Imprint of relativistic particles on the anisotropies of the stochastic gravitational-wave background

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The stochastic gravitational-wave background (SGWB) is expected to be a key observable for gravitational-wave interferometry. Its detection will open a new window on early Universe cosmology, on the astrophysics of compact objects, and, as shown in this paper, on the particle physics content of the Universe. In this article we show that, besides their effects on the cosmic microwave background and on large-scale structure, relativistic particles in the early Universe leave a clear imprint on the anisotropies of the SGWB. In particular we show that a change in the number of decoupled relativistic particles shifts the angular power spectrum of the SGWB, as both the Sachs-Wolfe and the integrated Sachs-Wolfe terms are affected. Being very large-angle effects, these lead to new testable predictions for future gravitational-wave interferometers.

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I. INTRODUCTION

Future gravitational-wave (GW) interferometers [1–5] will probe the stochastic gravitational-wave background (SGWB) from late-time unresolved astrophysical sources and early Universe cosmological sources (see, e.g., [6,7] for reviews). Besides the important information about astrophysics and cosmology [8–10], such detections will allow to extract important information about particle physics within or beyond the Standard Model [11]. For many years the interplay between cosmology and particle physics has been pursued vigorously; one example being the constraints on the effective number of neutrinos N_ν and on neutrino masses, which have been largely investigated using cosmic microwave background (CMB) and large-scale structure data [12–14]. In this article, we will show that the cosmological SGWB will offer a new powerful tool to constrain the abundance of relativistic species in the early Universe.

As recently shown [15–17], using a Boltzmann equation approach, it is possible to characterize angular anisotropies of the GW energy density, thus providing an important tool to disentangle the different cosmological and astrophysical contributions to the SGWB [18–20]. Anisotropies in the cosmological background are imprinted both at its production, and by GW propagation through the large-scale scalar [15–17,21] and tensor perturbations of the Universe [15–17]. In considering the SGWB there is a crucial difference with respect to the CMB: while CMB temperature anisotropies are generated at the last scattering surface [22,23], the Universe is transparent to GWs at all energies below the Planck scale [24,25]. Therefore, the SGWB provides a snapshot of the Universe at the epoch of its production, and its anisotropies retain precious information about the primordial Universe, the mechanisms for GW formation, and the presence of extra particle species in such an era.

In this paper, we will focus on the impact of the effective number of relativistic degrees of freedom $N_{\rm eff}$ on the anisotropies of the stochastic background of cosmological origin generated from the propagation of GWs in the perturbed Universe. Although this effect is present also for different cosmological sources of GWs (e.g., phase transition, cosmic strings, preheating, etc.), we will consider the SGWB generated during inflation in the early Universe. The number of relativistic degrees of freedom will have a direct impact on the angular power-spectrum of the SGWB, on scales accessible to GW interferometers, mainly through the Sachs-Wolfe (SW) and integrated Sachs-Wolfe (ISW) effects. The overall effect of increasing the number of relativistic species on the CMB angular power spectrum is a horizontal and a vertical shift of the peak positions, respectively: by increasing the value of $N_{\rm eff}$ the height of the first peak is enhanced and the positions of the acoustic peaks are shifted to higher multipoles [26,27]. The first effect derives from the fact that, by increasing (decreasing) the value of $N_{\rm eff}$, the matter-radiation equality occurs later (earlier). A similar effect comes from a decrease (increase) of the matter energy

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density [26]. The second effect is again related to the change of the sound horizon (at the recombination epoch), which becomes smaller when $N_{\rm eff}$ increases [28,29].

Another well-known effect given by decoupled relativistic species, in particular neutrinos, on the CMB angular power-spectrum, is the damping due to their anisotropic stress of the amplitude of the GW spectrum by 35% [30,31]. Such an effect is quite enhanced in the frequency region between 10^{-16} and 10^{-10} Hz, while it is less significant below 10^{-16} Hz, since this frequency region probes the Universe when it was matter dominated [30,32,33].

