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Many-valued logics
and bivalent modalities

Abstract. In this paper, we investigate the family $\mathfrak{L}^{\mathsf{S0.5}}$ of many-valued modal logics $\mathsf{L}^{\mathsf{S0.5}}$'s. We prove that the modalities $\Box$ and $\Diamond$ of the logics $\mathsf{L}^{\mathsf{S0.5}}$'s capture well-defined bivalent concepts of logical validity and logical consistency. We also show that these modalities can be used as recovery operators.

Keywords: modal logic; many-valued logic; non-normal modal logic; bivalence

Introduction

The last few years have witnessed a growth of interest in many-valued logics (MVLs). Examples can be found in their application to the analysis of semantic paradoxes ((Priest, 1979), (Da Ré et al, 2018)) or in the study of rationality ((Belnap, 1977), (Kubyshkina, 2016), (Bezerra, 2020)). These fruitful exercises have indirectly responded to the criticism that MVLs have received in the literature, due to the conceptual difficulties in characterizing the meaning of their intermediate logical values (Pogorzelski, 1994).

Despite their philosophical significance, MVLs have been challenged on their own ground, due to their *metatheoretical bivalence*. As Suszko (1977) observed, the concepts of tautology and logical consequence only take into account whether a value t is *designated* (truth-like) or *non-designated* (false-like). In other words, due to the bi-partition of the set of truth values, the notions of tautology and logical consequence acquire a classical character, contrary to the multiplicity of truth values they display.

In this paper, we develop Suszko's observation by capturing this bivalent character of the meta-theory in modal terms. Concretely, we will extend MVLs with modalities that we shall call *Suszkian modalities*, which are able to formally capture the notions of tautology and logical consistency. These modalities, that we will indicate with the familiar

symbols $\square$ and $\Diamond$, are intended to interpret the concepts of "it is logically valid that", respectively, "it is logically consistent that". Since the formulas $\square\varphi$ and $\Diamond\varphi$ only receive classical values they will therefore capture the bivalent character of these meta-theoretical notions. Modalities intended to capture these notions have already appeared in the literature (Lemmon, 1957). They were introduced by Lemmon, together with the modal logic S0.5 to offer a meta-theoretical analysis of validity and consistency of *classical* logic. In this work we will extend Lemmon's ideas to a general framework, considering MVLs, and we will use these modalities to capture the classical aspects of the meta-theory of non-classical logics. By analysing these modalities in a broad family of logics, we will thus account for the most general properties of these meta-theoretical notions in modal terms.

The internalization of meta-theoretical concepts within the object language by means of modal tools is a fairly standard procedure. Its most successful example is the modal formalization of the notion of provability as developed in *provability logics* (Boolos, 1993). However, in this paper we will analyze a more semantic/model-theoretical concept of validity.

The paper is structured as follows. In §1 we introduce the basic notation which will be used in this paper. In §2 we introduce the family $\mathfrak{L}^{\mathsf{S0.5}}$ of many-valued modal logics, consisting of modal counterparts of n-valued modal logics. We prove a theorem of adequacy between validity and consistency in these logics and their modal characterizations. In §3 we present the logic $Ł_3^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$, the modal extension of the three-valued Łukasiewicz logic $Ł_3$, and discuss the possibility of using Suszkian modalities as recovery operators. After presenting, in §4, possible lines of future research, we end the paper with Appendix A, where we present a few technical results about $Ł_3^{\mathsf{S0.5}}$.

1. Matrix semantics

An n-valued logic L, for $n \in \mathbb{N}$, is a logic expressed in a language $\mathcal{L}_\mathsf{L} = \{\mathcal{V}, c_1^{k_1}, \ldots, c_m^{k_m}\}$, where $\mathcal{V} = \{p_i \mid i \in \mathbb{N}\}$ is a set of propositional variables[1] and $c_1^{k_1}, \ldots, c_m^{k_m}$ are connectives of such that the arity $c_i^{k_i}$ is k_i. The set of formulas of $\mathcal{L}_\mathsf{L}$, $For(\mathcal{L}_\mathsf{L})$ is defined inductively as usual:

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[1] For the sake of simplicity, we use the variables $p,q,r,\ldots$ instead of $p_0,p_1,p_2,p_3,$ $\ldots$

(i) $p_i \in For(\mathcal{L}_\mathsf{L})$; and (ii) if $\varphi_1, \ldots, \varphi_k \in For(\mathcal{L}_\mathsf{L})$, then $c_i^{k_i}(\varphi_1, \ldots, \varphi_k)$ for $1 \leq i \leq m$.

DEFINITION 1.1. A *matrix* for L is a structure $M_\mathsf{L} = \langle V_n, o_1^{k_1}, \ldots, o_m^{k_m}, D_\mathsf{L} \rangle$ where $V_n = \{\frac{m}{n-1}\mid 0 \leq m \leq n-1, m, n \in \mathbb{N}\}$ is the set of truth-values, $o_1^{k_1}, \ldots, o_n^{k_n}$ are operations on V_n such that the arity of $o_i^{k_i}$ is k_i, and $D_\mathsf{L} \subset V_n$ is the set of *designated values* $\{\frac{r}{n-1}, \ldots, 1\}$, for $r > 0$. We will assume the values 1 and 0 to denote the classical values of *truth* and *falsity*. A *valuation for* L v is a homomorphism $v : \mathcal{V} \to V_n$ which is extended to $For(\mathcal{L}_\mathsf{L})$ as usual.

$$v(c_m^{k_m}(\varphi_1, \ldots, \varphi_k)) = o_m^{k_m}(v(\varphi_1), \ldots, v(\varphi_k)).$$

The set of valuations $v : For(\mathcal{L}_\mathsf{L}) \to V_n$ is called the *semantics* of L, sem_L.

DEFINITION 1.2. For $\varphi \in For(\mathcal{L}_\mathsf{L})$, we say that v_L is a *model* for φ if $v_\mathsf{L}(\varphi) \in D_\mathsf{L}$. For $\Gamma \subseteq For(\mathcal{L}_\mathsf{L})$, we say that v_L is a *model* of Γ if v_L is a model of each $\gamma \in \Gamma$. If $v_\mathsf{L}(\varphi) \in D_\mathsf{L}$, for some (resp., for every) $v_\mathsf{L} \in sem_\mathsf{L}$, we say that φ is *satisfiable* (resp., a *tautology* of) in L. If $v_\mathsf{L}(\varphi) \notin D_\mathsf{L}$ for every $v_\mathsf{L} \in sem_\mathsf{L}$, then φ is a *contradiction* of L. The *semantic consequence relation* $\models_\mathsf{L}$ is the relation on $\wp(For(\mathcal{L}_\mathsf{L})) \times For(\mathcal{L}_\mathsf{L})$ given by: $\Gamma \models_\mathsf{L} \alpha$ iff whenever $v_\mathsf{L}(\gamma) \in D_\mathsf{L}$, for every $\gamma \in \Gamma$, then $v_\mathsf{L}(\alpha) \in D_\mathsf{L}$.

NOTATION 1.3. *(Rescher, 1969) A n-valued connective $c_m^{k_m}$ is called normal with respect to a matrix M_L if its corresponding interpretation $o_m^{k_m}$ in M_L agrees with a classical connective $\star$ (for $\star \in \{\wedge, \vee, \to, \leftrightarrow, \neg\}$) when restricted to the truth-values 1 and 0. A logic L is normal if all its connectives are normal.*

The n-valued logics that we investigate here are normal. This assumption guarantees that when we consider only classical values, we obtain classical propositional logic (CPL).

2. n-valued modal logics and bivalent modalities

The modal logics we consider contain two modal operators $\{\Box, \Diamond\}$, where $\Box$ and $\Diamond$ are unary operators.

DEFINITION 2.1. Given a language $\mathcal{L}_\mathsf{L}$, we define its modal extension by $\mathcal{L}_\mathsf{L}^{\Box \Diamond} = \mathcal{L}_\mathsf{L} \cup \{\Box, \Diamond\}$.

DEFINITION 2.2. Fix an n-valued normal logic L, with corresponding language $\mathcal{L}_\mathsf{L}$ and matrix $M_\mathsf{L} = \langle V_n, o_1^{k_1}, \ldots, o_m^{k_m}, D_\mathsf{L}\rangle$. An M_L-modal model is a structure of the form $\mathcal{M}_\mathsf{L} = \langle W, N, R, v\rangle$ where W is a set of worlds, $N \subseteq W$ is a proper subset of W of normal worlds, R is a relation such that (a) wRw for all $w \in N$ and (b) for every $y \in W$ there is some $w \in N$ such that wRy, and v is an assignment such that for every $w \in W$, $v_w(p) \in V_n$. The function v is recursively extended in the standard way for the connectives that are not modalities:

1 $v_w(c_m^{k_m}(\varphi_1, \ldots, \varphi_k)) = o_m^{k_m}(v_w(\varphi_1), \ldots, v_w(\varphi_k))$.

The interpretation of the modal operators runs as follows:

2.1 For any $w \in W$, $v_w(\Box\varphi) = 1$ if $w \in N$ and for all $y \in W$ such that wRy, $v_y(\varphi) \in D_\mathsf{L}$; otherwise $v_w(\Box\varphi) = 0$;
2.2 If $w \notin N$, the value of $v_w(\Box\varphi)$ is arbitrary in V_n;
3.1 For any $w \in W$, $v_w(\Diamond\varphi) = 1$ if $w \in N$ and for some $y \in W$ such that wRy, $v_y(\varphi) \in D_\mathsf{L}$; otherwise $v_w(\Diamond\varphi) = 0$;
3.2 If $w \notin N$, the value $v_w(\Diamond\varphi)$ is arbitrary in V_n.

