

# ILL POSEDNESS FOR THE FULL EULER SYSTEM DRIVEN BY MULTIPLICATIVE WHITE NOISE

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**ABSTRACT.** We consider the Euler system describing the motion of a compressible fluid driven by a multiplicative white noise. We identify a large class of initial data for which the problem is ill posed - there exist infinitely many global in time weak solutions. The solutions are adapted to the noise and satisfy the entropy admissibility criterion.

## 1. INTRODUCTION

Problems in continuum fluid mechanics involving perfect (inviscid) fluids are in general *ill posed* in the class of weak solutions. The adaptation of the method of convex integration, developed in the pioneering work of De Lellis and Székelyhidi [5] that culminated by the final proof of Onsager's conjecture for the incompressible Euler system, see Isett [12], Buckmaster et al. [2], produced a number of examples of non-uniqueness even in the context of compressible fluids, see [3], [4], [8], and Markfelder, Klingenberg [15], among others. In accordance with the results, the initial-value problem for the Euler system is ill posed even in the class of weak solutions satisfying various kinds of admissibility criteria as the energy and/or entropy inequality.

There is a piece of evidence that *stochastic* perturbations may provide a regularizing effect on deterministically ill-posed problems, in particular those involving transport, see e.g. [7], [10], [11]. On the other hand, as shown in [1], the isentropic Euler system driven by a general additive/multiplicative white noise is ill posed in the class of weak solutions. More specifically, there is a large class of initial data giving rise to infinitely many solutions defined up to a positive stopping time. These solutions, however, may experience an initial energy jump and as such can be discarded as physically irrelevant.

Our goal in the present paper is to show the existence of infinitely many global-in-time weak solutions to a stochastically driven Euler system that are physically admissible – they conserve the total energy and satisfy the differential version of the entropy

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*Date:* April 18, 2019.

The research of E.F. leading to these results has received funding from the Czech Sciences Foundation (GAČR), Grant Agreement 18-05974S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

inequality. Specifically, we consider the *full Euler system*:

$$(1.1) \quad \begin{aligned} d\rho + \operatorname{div}_x \mathbf{m} dt &= 0 \\ d\mathbf{m} + \operatorname{div}_x \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) dt + \nabla_x p dt &= -\frac{1}{2} \mathbf{m} \circ dW \\ dE + \operatorname{div}_x \left( (E + p) \frac{\mathbf{m}}{\rho} \right) dt &= -E \circ dW, \end{aligned}$$

where

$$E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\rho} + \rho e, \quad p = (\gamma - 1)\rho e, \quad \gamma > 1.$$

Introducing the temperature  $\vartheta$  via Boyle–Mariotte law,

$$(1.2) \quad p = \rho \vartheta, \quad e = \frac{1}{\gamma - 1} \vartheta \equiv c_v \vartheta,$$

we obtain the entropy inequality

$$(1.3) \quad d(\rho s) + \operatorname{div}_x (s \mathbf{m}) dt \geq -c_v \rho \circ dW.$$

For definiteness, we impose the impermeability condition

$$(1.4) \quad \mathbf{m} \cdot \mathbf{n}|_{\partial Q} = 0.$$

Here,  $W$  denotes the standard scalar valued Wiener process while the symbol  $\circ$  indicates that the stochastic integral in the weak formulation of the problem is interpreted in the Stratonovich sense.

We show the existence of infinitely many solutions following the strategy of Luo, Xie, Xin [14] used also in [9]. Specifically, we choose arbitrary *piece-wise* constant initial distributions of the density and the absolute temperature and we transform the problem into a family of partial differential equations with random parameters. Then we apply the result of De Lellis and Székelyhidi [5] for the incompressible Euler system with constant pressure on each domain where the initial data are constant. Finally, we pass back to the original system “pasting” together the solutions previously obtained. The issue of progressive measurability of the oscillatory solutions, that was absolutely crucial for the analysis in [1], is handled here by introducing a new stochastically rescaled time variable.

The paper is organized as follows. In Section 2, we introduce the necessary preliminary material and state our main result. The main ideas of the proof are described in Section 3.1, where the transformation into a system with random coefficients is performed. In Section 3.2, we apply the nowadays standard tools of convex integration to the transformed problem. In Section 3.4, we introduce a new “random” time variable. The existence proof is completed in Section 3.5.

## 2. MAIN RESULT

Let  $\{\Omega, \mathfrak{F}, \mathcal{P}\}$  be a probability basis, with a right continuous complete filtration  $\{\mathfrak{F}_t\}_{t \geq 0}$ , and a Wiener process  $W$ .