In this paper we compute the imprints of relativistic particles on the SGWB angular power spectrum: in particular, as far as the SW term is concerned, the effect is mainly due to the fact that gravitons decoupled at much larger energy scales (with respect to CMB photons) and the consequence of adding new species is to suppress its contribution. For the ISW effect we know that it is determined by the variation of the gravitational potentials Φ and Ψ from graviton decoupling until the present epoch. As we know from the CMB [28], there are two different ISW contributions: the early ISW, generated by the time variation of the modes when they cross the equality epoch, and the late ISW, due to the drastic changes of Φ and Ψ in the dark-energy-dominated era. Photons decoupled at last scattering, during the matter-dominated era, while gravitons decoupled long before (as detailed below); therefore the integration for computing the ISW effect for the gravitons started much earlier. This essentially does not modify the late ISW, while it changes considerably the early ISW. Following [15–17,26,28] we start by defining the distribution function f for the gravitons and we write down the Boltzmann equation for f in a perturbed spatially flat Friedmann-Lemaître-Robertson-Walker metric, accounting that gravitons move along null geodesics defined by the background metric (which should be understood including large-scale perturbations); we consider gravitons as collisionless particles, under the assumption that they decouple at early times. We include the effect of extra relativistic degrees of freedom on the scalar gravitational potentials through their contribution to the anisotropic stress, and then we quantify the impact on the angular SGWB power spectrum. We will briefly comment also on the effect on the tensor contribution to the SGWB spectrum, even if, as we will see, such effects will be important at much smaller scales.

II. BOLTZMANN EQUATION FOR GWs

To study the effect of new particle species on SGWB anisotropies we start from the Boltzmann equation for the gravitons. The collisions among GWs can be disregarded since they affect the distribution at higher orders (in a series expansion in the gravitational strength $1/M_{\rm P}$, see [34]). The emissivity can be related to some astrophysical processes (such as merging of compact objects) in the late Universe, as well as cosmological processes, so we treat the emissivity term as an initial condition on the GW distribution. The background metric on which our gravitons propagate is defined by $ds^2 = a^2(\eta)[-e^{2\Phi}d\eta^2 + (e^{-2\Psi}\delta_{ij} + h_{ij})dx^i dx^j]$, where $a(\eta)$ is the scale factor, η is conformal time, and we consider only scalar (Φ and Ψ) and tensor (h_{ij}, taken to be transverse and traceless) perturbations in the so-called Poisson gauge. The Boltzmann equation for the SGWB of cosmological origin can be computed in a similar way to what is done for the CMB [28,35,36]. It is convenient to rescale the perturbed part of the distribution function using the following redefinition $\delta f \equiv -q(\partial\bar{f}/\partial q)\Gamma(\eta, \vec{x}, q, \hat{n})$, $\bar{f}$ being the homogeneous and isotropic contribution, and the Boltzmann equation in Fourier space, keeping only the terms up to first order in the perturbations, reads [16,17]

$$\Gamma' + ik\mu\Gamma = S(\eta, \vec{k}, \hat{n}), \tag{1}$$

where $\hat{n} \equiv \hat{p}$ is the direction of motion of the GWs, while the source function is $S = \Psi' - ik\mu\Phi - n^i n^j h'_{ij}/2$ (primes denoting differentiation with respect to conformal time and $\mu \equiv \hat{k}\cdot\hat{n}$). The quantity Γ can be immediately related to the perturbation of the GW energy density, specifically to the SGWB density contrast $\delta_{\rm GW}$ and to the GW energy density fractional contribution $\Omega_{\rm GW}$ [16,17],

$$\delta_{\rm GW} = \left[4 - \frac{\partial \ln \bar{\Omega}_{\rm GW}(\eta, q)}{\partial \ln q}\right]\Gamma(\eta, \vec{x}, q, \hat{n}). \tag{2}$$

As shown in [32], $\bar{\Omega}_{\rm GW}$, the homogeneous, isotropic component of $\Omega_{\rm GW}$, is sensitive to the evolution of the relativistic degrees of freedom g_* before matter-radiation equality. From the end of inflation until the present epoch, the temperature of the different particle species decreases, and many of them become nonrelativistic, $T_\alpha \lesssim m_\alpha$, giving no more contribution to g_*, which changes from $g_*(T \gtrsim 10^4\ \text{MeV}) \simeq 106$, when all the Standard Model particles contribute, to $g_*(T \lesssim 0.1\ \text{MeV}) = 3.36$, when only photons and relativistic neutrinos contribute [29,37].

III. EFFECTS ON THE SCALAR PERTURBATIONS

The main role of relativistic particles is played on the "scalar" part of the anisotropic stress

$$k^2(\Phi - \Psi) = -32\pi G a^2 \rho_r \mathcal{N}_2, \tag{3}$$

where $\mathcal{N}_2$ is the quadrupole moment generated by the relativistic particles. The fractional energy density of decoupled relativistic particles can be described in terms of degrees of freedom as

$$f_{\rm dec}(\eta_i) \equiv g_*^{\rm dec}(T_i)/g_*(T_i), \tag{4}$$

where $g_*^{\rm dec}(T_i)$ are the relativistic degrees of freedom of decoupled particles evaluated at temperature T_i at the end of inflation, corresponding to conformal time η_i. This influences the initial conditions for the scalar metric perturbations at the end of inflation η_i [38,39]:

