSTEIN

We introduce a simple calculus, extending a variant of the Steenbrink spectrum, to describe Hodge-theoretic invariants for smoothings of isolated singularities with relative automorphisms. After computing these "eigenspectra" in the quasihomogeneous case, we give three applications to singularity bounding and monodromy of variations of Hodge structure (VHS).

Introduction

Recent work of M. Kerr and R. Laza on the Hodge theory of degenerations [Kerr et al. 2021; Kerr and Laza 2023] reexamined the mixed Hodge theory of the Clemens–Schmid and vanishing-cycle sequences, with an emphasis on applications to limits of period maps and compactifications of moduli. When a degenerating family of varieties has a finite group G acting on its fibers, these become exact sequences in the category of mixed Hodge structures with $G \times \mu_k$-action, where k is the order of T_{ss} (the semisimple part of monodromy). These kinds of situations often show up in generalized Prym or cyclic-cover constructions; for instance, instead of using the period map attached to a family of varieties, one may want to use the "exotic" period map arising from a cyclic cover branched along the family (e.g., [Allcock et al. 2002; 2011; Casalaina-Martin et al. 2012; Deligne and Mostow 1986; Dolgachev and Kondō 2007]).

In this note we explain how to encode the contributions of isolated singularities with G-action to the vanishing cohomology in terms of *G-spectra* (Definition 1.11). These are formal sums (with positive integer coefficients) of triples in $\mathbb{Z} \times \mathbb{Q} \times \mathfrak{R}$, where $\mathfrak{R}$ is the set of irreducible representations of G. The term *eigenspectrum* (Definition 1.12) refers to the specific case of a cyclic group $\langle g \rangle$ with fixed generator. (At the end of Section 3 and in most of Section 5 a larger group $\mathcal{G}$ nontrivially permutes the singularities; G always denotes a subgroup stabilizing them.)

In Section 1 this formalism emerges naturally from the general setting of a proper morphism of 1-parameter degenerations over a disk, by specializing the

MSC2020: 14D06, 14D07, 14J17, 32S25, 32S35.
Keywords: isolated singularity, nodes, spectrum, eigenspectrum, quasihomogeneous singularity, Calabi–Yau variety, variation of Hodge structure, monodromy.

morphism to an automorphism $g \in \mathrm{Aut}(\mathcal{X}/\Delta)$ fixing a singularity $x \in X_0$. The eigenspectrum $\sigma^g_{f,x}$ simply records the dimensions of simultaneous eigenspaces of g^* and T_{ss} in the (p, q)-subspaces of V_x (Definition 1.12). We give a general computation in Section 2 of $\sigma^g_{f,x}$ in the case of a quasihomogeneous singularity, in terms of a monomial basis for the associated Jacobian ring (Corollary 2.7).

In the remaining sections, we give three applications. The first, in Section 3, is to bounding the number of nodes on Calabi–Yau hypersurfaces in weighted projective spaces (Theorem 3.6) by passing to cyclic covers. There is already a large literature on node-bounding, including [Jaffe and Ruberman 1997; Kerr and Laza 2023; Miyaoka 1984; Schoen 1985; Varchenko 1983; van Straten 2020]. In the case of $\mathbb{P}^{n+1}$, our approach does not improve Varchenko's bound (e.g., 135 nodes for a quintic hypersurface in $\mathbb{P}^4$), but does yield a simpler proof. However, we do obtain the interesting result (in Theorem 3.11) that a CY hypersurface in $\mathbb{P}^{n+1}$ with isolated singularities and symmetric under $\mathfrak{S}_{n+2}$ cannot contain a node whose $\mathfrak{S}_{n+2}$-orbit has cardinality $(n+2)!$ (i.e., trivial stabilizer).

The other two applications concern codimension-one monodromy phenomena for VHSs over moduli of configurations of points and hyperplanes. In Section 4, the moduli space is $M_{0,2n}$, with the VHS arising from cyclic covers of $\mathbb{P}^1$ branched along the $2m$ ordered points. Propositions 4.5–4.6 and Example 4.7 describe the eigenspectra, LMHS and monodromy types along boundary strata of certain compactifications $\overline{M}^H_{0,2n}$ due to Hassett [2003], generalizing a computation of [Gallardo et al. 2021]. The cases $m = 2, 3, 4$, and 6 go back to work of Deligne and Mostow [1986] and feature a period map (isomorphism) to an arithmetic ball quotient. While the global/extended period map is not as elegant in the remaining cases, the point is that the codimension-one boundary behavior can be dealt with uniformly and efficiently using our calculus.

Our other main example, treated in Section 5, is the VHS $\mathcal{H} \to \mathcal{S}$ on the moduli space of general configurations of $(2n+2)$ hyperplanes in $\mathbb{P}^n$, arising from the middle (intersection) cohomology of a $2:1$ cover $X \to \mathbb{P}^n$ branched along these hyperplanes. These are singular Calabi–Yau n-folds admitting a crepant resolution, and have been studied in [Dolgachev and Kondō 2007; Gerkmann et al. 2007a; 2013; Sheng et al. 2015]. By passing to a smooth complete intersection 2^{2n}-cover of X and applying the Cayley trick (see [Kerr 2003, Section 4.5]), we replace X by a smooth hypersurface

$$Y \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^{2n+1}}(2)^{\oplus(n+1)})$$

with automorphisms by a group of order 2^{2n}. In codimension-one in moduli, Y acquires nodes, and a variant of Schoen's [1985] result ensures that these produce nontrivial symplectic transvections for $\mathcal{H}$ when n is odd. This gives an easy proof that the geometric monodromy group of $\mathcal{H}$ is maximal (for all n), and the period map “nonclassical”, a fact first proved by Gerkmann et al. [2013] for $n = 3$ and by Sheng et al. [2015] in general.

Notation. In this paper MHS stands for $\mathbb{Q}$-mixed Hodge structure. We shall make frequent use of the *Deligne bigrading* on a MHS V [Deligne 1971]. This is (by definition) the unique decomposition $V_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}$ with the properties that

$$F^k V_{\mathbb{C}} = \bigoplus_{\substack{p,q\\ p\geq k}} V^{p,q}, \quad W_\ell V_{\mathbb{C}} = \bigoplus_{\substack{p,q\\ p+q\leq \ell}} V^{p,q}, \quad \text{and} \quad \overline{V^{q,p}} \equiv V^{p,q} \mod \bigoplus_{\substack{a<p\\ b<q}} V^{a,b}.$$

We shall make free use of standard multiindex notation (for n-tuples of variables or field-elements) to simplify formulas, viz. $\underline{z} = (z_1, \dots, z_n)$, $\mathbb{C}[\underline{z}] = \mathbb{C}[z_1, \dots, z_n]$, $\underline{z}^{\underline{m}} = \prod_i z_i^{m_i}$, $\underline{m}\cdot\underline{w} = \sum_i m_i\, w_i$, $|\underline{m}| = \sum_i m_i$, $\underline{e}^{(i)} = i$-th standard basis vector, etc.

1. G-spectra and eigenspectra

Morphisms and mixed spectra. We begin in the general setting of a proper morphism

(1.1)
$$\begin{tikzcd} \mathcal{Y} \arrow[rr, "\pi"] \arrow[dr, "f'"'] & & \mathcal{X} \arrow[dl, "f"] \\ & \Delta & \end{tikzcd}$$

of complex analytic spaces over a disk, which we assume is the restriction to Δ of a proper morphism of quasiprojective varieties over an algebraic curve. (In particular, at the level of fibers we have that $\pi_t : Y_t \to X_t$ is a proper algebraic morphism of quasiprojective varieties.) Let $\mathcal{K}^\bullet \in D^b\,\mathrm{MHM}(\mathcal{X})$ and $\mathcal{L}^\bullet \in D^b\,\mathrm{MHM}(\mathcal{Y})$ be given, with a morphism $\rho : \mathcal{K}^\bullet \to R\pi_*\mathcal{L}^\bullet$. Writing $\imath : X_0 \hookrightarrow \mathcal{X}$ for the inclusion, the vanishing cycle triangle

(1.2)
$$\imath^* \xrightarrow{\mathrm{sp}} \psi_f \xrightarrow{\mathrm{can}} \phi_f \xrightarrow[{[+1]}]{\delta}$$

consists of functors from $D^b\,\mathrm{MHM}(\mathcal{X})$ to $D^b\,\mathrm{MHM}(X_0)$, with natural transformations between them; also, monodromy $T = T_{\mathrm{ss}}\, e^N$ induces natural automorphisms of ψ_f and ϕ_f. By proper base-change and faithfulness of $\mathrm{rat} : D^b\,\mathrm{MHM}(X_0) \to D^b_c(X_0)$, $R\pi_* : D^b\,\mathrm{MHM}(Y_0) \to D^b\,\mathrm{MHM}(X_0)$ intertwines the corresponding triangle (and monodromy actions) for $(\mathcal{Y}, f')$. Taking hypercohomology on X_0 yields:

1.3. Proposition. *We have the commutative diagram*

$$\begin{tikzcd} \to \mathbb{H}^k(X_0, \imath^*\mathcal{K}^\bullet) \arrow[r, "\mathrm{sp}"] \arrow[d, "\rho"] & \mathbb{H}^k(X_0, \psi_f\mathcal{K}^\bullet) \arrow[r, "\mathrm{can}"] \arrow[d, "\rho"] & \mathbb{H}^k(X_0, \phi_f\mathcal{K}^\bullet) \arrow[r, "\delta"] \arrow[d, "\rho"] & \mathbb{H}^{k+1}(X_0, \imath^*\mathcal{K}^\bullet) \to \arrow[d, "\rho"] \\ \to \mathbb{H}^k(Y_0, \imath^*\mathcal{L}^\bullet) \arrow[r, "\mathrm{sp}"] & \mathbb{H}^k(Y_0, \psi_{f'}\mathcal{L}^\bullet) \arrow[r, "\mathrm{can}"] & \mathbb{H}^k(Y_0, \phi_{f'}\mathcal{L}^\bullet) \arrow[r, "\delta"] & \mathbb{H}^{k+1}(Y_0, \imath^*\mathcal{L}^\bullet) \to \end{tikzcd}$$

with rows the vanishing-cycle (*long-exact*) *sequences*, *in which all arrows are morphisms of MHS. Moreover*, *the diagram intertwines the actions of* T_{ss} (*by automorphisms of MHS*) *and* N (*by nilpotent* $(-1,-1)$*-endomorphisms of MHS*), *which are trivial* (Id *resp.* 0) *on the end terms.*

1.4. Remark. If f, f' are themselves projective (hence proper), and $\mathcal{K}^\bullet$, $\mathcal{L}^\bullet$ semisimple with respect to the perverse t-structure (e.g., $\mathcal{K}^\bullet = \mathcal{IC}^\bullet_{\mathcal{X}}$, $\mathcal{L}^\bullet = \mathcal{IC}^\bullet_{\mathcal{Y}}$), then the decomposition theorem applies, yielding Clemens–Schmid sequences (see [Kerr et al. 2021, Section 5]) which are then automatically compatible under ρ. The main consequence is that the local invariant cycle theorem holds, i.e., sp surjects onto the T-invariants.

Next, assume $\mathcal{X}$, $\mathcal{Y}$, $\{X_t\}_{t\neq 0}$, and $\{Y_t\}_{t\neq 0}$ are smooth, and take $\mathcal{L}^\bullet = \mathbb{Q}_{\mathcal{Y}}$ and $\mathcal{K}^\bullet = \mathbb{Q}_{\mathcal{X}}$; then the diagram in Proposition 1.3 becomes

$$\begin{tikzcd}[ampersand replacement=\&] \to H^k(X_0) \arrow[r, "\mathrm{sp}"] \arrow[d, "\pi^*"] \& H^k_{\mathrm{lim}}(X_t) \arrow[r, "\mathrm{can}"] \arrow[d, "\pi^*"] \& H^k_{\mathrm{van}}(X_t) \arrow[r, "\delta"] \arrow[d, "\pi^*"] \& H^{k+1}(X_0) \to \arrow[d, "\pi^*"] \\ \to H^k(Y_0) \arrow[r, "\mathrm{sp}"] \& H^k_{\mathrm{lim}}(Y_t) \arrow[r, "\mathrm{can}"] \& H^k_{\mathrm{van}}(Y_t) \arrow[r, "\delta"] \& H^{k+1}(Y_0) \to \end{tikzcd} \tag{1.5}$$

Now if $n = \dim X_0$ and $\Sigma := \mathrm{sing}(X_0)$ is finite, then $H^k_{\mathrm{van}}(X_t) = \{0\}$ for $k \neq n$ and, defining $V_x := H^0 \imath_x^* \phi_f \mathbb{Q}_{\mathcal{X}}[n]$,

$$H^n_{\mathrm{van}}(X_t) \cong \bigoplus_{x\in\Sigma} V_x \tag{1.6}$$

as MHS. We have of course $\pi^{-1}(\Sigma) \subset \widetilde{\Sigma} := \mathrm{sing}(Y_0)$, and if $\dim Y_0 = n$ and $|\widetilde{\Sigma}| < \infty$ then, writing $\widetilde{V}_y := H^0 \imath_y^* \phi_{f'} \mathbb{Q}_{\mathcal{Y}}[n]$ for $y \in \widetilde{\Sigma}$, π^* restricts to morphisms

$$[\pi^*]_x : V_x \to \bigoplus_{y\in\pi^{-1}(x)} \widetilde{V}_y \tag{1.7}$$

of T-MHS — i.e., morphisms of MHS intertwining T (hence T_{ss} and N). These are local invariants.