**Definition 2.1.** We say that  $[\varrho, \mathbf{m}, E]$  is a weak solution of the Euler system (1.1), with the boundary condition (1.4), and the initial condition

$$(2.1) \quad \varrho(0, \cdot) = \varrho_0, \quad \mathbf{m}(0, \cdot) = \mathbf{m}_0, \quad E(0, \cdot) = E_0,$$

if:

- $\varrho \geq 0$   $\mathcal{P}$ -a.s., the functions

$$t \mapsto \int_Q \varrho(t, \cdot) \varphi \, dx, \quad t \mapsto \int_Q \mathbf{m}(t, \cdot) \cdot \varphi \, dx$$

are continuous  $\{\mathfrak{F}_t\}_{t \geq 0}$ -adapted semimartingales for any  $\varphi \in C^1(\overline{Q})$ ,  $\varphi \in C^1(\overline{Q}; R^N)$ , respectively,

$$(2.2) \quad \int_Q \varrho(\tau, \cdot) \varphi \, dx = \int_Q \varrho_0 \varphi \, dx + \int_0^\tau \int_Q \mathbf{m} \cdot \nabla_x \varphi \, dx$$

for any  $\tau \geq 0$  and any  $\varphi \in C^1(\overline{Q})$ ;

- $E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \geq 0$   $\mathcal{P}$ -a.s., the function

$$t \mapsto \int_Q E(t, \cdot) \varphi \, dx$$

is a continuous  $\{\mathfrak{F}_t\}_{t \geq 0}$ -adapted semimartingale for any  $\varphi \in C^1(\overline{Q})$ ;

(2.3)

$$\begin{aligned} & \int_Q \mathbf{m}(\tau, \cdot) \cdot \varphi \, dx \\ &= \int_Q \mathbf{m}_0 \cdot \varphi \, dx + \int_0^\tau \int_Q \left[ \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \varphi + p \operatorname{div}_x \varphi \right] \, dx - \frac{1}{2} \int_0^\tau \left( \int_Q \mathbf{m} \cdot \varphi \, dx \right) \circ dW \end{aligned}$$

for any  $\tau \geq 0$  and any  $\varphi \in C^1(\overline{Q}; R^N)$ ,  $\varphi \cdot \mathbf{n}|_{\partial Q} = 0$ , where  $p = (\gamma - 1) \left[ E - \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} \right]$ ;

- the energy equality

(2.4)

$$\int_Q E(\tau, \cdot) \varphi \, dx = \int_Q E_0 \varphi \, dx + \int_0^\tau \int_Q (E + p) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi \, dx \, dt - \int_0^\tau \left( \int_Q E \varphi \, dx \right) \circ dW$$

holds for any  $\tau \geq 0$  and any  $\varphi \in C^1(\overline{Q})$ ;

- the entropy inequality

(2.5)

$$\int_Q \varrho s(\tau, \cdot) \varphi \, dx \geq \int_Q \varrho_0 s(\varrho_0, E_0) \varphi \, dx + \int_0^\tau \int_Q \mathbf{sm} \cdot \nabla_x \varphi \, dx - \int_0^\tau \left( \int_Q c_v \varrho \varphi \, dx \right) \circ dW$$

holds for any  $\tau \geq 0$  and any  $\varphi \in C^1(\overline{Q})$ ,  $\varphi \geq 0$ .

It is worth noting that the solutions introduced above are weak in the PDE sense - partial derivatives are interpreted in the sense of distributions - but strong in the stochastic sense - stochastic integral is considered on the original probability space. Our goal is to show the following result.

**Theorem 2.2.** *Let  $Q \subset R^N$ ,  $N = 2, 3$  be a bounded domain,*

$$Q = \cup_{i=1}^\infty \overline{Q}_i, \quad Q_i \text{ domains, } \quad Q_i \cap Q_j = \emptyset \text{ for } i \neq j.$$

*Suppose that  $\varrho_0, \vartheta_0 \in L^1(Q)$  are  $\mathfrak{F}_0$ -adapted random variables satisfying*

$$0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho}, \quad 0 < \underline{\vartheta} \leq \vartheta_0 \leq \overline{\vartheta} \quad \mathcal{P} - a.s.$$

*for some deterministic constants  $\underline{\varrho}, \overline{\varrho}, \underline{\vartheta}, \overline{\vartheta}$  and such that*

$$\varrho_0|_{Q_i} = \varrho_{0,i}, \quad \vartheta_0|_{Q_i} = \vartheta_{0,i} \quad i = 1, 2, \dots, \quad \text{where } \varrho_{0,i}, \vartheta_{0,i} \text{ are constant.}$$

*Then there exists a deterministic constant  $\mathcal{E}_0$  such that for any  $\mathcal{E} > \mathcal{E}_0$  there exists an  $\mathfrak{F}_0$ -adapted random field  $\mathbf{m}_0 \in L^\infty(Q; R^N)$ , such that*

$$\int_\Omega \left[ \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + c_v \varrho_0 \vartheta_0 \right] dx \geq \mathcal{E} \quad \mathcal{P} - a.s.,$$

*and the problem (1.1), (1.4), (2.1) admits infinitely many weak solutions in  $(0, \infty) \times Q$  with the initial data*

$$\varrho_0, \quad \mathbf{m}_0, \quad E_0 = \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + c_v \varrho_0 \vartheta_0.$$

The rest of the paper is devoted to the proof of Theorem 2.2.

### 3. PROOF OF THEOREM 2.2

**3.1. Constant initial data.** We first assume that  $\varrho_0, \vartheta_0$  are positive (random) constants admitting deterministic lower and upper bounds as in Theorem 2.2. Later we extend the result to piecewise constant data by “pasting” solutions together.