A formula $\varphi \in For(\mathcal{L}_\mathsf{L}^{\Box\Diamond})$ is true in a M_L-modal model $\mathcal{M}$ iff for every $w \in N$ $v_w(\varphi) \in D_\mathsf{L}$. A formula $\varphi \in For(\mathcal{L}_\mathsf{L}^{\Box\Diamond})$ is M_L-valid iff it is true in every M_L-modal model.

DEFINITION 2.3. The *Suszkian modal counterpart of* L, which we indicate by $\mathsf{L}^{\mathsf{S0.5}}$, is the set of all M_L-valid formulas in the language $\mathcal{L}_\mathsf{L}^{\Box\Diamond}$.

DEFINITION 2.4. We denote the family of n-valued logics $\mathsf{L}^{\mathsf{S0.5}}$ as $\mathfrak{L}^{\mathsf{S0.5}}$.

When M_L is the two-valued classical matrix, $\mathsf{L}^{\mathsf{S0.5}}$ corresponds to $\mathsf{S0.5}$, introduced by Lemmon (1957), where the modal operator $\Box$ was interpreted as "it is tautologous (by truth-tables) that".

DEFINITION 2.5. (Lemmon, 1957) $\mathsf{S0.5}$ is a non-normal modal logic which has the following axioms and rules:
(CPL) all propositional tautologies and inference rules;
(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
(T) $\Box\varphi \to \varphi$;
(D) $\Box\varphi \to \Diamond\varphi$;
(Df^1) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$;

(Df^2) $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$;[2]
(Nec) If φ is a classical tautology, we infer $\vdash \Box\varphi$.

Let us notice that Definition 2.2 only comprehends finitely valued logics, *i.e.*, logics which are characterized by matrices where V_n is finite. So, many paraconsistent logics (Carnielli et al, 2005) as well as intuitionistic logic (Gödel, 1933), for example, are beyond the scope of the present work, because they cannot be characterized by finite matrices. Such logics require a richer structure than the structure of Definition 2.2 in order to accommodate their non-truth-functional connectives.[3]

In (Cresswell, 1966), Cresswell presents a simplified semantic structure for S0.5 of the form $\langle w^*, W, v\rangle$, where w^* is the unique normal world and the worlds $y \in W$ are the non-normal.[4] Cresswell then proves that these models are sound and complete with respect to the axioms and rules of the Definition 2.5. Note that our construction of the logics in $\mathfrak{L}^{\mathsf{S0.5}}$ differs from Priest's extension (Priest, 2008) of n-valued logics to modal logic. In his paper, Priest considers modal logics with an n-valued logic as the underlying non-modal logic. Thus, formulas $\Box\varphi$ and $\Diamond\varphi$ are allowed to receive intermediate truth-values.[5] On the other hand, Definition 2.2 imposes that these formulas can only be true or false in the

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[2] In normal modal logics $\Box$ is usually introduced as primitive and $\Diamond$ can be defined from $\Box$. Although S0.5 does not obey the principle of substitutivity of equivalents (Milberger, 1978), we can introduce the operator $\Diamond$ by means of the usual definition, still validating **Df^1** and **Df^2**. But, since we are considering logics which are weaker than the classical S0.5, **Df^1** and **Df^2** are not valid in some of them. For this reason, we introduce $\Diamond$ as a primitive in the language $\mathcal{L}_{\mathsf{L}}^{\Box\Diamond}$ as well as the axioms bridging $\Box$ and $\Diamond$ in S0.5.

[3] For non-modal logics based on intuitionistic logic, we refer the reader to (Dalmonte et al, 2020). In their work, Dalmonte *et al* use a semantic structure which contains neighborhood functions to deal with the modalities $\Box$ and $\Diamond$, and an ordering relation $\preceq$ to deal with implication. We conjecture that we should include such an accessibility relation to accommodate intuitionistic implication in our framework. We leave this possibility for further investigation.

[4] Pietruszczak (2009) provides a series of characterization results for non-normal modal logics by means of this simplified semantics. Among these systems, he proves soundness and completeness for two fragments of S0.5: S0.5^0 and S0.5^{0+}. S0.5^0 is obtained by S0.5 by dropping off the axiom **T** and **D**; and S0.5^{0+} is obtained by S0.5 dropping off the axiom **T**.

[5] In (Schotch et al, 1978), Schotch *et al* introduce a study of non-classically based modal logic, where they consider the three-valued logic Ł$_3$. In this work, they provide an axiomatization in the class of all (standard) models $\langle W, R, v\rangle$ for Ł$_3 M_2$, which is obtained by extending Ł$_3$ to the modal language where formulas $\Box\varphi$ only

worlds $w \in N$, the *normal* worlds, where thus the modal formulas only receive classical values; in this sense these are Suszkian modalities.

The intended meanings of $\Box\varphi$ and $\Diamond\varphi$ are "φ is a tautology in L" and "φ is logically consistent for L". Notice that the formulas $\Box\varphi$ and $\Diamond\varphi$ are not formulas of L and therefore cannot be tautologies of, or consistent with, L. Therefore, the formulas in $\mathsf{L}^{\mathsf{S0.5}}$ do not display iterated modalities. This property of the interpretation is reflected in Definition 2.2 by the distinction between normal and non-normal worlds, which has the effect of invalidating all iterated modalities.

Under this meta-theoretical interpretation of the modalities, the axiom **K** says that being a tautology is preserved under *modus ponens*. The axiom **T** says that the $\Box$ operator captures tautologies. The axiom **D**, then, says that if φ is a tautology, then φ is consistent. That is, there is at least one line of the truth-table where φ receives a designated value. Therefore, we can see that $\mathsf{S0.5}$ is sound with respect to its intended interpretation. Although $\mathsf{S0.5}$ captures tautological validity and consistency, however, it does not display all meta-theoretical properties of these classical notions. For example, $\mathsf{S0.5}$ is not able to capture invalidity, contrary to the decidability of CPL, since not all formulas of the form $\neg\Box p$ are theorems of $\mathsf{S0.5}$. It has been argued, in (Routley, 1968), that this constitutes a semantic form of incompleteness of $\mathsf{S0.5}$ with respect to this intended interpretation.[6] A possible response to this objection is that we are asking from $\mathsf{S0.5}$ too much, since we cannot expect it to mirror all meta-theoretical properties at once, especially those global properties that hardly fit with a local modal formalization.

Let us show, then, in which sense $\mathsf{L}^{\mathsf{S0.5}}$ captures the intended interpretation of the modalities.

2.1. Logics $\mathsf{L}^{\mathsf{S0.5}}$'s and tautological validity

Our strategy is to define a *theory of validity* for L, in the sense of Skyrms (1978). That is, we extend the language $\mathcal{L}_{\mathsf{L}}$ with a predicate Val, for validity, a predicate Con, for consistency, and a *sentence name* $\overline{\varphi}$, for each $\varphi \in \mathcal{L}_{\mathsf{L}}$. We name the resulting language $\mathcal{L}_{\mathsf{L}}^{VC}$. Then, the set of

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receive classical values. They suggest that this logic captures the idea that the modal discourse is essentially two-valued.

[6] In (Urquhart, 2010), one can find a logical system, called TS which contains validities of the form $\neg\Box\Box p_i$ as well as $\neg\Box p_i$. However, contrary to $\mathsf{S0.5}$, TS is not finitely axiomatizable.

formulas $For(\mathcal{L}_\mathsf{L}^{VC})$ is defined as follows: (i) $For(\mathcal{L}_\mathsf{L}) \subseteq For(\mathcal{L}_\mathsf{L}^{VC})$; (ii) if $\varphi \in For(\mathcal{L}_\mathsf{L})$ and $\overline{\varphi}$ is a sentence name of φ, then $Val(\overline{\varphi}), Con(\overline{\varphi}) \in For(\mathcal{L}_\mathsf{L}^{VC})$.

DEFINITION 2.6. A model for $\mathcal{L}_\mathsf{L}^{VC}$ is a structure $M_\mathsf{L}^{VC} = \langle v_0^+, M_\mathsf{L}, V \rangle$, where M_L is a matrix for L, V is a set of valuations $v_i \in sem_\mathsf{L}$ (for $i \in |sem_\mathsf{L}|$) and $v_0^+ : For(\mathcal{L}_\mathsf{L}^{VC}) \to \{1, 0\}$ is such that:

1. $v_0^+(Val(\overline{\varphi})) = v_0^+(\varphi) = 1$ if for all $v_i \in V$, $v_i(\varphi) \in D_\mathsf{L}$;
   $\qquad\qquad$ otherwise, $v_0^+(Val(\overline{\varphi})) = v_0^+(\varphi) = 0$;
2. $v_0^+(Con(\overline{\varphi})) = 1$ if for some $v_i \in V$, $v_i(\varphi) \in D_\mathsf{L}$;
   $\qquad\qquad$ otherwise, $v_0^+(Con(\overline{\varphi})) = 0$;

A formula $\varphi \in For(\mathcal{L}_\mathsf{L}^{VC})$ is true in M_L^{VC} iff either $v_0^+(\varphi) = 1$, or $v_i(\varphi) \in D_\mathsf{L}$, for all $v_i \in V$. A formula $\varphi \in For(\mathcal{L}_\mathsf{L}^{VC})$ is valid if it is true in every model M_L^{VC}.