Recall that T_{ss} acts through finite cyclic groups on each V_x (and $\widetilde{V}_y$), and let κ be the least common multiple of their orders. Write $\zeta_\kappa := e^{2\pi \boldsymbol{i}/\kappa}$ and $V^{p,q}_{x,\boldsymbol{e}(a/\kappa)}$ for the $\boldsymbol{e}(a/\kappa) := e^{2\pi\boldsymbol{i}(a/\kappa)} = \zeta_\kappa^a$-eigenspace of T_{ss} in $V_x^{p,q} \subset V_{x,\mathbb{C}}$. The explicit choice of $\zeta_\kappa \in \mathbb{C}$ is needed to make the following.

1.8. Definition. The *mixed spectrum* $\sigma_{f,x}$ of the isolated singularity $x \in \Sigma$ is the element $\sum_{\alpha,w} m^{f,x}_{\alpha,w}(\alpha, w)$ of the free abelian group $\mathbb{Z}\langle\mathbb{Q}\times\mathbb{Z}\rangle$, where we put $m^{f,x}_{\alpha,w} = \dim(V^{\lfloor\alpha\rfloor, w-\lfloor\alpha\rfloor}_{x,\boldsymbol{e}(\alpha)})$.[1]

[1] Here $\lfloor\cdot\rfloor$ is the greatest integer (floor) function; note also that $\boldsymbol{e}(\alpha)$ is equivalent to taking the fractional part $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$ of α.

Evidently (1.7) must be compatible with the decompositions recorded by the mixed spectra.

Automorphisms and eigenspectra. Now let $\mathcal{G} \leq \mathrm{Aut}(\mathcal{X}/\Delta)$, with $\mathcal{X}$ and $\{X_t\}_{t\neq 0}$ smooth and $|\Sigma| < \infty$. Applying the foregoing results with $\mathcal{Y} = \mathcal{X}$, $f = f'$, and $\pi := g \in \mathcal{G}$, together with [Kerr et al. 2021, Proposition 5.5(i)], yields:

1.9. Corollary. *The vanishing-cycle sequence*

$$0 \to H^n(X_0) \xrightarrow{\mathrm{sp}} H^n_{\lim}(X_t) \xrightarrow{\mathrm{can}} \bigoplus_{x\in\Sigma} V_x \xrightarrow{\delta} H^{n+1}_{\mathrm{ph}}(X_0) \to 0 \tag{1.10}$$

is an exact sequence of $\mathcal{G} \times \mu_\kappa$-MHS,[2] *where the $\langle T_{ss}\rangle \cong \mu_\kappa$-action on the end terms is trivial. If $\mathcal{X}/\Delta$ is proper, then $H^{n+1}_{\mathrm{ph}}(X_0) := \ker(\mathrm{sp}) \subseteq H^{n+1}(X_0)$ is pure of weight $n+1$.*

The decomposition of terms in (1.10) into irreducible representations for $\mathcal{G} \times \mu_\kappa$ only becomes useful if we understand the action on the vanishing cohomology $\bigoplus_{x\in\Sigma} V_x$ for a given collection of singularities. In particular, if $gx = x$ then we need to further refine the spectrum under the resulting automorphism $g^* : V_x \to V_x$ of T-MHS.

1.11. Definition. Write $G \leq \mathrm{stab}(x) \leq \mathcal{G}$, and $\mathcal{R}_G$ for the set of complex irreducible representations of G. The *G-spectrum* $\sigma^G_{f,x}$ of x is the element

$$\sum_{(\alpha,w,U)} m^{f,x,G}_{\alpha,w,U} (\alpha, w, U)$$

of the free abelian group $\mathbb{Z}\langle \mathbb{Q} \times \mathbb{Z} \times \mathcal{R}_G\rangle$, where (for each (α, w))

$$V^{\lfloor\alpha\rfloor, w-\lfloor\alpha,\rfloor}_{x,\boldsymbol{e}(\alpha)} \cong \bigoplus_{U\in\mathcal{R}_G} U^{\oplus m^{f,x,G}_{\alpha,w,U}}$$

as G-representations.

In the special case where $G = \langle g\rangle \cong \mu_\ell$ is cyclic, the $\mathbb{C}$-irreps are characters indexed by the power $\zeta_\ell^c = e^{2\pi \boldsymbol{i}(c/\ell)}$ of ζ_ℓ to which g is sent.

1.12. Definition. The *eigenspectrum* of an isolated singularity x with automorphism g is the element

$$\sigma^g_{f,x} = \sum_{(\alpha,w,\gamma)} m^{f,x,g}_{\alpha,w,\gamma} (\alpha, w, \gamma) \in \mathbb{Z}\langle \mathbb{Q} \times \mathbb{Z} \times \mathbb{Q}/\mathbb{Z}\rangle,$$

where $m^{f,x,g}_{\alpha,w,\gamma}$ is the dimension of the eigenspace $(V^{\lfloor\alpha\rfloor, w-\lfloor\alpha,\rfloor}_{x,\boldsymbol{e}(\alpha)})^{\boldsymbol{e}(\gamma)} \subseteq V^{\lfloor\alpha\rfloor, w-\lfloor\alpha\rfloor}_{x,\boldsymbol{e}(\alpha)}$ for g^* with eigenvalue $\boldsymbol{e}(\gamma) = e^{2\pi\boldsymbol{i}\gamma}$.

[2] Again, this means that the action of $\mathcal{G}$ and T_{ss} on the MHSs (as automorphisms of MHS) commute with each other and with sp, can, and δ.

1.13. Remark. For $\mathcal{X}/\Delta$ proper (with hypotheses as in Corollary 1.9), $H^n(X_t)$ is a VHS on Δ^* whose automorphism group contains $\mathcal{G}$. For any field extension $K/\mathbb{Q}$, this decomposes as K-VHS into a direct sum of $\mathcal{G}$-isotypical components, corresponding to K-irreps of $\mathcal{G}$. The $\mathcal{G}$-action on and decomposition of $H^n_{\lim}(X_t)$ obtained by taking limits are the same as those arising from the $\mathcal{G}$-MHS structure on $H^n_{\lim}(X_t)$ in Corollary 1.9.

We now turn to the explicit computation of these eigenspectra in the simplest case.

2. Quasihomogeneous singularities with automorphism

Let $F \in \mathbb{C}[z_1, \dots, z_{n+1}]$ (with $n > 0$) be a *quasihomogeneous polynomial* with an isolated singularity at the origin $\underline{0}$. That is to say, choosing a weight vector $\underline{w} = (w_1, \dots, w_{n+1}) \in \mathbb{Q}^{n+1}_{>0}$ and setting

$$\mathfrak{M}_{\underline{w}} := \{\underline{m} \in \mathbb{Z}^{n+1}_{\geq 0} \mid \underline{m} \cdot \underline{w} = 1\},$$

we have

$$F = \sum_{\underline{m} \in \mathfrak{M}_{\underline{w}}} a_{\underline{m}}\, \underline{z}^{\underline{m}} \tag{2.1}$$

for some $a_{\underline{m}} \in \mathbb{C}$. We recall that the *degree* κ_F of F is the least integer such that $\kappa_F\, w_i \in \mathbb{N}$ for $i = 1, \dots, n+1$; define $w_i := \kappa_F\, w_i$ and set $\underline{\kappa} := (\kappa_1, \dots, \kappa_{n+1})$.

Next recall the setting of Definition 1.8, where $f : \mathcal{X} \to \Delta$ is a holomorphic map with quasiprojective fibers and smooth total space, with X_t smooth for $t \neq 0$ and $\mathrm{sing}(X_0) =: \Sigma$ finite. A singularity $x \in \Sigma \subset X_0$ is *quasihomogeneous* if f can be locally analytically identified with (2.1) for some $\underline{w}$. In that case, V_x and $\sigma_{f,x}$ identify with the vanishing cohomology

$$V_F := H^0\, \imath_{\underline{0}}^*\, \phi_F\, \mathbb{Q}_{\mathbb{C}^{n+1}} \tag{2.2}$$

of $F : \mathbb{C}^{n+1} \to \mathbb{C}$ at $\underline{0}$, and its mixed spectrum σ_F. These were first computed by Steenbrink [1977], and we briefly review the treatment from [Kerr and Laza 2023, Section 2] before passing to eigenspectra.

Writing

$$J_F := \left(\frac{\partial F}{\partial z_1}, \dots, \frac{\partial F}{\partial z_{n+1}}\right) \subseteq \mathbb{C}[\underline{z}]$$

for the Jacobian ideal, let $\mathcal{B} \subset \mathbb{Z}^{n+1}_{\geq 0}$ be chosen so that the monomials $\{\underline{z}^{\underline{\beta}}\}_{\underline{\beta} \in \mathcal{B}}$ provide a basis of $\mathbb{C}[\underline{z}]/J_F$. Write $\mu_F := |\mathcal{B}|$ for the *Milnor number* of F, and $\ell(\underline{\beta}) := \frac{1}{\kappa_F} \sum_{i=1}^{n+1} \kappa_i (\beta_i + 1) = \sum_{i=1}^{n+1} w_i (\beta_i + 1)$.

2.3. Proposition. *We have* $\mu_F = \dim V_F$ *for the Milnor number and*

$$\sigma_F = \sum_{\underline{\beta} \in \mathcal{B}} (\alpha(\underline{\beta}), w(\underline{\beta})) \in \mathbb{Z}\langle \mathbb{Q} \times \mathbb{Z} \rangle$$

for the mixed spectrum, where $\alpha(\underline{\beta}) := n+1-\ell(\underline{\beta})$ *and* $w(\underline{\beta}) := n$ (*resp.* $n+1$) *if* $\alpha(\underline{\beta}) \notin \mathbb{Z}$ (*resp.* $\in \mathbb{Z}$).

Sketch. Perform a base-change followed by weighted blow-up at $\underline{0}$:

$$\begin{tikzcd} \mathbb{C}^{n+1} \arrow[d, "F"'] & \mathfrak{X} \arrow[l] \arrow[d, "\hat{F}"'] & \mathfrak{Y} \arrow[l, "\mathrm{Bl}_{\underline{\kappa}}"'] \arrow[dl, "\tilde{F}"] \\ \mathbb{C} & \Delta \arrow[l] & \end{tikzcd} \tag{2.4}$$

$$t^{\kappa_F} \longleftarrow\!\shortmid\; t$$

with exceptional divisor $\mathcal{E} = \{T^{\kappa_F} = F(\underline{Z})\} \subset \mathbb{WP}[1 : \underline{\kappa}] =: \boldsymbol{P}$ (in weighted homogeneous coordinates $T, Z_1, \dots, Z_{n+1}$). The singular fiber $\mathbb{Y}_0 := \tilde{F}^{-1}(0)$ is the union of $\mathcal{E}$ and the proper transform $\widetilde{\mathbb{X}}_0$ of $\mathbb{X}_0 := F^{-1}(0) = \hat{F}^{-1}(0)$, meeting in

$$E := \mathcal{E} \cap \widetilde{\mathbb{X}}_0 = \{F(\underline{Z}) = 0\} \subset \boldsymbol{H} := \{T = 0\}\ (\cong \mathbb{WP}[\underline{\kappa}]) \subset \boldsymbol{P}.$$

The claim is then that $V_F \cong H^n(\mathcal{E} \setminus E)$, which can be checked using (1.5) with $\pi = \mathrm{Bl}_{\underline{\kappa}}$. Since E [resp. $\underline{0}$] is a deformation retract of $\mathbb{Y}_0$ [resp. $\mathbb{X}_0$], while $\mathbb{Y}_t = \mathbb{X}_t$ for $t \neq 0$, and $\phi_{\tilde{F}}\, \mathbb{Q}_{\mathcal{Y}} \simeq \iota_*^E \mathbb{Q}_E(-1)[-1]$ (see [Kerr et al. 2021, 6.3 and 8.3–8.4]), the diagram becomes

$$\begin{tikzcd} & 0 \arrow[r] & H^n_{\lim}(\mathbb{X}_t) \arrow[r, "\cong"] \arrow[d, equal] & V_F \arrow[r] \arrow[d, "\mathrm{Bl}^*"] & 0 \\ H^{n-2}(E)(-1) \arrow[r] & H^n(\mathcal{E}) \arrow[r] & H^n_{\lim}(\mathbb{Y}_t) \arrow[r] & H^{n-1}(E)(-1) \arrow[r] & H^{n+1}(\mathcal{E}) \end{tikzcd}$$

whence the result.