3.1.1. *Solenoidal fields.* We look for solenoidal momentum fields  $\mathbf{m}$  with vanishing normal trace, meaning

$$(3.1) \quad \int_Q \mathbf{m} \cdot \nabla_x \varphi \, dx = 0 \text{ for any } \varphi \in C^1(\overline{Q}).$$

If we then set  $\varrho(t, \cdot) = \varrho_0$  for any  $t \geq 0$ , in particular, the equation of continuity (2.2) is automatically satisfied and  $\int_Q \varrho(t, \cdot) \, dx$  is a semimartingale.

3.1.2. *Temperature field.* Writing the internal energy equation as

$$d(\varrho e) + \operatorname{div}_x(e\mathbf{m}) \, dt = -(\varrho e) \circ dW - p \operatorname{div}_x \left( \frac{\mathbf{m}}{\varrho} \right) dt$$

we realize that, since  $\operatorname{div}_x \mathbf{m} = 0$  and  $\varrho = \varrho_0$  constant, then the unique solution is given by

$$(3.2) \quad e = \varrho \vartheta, \quad \vartheta = \vartheta_0 \exp(-W(t)),$$

where  $\vartheta_0$  is the constant initial temperature. Obviously both  $\varrho$  and  $\vartheta$  are continuous  $\{\mathfrak{F}_t\}_{t \geq 0}$ -adapted semimartingales. When we have established that  $\int_Q |\mathbf{m}|^2 \varphi \, dx$  is a semimartingale, then also  $\int_Q E(t, \cdot) \varphi \, dx$  is a semimartingale.

3.1.3. *Momentum equation.* The computations of this subsection have to be computed rigorously in the reversed order, using the rules of Stratonovich calculus and starting from the  $\mathfrak{F}_0$ -measurable process  $\mathbf{v}(t, \cdot)$ . We first present them in this order for convenience of intuition.

In view of (3.2), the momentum equation reads

$$d\mathbf{m} + \frac{1}{2} \mathbf{m} \circ dW + \operatorname{div}_x \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho_0} \right) dt + \nabla_x \left( \varrho_0 \vartheta_0 \exp(-W(t)) \right) dt = 0.$$

Using the chain rule for the Stratonovich integral, we obtain

$$\begin{aligned} d \left[ \mathbf{m} \exp \left( \frac{1}{2} W(t) \right) \right] + \exp \left( \frac{1}{2} W(t) \right) \operatorname{div}_x \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho_0} \right) dt \\ + \nabla_x \left( \varrho_0 \vartheta_0 \exp \left( -\frac{1}{2} W(t) \right) \right) dt = 0. \end{aligned}$$

In order to apply the convex integration argument, we need to recast the equation in a suitable way. Thanks to the computations above based on Stratonovich calculus, it is easy to observe that by introducing a new variable  $\mathbf{w}$ ,

$$\mathbf{w} = \mathbf{m} \exp \left( \frac{1}{2} W(t) \right),$$

we obtain the following PDE with random coefficients

$$\partial_t \mathbf{w} + \exp\left(-\frac{1}{2}W(t)\right) \left[ \operatorname{div}_x \left( \frac{\mathbf{w} \otimes \mathbf{w}}{\varrho_0} \right) + \nabla_x (\varrho_0 \vartheta_0) \right] = 0.$$

Moreover, introducing a new time variable

$$t \approx \int_0^t \exp\left(-\frac{1}{2}W(s)\right) \, ds$$

we obtain the system

$$(3.3) \quad \partial_t \mathbf{v} + \operatorname{div}_x \left( \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} + \varrho_0 \vartheta_0 \mathbb{I} \right) = 0, \quad \operatorname{div}_x \mathbf{v} = 0, \quad \mathbf{v}(0, \cdot) = \mathbf{v}_0,$$

which can be now treated in the ‘‘convex integration framework’’. To allow the ‘‘pasting’’ of piecewise constant solutions, the problem (3.3) should be supplemented with ‘‘do nothing’’ boundary conditions, specifically, its weak formulation reads:

$$(3.4) \quad \int_0^\infty \int_Q \left[ \mathbf{v} \cdot \partial_t \varphi + \left( \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} + \varrho_0 \vartheta_0 \mathbb{I} \right) : \nabla_x \varphi \right] \, dx \, dt = - \int_\Omega \mathbf{v}_0 \cdot \varphi(0, \cdot) \, dx$$

for any  $\varphi \in C_c^1([0, \infty) \times \overline{Q}; R^N)$ . As shown in the forthcoming section, problem (3.4) admits infinitely many solutions for suitable initial data provided by the method of convex integration.