It is important to remark that the sentence names $\overline{\varphi}$ introduced above are not Gödel's codes $\ulcorner \varphi \urcorner$. While the latter are defined within an arithmetical theory, the former are introduced as primitive objects in $\mathcal{L}_\mathsf{L}^{VC}$. The reason to introduce sentence names instead of Gödel names is due to the expressive limitation of the logical theory considered here. By Definition 2.6, $Val(\overline{\varphi})$ is true whenever φ is a tautology of L. In stronger theories, $Val(\overline{\varphi})$ is true when φ is a valid formula. In these stronger theories, a codification of the formulas *a la* Gödel would result in a modal formalization of meta-theoretical notions that are incompatible with the **T** axiom, as showed by Montague's Theorem (Montague, 1963).

Now consider the following translation.

DEFINITION 2.7. Let $t : \mathcal{L}_\mathsf{L}^{\square\Diamond} \to \mathcal{L}_\mathsf{L}^{VC}$ be a function defined as follows:

$$
\begin{array}{lcl}
t(p) & = & p \\
t(c_m^k(\varphi_1, \ldots, \varphi_k)) & = & c_m^k(t(\varphi_1), \ldots, t(\varphi_k)) \\
t(\square\varphi) & = & Val(\overline{t(\varphi)}) \\
t(\Diamond\varphi) & = & Con(\overline{t(\varphi)})
\end{array}
$$

We call this function the *t*-translation.

Notice that the *t*-translation is defined only for modal formulas without iterations of modalities. However, since the logics we consider do not allow such formulas among their validates, this is a harmless restriction.

Proposition 2.8. *Let t be the translation defined in Definition 2.7. For every $\varphi, \psi \in For(\mathcal{L}_{\mathsf{L}}^{\square\lozenge})$:*

$$\text{If } t(\varphi) = t(\psi), \text{ then } \varphi = \psi. \tag{1}$$

Proof. The proof runs by induction on the complexity of formulas.

For the atomic case, suppose that $\varphi = p$ and $\psi = q$, where $p \neq q$. By definition of t, we have that $t(p) \neq t(q)$.

For the case where $\varphi = c_i^{k_i}(\gamma_1, \ldots, \gamma_k)$ and $\psi = c_i^{k_i}(\gamma_1', \ldots, \gamma_k')$, suppose that $t(c_i^{k_i}(\gamma_1, \ldots, \gamma_k)) = t(c_i^{k_i}(\gamma_1', \ldots, \gamma_k'))$. By the definition of t, we obtain $c_i^{k_i}(t(\gamma_1), \ldots, t(\gamma_1)) = c_i^{k_i}(t(\gamma_1'), \ldots, t(\gamma_1'))$. By I.H., we obtain $\gamma_i = \gamma_i'$, for $1 \leq i \leq k$. Therefore, $c_i^{k_i}(\gamma_1, \ldots, \gamma_k) = c_i^{k_i}(\gamma_1', \ldots, \gamma_k')$. The case where $\varphi = c_i^{k_i}(\gamma_1, \ldots, \gamma_k)$ and $\psi = c_j^{k_i}(\gamma_1', \ldots, \gamma_k')$, for $i \neq j$, is straightforward.

For the case where $\varphi = \square\gamma$ and $\psi = \square\gamma'$, suppose that $t(\square\gamma) = t(\square\gamma')$. Since t is defined only for formulas without iteration of modalities, γ and γ' are also formulas of $\mathcal{L}_{\mathsf{L}}$. Moreover, for every formula α of $\mathcal{L}_{\mathsf{L}}$, each $\overline{\alpha}$ is a sentence name of α. By I.H., $\gamma = \gamma'$. Therefore, $Val(\overline{t(\gamma)}) = Val(\overline{t(\gamma)}')$.

$\square$

Then, t is an injective function whose inverse t^{-1} is also injective over its co-domain.

Lemma 2.9. *For every model $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V \rangle$ for $\mathcal{L}_{\mathsf{L}}^{VC}$ there is $\mathcal{M} = \langle W, N, R, v \rangle$ for $\mathsf{L}^{\mathsf{S0.5}}$ such that, for every $v \in V \cup \{v_0^+\}$ there is a $x \in W$, so that the following holds:*

$$v_x(\varphi) = v(t(\varphi))$$

For all $\varphi \in For(\mathcal{L}_{\mathsf{L}}^{\square\lozenge})$.

Proof. Given a model $M_{\mathsf{L}}^{VC} = \langle v_0^+, M_{\mathsf{L}}, V \rangle$ we define $\mathcal{M} = \langle W, N, R, v \rangle$ as follows:

- W is the collection of words w_i such that $v_{w_i}(p) = v_i(p)$, for $v_i \in V$, together with another world w_0^+ such that $v_{w_0^+}(\varphi) = 1$ iff for every $v_i \in V$, $v_{w_i}(\varphi) \in D_{\mathsf{L}}$,
- $N = \{w_0^+\}$,
- $R = \langle (w_0^+, w_i) | w_i \in W \rangle \cup \langle (w_0^+, w_0^+) \rangle$.

The proof that $v_{w_i}(\varphi) = v_i(t(\varphi))$ is a straightforward consequence of the Recursion Theorem: there is only one evaluation of the formulas that extends a fixed evaluation of the propositional variables.

For what concerns $v_{w_0^+}(\varphi) = v_0^+(t(\varphi))$ we only deal with the modal cases. We have that $v_{w_0^+}(\Box\varphi) = 1$ iff for every $v_i \in V$, $v_{w_i}(\varphi) \in D_\mathsf{L}$ iff (since φ has no modalities and by the previous case $v_{w_i}(\varphi) = v_i(t(\varphi))$) for every $v_i \in V$, $v_i(t(\varphi)) \in D_\mathsf{L}$ iff $v_0^+(Val(\overline{t(\varphi)})) = 1$; otherwise we get 0. The case of $\Diamond\psi$ equally depends on the definitions and the inductive hypothesis.

$\square$

LEMMA 2.10. *For every* $\mathcal{M} = \langle W, N, R, v \rangle$ *for* $\mathsf{L}^{S0.5}$ *there is* $M_\mathsf{L}^{VC} = \langle v_0^+, M_\mathsf{L}, V \rangle$ *for* $\mathcal{L}_\mathsf{L}^{VC}$ *such that, for every* $w \in W$ *there is a* $v \in V \cup \{v_0^+\}$ *the following holds:*

$$v(\varphi) = v_w(t^{-1}(\varphi))$$

For all $\varphi \in For(\mathcal{L}_\mathsf{L}^{VC})$.

PROOF. Let $\mathcal{M} = \langle W, N, R, v \rangle$ be a M_L-modal model for $\mathsf{L}^{S0.5}$. Without loss of generality we can assume that $N \neq \varnothing$; otherwise there are no modal formulas that are valid in $\mathcal{M}$ and, thus, the proof is trivial. Given a $w_0^* \in N$, we know that $w_0^* R w_i$, for all $w_i \in W$, by Definition 2.2. Notice that the normal worlds display the same set of modal validities, since they are all connected with all non-normal worlds. Then fix, w_0^* a world in N. We now define a model $M_\mathsf{L}^{VC} = \langle v_0^+, M_\mathsf{L}, V \rangle$ as follows:

- V is the collection of all valuations $v_x \in sem_\mathsf{L}$, for $x \in W \setminus N$,
- v_0^+ is a valuation of the whole language $\mathcal{L}_\mathsf{L}^{VC}$ such that $v_0^+(p) = v_{w_0^*}(p)$

It is straightforward to see that for all $v \in V$ there is an $x \in W$ such that $v(\varphi) = v_w(t^{-1}(\varphi))$. Thus, consider the case $v = v_0^+$ and when $\varphi = Val(\overline{\psi})$. As before we only need to deal with the modal case. Then, $v_0^+(Val(\overline{\psi})) = 1$ iff $v_x(\psi) \in D_\mathsf{L}$ for every $v_x \in V$ iff (for the previous case) $v_x(\psi)) \in D_\mathsf{L}$ for every $x \in W$ iff $v_{w_0^*}(\Box\psi) = 1$, which, by definition of the t-translation, is equivalent to say $v_{w_0^*}(t^{-1}(Val(\overline{\psi})) = 1$. $\square$

2.2. Some principles of logics $\mathsf{L}^{S0.5}$'s

Because $\mathfrak{L}^{S0.5}$ includes a wide class of many-valued logics, the majority of the characteristic modal principles are not valid for this family of logics. This happens because we have to take into consideration the

idiosyncrasies of each system $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$. The next theorem illustrates this point.