Next, one constructs a basis of $H^n(\mathcal{E} \setminus E)$ from $\mathcal{B}$, using residue theory. Writing (with $T := Z_0$)

$$\Omega_{\boldsymbol{P}} = \sum_{j=0}^{n+1} (-1)^j Z_j\, dZ_0 \wedge \cdots \wedge \widehat{dZ_j} \wedge \cdots \wedge dZ_{n+1},$$

for each $\underline{\beta} \in \mathcal{B}$ we set (with $\underline{Z}^{\underline{\beta}} = Z_1^{\beta_1} \cdots Z_{n+1}^{\beta_{n+1}}$)

$$\Omega_{\underline{\beta}} := \frac{T^{\kappa_F}\, \underline{Z}^{\underline{\beta}}\, \Omega_{\boldsymbol{P}}}{T(F(\underline{Z}) - T^{\kappa_F})^{\lceil \ell(\underline{\beta}) \rceil}} \in \Omega^{n+1}(\boldsymbol{P} \setminus \mathcal{E} \cap \boldsymbol{H}) \tag{2.5}$$

and $\omega_{\underline{\beta}} := \mathrm{Res}_{\mathcal{E} \setminus E}([\Omega_{\underline{\beta}}]) \in H^n(\mathcal{E} \setminus E)$. See [Kerr and Laza 2023, Theorem 2.2] for the proof that this has (p,q)-type $(\lfloor \alpha(\underline{\beta}) \rfloor, \lfloor \ell(\underline{\beta}) \rfloor)$, and [Steenbrink 1977, Theorem 1] for the assertion that the $\{\omega_{\underline{\beta}}\}$ give a basis. Note that $\lfloor \alpha(\underline{\beta}) \rfloor + \lfloor \ell(\underline{\beta}) \rfloor = w(\underline{\beta})$.

Finally, the action of T_{ss} is computed by $T \mapsto \zeta_{\kappa_F} T$, or equivalently (in weighted projective coordinates) by $Z_i \mapsto \zeta_{\kappa_F}^{-\kappa_i} Z_i = e^{-2\pi \boldsymbol{i} w_i} Z_i$. Clearly the effect of this on (2.5) is to multiply it by $e^{2\pi \boldsymbol{i} \sum w_i(\beta_i+1)} = e^{2\pi \boldsymbol{i} \alpha(\underline{\beta})}$, as desired. $\square$

Now given a finite group $G \leq \mathrm{Aut}(\mathcal{X}/\Delta)$ fixing $x \in \Sigma$, we can always choose local holomorphic coordinates on which the action is linear [Cartan 1954]. So for a given $g \in G$, we can choose coordinates to make the action diagonal, through roots of unity. Accordingly, we shall compute the eigenspectrum in the case where $g \in \mathrm{Aut}(\mathbb{C}^{n+1}, \underline{0})$ is given by

$$g(z_1, \dots, z_{n+1}) := (\zeta_\ell^{c_1} z_1, \dots, \zeta_\ell^{c_{n+1}} z_{n+1}) \tag{2.6}$$

and $F \in \mathbb{C}[\underline{z}]^{\langle g \rangle}$ is a g-invariant quasihomogeneous polynomial. In fact, taking $\mathcal{B} \subset \mathbb{Z}_{\geq 0}^{n+1}$ as above, we have:

2.7. Corollary. *The eigenspectrum σ_F^g is given by*

$$\sum_{\underline{\beta} \in \mathcal{B}} (\alpha(\underline{\beta}), w(\underline{\beta}), \gamma(\underline{\beta})) \in \mathbb{Z}\langle \mathbb{Q} \times \mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \rangle,$$

where $\gamma(\underline{\beta}) := \frac{1}{\ell} \sum_{i=1}^{n+1} c_i (\beta_i + 1)$.

Proof. We only need to compute the action of g^* on $\omega_{\underline{\beta}}$, which is to say the effect of $Z_i \mapsto \zeta_\ell^{c_i} Z_i$ on $\underline{Z}^{\underline{\beta}}\, \Omega_{\underline{\beta}}$. This is just multiplication by $\zeta_\ell^{\sum c_i(\beta_i+1)} = e^{2\pi \mathbf{i} \gamma(\underline{\beta})}$. □

2.8. Example. For a Brieskorn–Pham singularity $F = \sum_{i=1}^{n+1} z_i^{\lambda_i}$, $\lambda_i = 1/w_i = \kappa_F/\kappa_i$, we have $\mathcal{B} = \times_{i=1}^{n+1} \{\mathbb{Z} \cap [0, d_i - 2]\}$. Hence, writing $\Gamma_m = \sum_{j=1}^{m-1} [j/m]$ in the group ring $\mathbb{Z}[\mathbb{Q}]$ (with product $*$), we have

$$\sum_{\underline{\beta} \in \mathcal{B}} [\alpha(\underline{\beta})] = \Gamma_{\lambda_1} * \cdots * \Gamma_{\lambda_{n+1}}.$$

This extends to

$$\sum_{\underline{\beta} \in \mathcal{B}} [(\alpha(\underline{\beta}), \gamma(\underline{\beta}))] = \widetilde{\Gamma}_{\lambda_1}\Big(\frac{c_1}{\ell}\Big) * \cdots * \widetilde{\Gamma}_{\lambda_{n+1}}\Big(\frac{c_{n+1}}{\ell}\Big)$$

in the group ring $\mathbb{Z}[\mathbb{Q} \times (\mathbb{Q}/\mathbb{Z})]$ if we write $\widetilde{\Gamma}_m\big(\frac{c}{\ell}\big) = \sum_{j=1}^{m-1} \big[\big(\frac{m-j}{m}, \frac{jc}{\ell}\big)\big]$.

2.9. Example. As a *specific* example, consider $F = z_1^2 + z_2^2 + z_3^{m+1} + z_4^3$, with $g(z_1, z_2, z_3, z_4) := (z_1, z_2, z_3, \zeta_3 z_4)$. Applying Example 2.8 to compute the eigenspectrum gives

$$\sum_{j=1}^{m} \Big[\Big(\tfrac{5}{3} + \frac{j}{m+1}, \tfrac{1}{3}\Big)\Big] + \sum_{j=1}^{m} \Big[\Big(\tfrac{4}{3} + \frac{j}{m+1}, \tfrac{2}{3}\Big)\Big].$$

We can interpret this scenario as a local snapshot of a $3:1$ cover of $\mathbb{P}^3$ branched over a cubic surface acquiring an A_m singularity. So the ζ_3-eigenspace of the $(1, 2)$-part of vanishing cohomology has rank equal to the number of j's for which $\frac{5}{3} + j/(m+1) < 2$. Since the ζ_3-eigenspace of the general fiber (= cubic 3-fold) has Hodge numbers $h^{1,2} = 1$ and $h^{2,1} = 4$, from $\frac{5}{3} + \frac{2}{7} < 2$ we see that m *cannot* be ≥ 6. This bound is sharp, since A_5 can occur on a cubic surface in the form $z_1^3 + z_2^3 - z_2 z_3^2$ (see, for example, [Sakamaki 2010]).

Applying the vanishing-cycle analysis directly on a cubic surface, without passing to a triple cover and using eigenspectra, does *not* rule out A_6. It was this sort of phenomenon that motivated this paper.

2.10. Remark. The eigenspectrum of a μ-constant (semiquasihomogeneous) deformation of (F, γ) remains constant. Even in the more general case of [Kerr and Laza 2023, Section 5.2], one can in principle still use the action of γ^* on the (local) Jacobian ring $\mathcal{O}_{n+1}/J_F$ to refine σ_F to σ_F^g. But Corollary 2.7 (and quasihomogeneous deformations of Example 2.8) will suffice for our purposes below.

3. Bounding nodes on Calabi–Yau hypersurfaces

It is a classical problem to bound the number of nodes (ordinary double points) on a projective hypersurface, especially for Calabi–Yau (CY) varieties. In this section, we use eigenspectra to produce such a bound for hypersurfaces in many weighted projective spaces (Example 3.8). Though our emphasis is on CY varieties for illustrative purposes, it is not limited to them. In the special case of projective space, our formula recovers the bound conjectured by Arnol'd [1981] and proved by Varchenko [1983] (see also [van Straten 2020]) by applying his semicontinuity theorem to the Bruce deformation. This includes the famous bound of 135 for a quintic threefold; see Examples 3.10.

Let $\mathbb{W} = \mathbb{WP}[e_0 : \cdots : e_{n+1}]$ be a weighted projective $(n+1)$-space with finitely many singularities.[3] Suppose we want to bound (numbers and types of) singularities on a hypersurface $X_0 = \{F_0(\underline{W}) = 0\} \subset \mathbb{W}$ of degree d, where a smooth such hypersurface would have Hodge numbers $\underline{h} = (h^{n,0}, h^{n-1,1}, \dots, h^{0,n})$. Write $d_i = d/e_i$ for $i = 0, \dots, n+1$.

We shall assume that the singularities of X_0 are all isolated. Taking a general deformation $F_t = F_0 + tG$ to produce a family of $f : \mathcal{X} \to \Delta$ with smooth total space, the vanishing-cycle sequence

$$0 \to H^n(X_0) \to H^n_{\lim}(X_t) \to \bigoplus_{x \in \Sigma} V_x \xrightarrow{\delta} H^{n+1}_{\mathrm{ph}}(X_0) \to 0 \tag{3.1}$$

offers a naive such bound: first, by Schmid's nilpotent orbit theorem, the rank of Gr_F^p remains constant in the limit, giving the second equality of

$$h^{p,n-p} = h^{p,n-p}(X_t) = \sum_q h^{p,q}_{\lim}(X_t) \geq \sum_q h^{p,q}(\ker(\delta)). \tag{3.2}$$

Moreover, the mixed spectrum $\sigma_{f,x}$ tells us the $h^{p,q}_\zeta(V_x) = \dim(V^{p,q}_{x,\zeta})$ (for each eigenvalue ζ of T_{ss}), and only the $V^{p,n+1-p}_{x,1}$ can map nontrivially under δ. Since

[3] We may assume (without loss of generality) that no $n+1$ of the e_i have a common factor.

the hyperplane class also has T_{ss}-eigenvalue 1, equation (3.2) forces

$$\sum_q \sum_{\zeta \neq 1} \dim(V_{x,\zeta}^{p,q}) \leq h_{\mathrm{pr}}^{p,n-p}.$$

When x is a node, i.e., $f \overset{\mathrm{loc}}{\sim} \sum_{i=1}^{n+1} z_i^2$, Proposition 2.3 gives $V_{x,\mathbb{C}} = V_{x,-1}^{(n/2),(n/2)}$ for n even and $V_{x,1}^{(n+1)/2,(n+1)/2}$ for n odd. In the latter case, (3.2) yields no immediate bound on the number of nodes (though one does have results like [Kerr and Laza 2023, Theorem 2.9 and Corollary 2.11]). For $n = 2m$ even, (3.2) yields[4]

$$h_{\mathrm{pr}}^{(n/2),(n/2)}(X_t) = \text{coefficient of } \left[\frac{n}{2} + 1\right] \text{ in } \Gamma_{d_0} * \Gamma_{d_1} * \cdots * \Gamma_{d_{n+1}} \tag{3.3}$$

as a bound, which while better than nothing is rather weak.

3.4. Example. The simplest nontrivial case is given by $\mathbb{W} = \mathbb{P}^3$ ($n = 2$) and ($d_0 = d_1 = d_2 = d_3$) $d = 4$, where

$$\begin{aligned} \Gamma_4^{*4} &= \left(\left[\tfrac{1}{4}\right] + \left[\tfrac{1}{2}\right] + \left[\tfrac{3}{4}\right]\right)^{*4} \\ &= [1] + 4\left[\tfrac{5}{4}\right] + 10\left[\tfrac{3}{2}\right] + 16\left[\tfrac{7}{4}\right] + 19[2] + 16\left[\tfrac{9}{4}\right] + 10\left[\tfrac{5}{2}\right] + 4\left[\tfrac{11}{4}\right] + [3] \end{aligned} \tag{3.5}$$

correctly gives $19 = h_{\mathrm{pr}}^{1,1}(X_t)$. This is also a poor bound for the number of nodes on a quartic surface (see Example 3.8).