**3.2. Convex integration.** To finally apply the method of convex integration, we reformulate the problem (3.4). Specifically, we replace (3.4) by

$$(3.5) \quad \int_0^\infty \int_Q \left[ \mathbf{v} \cdot \partial_t \varphi + \left( \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} - \frac{1}{N} \frac{|\mathbf{v}|^2}{\varrho_0} \mathbb{I} \right) : \nabla_x \varphi \right] \, dx \, dt = - \int_Q \mathbf{v}_0 \cdot \varphi(0, \cdot) \, dx$$

for any  $\varphi \in C_c^\infty([0, \infty) \times R^N; R^N)$ . In addition, we prescribe the energy

$$\frac{1}{2} \frac{|\mathbf{v}|^2}{\varrho_0} = K_0,$$

where  $K_0 > 0$  is a positive random variable adapted to  $\mathfrak{F}_0$ .

If  $\varrho_0, K_0$  were deterministic quantities, the nowadays standard method developed by De Lellis and Székelyhidi in [6] would yield the existence of an initial datum  $\mathbf{v}_0 \in L^\infty(\Omega; R^N)$  such that:

•

$$\operatorname{div}_x \mathbf{v}_0 = 0, \quad \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0;$$

- the problem (3.5) admits infinitely many solutions  $\mathbf{v}$  satisfying

$$(3.6) \quad \int_0^\infty \int_Q \mathbf{v} \cdot \nabla_x \varphi \, dx \, dt = 0$$

for all  $\varphi \in C_c^1([0, \infty) \times \overline{Q}; R^N)$ ;

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$$(3.7) \quad \frac{1}{2} \frac{|\mathbf{v}_0|^2}{\varrho_0} = \frac{1}{2} \frac{|\mathbf{v}(t, \cdot)|^2}{\varrho_0} = K_0 \text{ a.a. in } \Omega \text{ for any } t \geq 0.$$

This result has been extended to the random setting in [1]. Indeed, if  $\varrho_0$ ,  $\vartheta_0$ , and  $K_0$  are  $\mathfrak{F}_0$ -adapted random variables, the stochastic version of the oscillatory lemma proved in [1, Lemma 5.7] can be applied to deduce that the solutions  $\mathbf{v}$ , obtained through process described in [5], are  $\mathcal{F}_0$  adapted as random variables considered in the space  $C_{\text{weak}}([0, \infty); L^2(Q; R^N))$ . More specifically, the random variable

$$t \mapsto \int_Q \mathbf{v}(t, \cdot) \cdot \varphi \, dx \in C[0, \infty) \text{ is } \mathfrak{F}_0 \text{ - adapted}$$

for any  $\varphi \in C^1(\overline{Q}; R^N)$ . Note that the present situation is much simpler than in [1] as the  $\sigma$ -field  $\mathcal{F}_0$  is independent of time.

Finally, we fix  $K_0$  in such a way that

$$(3.8) \quad \frac{N}{2} K_0 = \Lambda_0 - \varrho_0 \vartheta_0 > 0$$

where  $\Lambda_0$  is a (random) constant. Note that, in view of our hypotheses imposed on the data  $\varrho_0$ ,  $\vartheta_0$ , the quantity  $\Lambda_0$  can be chosen in a deterministic way.

Summarizing we obtain the following result.

**Proposition 3.1.** *Let  $Q \subset R^N$ ,  $N = 2, 3$  be a bounded domain. Suppose that  $\varrho_0$ ,  $\vartheta_0$ ,  $\Lambda_0$  are given (real valued) random variables that are  $\mathfrak{F}_0$ -adapted and satisfy*

$$\varrho_0, \vartheta_0, \Lambda_0 - \varrho_0 \vartheta_0 > 0 \text{ } \mathcal{P} \text{ - a.s.}$$

*Then there exists a random variable  $\mathbf{v}_0 \in L_{\text{weak}(\ast)}^\infty(Q; R^N)$  and infinitely many*

$$\mathbf{v} \in C_{\text{weak}}([0, \infty); L^2(Q; R^N))$$

*satisfying:*

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$$(3.9) \quad t \mapsto \int_Q \mathbf{v}(t, \cdot) \cdot \varphi \, dx \in C[0, \infty) \text{ are } \mathfrak{F}_0 \text{ - adapted}$$

*for any  $\varphi \in C^1(\overline{Q}; R^N)$ ;*

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$$(3.10) \quad \int_0^\infty \int_Q \mathbf{v} \cdot \nabla_x \varphi \, dx \, dt = 0$$

for all  $\varphi \in C_c^1([0, \infty) \times \overline{Q}; R^N)$ ;

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$$(3.11) \quad \frac{1}{2} \frac{|\mathbf{v}_0|^2}{\varrho_0} = \frac{1}{2} \frac{|\mathbf{v}(t, \cdot)|^2}{\varrho_0} = \frac{2}{N} (\Lambda_0 - \varrho_0 \vartheta_0) \quad \text{a.a. in } Q \text{ for any } t \geq 0;$$

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$$(3.12) \quad \int_0^\infty \int_Q \left[ \mathbf{v} \cdot \partial_t \varphi + \left( \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} + (\varrho_0 \vartheta_0 - \Lambda_0) \mathbb{I} \right) : \nabla_x \varphi \right] dx \, dt = - \int_Q \mathbf{v}_0 \cdot \varphi(0, \cdot) \, dx$$

for any  $\varphi \in C_c^1([0, \infty) \times \overline{Q}; R^N)$ .