THEOREM 2.11. *Let* $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$. *Then:*

1. **K** *is not valid in* $\mathsf{LP}^{\mathsf{S0.5}}$;
2. **Nec** *is not valid in* $\mathsf{K_3}^{\mathsf{S0.5}}$;
3. **T** *is not generally valid in* $\mathsf{RM_3}^{\mathsf{S0.5}}$;
4. $\square(\varphi \wedge \psi) \to (\square\varphi \wedge \square\psi)$ *and* $\Diamond(\varphi \wedge \psi) \to (\Diamond\varphi \wedge \Diamond\psi)$ *are not valid in* $\mathsf{L}^{\mathsf{S0.5}}$ *which have infectious designated values.*[7].
5. $(\square\varphi \wedge \square\psi) \to \square(\varphi \wedge \psi)$ *is not valid in logics whose connective of conjunction is such that* $v_w(\varphi \wedge \psi) = 0$ *whenever* $v_w(\varphi) \neq 1$ *or* $v_w(\psi) \neq 1$.
6. *The substitutivity of equivalents is not generally valid in logics* $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$.
7. $\Diamond\varphi \leftrightarrow \neg\square\neg\varphi$ *is not generally valid in logics* $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$.

PROOF. *1.* Let $M_{\mathsf{LP}} = \langle \{1, \frac{1}{2}, 0\}, \neg, \wedge, \{1, \frac{1}{2}\} \rangle$ be the matrix for LP (Priest, 1979), where $v_w(\neg\varphi) = 1 - v_w(\varphi)$; and $v_w(\varphi \wedge \psi) = min(v_w(\varphi), v_w(\psi))$, where $\varphi \to \psi := \neg(\varphi \wedge \neg\psi)$. Let $\mathcal{M} = \langle W, N, R, v \rangle$ be a M_{LP}-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$ and $v_w(q) = v_y(q) = 0$. Then, $v_w(p \to q) = v_y(p \to q) = \frac{1}{2}$. Since p and $p \to q$ take a designated value in every world of W, then $v_w(\square p) = v_w(\square(p \to q)) = 1$. On the other hand, $v_w(\square q) = 0$. Therefore, $v_w(\square(p \to q) \to (\square p \to \square q)) = 0$.

2. Let $M_{\mathsf{K_3}} = \langle \{1, \frac{1}{2}, 0\}, \neg, \wedge, \{1\} \rangle$ be the matrix of $\mathsf{K_3}$ (Kleene, 1938) where $\neg$ and $\wedge$ are defined as in LP. It is a well known fact that the matrix of $M_{\mathsf{K_3}}$ of $\mathsf{K_3}^{\mathsf{S0.5}}$ has no truth-functional operation o_k^2 such that $o_m^2(\frac{1}{2}, \frac{1}{2}) \in \{1, 0\}$. Since $\frac{1}{2} \notin \{1\}$, then there is no tautology in $\mathsf{K_3}^{\mathsf{S0.5}}$. Then $\mathsf{K_3}^{\mathsf{S0.5}}$ has no theorems of the form $\square\varphi$.

3. Let $M_{\mathsf{RM_3}} = \langle \{1, \frac{1}{2}, 0\}, \neg, \to, \{1, \frac{1}{2}\} \rangle$ be the matrix for $\mathsf{RM_3}$ (Anderson and Belnap, 1975) where $v_w(\varphi \to \psi) = 0$ whenever $v_w(\varphi) > v_w(\psi)$. Let $\mathcal{M} = \langle W, N, R, v \rangle$ be a $M_{\mathsf{RM_3}}$-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) =$

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[7] The first infectious logic in the literature was proposed by Bochvar (1981). For an example of infectious logic which has a designated value, check *Paraconsistent Weak Kleene* (Bonzio et al, 2017). We invite the reader to check Szmuc (2016) for a systematic investigation of these logics.

$v_y(p) = \frac{1}{2}$. By the definition of $\square$, we obtain $v_w(\square p) = 1$. By the definition of $\rightarrow$, we obtain $v_w(\square p \rightarrow p) = 0$.

4. A truth-value r is called *infectious* if, whenever it is an input of a truth-function, r is an output, for every truth-function of a given matrix. A logic L is called infectious if its characteristic matrix has at least one infectious value. Let M_{L} be a matrix for an infectious logic L and let $\mathcal{M} = \langle W, N, R, v \rangle$ be a M_{L}-modal model such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = t^i$, where $t^i \in D_{\mathsf{L}}$ is an infectious value, and $v_w(q) = v_y(q) = 0$. So $v_w(p \wedge q) = v_y(p \wedge q) = t_i$. By the truth-definition of $\square$, we obtain $v_w(\square(p \wedge q)) = v_w(\square p) = 1$ and $v_w(\square q) = 0$. Then, $w_w(\square p \wedge \square q) = 0$. Therefore, $v_w(\square(p \wedge q) \rightarrow (\square p \wedge \square q)) = 0$. The case of $\Diamond(p \wedge q) \rightarrow (\Diamond p \wedge \Diamond q)$ is similar.

5. A truth-value r is called *immune* (Da Ré and Szmuc, 2021) if, whenever it is an input of a truth-function along with a truth-value r', r' is the output. M_{L} be a matrix for an immune logic L such that 1 and d are the designated values of the matrix such that $v(\varphi \wedge \psi) = 1$ iff $v(\varphi) = 1$ and $v(\psi) = 1$, $v(\varphi \wedge \psi) = 0$ otherwise; and let $\mathcal{M}_{\mathsf{L}} = \langle W, N, R, v \rangle$ be a M_{L}-modal model. Suppose that $v_w(\square p) = v_w(\square q) = 1$, for $w \in N$. Then, for every $y \in W$ such that wRy, $v_y(p) \in \{1, d\}$ and $v_y(q) \in \{1, d\}$. If, for some $z \in W$ such that wRz, $v_z(p) = 1$ and $v_z(q) = t$, then $v_z(p \wedge q) = 0$. Therefore, $v_w(\square(p \wedge q)) = 0$.[8]

6. The classical $\mathsf{S0.5}$ does not validate the substitutivity of equivalents. To see why this principle fails, consider a model $\mathcal{M} = \langle \{w_0, y\}, \{w_0\}, \{(w_0, w_0), (w_0, y)\}, v \rangle$ such that $v_y(\square\varphi) = 1$ and $v_y(\square\neg\neg\varphi) = 0$. So, we obtain $v_y(\square\varphi \leftrightarrow \square\neg\neg\varphi) = 0$. Therefore, $v_{w_0}(\square(\square\varphi \leftrightarrow \square\neg\neg\varphi)) = 0$. Since all logics $\mathsf{L}^{\mathsf{S0.5}} \in \mathfrak{L}^{\mathsf{S0.5}}$ are fragments of classical $\mathsf{S0.5}$, the same holds for them.

7. Let $\mathsf{LP}^{\mathsf{S0.5}}$ be the logic presented in item 1. $W = \{w, y\}$, $N = \{w\}$, $R = \{(w, w), (w, y)\}$ and v an assignment such that $v_w(p) = v_y(p) = \frac{1}{2}$. Then, $v_w(\square\neg p) = v_w(\Diamond p) = 1$. By applying the negation, we obtain $v_w(\neg\square\neg p) = 0$. Therefore , $v_w(\Diamond p \rightarrow \neg\square\neg p) = 0$. $\square$

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[8] *Immune logics* are new in the literature. We refer the reader to (Da Ré and Szmuc, 2021) where they were first introduced.

The items 6 and 7 of Theorem 2.11 justify the introduction of both modal operators as primitive. From a conceptual point of view one could argue that logical validity and logical consistency are, in this context, two independent notions. The item 2 of Theorem 2.11 says that the logic K_3 does not have any tautologies. But this does not constitute a problem. It reinforces the claim that the meaning of logical validity is local, depending on the formal system where it is defined.

Definition 2.2 covers a myriad of many-valued modal systems $L^{S0.5}$s. So an axiomatization *à la* Hilbert of the most general modal principles which all systems $L^{S0.5} \in \mathfrak{L}^{S0.5}$ satisfy would constitute an important result about validity and consistency of logics L. But, as Theorem 2.11 shows, many modal principles interact with the truth-functional connectives and these significantly vary according to the logic L. So, it is not immediate for us how to obtain such a general axiomatization. Two general possible routes towards the proof-theoretical characterization of the logics $L^{S0.5} \in \mathfrak{L}^{S0.5}$ would be the modal extension of *n-sided sequents* provided by Baaz et al (1993) and the modal extension of labelled tableaux provided by Carnielli (1987). Such proof-theoretical characterizations will be investigated in a further work.

Although we do not have a general axiomatization for all logics $L^{S0.5}$, we can establish the following semantic fact about the modal logics of Definition 2.2:

PROPOSITION 2.12. *The following principles hold for any* $L^{S0.5} \in \mathfrak{L}^{S0.5}$.

1. $\models_{L^{S0.5}} \Box\varphi \to \Diamond\varphi$;
2. $\Box\varphi \models_{L^{S0.5}} \varphi$;
3. $\varphi \models_{L^{S0.5}} \Diamond\varphi$;
4. $\Box\varphi \to \Box\psi, \Box\varphi \models_{L^{S0.5}} \Box\psi$
5. *If* ψ *is a* $L^{S0.5}$-*tautological consequence of* φ, *then* $\Box\varphi \models_{L^{S0.5}} \Box\psi$.

PROOF. *1.* Suppose that every M_L-modal model $\mathcal{M}$ is such that $v_w(\Box\varphi) = 1$, for every $w \in N$. Then, for every $y \in W$, such that wRy, $v_y(\varphi) \in D_L$. Then, since R is reflexive over N, $v_w(\varphi) \in D_L$. So there is $y \in W$ such that $v_y(\varphi) \in D_L$. Therefore, $v_w(\Diamond\varphi) = 1$. By Notation 1.3, we obtain $v_w(\Box\varphi \to \Diamond\varphi) = 1$.