$$\Psi(\eta_i,k) = \left(1+\frac{2}{5}f_{\rm dec}(\eta_i)\right)\Phi(\eta_i,k), \tag{5}$$

where the initial value of Φ is related to the value of the gauge-invariant curvature perturbation ζ of comoving spatial hypersurfaces at the end of inflation, $\zeta(\eta_i,k) = \zeta_I(k)$,

$$\Phi(\eta_i,k) = -\frac{2}{3}\left(1+\frac{4}{15}f_{\rm dec}(\eta_i)\right)^{-1}\zeta_I(k). \tag{6}$$

The fractional energy density of decoupled relativistic particles varies since η_i down to temperatures around 0.1 MeV, when it reaches a constant value which depends on the chosen $N_{\rm eff}$: for instance for three light neutrino species it corresponds to $f_{\rm dec}(\eta_{T<0.1\ {\rm MeV}}) = 0.4$ [different evolutions of $f_{\rm dec}(\eta)$ for different particle candidates are shown for instance in [40]]. In this interval Φ and Ψ evolve following Eqs. (5) and (6) for different $f_{\rm dec}(\eta)$ values. At lower temperatures the features of large-scale, small-scale, and scales with $k\approx k_{\rm eq}$ evolutions are described in [28,41,42]. Until $\eta \gtrsim \eta_{\rm eq}$, decoupled relativistic particles make a substantial contribution to the total energy density and Eq. (3) shows that Φ and Ψ evolve differently. For $\eta \gg \eta_{\rm eq}$ no more species contribute considerably to the anisotropic stress and Φ and Ψ become approximately equal because the Universe is matter dominated.

IV. CORRELATORS OF GW ANISOTROPIES AND EXTRA SPECIES CONTRIBUTION

Following the treatment adopted for CMB anisotropies, we expand the solution in spherical harmonics, $\Gamma(\hat{n}) = \sum_\ell \sum_{m=-\ell}^{\ell}\Gamma_{\ell m}Y_{\ell m}(\hat{n})$. We focus on two contributions, even though, as shown in [16,17] there are three contributions to the anisotropies (the third contribution being an intrinsic initial perturbation of the distribution function that is not relevant here for our purposes). There is a first contribution $\Gamma_{\ell m,S}$ due to the scalar sources in Eq. (1), given by the sum of a SW term, similar to CMB photons, plus an ISW term, which are both affected by the presence of extraparticle species after inflation. Then there is a second contribution $\Gamma_{\ell m,T}$ due to the tensor modes in Eq. (1), which is not important for our purposes. More details on the tensor source sector can be found in [16,17]. Therefore the SGWB angular power spectrum reads $\langle\Gamma_{\ell m}\Gamma^*_{\ell' m'}\rangle \equiv \delta_{\ell\ell'}\delta_{mm'}\tilde{C}_\ell = \delta_{\ell\ell'}\delta_{mm'}[\tilde{C}_{\ell,S}+\tilde{C}_{\ell,T}]$, where we denote the correlators with a tilde to distinguish them from the CMB case. Focusing only on the scalar contribution to the angular power spectrum we have

$$\frac{\tilde{C}_{\ell,S}(\eta_0)}{4\pi} = \int\frac{dk}{k}P^{(0)}(k)\Big\{T_\Phi(\eta_i,k)j_\ell[k(\eta_0-\eta_i)] + \int_{\eta_i}^{\eta_0}d\eta[T'_\Phi(\eta,k)+T'_\Psi(\eta,k)]j_\ell[k(\eta_0-\eta)]\Big\}^2, \tag{7}$$

where $P^{(0)}(k)$ is the primordial scalar power spectrum, j_ℓ are the spherical Bessel functions of order ℓ and η_0 is the conformal time at the present epoch. It is important to notice that in this case η_i corresponds to the time at which gravitons decoupled (the end of inflation). In fact, even if gravitational interactions decoupled around the Planck energy scale, the SGWB has been produced after the Planck epoch, i.e., during inflation, thus we can state that the cosmological GWs decoupled at the end of inflation, because at that time they started their free streaming. For the CMB the situation is different: the initial integration time corresponds to recombination, $T_{\rm rec}\simeq 0.3$ eV. In the following, we are going to quantify the effect of the extra relativistic species on such terms, which dominate on large scales, and as such they are the ones which can be probed by GW interferometers due to their limited angular resolution [43–46].