**Remark 3.2.** Note that the original result of De Lellis and Székelyhidi [5] would apply without modification should the initial data  $\varrho_0, \vartheta_0$  be deterministic.

The conclusion of Proposition 3.1 should be seen as a starting point of the existence of infinitely many solutions claimed in Theorem 2.2. Note that, at this level, the density  $\varrho$ , as well as the kinetic energy  $\frac{|\mathbf{m}|^2}{\varrho}$  are in fact constants independent of the time variable.

**3.3. Piecewise constant data.** We claim that the conclusion of Proposition 3.1 remains valid if the quantities  $\varrho_0, \vartheta_0$ , and  $\Lambda_0$  are piecewise constant random variables as required in Theorem 2.2. Specifically, we suppose that

$$\overline{Q} = \cup_{i=1}^\infty \overline{Q}_i, \quad Q_i \cap Q_j = \emptyset \text{ if } i \neq j,$$

and

$$\varrho_0 = \varrho_{0,i}, \quad \vartheta_0 = \vartheta_{0,i}, \quad \Lambda_0 = \Lambda_{0,i} \text{ in } Q_i, \quad i = 1, \dots$$

Indeed such a generalization is possible as the integrals in (3.10), (3.12) are additive, and the test functions need not vanish on  $\partial Q_i$ . We simply apply Proposition 3.1 on each  $Q_i$  and take the sum of the corresponding integrals in (3.10), (3.12).

In addition, if  $\varrho_0, \vartheta_0$  are bounded by deterministic constants as in Theorem 2.2, we can choose the constants  $\Lambda_{0,i} = \Lambda_0$  the same on each  $Q_i$ . In particular, equation (3.12) gives rise to

$$(3.13) \quad \int_0^\infty \int_Q \left[ \mathbf{v} \cdot \partial_t \varphi + \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} : \nabla_x \varphi + \varrho_0 \vartheta_0 \operatorname{div}_x \varphi \right] dx \, dt = - \int_Q \mathbf{v}_0 \cdot \varphi(0, \cdot) \, dx$$

for any  $\varphi \in C_c^\infty([0, \infty) \times \overline{Q}; R^N)$  as long as  $\varphi \cdot \mathbf{n}|_{\partial Q} = 0$ .



Next, we derive from (3.6) that

$$(3.14) \quad \int_0^\infty \int_Q \mathbf{v} \cdot \nabla_x \varphi \, dx \, dt = 0$$

for any  $\varphi \in C_c^1([0, \infty) \times \overline{Q})$ .

Finally, the kinetic energy is piecewise constant and independent of time,

$$(3.15) \quad \frac{1}{2} \frac{|\mathbf{v}(t, \cdot)|^2}{\varrho} = \frac{1}{2} \frac{|\mathbf{v}_0|^2}{\varrho_0} = \frac{2}{N} (\Lambda_0 - \varrho_0 \vartheta_0) \text{ a.a. in } Q \text{ for any } t \geq 0.$$

Thus Proposition 3.1 can be extended to piece-wise constant data as follows.

**Proposition 3.3.** *Let  $Q \subset R^N$ ,  $N = 2, 3$  be a bounded domain,*

$$\overline{Q} = \cup_{i=1}^\infty \overline{Q}_i, \quad Q_i \cap Q_j = \emptyset \text{ if } i \neq j.$$

*Suppose that  $\varrho_0, \vartheta_0 \in L^1(Q)$ ,  $\Lambda_0 \in R$  are given random variables that are  $\mathfrak{F}_0$ -adapted, with  $\varrho_0, \vartheta_0$  piecewise constant, meaning*

$$\varrho_0 = \varrho_{0,i} > 0, \quad \vartheta_0 = \vartheta_{0,i} > 0 \text{ in } Q_i, \quad \Lambda_0 - \varrho_0 \vartheta_0 > 0 \text{ } \mathcal{P} - \text{a.s.}$$

*Then there exists an  $\mathfrak{F}_0$ -measurable random variable  $\mathbf{v}_0 \in L_{\text{weak}^*}^\infty(Q; R^N)$  and infinitely many*

$$\mathbf{v} \in C_{\text{weak}}([0, \infty); L^2(Q; R^N))$$

*satisfying:*

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- (3.16) 
$$t \mapsto \int_Q \mathbf{v}(t, \cdot) \cdot \boldsymbol{\varphi} \, dx \in C[0, \infty) \text{ are } \mathfrak{F}_0 - \text{adapted}$$

*for any  $\boldsymbol{\varphi} \in C^1(\overline{Q}; R^N)$ ;*

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- (3.17) 
$$\int_0^\infty \int_Q \mathbf{v} \cdot \nabla_x \boldsymbol{\varphi} \, dx \, dt = 0, \quad \int_0^\infty \int_{Q_i} \mathbf{v} \cdot \nabla_x \boldsymbol{\varphi} \, dx \, dt = 0, \quad i = 1, \dots$$

*for all  $\boldsymbol{\varphi} \in C_c^1([0, \infty) \times \overline{Q}; R^N)$ ;*