2. Suppose that every model $\mathcal{M}$ for $L^{S0.5}$ is such that $v_w(\Box\varphi) = 1$, for every $w \in N$. Then, for every $y \in W$, such that wRy, $v_y(\varphi) \in D_L$.

Then, since R is reflexive over N, we obtain $v_w(\varphi) \in D_{\mathsf{L}}$.

3. Same as *2.*

4. This follows from Notation 1.3, since the reasoning only involves the classical values 1 and 0.

5. Suppose that ψ is a tautological consequence of φ in $\mathsf{L}^{\mathsf{S0.5}}$. Then, for every M_{L}-modal model $\mathcal{M}$, for all $w \in N$, $v_w(\varphi) \in D_{\mathsf{L}}$ implies $v_w(\varphi) \in D_{\mathsf{L}}$. Since ψ is a tautological consequence of φ, it is the case all worlds $y \in W$. By definition of models for $\mathsf{L}^{\mathsf{S0.5}}$, every $y \in W$ is accessed by a normal world $w \in N$. Then, $v_w(\Box\varphi) = 1$ implies $v_w(\Box\psi) = 1$. $\qquad\square$

Even if Proposition 2.12 establishes general facts for logics $\mathsf{L}^{\mathsf{S0.5}}$, they are certainly not complete for all logics $\mathsf{L}^{\mathsf{S0.5}}$, as the classical $\mathsf{S0.5}$ (Definition 2.5) witnesses.

3. Recovery operators and modalities

Because of the bivalence of the modalities investigated, we argue that $\Box$ and $\Diamond$ can also work as *recovery operators*.[9] Recovery operators are devices to recover inferences which we lose when we depart from classical logic. They were carefully investigated in the fields of paraconsistent and paracomplete logics. We now exemplify the use of the operators $\Box$ and $\Diamond$ to recover inferences from $\mathsf{S0.5}$ in a many-valued modal logic.

DEFINITION 3.1. The three-valued Łukasiewicz logic Ł_3 is characterized by the matrix $M_{\text{Ł}_3} = \langle\{1, \frac{1}{2}, 0\}, \neg, \to, \{1\}\rangle$ whose connectives $\neg$ and $\to$ are interpreted by the following truth-tables:

	$\neg$		$\to$	1	$\frac{1}{2}$	0
1	0		1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1		0	1	1	1

---

[9] On this topic see Coniglio and Peron (2013) for an application to the paraconsistent case and Marcos (2005) for non-classical negations in general. Here we are dealing with the even more general case of many-valued logics.

The modal three-valued Łukasiewicz logic $Ł_3^{S0.5}$ is characterized by the $M_{Ł_3}$-modal model $\mathcal{M} = \langle W, N, R, v \rangle$ in accordance with Definition 2.2.

In the definition below, we present an axiomatic system for $Ł_3^{S0.5}$. Our axiomatization is inspired by Schotch et al (1978) with some obvious modifications, given that we are dealing with non-normal modalities.[10]

DEFINITION 3.2. The logic $Ł_3^{S0.5}$ has the following axioms and rules:

(I) Propositional axioms of $Ł_3$ (Wajsberg, 1931):

(Ł₃-1) $\varphi \to (\psi \to \varphi)$;
(Ł₃-2) $(\varphi \to \psi) \to ((\psi \to \gamma) \to (\varphi \to \gamma))$;
(Ł₃-3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$;
(Ł₃-4) $((\varphi \to \neg\varphi) \to \varphi) \to \varphi$.
(MP) From $\vdash_{Ł_3^{S0.5}} \varphi$ and $\vdash_{Ł_3^{S0.5}} \varphi \to \psi$ we infer $\vdash_{Ł_3^{S0.5}} \psi$;

(II) Modal axioms:

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \psi)$;
(T) $\Box\varphi \to \varphi$;
(D) $\Box\varphi \to \Diamond\varphi$;
(D$^{\Box\neg}$) $\Box\neg\varphi \to \neg\Diamond\varphi$;
(T$^\Diamond$) $\varphi \to \Diamond\varphi$;
(Biv₁) $\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$;
(Biv₂) $\neg(\Diamond\varphi \leftrightarrow \neg\Diamond\varphi)$;
(Nec$_{Ł_3}$) If $\vdash_{Ł_3^{S0.5}} \varphi$ and φ is a $Ł_3$-tautology, we infer $\vdash_{Ł_3^{S0.5}} \Box\varphi$;
(RK$_{Ł_3}$) If $\vdash_{Ł_3^{S0.5}} \varphi \to \psi$ and $\varphi \to \psi$ is a $Ł_3$-tautology, we infer $\vdash_{Ł_3^{S0.5}} \Diamond\varphi \to \Diamond\psi$.

The axioms **Biv₁** and **Biv₂** say that the modalities do not receive intermediate values. They are easily provable in S0.5 as the following result shows:

THEOREM 3.3. *The following formulas are theorems of* S0.5*:*

---

[10] It is worth noticing that Schotch *et al*'s axiomatization of the modal counterpart of $Ł_3$ is $\Diamond$-free. As we will see, the bivalent modalities of the modal counterpart of $Ł_3$ break the interdefinability between $\Box$ and $\Diamond$. Therefore, it is necessary to introduce both operators as primitive in order to give a completeness proof for the full language.

1. $\vdash_{\mathsf{S0.5}} \neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$;
2. $\vdash_{\mathsf{S0.5}} \neg(\Diamond\varphi \leftrightarrow \neg\Diamond\varphi)$.

PROOF. For (1), consider the formal derivation:

1. $\vdash_{\mathsf{S0.5}} (\Box\varphi \leftrightarrow \neg\Box\varphi) \to (\varphi \wedge \neg\varphi)$ CPL
2. $\vdash_{\mathsf{S0.5}} \neg(\varphi \wedge \neg\varphi) \to \neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$ CPL, 1
3. $\vdash_{\mathsf{S0.5}} \neg(\varphi \wedge \neg\varphi)$ CPL
4. $\vdash_{\mathsf{S0.5}} \neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$ MP 2,3

The reasoning for (2) is exactly the same. This concludes the proof.
□

THEOREM 3.4. *In* $\text{Ł}_3^{\mathsf{S0.5}}$ *the following schemas are not valid:*

1. $\neg\Box\neg\varphi \to \Diamond\varphi$;
2. $\neg\Diamond\neg\varphi \to \Box\varphi$.

PROOF. For (1), consider the model $\mathcal{M} = \langle W, N, R, v \rangle$ such that $W = \{w, y\}$, $N = \{w\}$, $R = \{(w,w), (w,y)\}$ and $v_w(\varphi) = v_y(\varphi) = \frac{1}{2}$. Then, $v_w(\neg\varphi) = v_y(\neg\varphi) = \frac{1}{2}$. So, we obtain $v_w(\Box\neg\varphi) = 0$ and $v_w(\neg\Box\neg\varphi) = 1$ and $v_w(\Diamond\varphi) = 0$. Therefore, $v_w(\neg\Box\neg\varphi \to \Diamond\varphi) = 0$. The reasoning for (2) is similar. This concludes the proof. □

In virtue of Theorem 3.4, $\mathbf{Df}^1$ and $\mathbf{Df}^2$ cannot be axioms of $\text{Ł}_3^{\mathsf{S0.5}}$. The characterization results for the system $\text{Ł}_3^{\mathsf{S0.5}}$ will be proved in Appendix A.

DEFINITION 3.5. Given the operations $\neg$ and $\to$ of Ł_3, we define the connectives $\vee, \wedge$, and the recovery operator $\star$ as follows:

$$
\begin{aligned}
\varphi \vee \psi &:= (\varphi \to \psi) \to \psi \\
\varphi \wedge \psi &:= \neg(\neg\varphi \vee \neg\psi) \\
\varphi \leftrightarrow \psi &:= (\varphi \to \psi) \wedge (\psi \to \varphi) \\
\star\varphi &:= \neg(\varphi \leftrightarrow \neg\varphi)
\end{aligned}
$$

They display the following truth-tables.

$\vee$	1	$\frac{1}{2}$	0		$\wedge$	1	$\frac{1}{2}$	0		$\leftrightarrow$	1	$\frac{1}{2}$	0			$\star$
1	1	1	1		1	1	$\frac{1}{2}$	0		1	1	$\frac{1}{2}$	0		1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$		$\frac{1}{2}$	0
0	1	$\frac{1}{2}$	0		0	0	0	0		0	0	$\frac{1}{2}$	1		0	1

Informally, the connective $\star$ is interpreted as "φ is determinated." Interestingly, some remarkable principles are not valid in Ł₃.

Proposition 3.6. *The following items hold for Ł₃:*

1. $\nvDash_{Ł_3} \neg(\varphi \wedge \neg\varphi)$
2. $\nvDash_{Ł_3} \varphi \vee \neg\varphi$
3. $\varphi \to (\varphi \to \psi) \nvDash_{Ł_3} \varphi \to \psi$

The verification of Proposition 3.6 is simple and it is left to the reader. With the presence of the connective $\star$, it is possible to recover the invalidities of Proposition 3.6.