However, a simple trick can improve the bound while also giving one for odd n:

3.6. Theorem. *The number of nodes on X_0 is bounded by the coefficient, in $\Gamma_{d_0} * \Gamma_{d_1} * \cdots * \Gamma_{d_{n+1}}$, of $\left[\frac{n+1}{2} + \frac{1}{2d}\right]$ if n is even and d is odd, or of $\left[\frac{n+1}{2} + \frac{1}{d}\right]$ otherwise.*

Proof. Let $Y_t = \{F_t(\underline{W}) + W_{n+2}^d = 0\} \subset \mathbb{WP}[\underline{e} : 1] =: \widetilde{\mathbb{W}}$ be the cyclic $d:1$-cover of $\mathbb{W}$ branched over X_t, with $g : W_{n+2} \mapsto \zeta_d W_{n+2}$ the cyclic automorphism. By Dolgachev's extension of the Griffiths residue theorem [Dolgachev 1982], a basis for the g^*-eigenspace $H_{\mathrm{pr}}^{n-q+1,q}(Y_t)^{\bar{\zeta}_d^j}$ ($t \neq 0$, $0 \leq j < d$) is given by the Poincaré residue classes

$$\mathrm{Res}_{Y_t}\left(\frac{\underline{W}^{\underline{k}-\underline{1}}\, W_{n+2}^{d-j-1}\, \Omega_{\widetilde{\mathbb{W}}}}{(F_t + W_{n+2}^d)^{q+1}}\right),$$

with $k_i \in \mathbb{Z} \cap (0, d_i)$ for $i = 0, \ldots, n+1$ and weights of numerator and denominator equal, that is, $\sum_{i=0}^{n+1} e_i k_i + (d - j) = (q+1)\, d$, or equivalently (dividing by d)

$$\sum_{i=0}^{n+1} \frac{k_i}{d_i} = q + \frac{j}{d}.$$

Hence $\dim \mathrm{Gr}_F^{n-q+1} H_{\lim}^{n+1}(Y_t)^{\bar{\zeta}_d^j} = h^{n-q+1,q}(Y_t)^{\bar{\zeta}_d^j}$ is given (for $0 < j < d$) by the coefficient of $[q + j/d]$ in $\Gamma_{d_0} * \cdots * \Gamma_{d_{n+1}}$.

[4] This is by the same residue theory as used in the proof of Theorem 3.6 below. The notation $*$ is from Example 2.8.

Each node $x \in X_0$ becomes an A_{d-1} singularity $y \in Y_0$, with eigenspectrum $\sum_{j=1}^{d-1}((n+1)/2+j/d, n+1, -j/d)$ unless n is even and d is even (in which case the middle entry is $n+2$ at $j=d/2$). If r is the number of nodes, applying equations (3.1)–(3.2) to $\mathcal{Y}$ and refining by g^*-eigenspaces therefore yields $h^{p_j,q_j}(Y_t)^{\bar{\zeta}_d^j} \geq r$ (for $0<j<d$), where $p_j = \lfloor (n+1)/2+j/d \rfloor$ and $q_j = n+1-p_j$. Taking $j=1$ if n is odd and $j = \lceil (d+1)/2 \rceil$ if n is even (so that $p_j = (n+1)/2$ resp. $n/2+1$) yields $q_j + j/d = (n+1)/2+1/d$ resp. $n/2+(1/d)\lceil (d+1)/2 \rceil$, hence the claimed bound. □

3.7. Remark. As mentioned above, when $\mathbb{W} = \mathbb{P}^{n+1}$ this recovers Varchenko's [1983] bound. While Varchenko also uses the "cyclic-cover trick", our approach avoids the use of deformations and semicontinuity.

3.8. Example. For CY hypersurfaces in $\mathbb{P}^{n+1}$ ($d=n+2$), Theorem 3.6 yields the bounds 3, 16, 135, 1506, and 20993 for $n = 1, 2, 3, 4, 5$, the first two of which are sharp.[5] (This is also better than what (3.3) yields for $n=2$ and 4, namely 19 and 1751.) It is still not known whether 135 is sharp for quintic 3-folds. The well-known Fermat pencil has fiber $W_0^5+\cdots+W_4^5 = 5W_0\cdots W_4$, with $125 = |(\mathbb{Z}/5\mathbb{Z})^3|$ nodes, while the example of van Straten [1993] with 130 nodes remains the record.

3.9. Remark. For $n=2$, the following bound by Miyaoka [1984] sometimes yields better results. If X is any smooth projective surface which is smooth except at r nodes, and K_X is nef, then $r \leq 8\chi(\mathcal{O}_X) - \frac{8}{9}K_X^2$.

(a) For $X \subset \mathbb{P}^3$ a surface of degree d, this yields the bound

$$\tfrac{4}{3}(d-1)(d-2)(d-3)+8-\tfrac{8}{9}d(d-4)^2 = \tfrac{4}{9}d(d-1)^2,$$

which is better than Theorem 3.6 for $d \geq 6$ even or $d \geq 15$ odd. A case in point is $d=6$, where we get 85 by (3.3), 68 by Theorem 3.6, and 66 by [Miyaoka 1984]; this was further reduced to 65 (which is sharp) by a clever use of coding theory [Jaffe and Ruberman 1997]. Another is $d=8$, where we get $r \leq 174$.

(b) As a weighted projective example, one can consider surfaces X of degree 10 in $\mathbb{WP}[1:1:1:2]$. We have $\chi(\mathcal{O}_X) = 1+h^2(\mathcal{O}_X) = 35$ and

$$(K_X \cdot K_X)_X = (X\cdot(X+K_{\mathbb{W}})^2)_{\mathbb{W}} = \frac{10(10-5)^2}{1\cdot 1\cdot 1\cdot 2} = 125,$$

and hence $r \leq \left\lfloor \frac{1520}{9} \right\rfloor = 168$.

3.10. Examples. We consider some CY 3-fold hypersurfaces with r nodes in weighted projective 4-folds.

[5] The union of 3 lines in $\mathbb{P}^2$ has 3 nodes, and a Kummer quartic $K3$ in $\mathbb{P}^4$ has 16 nodes. The bounds here are the coefficients of $\left[\frac{n+1}{2}+\frac{1}{n+2}\right]$ in $\Gamma^{*(n+2)}_{n+2}$, e.g., 16 is the coefficient of $\left[\frac{7}{4}\right]$ in (3.5).

(i) $X_0 \subset \mathbb{WP}[1:1:1:1:2]$ of degree 6: Theorem 3.6 yields $r \leq 137$, while the "Fermat pencil" type example $W_0^6 + \cdots + W_3^6 + W_4^3 = 3 \cdot 2^{2/3} W_0 \cdots W_4$ has $|((\mathbb{Z}/6\mathbb{Z})^3 \times \mathbb{Z}/3\mathbb{Z})/(\mathbb{Z}/6\mathbb{Z})| = 108$ nodes.

(ii) $X_0 \subset \mathbb{WP}[1:1:1:1:4]$ of degree 8: the Theorem yields $r \leq 180$, while $W_0^8 + \cdots + W_3^8 + W_4^2 = 4W_0 \cdots W_4$ has $|((\mathbb{Z}/8\mathbb{Z})^3 \times \mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/8\mathbb{Z})| = 128$ nodes. Here we can improve both the bound and example, since X_0 is (by the quadratic formula) a double-cover of $\mathbb{P}^3$ branched along an r-nodal octic surface. So Remark 3.9(a) gives $r \leq 174$, while the Endrass [1997] example has $r = 168$.

(iii) $X_0 \subset \mathbb{WP}[1:1:1:2:5]$ of degree $d = 10$: Theorem 3.6 yields $r \leq 169$, but because these are double covers of $\mathbb{WP}[1:1:1:2]$ branched along an r-nodal dectic surface, Remark 3.9(b) reduces the bound to 168. The standard example is $W_0^{10} + W_1^{10} + W_2^{10} + W_3^5 + W_4^2 = 2^{4/5}\, 5^{1/2}\, W_0 \cdots W_4$, but this has only 100 nodes. One can do somewhat better by taking the preimage of a Togliatti quintic [Beauville 1980] (with 31 nodes avoiding the coordinate axes) under

$$\mathbb{WP}[1:1:1:2] \xrightarrow{1:2} \mathbb{WP}[1:1:2:2] \xrightarrow{1:2} \mathbb{WP}[1:2:2:2] \cong \mathbb{P}^3,$$

to get $4 \cdot 31 = 124$.

In the case of a symmetric hypersurface $X_0 \subset \mathbb{P}^{n+1}$, cut out by $F_0 \in \mathbb{C}[\underline{W}]^{\mathfrak{S}_{n+2}}$ (homogeneous of degree d), one can consider the family $\mathcal{Y} \to \Delta$ of d-fold cyclic covers branched along an $\mathfrak{S}_{n+2}$-invariant smoothing $\mathcal{X} \to \Delta$. A full accounting of this story gets into G-spectra ($G \cong \mu_d \times \mathrm{stab}_{\mathfrak{S}_{n+2}}(x)$) of the resulting A_{d-1} singularities of Y_0. This leads to constraints, via character theory of $\mathfrak{S}_{n+2}$, on how Σ can be built out of $\mathfrak{S}_{n+2}$-orbits. (However, it does not, for example, *rule out* the possibility of 135 nodes on an $\mathfrak{S}_5$-symmetric quintic threefold.) Here we shall only give the simplest result in this direction:

3.11. Theorem. *A symmetric CY hypersurface in $\mathbb{P}^{n+1}$ (of degree $d = n+2$) with isolated singularities cannot contain a node with trivial stabilizer in $\mathfrak{S}_{n+2}$.*

Proof. Suppose otherwise; then Y_0 has a set of $(n+2)!$ A_{n+1} singularities with eigenspectra

$$\sum_{j=1}^{n+1} \left(\frac{n+1}{2} + \frac{j}{n+2},\, n+1,\, \frac{-j}{n+2} \right),$$

contributing a subspace V of dimension $(n+2)!$ to the g^*-eigenspace[6] $H^{n+1}_{\mathrm{van}}(Y_t)^{\zeta_{n+2}}$. It is closed under the action of $\mathfrak{S}_{n+2}$, and the triviality of the stabilizers of these A_{n+1} singularities means that the trace of any $\sigma \in \mathfrak{S}_{n+2} \setminus \{1\}$ is zero. So V is a copy of the regular representation of $\mathfrak{S}_{n+2}$, which belongs to

$$\ker(\delta) \subseteq H^{(n+1)/2,(n+1)/2}_{\mathrm{van}}(Y_t)^{\zeta_{n+2}}.$$

[6] As before, $g : W_{n+2} \mapsto \zeta_{n+2} W_{n+2}$ denotes the cyclic automorphism of Y_t.

By the compatibility[7] of the vanishing-cycle sequence for $\mathcal{Y}$ with g^* and $\mathfrak{S}_{n+2}$, this forces a copy of the regular representation in $H_{\lim}^{(n+1)/2,(n+1)/2}(Y_t)^{\zeta_{n+2}}$, hence $H^{(n+1)/2,(n+1)/2}(Y_t)^{\zeta_{n+2}}$ for $t \neq 0$ (as $\mathfrak{S}_{n+2}$ acts on the VHS, compatibly with taking limits, see Remark 1.13).

Now $U := \mathcal{H}^{(n+1)/2,(n+1)/2}(Y_t)^{\zeta_{n+2}}$ has a basis of the form

$$\eta_{\underline{k}} := \mathrm{Res}_{Y_t}\left(\frac{\underline{W}^{\underline{k}-\underline{1}}\,\Omega_{\mathbb{P}^{n+2}}}{\left(F_0(\underline{W}) + W_{n+2}^{n+2}\right)^{(n+3)/2}}\right),$$

where $0 < k_i < n+2$ (for $i = 0, \dots, n+1$) and (for equality of weights of numerator and denominator) $\left(\sum_{i=0}^{n+1} k_i\right) + 1 = \frac{n+3}{2}(n+2)$. Here $\mathfrak{S}_{n+2}$ acts trivially on the denominator, through the sign representation χ on $\Omega_{\mathbb{P}^{n+2}}$, and by permutations of the W_i on $\underline{W}^{\underline{k}-\underline{1}}$. We claim that U contains no copy of the trivial representation, *a fortiori* of the regular representation, furnishing the desired contradiction.