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- (3.18) 
$$\frac{1}{2} \frac{|\mathbf{v}_0|^2}{\varrho_0} = \frac{1}{2} \frac{|\mathbf{v}(t, \cdot)|^2}{\varrho_0} = \frac{2}{N} (\Lambda_0 - \varrho_0 \vartheta_0) \text{ a.a. in } Q \text{ for any } t \geq 0;$$

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- (3.19) 
$$\int_0^\infty \int_Q \left[ \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \left( \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} + \varrho_0 \vartheta_0 \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \right] \, dx \, dt = - \int_Q \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, dx$$

*for any  $\boldsymbol{\varphi} \in C_c^1([0, \infty) \times \overline{Q}; R^N)$ ,  $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial Q} = 0$ .*

**3.4. Rescaling time.** In this last part of the proof, we show how to go back from the “convex integration constructed  $\mathbf{v}$ ” to solutions  $\mathbf{m}$  of the original system (1.1). This can be done by computing and justifying formally the reversed transformations of the ones performed in Section 3.1.3 to obtain the system for  $\mathbf{v}$ . As a first step, we need to rescale time. Consider the function of time,

$$\langle \mathbf{v}, \phi \rangle \equiv t \mapsto \int_Q \mathbf{v}(t, \cdot) \cdot \phi \, dx, \quad t \in [0, \infty), \quad \phi \in C^1(\overline{Q}, R^N), \quad \phi \cdot \mathbf{n}|_{\partial Q} = 0.$$

It follows from (3.19) that  $\langle \mathbf{v}, \phi \rangle$  is globally Lipschitz on  $[0, \infty)$  with the time derivative

$$\frac{d}{dt} \langle \mathbf{v}, \phi \rangle = \int_Q \left[ \frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_0} : \nabla_x \phi + \varrho_0 \vartheta_0 \operatorname{div}_x \phi \right] \, dx \text{ for a.a. } t \in (0, \infty).$$

We introduce a new function  $\mathbf{w} \in C_{\text{weak}}([0, \infty); L^2(\Omega; R^N))$ ,

$$\begin{aligned} \langle \mathbf{w}, \phi \rangle &\equiv \int_Q \mathbf{w}(t, \cdot) \cdot \phi \, dx = \int_Q \mathbf{v} \left( \int_0^t \exp \left( -\frac{1}{2} W(s) \right) ds, x \right) \phi(x) \, dx, \\ \phi &\in C^1(\overline{Q}; R^N), \quad \phi \cdot \mathbf{n}|_{\partial Q} = 0. \end{aligned}$$

Note carefully that  $\mathbf{w}$  is  $(\mathfrak{F})_{t \geq 0}$ -adapted for any  $\phi$ , where  $(\mathfrak{F})_{t \geq}$  is the filtration associated to the noise  $W$ .

Since  $\langle \mathbf{v}, \varphi \rangle$  is Lipschitz function of time, we can use the abstract chain rule (see e.g. Ziemer [17]) to deduce that,  $\mathcal{P}$ -a.s.,

$$(3.20) \quad \frac{d}{dt} \int_Q \mathbf{w} \cdot \varphi \, dx = \exp \left( -\frac{1}{2} W(t) \right) \int_Q \left[ \frac{\mathbf{w} \otimes \mathbf{w}}{\varrho_0} : \nabla_x \varphi + \varrho_0 \vartheta_0 \operatorname{div}_x \varphi \right] \, dx$$

for any  $\varphi \in C^1(\overline{Q}; R^N)$ ,  $\varphi \cdot \mathbf{n}|_{\partial Q} = 0$ ,  $\mathbf{w}(0) \equiv \mathbf{w}_0 = \mathbf{v}_0$ .

Finally, we observe that the relations (3.14), (3.15) remain valid for  $\mathbf{w}$ , specifically,

$$(3.21) \quad \int_0^\infty \int_{Q_i} \mathbf{w} \cdot \nabla_x \varphi \, dx \, dt = 0, \quad i = 1, \dots$$

for any  $\varphi \in C_c^1([0, \infty) \times \overline{Q})$ , and

$$(3.22) \quad \frac{1}{2} \frac{|\mathbf{w}(t, \cdot)|^2}{\varrho} = \frac{1}{2} \frac{|\mathbf{w}_0|^2}{\varrho_0} = \frac{2}{N} (\Lambda_0 - \varrho_0 \vartheta_0) \text{ a.a. in } Q \text{ for any } t \geq 0.$$

**3.5. Chain rule for Stratonovich integral.** Finally, we can introduce the momentum  $\mathbf{m}$  in terms of the rescaled  $\mathbf{v}$  (i.e. of  $\mathbf{w}$ ), so to get solutions to system (1.1). To this aim it is essential the use of Stratonovich calculus. We introduce the momentum

$$\mathbf{m} = \mathbf{w} \exp \left( -\frac{1}{2} W(t) \right)$$

noting that

$$\mathbf{m}(0, \cdot) \equiv \mathbf{m}_0 = \mathbf{w}_0 = \mathbf{v}_0.$$

Obviously, the relation (3.21) applies to  $\mathbf{m}$ ,

$$(3.23) \quad \int_0^\infty \int_{Q_i} \mathbf{m} \cdot \nabla_x \varphi \, dx \, dt = 0, \quad i = 1, \dots$$

for any  $\varphi \in C_c^1([0, \infty) \times \overline{Q})$ .