Proposition 3.7. *The following items hold for Ł₃:*

1. $\star\varphi \models_{Ł_3} \neg(\varphi \wedge \neg\varphi)$
2. $\star\varphi \models_{Ł_3} \varphi \vee \neg\varphi$
3. $\star\varphi, \star\psi, \varphi \to (\varphi \to \psi) \models_{Ł_3} \varphi \to \psi$

In the light of Proposition 3.6 and Proposition 3.7 we can say that Ł₃ is a *Logic of Formal Undeterminedness* (LFU, for short) Marcos (2005). Roughly speaking, L is a LFU if it does not validate excluded middle, while it validates a restricted version of such principle, such as item 2 of Proposition 3.7. In fact, it is possible to prove that the connective $\star$ recovers classical inferences which are lost in Ł₃ when "determined assumptions" are made. The following theorem is a version, for Ł₃, of da Costa's *Derivability Adjustment Theorem* (Da Costa, 1974).

Theorem 3.8. *For every $\Gamma \subseteq For(\mathcal{L}_{Ł_3})$, for every $\varphi \in For(\mathcal{L}_{Ł_3})$,*

$$\Gamma \models_{\mathsf{CPL}} \varphi \text{ iff } \Gamma, \{\star p_1, \ldots, \star p_n\} \models_{Ł_3} \varphi \tag{2}$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$.[11]

Now, in the case of $Ł_3^{\mathsf{S0.5}}$, it is possible to recover the inferences of the classical S0.5 without the use of connective $\star$. Given the modal operator $\Box$, the formula

$$\Box\varphi \vee \Box\neg\varphi \tag{3}$$

---

[11] A general version of this theorem for finite many-valued logics can be found in Ciuni and Carrara (2020).

expresses a form of non-contingency.

The modal notion of non-contingency was introduced by Montgomery and Routley (1966) and investigated by Humberstone (1995) and Cresswell (1988). By the semantic condition of $\square$ in $Ł_3^{\mathsf{S0.5}}$, $\square\varphi \vee \square\neg\varphi$ receives the truth-value 1 in normal worlds w if and only if φ receives 1 or 0 in all worlds y and z accessible to w. Thus, the formula 3 reflects some intuitions of the truth-functional connective $\star$, but now from a modal perspective. So, formula 3 says that φ only receives classical values.

The fact that the truth-functional connectives of $Ł_3^{\mathsf{S0.5}}$ are normal will play a significant role in the next result. The next theorem states the possibility of recovering inferences of $\mathsf{S0.5}$ in $Ł_3^{\mathsf{S0.5}}$ under certain "non-contingency assumptions". In order to state the next result we need the following definition.

DEFINITION 3.9. Let $\varphi \in For(\mathcal{L}_{Ł_3^{\mathsf{S0.5}}}^{\square\Diamond})$ be a $Ł_3^{\mathsf{S0.5}}$ formula. The modal degree of φ, $md(\varphi)$, is defined as follows:

1. if $\varphi = p$, then $md(p) = 0$;
2. if $\varphi = \neg\psi$, then $md(\neg\psi) = md(\psi)$;
3. if $\varphi = \psi \to \gamma$, then $md(\psi \to \gamma) = max(md(\psi), md(\gamma))$;
4. if $\varphi = \square\psi$, then $md(\square\psi) = md(\psi) + 1$;
5. if $\varphi = \Diamond\psi$, then $md(\Diamond\psi) = md(\psi) + 1$.

THEOREM 3.10. *For every $\Gamma \subseteq For(\mathcal{L}_{Ł_3}^{\square\Diamond})$ containing a finite number of propositional variables, such that $md(\gamma) \leq 1$, for every $\gamma \in \Gamma$, and for every $\varphi \in For(\mathcal{L}_{Ł_3}^{\square\Diamond})$ such that $md(\varphi) \leq 1$,*

$$\Gamma \models_{\mathsf{S0.5}} \varphi \text{ iff } \Gamma, \{\square p_1 \vee \square\neg p_1, \ldots, \square p_n \vee \square\neg p_n\} \models_{Ł_3^{\mathsf{S0.5}}} \varphi \qquad (4)$$

where $\{p_1, \ldots, p_n\}$ is the set of propositional variables which occur in $\Gamma \cup \{\varphi\}$. $\square$

The proof is a routine exercise, based on the fact that the non-contingency of the propositional variables force the (non-vacuous) valuation in $Ł_3^{\mathsf{S0.5}}$ to be either the classical top element of or the bottom element of Łukasiewicz's logic $Ł_3$. Moreover, notice that Theorem 3.10 does not generalise to any modal degree, as one can easily see by considering the formula $\square(\square p \vee \neg\square p)$.

COROLLARY 3.11. *The following items hold for $Ł_3^{\mathsf{S0.5}}$:*

1. $\square p \vee \square \neg p \models_{\text{Ł}_3^{\text{S0.5}}} \neg(p \wedge \neg p)$
2. $\square p \vee \square \neg p \models_{\text{Ł}_3^{\text{S0.5}}} p \vee \neg p$
3. $\square p \vee \square \neg p, \square q \vee \square \neg q, p \to (p \to q) \models_{\text{Ł}_3^{\text{S0.5}}} p \to q$

By Suszko's reduction result, we could have worked with the bivalent counterpart of the logic Ł$_3$ and define modal structures for Ł$_3^{\text{S0.5}}$.[12] However, the reason for presenting here a matricial semantics for Ł$_3$ stems from the fact that matrix semantics are more user-friendly than bivalent semantics. Besides, not every many-valued semantics can directly define a recovery operator, because of their lack of expressive power. In this case, the signature of the logic needs to be extended with a new symbol for the recovery operator.[13] In this sense, the meta-theoretical move of adding Suszkian modalities is not completely alien to the study of many valued logics. Moreover, the possibility that these modalities offer in recovering classical inferences confirms both their connection with Suszko's thesis and their (classical) meta-theoretical interpretation of validity and consistency.

4. Concluding remarks

We have seen that the modalities $\square$ and $\lozenge$ can capture a form of validity and consistency for many-valued logics L, when we consider sets of valuations in the meta-theory. The results proved in Subsections 2.1 show that the logics L$^{\text{S0.5}}$ capture a well-grounded notion of model-theoretical validity. We saw that one of the fundamental properties of the predicate of validity is reflexivity. That is, if φ is valid, then φ is the case. Moreover, the modalities $\square$ and $\lozenge$ of the logics L$^{\text{S0.5}}$ capture the predicates Val and Con of tautological validity and logical consistency, respectively, in the language $\mathcal{L}_{\text{L}}^{VC}$.

In the case of $\text{L} = \text{CPL}$, we have by the results of Subsection 2.1 the following valid schemas:

$\mathbf{K}^t$ $Val(\overline{t(\varphi) \to t(\psi)}) \to (Val(\overline{t(\varphi)}) \to Val(\overline{t(\psi)}))$;
$\mathbf{T}^t$ $Val(\overline{t(\varphi)}) \to t(\varphi)$;

---

[12] We refer the reader to (Malinowski, 1993, Chapter 10) for a bivalent semantics for Ł$_3$.

[13] The problem of the bivalent reduction of MVLs with weak expressive power is discussed in (Caleiro et al, 2005).

$\mathbf{D}^t$ $Val(\overline{t(\varphi)}) \to Con(\overline{t(\varphi)})$;
$\mathbf{Nec}^t$ If $t(\varphi)$ is a tautology, we infer $\vdash Val(\overline{t(\varphi)})$.

Let Val^* be a validity predicate which satisfies the principles $\mathbf{K}$, $\mathbf{T}$, $\mathbf{D}$ and $\mathbf{Nec}$, and has Gödel numbers instead of sentence names as arguments. Ketland (2012) showed that Val^* is consistent with PA. Because of the similarities between Val^* and Val, we conjecture that S0.5 is also the logic of the predicate Val^*. But, we leave this question for future work.

Now we turn to the relation between the modalities investigated here and the meaning of recovery operators. It is not difficult to find in the literature arguments defending that recovery operators, such as $\star$, internalize metalogical concepts in the object language of the logic. In the case of $Ł_3$ it is said that $\star$ internalizes a form of decidability. But it is clear that $\star\varphi$ and $\Box\varphi \vee \Box\neg\varphi$ differ in meaning, as the following validities show. First, let $\Delta\varphi$ be an abbreviation of $\Box\varphi \vee \Box\neg\varphi$.

1. $\models_{Ł_3^{S5}} (\star(\varphi \vee \psi) \wedge (\varphi \vee \psi)) \to ((\star\varphi \wedge \varphi) \vee (\star\psi \wedge \psi))$
2. $\nvDash_{Ł_3^{S5}} (\Delta(\varphi \vee \psi) \wedge (\varphi \vee \psi)) \to ((\Delta\varphi \wedge \varphi) \vee (\Delta\psi \wedge \psi))$

As a consequence, $\star\varphi$ does not have the same interpretation as the modal formula $\Box\varphi \vee \Box\neg\varphi$. Since the results of Subsection 2.1 show that $\Box$ and $\Diamond$ have well justified interpretations of validity and consistency, we cannot say the same with respect to $\star$. That is, it is not obvious that $\star$ incorporates a metalogical notion in the object language of the logic.

Another interesting question we leave for further investigation is about the provability interpretation of many-valued logics. Indeed, an investigation of many-valued counterparts of the modal logic GL may establish general provability principles of arithmetical theories based on many-valued logics.