Clearly it is equivalent to show that the representation of $\mathfrak{S}_{n+2}$ on the $\mathbb{C}$-span $\tilde{U}$ ($\cong U \otimes \chi$) of the monomials $\{\underline{W}^{\underline{k}}\}_{\underline{k}\text{ as above}}$ contains no copy of χ. Suppose $o := \mathfrak{S}_{n+2}.\underline{W}^{\underline{k}}$ is an orbit and $\tilde{U}_o \subseteq \tilde{U}$ its span. By Burnside's lemma,

$$\frac{1}{(n+2)!} \sum_{g \in \mathfrak{S}_{n+2}} |o^g| = 1.$$

On the other hand, $\underline{k} = (k_0, \dots, k_{n+1})$ contains a repeated entry since there are only $n+1$ choices for each k_i; hence for some transposition τ, $|o^\tau| \neq 0$. Since $\mathrm{sgn}(\tau) = -1$, this forces

$$\frac{1}{(n+2)!} \sum_{g \in \mathfrak{S}_{n+2}} \mathrm{sgn}(g)\,|o^g|,$$

which computes the number of copies of χ in $\tilde{U}_o$, to be zero. □

For $n = 1$ or 2 this result is obvious (since $6 > 3$ and $24 > 16$), but for $n = 3$, 4, or 5 it is less so (as $120 < 135$, $720 < 1506$, and $5040 < 20993$). In particular, since the examples of quintic 3-folds with 125 and 130 nodes are $\mathfrak{S}_5$-symmetric, and the latter has a 60-node orbit, it is interesting that a 120-node orbit is impossible.

4. Cyclic covers of $\mathbb{P}^1$

In the final two sections we turn to "codimension-one" monodromy phenomena for period maps arising from cyclic covers. We begin with a story that generalizes elliptic curves and goes back to Deligne and Mostow [1986] (see also [Moonen 2018]). Given distinct points $t_1, \dots, t_{2m} \in \mathbb{P}^1$ (with projective coordinates $[S_i : T_i]$), define

$$C_{\underline{t}} := \left\{[Z_0 : Z_1 : Z_2] \in \mathbb{P}[1:1:2] \mid Z_2^m = \prod_{i=1}^{2m}(S_i Z_1 - T_i Z_0)\right\},$$

[7]This is nothing but Corollary 1.9 with $\mathcal{G} = \langle g \rangle \times \mathfrak{S}_{n+2}$.

with automorphism $g([Z_0 : Z_1 : Z_2]) := [Z_0 : Z_1 : \zeta_m Z_2]$. For $m = 2, 3, 4$, or 6, the sum of g^*-eigenspaces $H^1(C_{\underline{t}})^{\zeta_m} \oplus H^1(C_{\underline{t}})^{\bar{\zeta}_m}$ produces a $\mathbb{Q}$-VHS over $M_{0,2m}$,[8] and hence a period map to an arithmetic ball quotient $\Gamma\backslash\mathbb{B}_{2m-3}$. This turns out to be injective,[9] and extends to an isomorphism between GIT resp. Hassett/KSBA compactifications of $M_{0,2m}$ and Baily–Borel resp. toroidal compactifications of the ball quotient [Deligne and Mostow 1986; Gallardo et al. 2021].

So what if $m \neq 2, 3, 4$, or 6? In the discussion that ensues, we will not be concerned with ball quotients or even the period map per se, but only with

- the $\mathbb{Q}$-VHS $\mathcal{V}$ over $M_{0,2m}$ arising from $H^1(C_{\underline{x}})$,
- its sub-$\mathbb{C}$-VHSs $\mathcal{V}^{\zeta_m^j} := \ker(g^* - \zeta_m^j I)$ $(1 \leq j \leq m-1)$, and
- their limiting behavior along the boundary of the Hassett compactifications $\overline{M}_{0,[(1/m)+\epsilon]_{2m}}$ (see below).

The point is that *these can be considered uniformly for all* $m \geq 2$, not just $m = 2, 3, 4$, and 6. Moreover, using eigenspectra, *we can easily compute LMHS and monodromy types along the Hassett boundary strata*, as we demonstrate in Propositions 4.5–4.6 and Example 4.7. This is the first step toward a global study of the extended period map for this series of examples, which will necessarily go beyond the arithmetic ball quotient setting (see Remark 4.8). We also refer the reader to [Deng and Gallardo 2023], where global partial compactifications of the period maps for some other non-Deligne–Mostow cases are constructed.

To begin with, in affine coordinates $x = Z_1/Z_0$, $y = Z_2/Z_0$, $C_{\underline{t}}$ takes the form

$$y^m = f_{\underline{t}}(x) := \prod_{i=1}^{2m}(x - t_i)$$

[resp. $\prod_{i \neq j}(x - t_i)$ if $t_j = \infty$]. While there are three possibilities for the Newton polytope Δ, they all have the same interior integer points

$$(\Delta \setminus \partial\Delta) \cap \mathbb{Z}^2 = \{(i, j) \mid 1 \leq j \leq m-1,\ 1 \leq i \leq 2(m-j)-1\},$$

which provide a basis of $\Omega^1(C_{\underline{t}})$ via

$$\omega_{(i,j)} := \mathrm{Res}_{C_{\underline{t}}}\left(\frac{x^{i-1}y^{j-1}\,dx \wedge dy}{y^m - f_{\underline{t}}(x)}\right).$$

Since $g^*\omega_{(i,j)} = \zeta_m^j \omega_{(i,j)}$, we find that

$$\begin{cases} \mathrm{rk}(\mathcal{V}^{\zeta_m^j})^{1,0} = 2(m-j)-1, & \mathrm{rk}(\mathcal{V}^{\zeta_m^j})^{0,1} = 2j-1, \\ \mathrm{rk}\,\mathcal{V}^{\zeta_m^j} = 2m-2, & \mathrm{rk}\,\mathcal{V} = 2(m-1)^2. \end{cases} \tag{4.1}$$

[8] $M_{0,n}$ parametrizes *ordered* n-tuples of distinct points on $\mathbb{P}^1$ modulo the action of $\mathrm{PSL}_2(\mathbb{C})$.

[9] For $m = 6$ one has to quotient $M_{0,12}$ by $\mathfrak{S}_{12}$; see [Gallardo et al. 2021].

For example, if $m=5$, then $C_{\underline{t}}$ has genus 12; and $\mathcal{V}_{\mathbb{C}}$ decomposes into four $\mathbb{C}$-VHSs $\{\mathcal{V}^{\zeta_5^j}\}_{j=1}^4$ with respective Hodge numbers (7, 1), (5, 3), (3, 5), and (1, 7).

4.2. Definition [Hassett 2003]. A *weighted stable rational curve* for the weight $\underline{\mu} := (\mu_1, \dots, \mu_n) \in \{(0,1] \cap \mathbb{Q}\}^{\times n}$ is a pair[10] $(\mathcal{C}, \sum \mu_i p_i)$ with:

- $\mathcal{C}$ a nodal connected projective curve of arithmetic genus 0.
- Each p_i a smooth point of $\mathcal{C}$.
- If $p_{i_1} = \dots = p_{i_r}$, then $\mu_{i_1} + \dots + \mu_{i_r} \le 1$.
- The $\mathbb{Q}$-divisor $K_{\mathcal{C}} + \sum_{i=1}^n \mu_i p_i$ is ample (i.e., on each irreducible component, the sum of weights plus number of nodes is > 2).

We will write $(\mu, \dots, \mu) =: [\mu]_n$ for repeated weights.

4.3. Theorem [Hassett 2003]. (i) *There exists a smooth projective fine moduli space $\overline{M}_{0,\underline{\mu}}$ parametrizing $\underline{\mu}$-weighted stable rational curves, and containing $M_{0,n}$ as a Zariski-open subset.*

(ii) *Given weights $\underline{\mu} = (\mu_1, \dots, \mu_n)$ and $\underline{\tilde{\mu}} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ with $\mu_i \le \tilde{\mu}_i$ ($\forall i$), there exists a birational* ***reduction morphism*** *$\pi_{\underline{\tilde{\mu}},\underline{\mu}} : \overline{M}_{0,\underline{\tilde{\mu}}} \twoheadrightarrow \overline{M}_{0,\underline{\mu}}$ contracting all components which violate the ampleness property in Definition 4.2 for the weight $\underline{\mu}$.*

4.4. Remark. (a) $\overline{M}_{0,[1]_n}$ reproduces the Deligne–Mumford–Knudsen compactification $\overline{M}_{0,n}$.

(b) Although the ampleness property forces $\sum \mu_i > 2$, if for $|\underline{\mu}| = 2$ we *define* $\overline{M}_{0,\underline{\mu}}$ to be the GIT quotient $(\mathbb{P}^1)^n /\!\!/_{\underline{\mu}} \mathrm{SL}_2$, then Theorem 4.3(ii) extends to this case; and if we take $\tilde{\mu}_i = \mu_i + \epsilon$ ($\epsilon \in \mathbb{Q}$, $0 < \epsilon \ll 1$) then $\pi_{\underline{\tilde{\mu}},\underline{\mu}}$ is Kirwan's partial desingularization which blows up the strictly semistable locus.

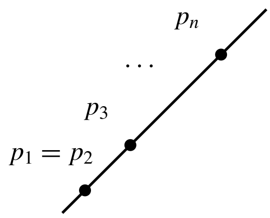
Our interest henceforth is in the equal-weight Hassett compactification

$$\overline{M}^H_{0,2m} := \overline{M}_{0,[(1/m)+\epsilon]_{2m}}$$

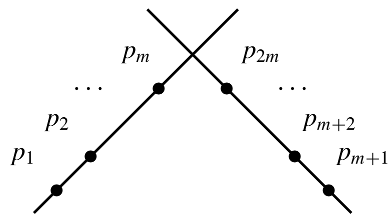
and its morphism π to $\overline{M}^{\mathrm{GIT}}_{0,2m} := \overline{M}_{0,[1/m]_{2m}}$. As the reader may check, the irreducible components of $\overline{M}^H_{0,2m} \setminus M_{0,2m}$ are of two types, parametrizing[11] stable weighted curves as shown (up to reordering of the $\{p_i\}$):

[10] Despite the sum notation, the order of points with equal weights is retained.

[11] More precisely, it is a dense open subset of each component that parametrizes the displayed objects.

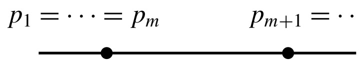


type (A)



type (B)

It is also clear that π preserves the type (A) strata whilst contracting the type (B) ones to a (strictly semistable) point parametrizing the object



The $\mathbb{C}$-VHSs $\mathcal{V}^{\zeta_m^j}$ admit canonical extensions across the smooth part of $\overline{M}^H_{0,2m} \backslash M_{0,2m}$, and we and we shall now compute the LMHS types there.

4.5. Proposition. *Along type* (*A*) *strata*:

- $\mathcal{V}^{\zeta_m^j}_{\lim}$ *is pure of weight* 1*, with* $h^{1,0} = 2m - 2j - 1$ *and* $h^{0,1} = 2j - 1$*, unless* $j = m/2$.
- *If* $j = m/2$*, then* $h^{1,1} = h^{0,0} = 1$*,* $h^{1,0} = h^{0,1} = m - 1$*, and* $T = e^N$ *(with* N *an isomorphism from the* $(1, 1)$ *to* $(0, 0$ *part*).
- *If* $j > m/2$ (*resp.* $< m/2$)*, then we have the decomposition*

$$\mathcal{V}^{\zeta_m^j}_{\lim} = \mathcal{V}^{\zeta_m^j}_{\lim,1} \oplus \mathcal{V}^{\zeta_m^j}_{\lim,\bar{\zeta}_m^{2j}}$$

into $T = T_{ss}$*-eigenspaces, where* $\mathcal{V}^{\zeta_m^j}_{\lim,\bar{\zeta}_m^{2j}}$ *is* 1*-dimensional of type* $(0, 1)$ (*resp.* $(1, 0)$).