V. SACHS-WOLFE EFFECT AND RELATIVISTIC PARTICLE SPECIES

Similarly to the CMB case, at large angular scales the dominant term of the scalar contribution to the angular power spectrum is the SW one. Taking into account the initial time η_i for the SGWB case, we modified the public code CLASS for the computation of CMB anisotropies [47] adapting it to the SGWB. In Fig. 1 we have plotted $\tilde{C}_{\ell,S}$, showing how different values of $f_{\rm dec}(\eta_i)$ (and thus different choices for the end of inflation energy scale and implicitly for the number of relativistic particles present at that time) affect differently the spectra. In the absence of a specific particle physics model for describing the decoupled relativistic species at η_i, we have varied $f_{\rm dec}(\eta_i)$ over all its domain, between 0 and 1, in this way we have determined the maximum and the minimum SW effect (values close to 1 are only considered for illustrative purposes; as such a large fraction is not physically achievable). We can also give a simple analytic estimate of the SW contribution starting from Eq. (6). Considering that the effect is generated by particles that are relativistic at their decoupling ($T_{\rm dec} > m$), a simple estimate of the damping at low ℓ is given by

$$\frac{\tilde{C}^{SW}_{\ell,S}(\eta_0)}{4\pi} = \frac{4}{9}\left[1+\frac{4}{15}f_{\rm dec}(\eta_i)\right]^{-2} \times \int\frac{dk}{k}P^{(0)}(k)j^2_\ell[k(\eta_0-\eta_i)]. \tag{8}$$

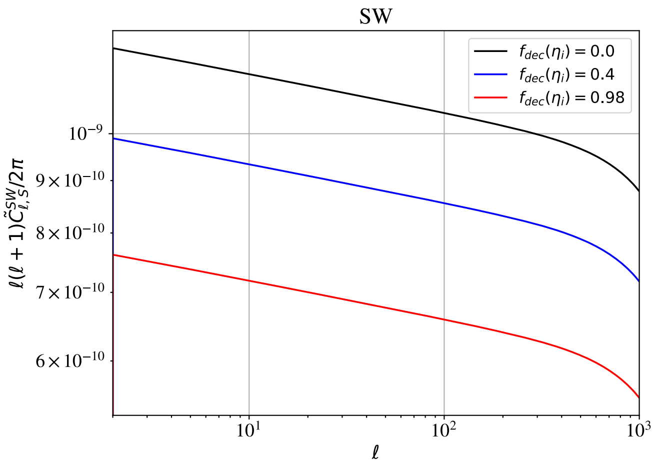

FIG. 1. SW contribution to the angular power spectrum of the SGWB. We can see that by increasing $f_{\rm dec}(\eta_i)$ we are decreasing more and more the amplitude of the angular power spectrum. For values of $f_{\rm dec}(\eta_i)$ close to 1 we observe a saturation.

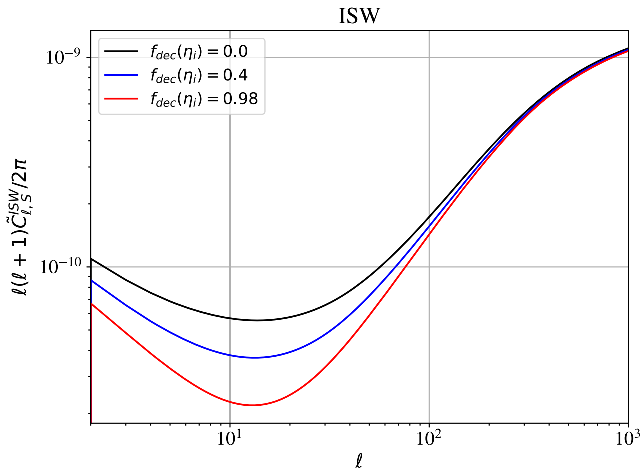

FIG. 2. ISW contribution to $\tilde{C}_{\ell,S}$. We observe a bump at large ℓ due to the fact that the potentials at large ℓ have the maximum variation.

Measuring deviations in the SGWB anisotropies at low ℓ from the angular power spectrum for $f_{\rm dec}(\eta_i) = 0$ would be a proof that at early epochs there were decoupled relativistic particle species contributing to the total energy density by an amount $f_{\rm dec}(\eta_i)$. Notice that η_i would correspond to temperatures so high that any Standard Model particle would be coupled at that epoch, thus any species responsible for this effect should arise in theories beyond the Standard Model. We can therefore conclude that this damping effect would provide precious information about new physics.