Using the basic properties of Stratonovich integral, we obtain

$$(3.24) \quad d(b \exp(W)) = \exp(W) db + b \exp(W) \circ dW$$

whenever  $b$  is a Lipschitz function.

At this stage, we are ready to finish the proof of Theorem 2.2.

3.5.1. *Equation of continuity.* Setting  $\varrho = \varrho_0$  and using (3.23) we easily deduce the equation of continuity

$$(3.25) \quad d \int_Q \varrho \varphi \, dx = \int_Q \mathbf{m} \cdot \nabla_x \varphi \, dx \, dt, \quad \int_\Omega \varrho(0, \cdot) \varphi \, dx = \int_Q \varrho_0 \varphi \, dx$$

for any  $\varphi \in C^1(\overline{Q})$ .

3.5.2. *Internal energy, entropy, total energy.* We define

$$\vartheta = \vartheta_0 \exp(-W(t))$$

and, using the relation (3.23), (3.24), we easily deduce the internal energy equation

$$(3.26) \quad \begin{aligned} d \int_Q c_v \varrho \vartheta \varphi \, dx &= \int_Q c_v \varrho \vartheta \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi \, dx \, dt - \int_Q c_v \varrho \vartheta \varphi \, dx \circ dW \\ \int_Q \varrho \vartheta(0, \cdot) \varphi \, dx &= \int_Q \varrho_0 \vartheta_0 \varphi \, dx \end{aligned}$$

for any  $\varphi \in C^1(\overline{Q})$ .

Similarly, seeing that the entropy is

$$s(\varrho, \vartheta) = c_v \log(\vartheta) - \log(\varrho) = c_v \log(\vartheta_0) - \log(\varrho_0) - c_v W,$$

we obtain the entropy equation

$$(3.27) \quad \begin{aligned} d \int_Q \varrho s(\varrho, \vartheta) \varphi \, dx &= \int_Q s(\varrho, \vartheta) \mathbf{m} \cdot \nabla_x \varphi \, dx \, dt - \int_Q c_v \varrho \varphi \, dx \circ dW \\ \int_Q \varrho s(\varrho, \vartheta)(0, \cdot) \varphi \, dx &= \int_Q \varrho_0 s(\varrho_0, \vartheta_0) \varphi \, dx \end{aligned}$$

for any  $\varphi \in C^1(\overline{Q})$ .

**Remark 3.4.** Note that we have shown the existence of infinitely many solutions that satisfy the entropy *equation* instead of the mere inequality required in Definition 2.1.

Finally, by virtue of (3.22), the total energy reads

$$\begin{aligned} E &= \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + c_v \varrho \vartheta = \exp(-W(t)) \left( \frac{1}{2} \frac{|\mathbf{w}|^2}{\varrho} + c_v \varrho_0 \vartheta_0 \right) \\ &= \exp(-W(t)) \left( \frac{2}{N} (\Lambda_0 - \varrho_0 \vartheta_0) + c_v \varrho_0 \vartheta_0 \right). \end{aligned}$$

Thus, similarly to the above, we deduce the total energy balance

$$\begin{aligned} (3.28) \quad d \int_Q E \varphi \, dx &= \int_Q (E + \varrho \vartheta) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi \, dx \, dt - \int_Q E \varphi \, dx \circ dW \\ \int_\Omega E(0, \cdot) \varphi \, dx &= \int_Q \left( \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + c_v \varrho_0 \vartheta_0 \right) \varphi \, dx \end{aligned}$$

for any  $\varphi \in C^1(\overline{Q})$ .

3.5.3. *Momentum equation.* We multiply (3.20) on  $\exp(-\frac{1}{2}W)$  obtaining

$$\exp\left(-\frac{1}{2}W(t)\right) \frac{d}{dt} \left[ \exp\left(\frac{1}{2}W(t)\right) \int_Q \mathbf{m} \cdot \boldsymbol{\varphi} \, dx \right] = \int_Q \left[ \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \varrho \vartheta \operatorname{div}_x \boldsymbol{\varphi} \right] \, dx,$$

which, in view of (3.24), gives rise to the desired conclusion

$$(3.29) \quad d \int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \, dx = \int_Q \left[ \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + \varrho \vartheta \operatorname{div}_x \boldsymbol{\varphi} \right] \, dx \, dt - \frac{1}{2} \int_Q \mathbf{m} \cdot \boldsymbol{\varphi} \, dx \circ dW$$

for any  $\boldsymbol{\varphi} \in C^1(\overline{Q}; R^N)$ ,  $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial Q} = 0$ .

We have shown Theorem 2.2.