A. Characterization results for $Ł_3^{S0.5}$

First we will show that the axiom system for $Ł_3^{S0.5}$ is sound with respect to the models of Definition 3.1.

THEOREM A.1. *$Ł_3^{S0.5}$ is sound with respect to the models of Definition 3.1.*

Proof. We will show that the axioms of $Ł_3^{\mathsf{S0.5}}$ are valid with respect to the models $\mathcal{M} = \langle W, N, R, v \rangle$ and that the rules of inferences preserve validity. We will only analyse the modal axioms. The validity of non-modal axioms can be found in Wajsberg (1931).

Axiom **K**. Suppose that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is not valid. Then there is a model $\mathcal{M} = \langle W, N, R, v \rangle$ such that $v_w(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)) \in \{0, \frac{1}{2}\}$, for some $w \in N$. Since modal formulas receive only classical values we will only analyse the case where $v_w(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)) = 0$. So, $v_w(\Box(\varphi \to \psi)) = v_w(\Box\varphi) = 1$ and $v_w(\Box\psi) = 0$. By definition, there is $y \in W$ such that wRy and $v_w(\psi) \in \{0, \frac{1}{2}\}$. Also, by definition, for every $z \in W$ such that wRz, $v_z(\varphi \to \psi) = v_z(\varphi) = 1$, including $y \in W$. By semantic modus ponens, we obtain $v_z(\psi) = 1$. Then so, $v_y(\psi) = 1$. Contradiction. Therefore, $v_w(\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)) = 1$, for all models $\mathcal{M} = \langle W, N, R, v \rangle$.

The reasoning for the axioms **T**, $\mathbf{T}^{\Diamond}$, **D** and $\mathbf{D}^{\Box\neg}$ is straightforward.

Axiom $\mathbf{Biv}_1$. Suppose that $\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)$ is not valid. Then there is a model $\mathcal{M} = \langle W, N, R, v \rangle$ such that $v_w(\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)) \in \{0, \frac{1}{2}\}$. For the same reason as **K**, we will only analyse the case where $v_w(\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)) = 0$. Then, $v_w(\Box\varphi \leftrightarrow \neg\Box\varphi) = 1$ and we have the following cases to consider: (i) $v_w(\Box\varphi) = v_w(\neg\Box\varphi) = 1$; (ii) $v_w(\Box\varphi) = v_w(\neg\Box\varphi) = 0$; and (iii) $v_w(\Box\varphi) = v_w(\neg\Box\varphi) = \frac{1}{2}$. The cases (i) and (ii) are impossible in virtue of the semantic definition of negation and (iii) is impossible because modal formulas do not receive the value $\frac{1}{2}$ in worlds in N. Then, $v_w(\neg(\Box\varphi \leftrightarrow \neg\Box\varphi)) = 1$, for all models $\mathcal{M} = \langle W, N, R, v \rangle$. The reasoning for $\mathbf{Biv}_2$ is the same.

Rule $\mathbf{Nec}_{Ł_3}$. Suppose that φ is a $Ł_3$-tautology. Then $v_y(\varphi) = 1$ for all $y \in W$, including $w \in N$. Therefore, by definition of R, $v_w(\Box\varphi) = 1$. The reasoning for $\mathbf{RK}_{Ł_3}$ is similar. This concludes the proof.

$\square$

Characterization results for propositional $Ł_3$ are easily found in the literature (*e.g.*, (Wajsberg, 1931), (Golberg et al, 1974), (Epstein, 1990)). So, the set of valid formulas of $Ł_3$ and its set of theorems coincide. Then, whenever we use a $Ł_3$-tautology or inference rule in a $Ł_3^{\mathsf{S0.5}}$-demonstration we will justify the step as $Ł_3$.

Let $\bot$ stand for any negated theorem of $Ł_3$ (*e.g.*, $\bot := \neg(p \to p)$). The following theorem presents some formulas which will be used for the results below:

THEOREM A.2. *The following formulas are theorems of $Ł_3^{S0.5}$.*

1. $\vdash_{Ł_3^{S0.5}} (\varphi \to \bot) \to (\varphi \to \neg\varphi)$
2. $\vdash_{Ł_3^{S0.5}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi);$
3. $\vdash_{Ł_3^{S0.5}} ((\varphi \wedge \psi) \to \gamma) \to (\varphi \to (\psi \to \gamma));$
4. $\vdash_{Ł_3^{S0.5}} \Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \leftrightarrow (\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n) \ (n \geq 2);$
5. $\vdash_{Ł_3^{S0.5}} \Diamond(\varphi_1 \wedge \ldots \wedge \varphi_n) \to (\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n) \ (n \geq 2).$

The proof of Theorem A.2 is the same as in the classical case and it will be left for the reader.

DEFINITION A.3. Let $\Gamma \subseteq For(\mathcal{L}_{Ł_3^{S0.5}}^{\Box\Diamond})$ be a set of formulas. Then we say that

1. Γ is *consistent* if there are no $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $(\gamma_1 \wedge \ldots \wedge \gamma_n) \to \bot \in \Gamma$ and $\neg\gamma \in \Gamma$. Γ is *inconsistent* if it is not consistent;
2. Γ is *$Ł_3$-consistent* if there are no $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $(\gamma_1 \wedge \ldots \wedge \gamma_n) \to \bot \in \Gamma$ and $(\gamma_1 \wedge \ldots \wedge \gamma_n) \to \bot$ is substitution instance of a $Ł_3$-tautology.

LEMMA A.4 (Lindenbaum's Lemma). *Let Δ be a consistent set of formulas in the language of $Ł_3^{S0.5}$. Then, there is a maximal consistent set of formulas Γ of $Ł_3^{S0.5}$ such that $\Delta \subseteq \Gamma$.*

DEFINITION A.5. The canonical model $\mathcal{M} = \langle W, N, R, v \rangle$ of $Ł_3^{S0.5}$ is defined as follows:

1. $w \in N$ is a maximal consistent set of $Ł_3^{S0.5}$ formulas;
2. Every $w \in W \setminus N$ is a maximal $Ł_3$-consistent set of formulas in the language of $Ł_3^{S0.5}$.
3. The relation R is defined as follows. Let $w, y \in W$.
   (a) For $w \in N$: if $\Box\varphi \in w$, then we let wRy iff $\lambda(w) \subseteq y$ ($\lambda(w) = \{\varphi | \Box\varphi \in w\}$);
   (b) For $w \in N$: if $\Diamond\varphi \in w$, then we let wRy iff $\mu(w) \subseteq w$ ($\mu(w) = \{\Diamond\varphi | \varphi \in y\}$).
4. For every $w \in W$, the assignment v is defined over atomic propositions as follows:

$$\begin{aligned} v_w(p) = 1 \quad &\text{iff} \quad p \in w; \\ v_w(p) = \tfrac{1}{2} \quad &\text{iff} \quad p \notin w \text{ and } \neg p \notin w; \\ v_w(p) = 0 \quad &\text{iff} \quad \neg p \in w. \end{aligned}$$

Since the operator $\Diamond$ was introduced as primitive and $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ and $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ are not valid in $Ł_3^{S0.5}$, then we separately introduced

the sets $\lambda(w)$ and $\mu(w)$. In the classical case, we could introduce only the set $\lambda(w)$ because of the validity of both biconditionals.

PROPOSITION A.6. *Let $w \in N$ be a maximal consistent set of $\mathrm{Ł}_3^{\mathsf{S0.5}}$ formulas such that $\neg\Box\varphi \in w$. Then there is a ($\mathrm{Ł}_3$-) consistent set such that $\varphi \notin \lambda(w)$.*

PROOF. Suppose that $\varphi \in \lambda(w)$ ($*'$) and let $\gamma_1, \ldots, \gamma_n$ be formulas of $\mathrm{Ł}_3^{\mathsf{S0.5}}$. Since $\varphi \to (\psi \to \varphi)$ is a instance of a $\mathrm{Ł}_3$ tautology, we obtain:[14]

1.	$\vdash \varphi \to ((\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi)$	$\mathrm{Ł}_3$
2.	$\vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi$	($*'$), MP
3.	$\vdash \Box((\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi)$	$\mathrm{Nec}_{\mathrm{Ł}_3}$
4.	$\vdash \Box((\gamma_1 \wedge \ldots \wedge \gamma_n) \to \varphi) \to (\Box(\gamma_1 \wedge \ldots \wedge \gamma_n) \to \Box\varphi)$	**K**
5.	$\vdash \Box(\gamma_1 \wedge \ldots \wedge \gamma_n) \to \Box\varphi$	MP 3,4
6.	$\vdash (\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \to \Box(\gamma_1 \wedge \ldots \wedge \gamma_n)$	Thm A.2
7.	$\vdash (\Box\gamma_1 \wedge \ldots \wedge \Box\gamma_n) \to \Box\varphi$	$\mathrm{Ł}_3$ 5,6

Since $\gamma_1, \ldots, \gamma_n \in \lambda(w)$, we obtain by the maximality of w:

8.	$\vdash \Box\gamma_1 \wedge \cdots \wedge \Box\gamma_n$	
9.	$\vdash \Box\varphi$	MP 7,8

Which contradicts the consistency of w. Therefore, $\varphi \notin \lambda(w)$. $\square$