The most promising candidate which can give important contributions to $f_{\rm dec}(\eta_i)$ is extraradiation (ER), parametrized as $\Delta N_{\rm eff}$, the excess from the standard value of 3.046 for the effective neutrino number $N_{\rm eff}$ [27]. These new species are relativistic at the present epoch, so they were relativistic at the end of inflation too. They cannot be Standard Model particles, therefore it is reasonable to suppose that they decoupled at temperatures higher than the energies reached in modern accelerators, $T^{\rm ER}_{\rm dec} \gtrsim 10^6$ GeV. This is consistent with the hypothesis that at the end of inflation they were decoupled too, i.e., $\eta_{T^{\rm ER}_{\rm dec}} \lesssim \eta_i$. On the other hand, if we fix a specific particle physics model, we are able to describe the evolution of the decoupled relativistic degrees of freedom, or, in other words, we know $f_{\rm dec}(\eta)$. Under such a hypothesis, a measurement of the SGWB angular power spectrum would allow us to determine a range for η_i, on the basis of the evolution of $f_{\rm dec}$.

VI. INTEGRATED SACHS-WOLFE EFFECT AND RELATIVISTIC PARTICLE SPECIES

As anticipated, the ISW effect is roughly proportional to the total variation of the potentials $\Delta\Phi + \Delta\Psi$, so, when we consider the total variation for the SGWB, we end up with larger variations with respect to the CMB, because in the CMB case we take the difference between the initial value (at recombination) and the present epoch, but at recombination the potentials were already damped, especially at small scales, therefore they would have a smaller impact on the ISW.

The ISW contribution depends upon the variation of the potentials between η_i and η_0; so it is sensitive to the evolution of $f_{\rm dec}(\eta_i)$ up to low energies scales ($T \lesssim 0.1$ MeV). Thus measurements of the anisotropies of the SGWB anisotropies can constrain extra particles species both at high and low energy scales. The effect of the change of number of relativistic degrees of freedom on the ISW contribution to the angular power spectrum is represented in Fig. 2. As anticipated, a higher number of relativistic species suppresses the ISW contribution at the largest angular scales through its effect on the early ISW contribution.

We can sum up the two "scalar" contributions to show the main effect on large angular scales that, in the future,

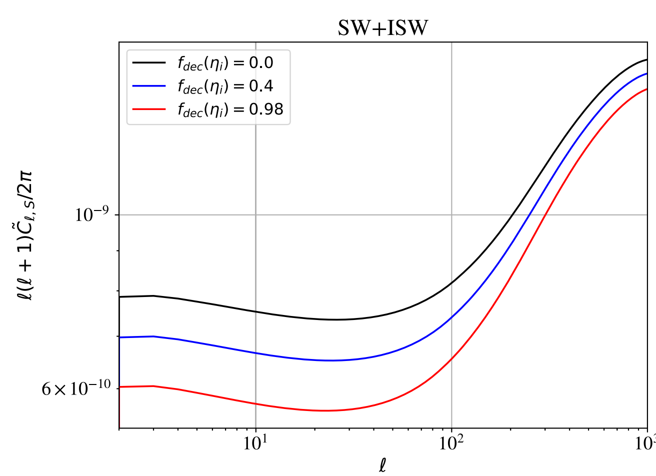

FIG. 3. Total scalar contribution to the SGWB angular power spectrum, sum of the SW and the ISW terms.

can be probed by GW interferometers. The result is given in Fig. 3, where the impact of a varying number of decoupled relativistic species is evident. We did not consider the contribution coming from the tensor background perturbations since we checked that they do not alter the spectrum at scales that can be probed in the future by GW direct detection experiments. As is well known [30], decoupled relativistic particles, and in particular neutrinos, create a damping on the amplitude of the tensor modes in the CMB. In a similar way relativistic particles have an impact also on the monopole amplitude of the GW energy density [32,33] and so on the amplitude of the angular power spectrum.

VII. CONCLUSIONS

In this paper we have shown that the future detection of the SGWB of cosmological origin has profound implications on our understanding of the physics of the early Universe and on high energy physics aspects not accessible by present-day particle accelerators. We have shown that the anisotropies of the SGWB inherited by the GW generated during their propagation in the Universe, from the time of their decoupling at the end of inflation until today, feel the effect of relativistic particle species that are decoupled from the thermal bath. Having in mind the poor angular resolution of future GW detectors, we have focused on the effects most relevant at very large scales. As for the CMB, also for the SGWB, such scales are affected by the SW effect and by the ISW effect. We have therefore quantified the effect of different particle species on both the SW and ISW, and we have computed the SGWB angular power spectrum. The cumulative effect of a larger number of decoupled relativistic particle species on the angular power spectrum of the SGWB is a suppression at large scales. This will clearly becomes a potential observable effect as soon as such anisotropies will be detected.

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