#### 4. APPENDIX

Since the use of Stratonovich integrals and calculus is an essential tool of this work, we summarize some definitions and rules; everything can be found in details in Kunita [13]. Given a probability basis  $(\Omega, \mathcal{F}, P)$  with a right-continuous complete filtration  $(\mathcal{F}_t)_{t \geq 0}$ , the general concept of *continuous semimartingale* can be found in many books, see e.g. Kunita [13], Revuz and Yor [16]. Examples of continuous semimartingales are the Brownian motion  $(\beta_t)_{t \geq 0}$ , the deterministic (Riemann type) integrals  $\int_0^t X_s ds$  of continuous semimartingales  $(X_t)_{t \geq 0}$  and three objects we now define. Given two continuous semimartingales  $(X_t)_{t \geq 0}$  and  $(Y_t)_{t \geq 0}$ , the following limits of finite Riemann type sums exist, understood as limits in probability. Given  $t > 0$ , let  $(\pi_n)_{n \in \mathbb{N}}$  be a

sequence of partitions of  $[0, t]$  and denote points of  $\pi_n$  generically by  $t_i$ . When we write  $\sum_{t_i \in \pi_n}$  we understand that the sum is extended to all indexes that are admissible in the expression (since also  $t_{i+1}$  appears). Then we set:

$$\begin{aligned} \int_0^t X_s dY_t &:= \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} X_{t_i} (Y_{t_{i+1}} - Y_{t_i}) \\ \int_0^t X_s \circ dY_t &:= \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} \frac{X_{t_i} + X_{t_{i+1}}}{2} (Y_{t_{i+1}} - Y_{t_i}) \\ [X, Y]_t &:= \lim_{n \rightarrow \infty} \sum_{t_i \in \pi_n} (X_{t_{i+1}} - X_{t_i}) (Y_{t_{i+1}} - Y_{t_i}). \end{aligned}$$

We call them Itô integral, Stratonovich integral and covariation, respectively. These limit, always in probability, can be also understood uniformly in time on finite intervals, with due modification of the notations, so that the partitions are not adapted to a single interval  $[0, t]$ . The class of continuous semimartingales is closed also under the previous three operations; and under sum, product and in general composition by functions  $f \in C^{1,2}([0, T] \times \mathbb{R}^d)$ .

When  $(Y_t)_{t \geq 0}$  is a Brownian motion  $(\beta_t)_{t \geq 0}$ , the integral  $\int_0^t X_s d\beta_t$  is the classical Itô integral. It exists also when  $X$  is just a continuous adapted process; and with an alternative definition it is well defined also in more general cases; in the framework of Itô calculus, functions  $f$  of class  $C^{1,2}([0, T] \times \mathbb{R}^d)$  suffice to write a chain rule. On the contrary, the two objects  $\int_0^t X_s \circ dY_t$  and  $[X, Y]_t$  are quite restrictive from the viewpoint of existence and the class of continuous semimartingales looks the right one for a general theory; and manipulations require functions  $f$  of class  $C^{1,3}([0, T] \times \mathbb{R}^d)$ . This is the price to work with them. The advantage are the rules of calculus. These rules (summarized by the multidimensional chain rule) based on Itô integrals are well known to be modified by the presence of a correction term. When Stratonovich integral is used, the rules are the same as deterministic calculus. For instance, in this work we use the fact that, for two continuous semimartingales  $(X_t)_{t \geq 0}$  and  $(Y_t)_{t \geq 0}$ ,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s \circ dY_s + \int_0^t Y_s \circ dX_s$$

which is the rigorous formulation of the identity commonly written as

$$d(X_t Y_t) = X_t \circ dY_t + Y_t \circ dX_t.$$

The analogous result with Itô integrals is

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

More generally, if  $f \in C^{1,2}([0, T] \times \mathbb{R}^d)$  and  $X_t = (X_t^1, \dots, X_t^n)$  is a vector of continuous semimartingales, then

$$df(t, X_t) = (\partial_t f)(t, X_t) dt + \sum_{i=1}^n (\partial_{x_i} f)(t, X_t) \circ dX_t^i$$

opposite to Itô formula

$$\begin{aligned} df(t, X_t) &= (\partial_t f)(t, X_t) dt + \sum_{i=1}^n (\partial_{x_i} f)(t, X_t) dX_t^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n (\partial_{x_i} \partial_{x_j} f)(t, X_t) d[X^i, X^j]_t. \end{aligned}$$

A technical remark: when Itô interpretation of integrals is given, the integrand is just required to be continuous adapted hence  $(\partial_{x_i} f)(t, X_t)$ ,  $(\partial_{x_i} \partial_{x_j} f)(t, X_t)$  are admissible integrands. When Stratonovich interpretation of integrals is chosen, the integrand must be a continuous semimartingale. Hence  $(\partial_{x_i} f)(t, X_t)$  must have such property and, by Itô calculus, we know it is true when  $\partial_{x_i} f \in C^{1,2}([0, T] \times \mathbb{R}^d)$ . This is why the property  $f \in C^{1,3}([0, T] \times \mathbb{R}^d)$  is required in Stratonovich calculus.

**Acknowledgement.** The work has been essentially discussed during the stay of E.F. at Scuola Normale Superiore in Pisa, whose support and hospitality is gladly acknowledged.

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