Proof. Begin by locally modeling (the effect on $C_{\underline{t}}$ of) the collision of two points by $y^m + z^2 = s$, as $s \to 0$. This has eigenspectrum

$$\sum_{j=1}^{m-1} \left(\tfrac{3}{2} - \frac{j}{m},\, w(j),\, \frac{j}{m} \right),$$

where $w(j) = 2$ if $j = m/2$ and 1 otherwise. Next, we apply the vanishing-cycle sequence (with $H^2_{\mathrm{ph}} = \{0\}$ since the degenerate curve remains irreducible) to compute the LMHS. Finally, we perform a base-change by $s \mapsto s^2$ to preserve ordering of points, which squares the eigenvalues of the T_{ss}-action; in other words, we replace $\frac{3}{2} - \frac{j}{m}$ by $\left\{2\left(\frac{3}{2} - \frac{j}{m}\right)\right\} + \left\lfloor \frac{3}{2} - \frac{j}{m} \right\rfloor$ ($\{\cdot\}$ denoting the fractional part), which gives the result. $\square$

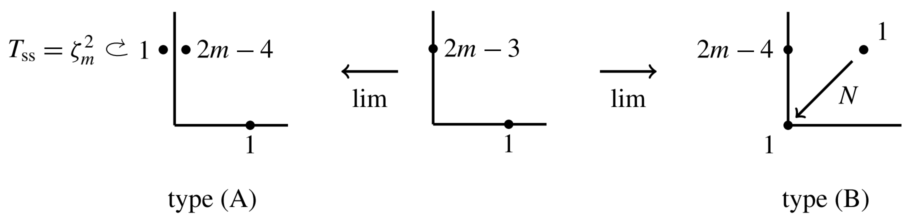
4.6. Proposition. *Along the type* (B) *strata, for each* $1 \le j \le m - 1$*,* $\mathcal{V}^{\zeta_m^j}_{\lim}$ *has Hodge numbers* $h^{1,1} = h^{0,0} = 1$*,* $h^{1,0} = 2m - 2j - 2$*, and* $h^{0,1} = 2j - 2$; N *is an isomorphism from the* $(1, 1)$ *to* $(0, 0)$ *part, and* $T = e^N$ *is unipotent.*

Proof. In the GIT compactification for *unordered* points, the degeneration is locally modeled by two copies of $y^m + x^m = s$, each with eigenspectrum

$$\sum_{j=1}^{m-1}\Big(1, 2, \frac{j}{m}\Big) + \sum_{j=2}^{m-1}\sum_{k=1}^{j-1}\Big(\frac{k+m-j}{m}, 1, \frac{j}{m}\Big) + \sum_{j=1}^{m-2}\sum_{k=j+1}^{m-1}\Big(\frac{k+m-j}{m}, 1, \frac{j}{m}\Big).$$

At this point one applies the vanishing-cycle sequence to deduce the form of the LMHS, noting that the degenerate curve is a union of m $\mathbb{P}^1$'s and $H^2_{\mathrm{ph}} \cong \mathbb{Q}(-1)^{\oplus m-1}$. For $\overline{M}^H_{0,2m}$, one then applies the base-change by $s \mapsto s^m$, which trivializes T_{ss}, allowing the extension class to vary along the type (B) stratum. □

4.7. Example. Combining (4.1) with the two propositions, $\mathcal{V}^{\bar{\zeta}_m}$ has Hodge–Deligne diagrams



For $m = 4$ (resp. 6), the monodromy in type (A) is thus given by a complex reflection (resp. "triflection").

4.8. Remark. For any m, we have that $\mathcal{V}^{\bar{\zeta}_m}$ ($\oplus\mathcal{V}^{\zeta_m}$) induces a map from the universal cover $\tilde{M}^{\mathrm{un}}_{0,2m}$ to a ball $\mathbb{B}_{2m-3}$. Moreover, both LMHS types have $2m-4$ complex moduli. However, for m different from 2, 3, 4, or 6, this does not lead to a tidy extended period map: as the projection of the monodromy to $U(1, 2m-3)$ is not discrete [Mostow 1988], the quotient of $\mathbb{B}_{2m-3}$ by this is not Hausdorff.

To circumvent this problem, we must replace $\mathbb{B}_{2m-3}$ by its product with other (nonball) symmetric domains, which receives the image of the period map for the $\mathbb{Q}$-VHS $\oplus_{(j,m)=1}\mathcal{V}^{\zeta_m^j}$. For instance, if $m = 5$ then the real points of the generic Mumford–Tate group of $\mathcal{V}$ take the form $U(1,7) \times U(3,5)$, and the full period map lands in a discrete quotient of the product $\mathbb{B}_7 \times \mathrm{I}_{3,5}$.

5. Hyperplane configurations and Dolgachev's conjecture

Both differential and asymptotic methods in Hodge theory can be used to establish that a VHS is "generic" in some sense. In [Gerkmann et al. 2013], differential methods (characteristic varieties and Yukawa couplings) were employed to show that the period map for the family of CY 3-folds $X \xrightarrow{2:1} \mathbb{P}^3$ branched over 8 planes does not factor through a locally symmetric variety of the form $\Gamma\backslash\mathrm{SU}(3,3)/K$. Indeed, the geometric monodromy and Mumford–Tate groups of the corresponding VHS turn out to be as large as they can be (with both equal to the symplectic group Sp_{20}). This was later extended to similarly constructed CY n-fold families [Sheng et al. 2015], see below. Our goal here is to quickly deduce these results using eigenspectra and local monodromy, demonstrating the effectiveness of the asymptotic approach.

Let $L_0, \dots, L_{2n+1} \subset \mathbb{P}^n$ be hyperplanes defined by linear forms ℓ_i, in general position in the sense that $\bigcup L_i$ is a normal crossing divisor. Consider the $2:1$ cover $X \overset{\pi}{\twoheadrightarrow} \mathbb{P}^n$ branched along $\bigcup L_i$, and the rank-1 $\mathbb{Q}$-local system $\mathbb{L}$ on

$$U = \mathbb{P}^n \setminus \big(\bigcup L_i\big) \overset{\jmath}{\hookrightarrow} \mathbb{P}^n,$$

with monodromy -1 about each L_i. Since X has finite quotient singularities, we have $\mathrm{IC}^\bullet_X = \mathbb{Q}_X[n]$ and[12]

$$H := H^n_{\mathrm{pr}}(X) := \frac{H^n(X)}{\pi^* H^n(\mathbb{P}^n)} \cong H^n(\mathbb{P}^n, \jmath_* \mathbb{L}) \cong \mathrm{IH}^n(\mathbb{P}^n, \mathbb{L}) \tag{5.1}$$

is a pure HS of weight n. By [Dolgachev and Kondō 2007, Lemma 8.2], it has Hodge numbers

$$h^{p,n-p}_{\mathrm{pr}}(X) = \binom{n}{p}^2 \implies h^n_{\mathrm{pr}}(X) = \binom{2n}{n}. \tag{5.2}$$

It is polarized by the intersection form Q, which presents no difficulties as X has a smooth finite cover.

Taking $\mathcal{S} \subset (\check{\mathbb{P}}^n)^{2n+2}/\mathrm{PGL}_{n+1}(\mathbb{C}) =: \bar{\mathcal{S}}$ to be the (n^2-dimensional) moduli space of $2n+2$ ordered hyperplanes in $\mathbb{P}^n$ in general position, this construction yields a $\mathbb{Z}$-PVHS $\mathcal{H} \to \mathcal{S}$ of CY-n type with H as reference fiber. Let

$$\rho : \pi_1(\mathcal{S}) \to \mathrm{Aut}(H, Q)^\circ =: M_{\max}$$

be the monodromy representation of $\mathcal{H}$,[13] Π its geometric monodromy group, and M its Hodge (special Mumford–Tate) group. Here Π is the identity connected component of $\widetilde{\Pi} := \overline{\rho(\pi_1(\mathcal{S}))}^{\mathbb{Q}\text{-Zar}}$, and $\Pi \le M \le M_{\max}$. A conjecture attributed by [Sheng et al. 2015] to Dolgachev states that the period map for $\mathcal{H}$ factors through a locally symmetric variety (also n^2-dimensional) of type $I_{n,n}$,[14] which would imply that $\mathfrak{m}_{\mathbb{R}} \cong \mathfrak{su}(n,n)$. This is equivalent to saying that,

(5.3) *up to finite data* (i.e., after passing to a finite cover), *$\mathcal{H}$ is the n-th wedge power of a VHS of weight* 1 *and rank* $2n$.

[12] See [Hotta et al. 2008, Proposition 8.2.30] for the statement that $\mathrm{IC}^\bullet_{\mathbb{P}^n} \mathbb{L} = \jmath_* \mathbb{L}[n]$.

[13] Here $(\,\cdot\,)^\circ$ means the identity component as algebraic group (i.e., $\mathrm{SO}(H)$ instead of $\mathrm{O}(H)$ if n is even).

[14] Note that the "tautological VHS" over $I_{n,n}$ is already geometrically realized by the n-th primitive cohomology of a universal family of Weil abelian $2n$-folds.

The conjecture does hold for $n = 1$ and $n = 2$, but this merely reflects exceptional isomorphisms of Lie groups in low rank, namely

$$\mathrm{SU}(1, 1) \cong \mathrm{SL}_2(\mathbb{R}) \quad \text{and} \quad \mathrm{SU}(2, 2) \cong \mathrm{Spin}(2, 4)^+.$$

That is, in both of these cases we *also* have $\Pi \cong M_{\max}$ ($= \mathrm{SL}_2$ resp. $\mathrm{SO}(2, 4)$). For $n \geq 3$, in contrast, the conjecture would have $\Pi < M_{\max}$ a proper algebraic subgroup. In [Sheng et al. 2015, Proposition 8.2.30] (and earlier works [Gerkmann et al. 2007a; 2007b; 2013]), it was shown via quite computationally involved differential methods that in fact the monodromy is maximal for all n, and the conjecture fails for $n \geq 3$:

5.4. Theorem. $\Pi = M = M_{\max}$ *for all* $n \geq 1$.

In the remainder of this section, we explain how asymptotic methods provide a much simpler approach to these results. First we will give a careful argument disproving the conjecture for $n \geq 3$ odd, which *a priori* is a weaker statement than the Theorem in that case. (The relation to the main theme of his paper — specifically, to the setting of Corollary 1.9 — enters when we pass to the smooth finite cover $\hat{X}$ of X.) Then we sketch a proof of Theorem 5.4 using a more topological and monodromy-theoretic approach.

Disproof of* (5.3) *for n odd. Most of the analysis that follows works for all n, though the last step is inconclusive for even n.

To begin, consider a pencil $\mathbb{P}^1 \overset{\varepsilon}{\hookrightarrow} \bar{\mathcal{S}}$ of hyperplane configurations given by fixing $L_0, \dots, L_{2n}$ (in general position) and letting $L_{2n+1} := L_s$ vary along a line in $\check{\mathbb{P}}^n$ (chosen to avoid linear spans of any $n-2$ L_i in $\check{\mathbb{P}}^n$).[15] Writing $\Sigma = \varepsilon^{-1}(\bar{\mathcal{S}} \setminus \mathcal{S})$, we have $|\Sigma| = \binom{2n+1}{n}$; and degenerations $\mathcal{X}_\sigma \to \Delta_\sigma$ of our double-covers at $\sigma \in \Sigma$ are locally modeled (with $t = s - \sigma$) by

$$w^2 \underset{\text{loc}}{=} x_1 \cdots x_n (t - x_1 - \cdots - x_n) \tag{5.5}$$

after a $\mathrm{PGL}_{n+1}(\mathbb{C})$-action. Accordingly, writing $X_0, \dots, X_n$ for projective coordinates on $\mathbb{P}^n$, we take $\ell_i = X_i$ for $0 \leq i \leq n$ and $\ell_{n+1} = tX_0 - \sum_{i=1}^n X_i$, and $\ell_{n+2}, \dots, \ell_{2n+1}$ "general".

Let $\underline{\ell} : \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1}$ denote the linear embedding

$$[X_0 : \cdots : X_n] \mapsto [\ell_0(\underline{X}) : \cdots : \ell_{2n+1}(\underline{X})]$$

and $\phi : \mathbb{P}^{2n+1} \to \mathbb{P}^{2n+1}$ denote the map sending

$$[Z_0 : \cdots : Z_{2n+1}] \mapsto [Z_0^2 : \cdots : Z_{2n+1}^2].$$

[15] It already follows from Zariski's theorem [Voisin 2003, Theorem 3.22] that $\rho(\pi_1(\mathbb{P}^1 \setminus \Sigma)) = \rho(\pi_1(\mathcal{S}))$ but we won't need this.

Then the variety $\hat{X} := \phi^{-1}(\underline{\ell}(\mathbb{P}^n)) \subset \mathbb{P}^{2n+1}$ is a smooth complete intersection on which[16] $\mathcal{A} := (\mathbb{Z}/2\mathbb{Z})^{2n+2}/\mathbb{\Delta}(\mathbb{Z}/2\mathbb{Z})$ acts via $\underline{e}^{(i)} \mapsto \{Z_i \mapsto -Z_i\}$, with quotient $\mathbb{P}^n$; explicitly, we have

$$\hat{X} = \bigcap_{k=0}^{n} \{0 = F_k(\underline{Z}) := -Z_{n+k+1}^2 + \ell_{n+k+1}(Z_0^2, \ldots, Z_n^2)\}. \tag{5.6}$$

Write $\chi \in \mathrm{X}^*(\mathcal{A})$ for the character sending each $\underline{e}^{(i)} \mapsto -1$, $\mathcal{A}^\circ := \ker(\chi) \leq \mathcal{A}$, and $q : \hat{X} \twoheadrightarrow X$ for the quotient by $\mathcal{A}^\circ$; then $H \cong q^* H^n_{\mathrm{pr}}(X) \cong H^n(\hat{X})^\chi$. Since

$$F_0(\underline{Z}) = tZ_0^2 - \sum_{i=1}^{n+1} Z_i^2,$$

we have thus replaced our original *non*-isolated degeneration (5.5) by a nodal one.