PROPOSITION A.7. *Let y be a maximal ($\mathrm{Ł}_3^{\mathsf{S0.5}}$-) consistent set of $\mathrm{Ł}_3^{\mathsf{S0.5}}$ formulas. For all y, if $\varphi \notin y$, then $\mu(w) \cup \{\neg\Diamond\varphi\}$ is consistent.*

PROOF. Suppose that $\mu(w) \cup \{\neg\Diamond\varphi\}$ is not consistent. Then by Definition A.3, there are $\Diamond\gamma_1, \ldots, \Diamond\gamma_n \in \mu(w)$ such that

1.	$\vdash (\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n \wedge \neg\Diamond\varphi) \to \bot$	Def A.3
2.	$\vdash (\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n) \to (\neg\Diamond\varphi \to \bot)$	$\mathrm{Ł}_3$ 1
3.	$\vdash (\neg\Diamond\varphi \to \bot) \to (\top \to \neg\neg\Diamond\varphi)$	$\mathrm{Ł}_3$
4.	$\vdash (\Diamond\varphi_1 \wedge \ldots \wedge \Diamond\varphi_n) \to (\top \to \neg\neg\Diamond\varphi)$	$\mathrm{Ł}_3$ 2,3
5.	$\vdash \Diamond(\gamma_1 \wedge \ldots \wedge \gamma) \to (\Diamond\gamma_1 \wedge \ldots \wedge \Diamond\varphi_n)$	Theorem A.2

Since $\Diamond\gamma_1, \ldots, \Diamond\varphi_n \in \mu(w)$, then $\gamma_1, \ldots, \gamma_n$. By the maximality of y, we obtain $\gamma_1 \wedge \ldots \wedge \gamma_n \in y$. Then, by definition of $\mu(w)$, $\Diamond(\gamma_1 \wedge \ldots \wedge \gamma_n)$. Then by modus ponens we obtain:

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[14] To facilitate the reading, we will drop the subscript in $\vdash$.

6. $\vdash \top \to \neg\neg\Diamond\varphi$

Since $\top \in \mu(w)$, we obtain by modus ponens:

7. $\vdash \neg\neg\Diamond\varphi$

Which amounts to

8. $\vdash \Diamond\varphi$

So $\Diamond\varphi \in w$. By definition $\mu(w)$, $\varphi \in y$, which contradicts the (Ł$_3$-) consistency of y which we supposed not to contain φ. Therefore, $\mu(w) \cup \{\neg\Diamond\varphi\}$ is consistent. This concludes the proof. $\qquad\square$

PROPOSITION A.8. *Let $w \in N$ be a maximal consistent set of $Ł_3^{\mathsf{S0.5}}$ formulas. Then:*

1. *If $\Box\varphi \notin w$, then $w \cup \{\neg\Box\varphi\}$ is consistent.*
2. *If $\neg\Box\varphi \notin w$, then $w \cup \{\Box\varphi\}$ is consistent.*
3. *If $\Diamond\varphi \notin w$, then $w \cup \{\neg\Diamond\varphi\}$ is consistent.*
4. *If $\neg\Diamond\varphi \notin w$, then $w \cup \{\Diamond\varphi\}$ is consistent.*

PROOF. We will prove only the statement (1). The others follow the same reasoning. Suppose that $w \cup \{\neg\Box\varphi\}$ is not consistent. Then, for $\gamma_1, \ldots, \gamma_n \in \lambda(w)$:

1. $\vdash (\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \neg\Box\varphi) \to \bot$ $\qquad$ Def.
2. $\vdash (\gamma_1 \wedge \ldots \wedge \gamma_n) \to (\neg\Box\varphi \to \bot)$ $\quad$ Ł$_3$ 1

Since $\gamma_1, \ldots, \gamma_n \in w$, then, by maximality of w, we obtain

3. $\vdash \gamma_1 \wedge \ldots \wedge \gamma_n$
4. $\vdash \neg\Box\varphi \to \bot$ $\hspace{5em}$ MP 2,3
5. $\vdash (\neg\Box\varphi \to \bot) \to (\neg\Box\varphi \to \neg\neg\Box\varphi)$ $\quad$ Ł$_3$
6. $\vdash \neg\Box\varphi \to \neg\neg\Box\varphi$ $\hspace{4em}$ MP 4,5
7. $\vdash \neg\Box\varphi$ $\hspace{6em}$ $\neg\Box\varphi \in w \cup \{\neg\Box\varphi\}$
8. $\vdash \neg\neg\Box\varphi$ $\hspace{5.5em}$ MP 6,7
9. $\vdash \neg\neg\Box\varphi \to \Box\varphi$ $\hspace{3.5em}$ Ł$_3$
10. $\vdash \Box\varphi$ $\hspace{6.5em}$ MP 8,9

Since w is maximal, $\Box\varphi \in w$, contradicting the consistency of w. Therefore, $w \cup \{\neg\Box\varphi\}$ is consistent. $\Box$

LEMMA A.9. *Let $\mathcal{M}$ be a canonical model for $\text{Ł}_3^{\text{S0.5}}$. Then, for every $w \in W$ and every formula φ of $\text{Ł}_3^{\text{S0.5}}$:*

$$
\begin{array}{lll}
v_w(\varphi) = 1 & \text{iff} & \varphi \in w; \\
v_w(\varphi) = \frac{1}{2} & \text{iff} & \varphi \notin w \text{ and } \neg\varphi \notin w; \\
v_w(\varphi) = 0 & \text{iff} & \neg\varphi \in w.
\end{array}
$$

PROOF. The proof runs by induction on the complexity of φ. We will only consider the modal cases. Moreover, since the semantic definition of modalities are defined only in normal worlds, we will only focus in the case where $w \in N$.

$\varphi = \Box\psi$. If $\Box\psi \in w$, then $\psi \in y$ for every $y \in W$ such that $\lambda(w) \subseteq y$. Moreover, since $\Box\psi \to \psi \in w$, we obtain $\psi \in w$. By I.H., we obtain $v_w(\psi) = 1$ and $v_y(\psi) = 1$, for all $y \in W$ such that wRy. Then, $v_w(\Box\psi) = 1$.

Conversely, if $\neg\Box\psi \in w$, then by Proposition A.6, there is a (Ł_3-) consistent set $\lambda(w)$ such that $\psi \notin \lambda(w)$. Now, we have two possibilities to consider: (i) $\neg\psi \in \lambda(w)$ and (ii) $\neg\psi \notin \lambda(w)$. In the case (i) $\lambda(w) \cup \{\neg\psi\} \subseteq y$, where y is a (Ł_3-) maximal consistent set of formulas. Since $\psi \notin y$ and $\neg\psi \in y$, $v_y(\psi) = 0$ for some $y \in W$ such that wRy. Then, $v_w(\Box\psi) = 0$. In the case (ii), $\lambda(w) \cup \{\psi \leftrightarrow \neg\psi\}$ is consistent, and then, $\lambda(w) \cup \{\psi \leftrightarrow \neg\psi\} \subseteq y$ and y is maximal (Ł_3-) consistent. Since $\psi \notin y$ and $\neg\psi \notin y$, we obtain by I.H. $v_y(\psi) = \frac{1}{2}$. Given that wRy, $v_w(\Box\psi) = 0$.

$\varphi = \Diamond\psi$. If $\neg\Diamond\psi \in w$, then $\psi \notin y$ for every $y \in W$ such that $\mu(w) \subseteq w$. Moreover, since $(\psi \to \Diamond\psi) \to (\neg\Diamond\psi \to \neg\psi) \in w$ and $\psi \to \Diamond\psi \in w$, we obtain $\neg\Diamond\psi \to \neg\psi \in w$. By modus ponens again, we obtain $\neg\psi \in w$. By I.H., $v_y(\psi) \in \{\frac{1}{2}, 0\}$ and $v_w(\psi) = 0$. Then, $v_w(\Diamond\psi) \in \{0, \frac{1}{2}\}$. By the semantic definition of the modal operators, we obtain $v_w(\Diamond\psi) = 0$.

Conversely, if $v_w(\Diamond\psi) = 0$, then for every $y \in W$ such that wRy, $v_y(\psi) \in \{\frac{1}{2}, 0\}$. By I.H., $\psi \notin w$ for every (Ł_3-) maximal consistent set of formulas. By Lemma A.7, $\mu(w) \cup \{\neg\Diamond\psi\}$ is consistent. By Lindenbaum's lemma, $\mu(w) \cup \{\neg\Diamond\psi\} \subseteq w$ and w is a maximal consistent. Then, $\neg\Diamond\psi \in w$.

Now we will show that modal formulas $\Box\psi$ and $\Diamond\psi$ cannot receive intermediate values. By Proposition A.8, if $\Box\psi \notin w$, then $\neg\Box\psi \in w$; and if $\neg\Box\psi \in w$, then $\Box\psi \notin w$. If it were the case that $\Box\psi, \neg\Box\psi \notin w$,

we would obtain, by propositional $Ł_3$, $\Box\psi \leftrightarrow \neg\Box\psi \in w$, which would contradict the axiom $\mathbf{Biv}_1$, $\neg(\Box\psi \leftrightarrow \neg\Box\psi) \in w$, which means that modal formulas $\Box\varphi$ are bivalent in normal worlds. The same reasoning applies to the case of $\Diamond\psi$. This concludes the proof.

$\Box$

Therefore, we have the following result:

THEOREM A.10. *If φ is valid in $M_{Ł_3}$-modal models, then $\vdash_{Ł_3^{S0.5}} \varphi$.*

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