Next, we use the "Cayley trick" to replace the complete intersection $\hat{X}$ by a hypersurface

$$Y := \left\{0 = F := \sum_{k=0}^{n} Y_k F_k(\underline{Z})\right\} \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^{2n+1}}(2)^{\oplus n+1}) =: \boldsymbol{P} \tag{5.7}$$

of dimension $3n$. We have an $\mathcal{A}$-equivariant isomorphism $H^{3n}(Y)(n) \cong H^n(\hat{X})$ of HSs, so that $H \cong H^{3n}(Y)^\chi(n)$. In affine coordinates $(z_1, \ldots, z_{2n+1}; y_1, \ldots, y_n)$, notice that $F = 0$ becomes[17]

$$0 = t - z_1^2 - \cdots - z_{n+1}^2 + \sum_{k=1}^{n} y_k(b_k - z_{n+k+1})(b_k + z_{n+k+1}) + \text{h.o.t.}, \tag{5.8}$$

where $b_k := \sqrt{F_k(1, 0, \ldots, 0)}$. So at $t = 0$, the singular fiber Y_σ has 2^n nodes at

$$\begin{aligned}(Z_0;\, &Z_1, \ldots, Z_{n+1};\, Z_{n+2}, \ldots, Z_{2n+1};\, Y_0;\, Y_1, \ldots, Y_n)\\ &= \big(1; 0, \ldots, 0; (-1)^{a_1} b_1, \ldots, (-1)^{a_n} b_n; 1; 0, \ldots, 0\big), \quad \underline{a} \in (\mathbb{Z}/2\mathbb{Z})^n,\end{aligned} \tag{5.9}$$

and the degeneration $\mathcal{Y}_\sigma \to \Delta_\sigma$ has smooth total space. The mixed spectrum of each node is $[((3n+1)/2, 3n+1)]$ for n odd and $[((3n+1)/2, 3n)]$ for n even; so T_σ acts through multiplication by $(-1)^{n+1}$ on

$$H^{3n}_{\mathrm{van}}(Y_t) \cong \mathbb{Q}\Big(-\Big\lfloor \frac{3n+1}{2} \Big\rfloor\Big)^{\oplus 2^n}. \tag{5.10}$$

Moreover, since the summands of (5.10) are represented by

$$\eta_{\underline{a}} = (-1)^{|\underline{a}|}(dz_1 \wedge \cdots \wedge dz_{2n+1} \wedge dy_1 \wedge \cdots \wedge dy_n)/F^{\lceil (3n+1)/2 \rceil}$$

near the nodes (5.9) (in the sense of [Kerr and Laza 2023, Section 2]), it has a 1-dimensional subspace (generated by $\eta_\chi := \sum (-1)^{|\underline{a}|} \eta_{\underline{a}}$) on which $\mathcal{A}$ acts through χ.

---

[16] Here $\mathbb{\Delta}$ denotes the diagonal embedding.

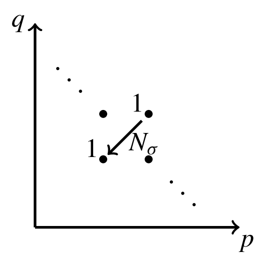
[17] Here "h.o.t." means terms vanishing to order 3 at the nodes.

Taking χ-eigenspaces of the vanishing-cycle sequence for $\mathcal{Y}_\sigma \to \Delta_\sigma$ and twisting by $\mathbb{Q}(n)$ now yields

$$\begin{aligned}(5.11)\quad 0 \to H^{3n}(Y_\sigma)^\chi(n) &\xrightarrow{\mathrm{sp}^\chi} \\ &\underbrace{H^{3n}_{\lim}(Y_t)^\chi(n)}_{\cong H_{\lim}} \xrightarrow{\mathrm{can}^\chi} \mathbb{Q}\Big(-\Big\lfloor \frac{n+1}{2} \Big\rfloor\Big) \xrightarrow{\delta^\chi} H^{3n+1}_{\mathrm{ph}}(Y_\sigma)^\chi(n) \to 0.\end{aligned}$$

We claim that $\delta = 0$. For n even, this is clear, since T_σ acts trivially on $H^{3n+1}_{\mathrm{ph}}(Y_\sigma)$ and by -1 on $\mathbb{Q}(-\lfloor (n+1)/2 \rfloor)$. So we conclude that T_σ acts on $H_{\lim}$ via an orthogonal reflection. This doesn't factor through $\bigwedge^n$ of any automorphism of $\mathbb{C}^{2n}$, but because it is finite (of order 2), this does not (yet) disprove the conjecture.

On the other hand, for n odd, it is not automatic that $\delta = 0$. (This is a well-known problem with nodal degenerations in odd dimensions, see [Kerr and Laza 2023, Section 2.2]; and as we saw in the proof of (5.5), our degenerations are finite quotients of nodal ones.) But if we can show $\delta = 0$, then the conjecture is *immediately* disproved (for odd $n \geq 3$). Here is why: by (5.6), $H_{\lim}$ then has a class of type $(n+1, n+1)$, which must go to an (n, n) class by N_σ,



forcing $\mathrm{rk}(N_\sigma) = 1$ (rather than 0). (In different terms, each T_σ is a nontrivial symplectic transvection.) But this is impossible for $\bigwedge^n$ of a nilpotent endomorphism of $\mathbb{C}^{2n}$.

To complete the (dis)proof, then, we apply [Kerr and Laza 2023, Theorem 2.9]: for a nodal degeneration $Y \rightsquigarrow Y_\sigma$ of an odd-dimensional hypersurface of a smooth projective variety $\boldsymbol{P}$ satisfying Bott vanishing, the rank of δ is the number m of nodes minus the rank of the map

$$\mathrm{ev} : H^0\Big(\boldsymbol{P}, K_{\boldsymbol{P}}\Big(\frac{3n+1}{2} Y_\sigma\Big)\Big) \to \mathbb{C}^m$$

given by evaluation at the nodes. The proof in [loc. cit.] is equivariant in $\mathcal{A}$, and so we find that $\delta^\chi = 0 \iff$ ev is nonzero on $H^0\big(\boldsymbol{P}, K_{\boldsymbol{P}}\big(\frac{3n+1}{2} Y_\sigma\big)\big)^\chi$, which can be checked at any node. Writing

$$\boldsymbol{e}_1 := \sum_{i=0}^{n} Y_i \frac{\partial}{\partial Y_i}, \quad \boldsymbol{e}_2 := \sum_{j=0}^{2n+1} Z_j \frac{\partial}{\partial Z_j} - 2\boldsymbol{e}_1, \quad \text{and} \quad \Omega := \langle \boldsymbol{e}_2, \langle \boldsymbol{e}_1, d\underline{Z} \wedge d\underline{Y} \rangle \rangle,$$

one checks that

$$Y_0 Z_0^2 \Omega / (F_{t=0})^{(3n+1)/2} \tag{5.12}$$

is a well-defined section of $K_{\boldsymbol{P}}\big(\frac{3n+1}{2} Y_\sigma\big)$ (see [Kerr 2003, Section 4.5]); and evidently $\mathcal{A}$ acts on it through χ. Clearly, it is nonzero on the fiber of $K_{\boldsymbol{P}}\big(\frac{3n+1}{2} Y_\sigma\big)$ at any of the nodes (5.9).

Sketch of proof of Theorem 5.4. Returning to the local picture (5.5), we now seek a more concrete topological description of the orthogonal reflections (n even) and symplectic transvections (n odd) through which T_σ acts on H. So let $U_0 \subset \mathbb{A}^n$ be the complement of the hyperplanes $x_1 = 0, \ldots, x_n = 0$ and $x_1 + \cdots + x_n = 1$, and $\mathbb{L}_0$ the rank-1 local system on U_0 with monodromies -1 about each of them. While the singularity $x_\sigma \overset{\imath_\sigma}{\hookrightarrow} X_\sigma$ "at $\underline{0}$" in (5.5) isn't isolated, the vanishing-cycle complex $\phi_t\, \mathbb{Q}_{\mathcal{X}}$ is nothing but $\imath_*^\sigma V[-n]$, where $V := \mathrm{IH}^n(\mathbb{A}^n, \mathbb{L}_0)$ (as MHS). We begin with a local analogue of the covering argument just seen.

5.13. Lemma. (i) $\mathrm{IH}^n(\mathbb{A}^n, \mathbb{L}_0) \cong \mathbb{Q}(-\lfloor (n+1)/2 \rfloor)$.

(ii) *Local monodromy* T_σ *acts on* V *through multiplication by* $(-1)^{n+1}$.

(iii) *The canonical map* $\mathrm{can}_\sigma : H_{\lim} \to V$ *is onto.*

Proof. Define maps

- $f_0 : \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}$ by $\underline{x} \mapsto \big(\underline{x}, 1 - \sum_{i=1}^n x_i\big)$ and
- $\phi_0 : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$ by squaring all coordinates z_i.

Then $\hat{X}_0 := \phi_0^{-1}(f_0(\mathbb{A}^n)) \subset \mathbb{A}^{n+1}$ is the quadric hypersurface $\sum_{i=1}^{n+1} z_i^2 = 1$. The group $\mathcal{A}_0 := (\mathbb{Z}/2\mathbb{Z})^{n+1}$ acts on $\hat{X}_0$ (multiplying coordinates by ± 1), with quotient $\mathbb{A}^n$. The quotient $q_0 : \hat{X}_0 \twoheadrightarrow X_0$ by the augmentation subgroup $\mathcal{A}_0^\circ$ yields the obvious $2:1$ branched cover of $\mathbb{A}^n$, with $H^n(X_0) \cong \mathrm{IH}^n(\mathbb{A}^n, \mathbb{L}_0)$.

By the localization sequence for $\hat{X}_0$ (relative to its closure $\bar{\hat{X}}_0 \subset \mathbb{P}^{n+1}$) and weak Lefschetz, one easily shows that $H^j(\hat{X}_0) = 0$ for $j \neq n$,[18] and

$$H^n(\hat{X}_0) \cong \mathbb{Q}\Big(-\Big\lfloor \frac{n+1}{2} \Big\rfloor\Big).$$

(Writing $\partial \hat{X}_0 = \bar{\hat{X}}_0 \setminus \hat{X}_0$, this is $H^n(\bar{\hat{X}}_0)/H^{n-2}(\partial \hat{X}_0)(-1)$ for n even, and for n odd $\ker\{H^{n-1}(\partial \hat{X}_0)(-1) \to H^{n+1}(\bar{\hat{X}}_0)\}$.) A generator for the *dual* group $H_c^n(\hat{X}_0)$ is given by the real (vanishing) n-sphere $S_1^n := \big\{\sum z_i^2 = 1\big\} \cap \mathbb{R}^{n+1}$, whose class is invariant under $\mathcal{A}_0^\circ$ hence comes from $H_c^n(X_0)$. This gives (i).

The degeneration is modeled by replacing $\sum z_i^2 = 1$ by $\sum z_i^2 = t$; as the spectrum of $\sum z_i^2$ is $[(n+1)/2]$, the monodromy is as described in (ii). Finally, (iii) follows from the last subsection since can_σ identifies with can^χ in (5.11). $\square$

[18] This simply recovers perversity of $\phi_f\, \mathbb{Q}_{\mathcal{X}}[n]$.

The vanishing sphere $S^n_t := \left\{\sum z_i^2 = t\right\} \cap \mathbb{R}^{n+1}$ in $\hat{X}_0$ has image in X_0 (by q_0) given by the double cover of $\left(\bigcap_{i=1}^n \{x_i \geq 0\}\right) \cap \left\{\sum x_i \leq t\right\}$. Let its image in X (essentially via $\mathrm{can}^{\chi} : H^n_c(X_0) \to H^n(X)$) be denoted by ν_σ; this is the vanishing cycle at σ, a "double simplex" branched along $\mathcal{H}_s$ and n additional hyperplanes. It follows from (iii) that T_σ is a transvection/reflection in ν_σ. More precisely, rescaling Q to have $Q(\nu_\sigma, \nu_\sigma) = \frac{1}{2}(1 + (-1)^n)$,

$$T_\sigma(u) = u - 2Q(u, \nu_\sigma)\, \nu_\sigma \tag{5.14}$$

for $u \in H$.

Now consider the general setting where $L_{2n+1} = L_s$, $L_0 = \{X_0 = 0\}$, and the remaining L_i are in general position. An easy extension of (5.1) gives

$$H \cong \mathrm{IH}^n_c(\mathbb{A}^n, \mathbb{L}) \cong H^n_c(X \setminus L_0),$$

whence $H^n_{\mathrm{pr}}(X)$ is spanned by double simplices branched along $n+1$ of the $L_{i\geq 0}$. Obviously all of these can be rewritten as $\mathbb{Z}$-linear combinations of double simplices branched along L_s and n of the $\{L_i\}_{1\leq i\leq 2n}$; call these ν_I, where $I \subset \{1, \ldots, 2n\}$ with $|I| = n$. Since $\mathrm{rk}\, H = \binom{2n}{n}$ and there are $\binom{2n}{n}$ of these vanishing cycles, they form a $\mathbb{Q}$-basis of $H = H^n_{\mathrm{pr}}(X)$. Write T_I for the corresponding monodromies, and $\Gamma \leq \mathrm{Aut}(H_{\mathbb{C}}, Q)$ for the smallest $\mathbb{C}$-algebraic group containing them; clearly $\Gamma \leq \widetilde{\Pi}_{\mathbb{C}}$. Moreover, we note that if $|I \cap I'| = n-1$, then $Q(\nu_I, \nu_{I'}) = \pm 1$ (rescaling as above, compatibly with (5.14)).

Suppose then that $|I \cap I'| = n-1$. If n is odd, then $T_I(\nu_{I'}) = \nu_I \pm \nu_{I'} = \pm T_{I'}^{-1}(\nu_I)$, whence $\nu_{I'}$ is in the Γ-orbit of ν_I; so *all* the ν_J are in the Γ-orbit of ν_I. If n is even, then reasoning as in [Deligne 1980, Section 4.4] (see the paragraph after Lemme 4.4.3$^{\mathrm{s}}$), $T_I T_{I'}^{\pm 1}$ is a transvection and its Zariski closure a $\mathbb{G}_a$ including transformations which send $\nu_I \mapsto \nu_{I'}$ and vice versa; once again, all the ν_J are in the Γ-orbit of a single ν_I.

Let $R := \Gamma.\nu_I$ denote this orbit. Obviously it spans $H_{\mathbb{C}}$. Furthermore, for any $\delta \in R$, we have that Γ contains the transvection/reflection T_δ: writing $\delta = \gamma.\nu_I$ ($\gamma \in \Gamma$), we have $T_\delta = T_{\gamma.\nu_I} = \gamma T_I \gamma^{-1} \in \Gamma$. So Γ is in fact the $\mathbb{C}$-algebraic closure of the $\{T_\delta\}_{\delta \in R}$, and we are exactly in the situation of [Deligne 1980, Lemme 4.4.2]. Conclude that $\Gamma = \mathrm{Aut}(H_{\mathbb{C}}, Q)$, and hence $\widetilde{\Pi} = \mathrm{Aut}(H, Q)$, and thus $\Pi = \mathrm{Aut}(H, Q)^\circ$, proving Theorem 5.4.

5.15. Remark. After writing this paper we encountered the article [Xu 2018] which treats the more general setting of r-covers of $\mathbb{P}^n$ branched along hyperplanes by considering local monodromies (as we have just done). The argument is necessarily more complicated and technical than ours. However, in the case $r = 2$ (i.e., our setting) it appears to be incomplete.

If $r = 2$ and n is odd, Proposition 3.4 of [Xu 2018] does not actually establish that, in the notation of [loc. cit.], $e_{(1)}$ is nonzero; this is exactly the issue regarding possible nonvanishing of δ dealt with above. One could read [Xu 2018, Proposition 4.2] as confirming this in retrospect, but this makes the argument quite convoluted.

If $r = 2$ and n is even, the proof of [Xu 2018, Proposition 4.2] is wrong, as it makes use of the (false) statement that $\mathrm{Sp}_{2n}(\mathbb{R})$ "does not admit any nontrivial one-dimensional invariant subspace" in its action on $\bigwedge^n \mathbb{R}^{2n}$.

Acknowledgements

We thank P. Gallardo and R. Laza for valuable discussions related to this paper, and the referee for helpful expository suggestions. This work was partially supported by Simons Collaboration grant 634268 and NSF grant DMS-2101482 (Kerr). Pearlstein's work was partially supported by the following grants: MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001 and Progetti di Rilevante Interesse Nazionale (PRIN), Geometry of Algebraic Structures: Moduli, Invariants, Deformations. Kerr and Pearlstein would also like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the program "K-theory, algebraic cycles, and motivic homotopy theory" (supported by EPSRC grant number EP/R014604/1) when work on this paper was undertaken.

References

[Allcock et al. 2002] D. Allcock, J. A. Carlson, and D. Toledo, "The complex hyperbolic geometry of the moduli space of cubic surfaces", *J. Algebraic Geom.* **11**:4 (2002), 659–724. MR Zbl

[Allcock et al. 2011] D. Allcock, J. A. Carlson, and D. Toledo, *The moduli space of cubic threefolds as a ball quotient*, Mem. Amer. Math. Soc. **209**, 2011. MR Zbl

[Arnol'd 1981] V. Arnol'd, "On some problems in singularity theory", *Proc. Indian Acad. Sci. Math. Sci.* **90**:1 (1981), 1–9. MR Zbl

[Beauville 1980] A. Beauville, "Sur le nombre maximum de points doubles d'une surface dans $\boldsymbol{P}^3$ ($\mu(5) = 31$)", pp. 207–215 in *Journées de Géometrie Algébrique d'Angers* (Angers, 1979), Sijthoff and Noordhoff, 1980. MR Zbl

[Cartan 1954] H. Cartan, "Quotient d'une variété analytique par un groupe discret d'automorphismes", *Séminaire Henri Cartan* **6**:12 (1954), 1–13. Zbl

[Casalaina-Martin et al. 2012] S. Casalaina-Martin, D. Jensen, and R. Laza, "The geometry of the ball quotient model of the moduli space of genus four curves", pp. 107–136 in *Compact moduli spaces and vector bundles*, Contemp. Math. **564**, Amer. Math. Soc., Providence, RI, 2012. MR Zbl

[Deligne 1971] P. Deligne, "Théorie de Hodge, II", *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57. MR Zbl

[Deligne 1980] P. Deligne, "La conjecture de Weil, II", *Inst. Hautes Études Sci. Publ. Math.* **52** (1980), 137–252. MR Zbl

[Deligne and Mostow 1986] P. Deligne and G. D. Mostow, "Monodromy of hypergeometric functions and nonlattice integral monodromy", *Inst. Hautes Études Sci. Publ. Math.* **63** (1986), 5–89. MR Zbl

[Deng and Gallardo 2023] H. Deng and P. Gallardo, "Eigenperiods and the moduli of points in the line", preprint, 2023. Zbl arXiv 2311.09352

[Dolgachev 1982] I. Dolgachev, "Weighted projective varieties", pp. 34–71 in *Group actions and vector fields* (Vancouver, B.C., 1981), Lecture Notes in Math. **956**, Springer, Berlin, 1982. MR Zbl

[Dolgachev and Kondō 2007] I. V. Dolgachev and S. Kondō, "Moduli of $K3$ surfaces and complex ball quotients", pp. 43–100 in *Arithmetic and geometry around hypergeometric functions*, Progr. Math. **260**, Birkhäuser, Basel, 2007. MR Zbl

[Endrass 1997] S. Endrass, "A projective surface of degree eight with 168 nodes", *J. Algebraic Geom.* **6**:2 (1997), 325–334. MR Zbl

[Gallardo et al. 2021] P. Gallardo, M. Kerr, and L. Schaffler, "Geometric interpretation of toroidal compactifications of moduli of points in the line and cubic surfaces", *Adv. Math.* **381** (2021), art. id. 107632. MR Zbl

[Gerkmann et al. 2007a] R. Gerkmann, S. Mao, and K. Zuo, "Disproof of modularity of moduli space of CY 3-folds of double covers of P3 ramified along eight planes in general positions", preprint, 2007. Zbl arXiv 0709.1051

[Gerkmann et al. 2007b] R. Gerkmann, M. Sheng, and K. Zuo, "Computational details on the disproof of modularity", preprint, 2007. arXiv 0709.1054

[Gerkmann et al. 2013] R. Gerkmann, M. Sheng, D. van Straten, and K. Zuo, "On the monodromy of the moduli space of Calabi–Yau threefolds coming from eight planes in $\mathbb{P}^3$", *Math. Ann.* **355**:1 (2013), 187–214. MR Zbl

[Hassett 2003] B. Hassett, "Moduli spaces of weighted pointed stable curves", *Adv. Math.* **173**:2 (2003), 316–352. MR Zbl

[Hotta et al. 2008] R. Hotta, K. Takeuchi, and T. Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics **236**, Birkhäuser, Boston, MA, 2008. MR Zbl

[Jaffe and Ruberman 1997] D. B. Jaffe and D. Ruberman, "A sextic surface cannot have 66 nodes", *J. Algebraic Geom.* **6**:1 (1997), 151–168. MR Zbl

[Kerr 2003] M. D. Kerr, *Geometric construction of regulator currents with applications to algebraic cycles*, Ph.D. thesis, Princeton University, 2003, available at https://www.proquest.com/docview/305317544. MR Zbl

[Kerr and Laza 2023] M. Kerr and R. Laza, "Hodge theory of degenerations, (II): vanishing cohomology and geometric applications", preprint, 2023. Zbl arXiv 2006.03953v2

[Kerr et al. 2021] M. Kerr, R. Laza, and M. Saito, "Hodge theory of degenerations, (I): consequences of the decomposition theorem", *Selecta Math.* (*N.S.*) **27**:4 (2021), art. id. 71. MR Zbl

[Miyaoka 1984] Y. Miyaoka, "The maximal number of quotient singularities on surfaces with given numerical invariants", *Math. Ann.* **268**:2 (1984), 159–171. MR Zbl

[Moonen 2018] B. Moonen, "The Deligne–Mostow list and special families of surfaces", *Int. Math. Res. Not.* **2018**:18 (2018), 5823–5855. MR Zbl

[Mostow 1988] G. D. Mostow, "On discontinuous action of monodromy groups on the complex n-ball", *J. Amer. Math. Soc.* **1**:3 (1988), 555–586. MR Zbl

[Sakamaki 2010] Y. Sakamaki, "Automorphism groups on normal singular cubic surfaces with no parameters", *Trans. Amer. Math. Soc.* **362**:5 (2010), 2641–2666. MR Zbl

[Schoen 1985] C. Schoen, "Algebraic cycles on certain desingularized nodal hypersurfaces", *Math. Ann.* **270**:1 (1985), 17–27. MR Zbl

[Sheng et al. 2015] M. Sheng, J. Xu, and K. Zuo, "The monodromy groups of Dolgachev's CY moduli spaces are Zariski dense", *Adv. Math.* **272** (2015), 699–742. MR Zbl

[Steenbrink 1977] J. Steenbrink, "Intersection form for quasi-homogeneous singularities", *Compositio Math.* **34**:2 (1977), 211–223. MR Zbl [van Straten 1993] D. van Straten, "A quintic hypersurface in $\boldsymbol{P}^4$ with 130 nodes", *Topology* **32**:4 (1993), 857–864. MR Zbl

[van Straten 2020] D. van Straten, "The Spectrum of Hypersurface Singularities", preprint, 2020. Zbl arXiv 2003.00519

[Varchenko 1983] A. N. Varchenko, "Semicontinuity of the spectrum and an upper bound for the number of singular points of the projective hypersurface", *Dokl. Akad. Nauk SSSR* **270**:6 (1983), 1294–1297. MR Zbl

[Voisin 2003] C. Voisin, *Hodge theory and complex algebraic geometry, II*, Cambridge Studies in Advanced Mathematics **77**, Cambridge University Press, Cambridge, 2003. MR Zbl

[Xu 2018] J. Xu, "Zariski density of monodromy groups via a Picard–Lefschetz type formula", *Int. Math. Res. Not.* **2018**:11 (2018), 3556–3586. MR Zbl

Received November 8, 2022. Revised December 21, 2023.

BEN CASTOR
DEPARTMENT OF MATHEMATICS
KENYON COLLEGE
GAMBIER, OH
UNITED STATES
castor1@kenyon.edu

HAOHUA DENG
MATHEMATICS DEPARTMENT
DUKE UNIVERSITY
DURHAM, NC
UNITED STATES
haohua.deng@duke.edu

MATT KERR
DEPARTMENT OF MATHEMATICS AND STATISTICS
WASHINGTON UNIVERSITY IN ST. LOUIS
ST. LOUIS, MO
UNITED STATES
matkerr@wustl.edu

GREGORY PEARLSTEIN
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PISA
PISA
ITALY
greg.pearlstein@unipi.it

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2023 is US $605/year for the electronic version, and $820/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 327 No. 1 November 2023