# Estimation of Dynamical Noise Power in Unknown Systems

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***Abstract*—Noise can be modeled as a sequence of random variables defined on a probability space that may be added to a given dynamical system $T$, which is a map on a phase space. In the non-trivial case of dynamical noise $\{\varepsilon_n\}_n$, where $\varepsilon_n$ follows a Gaussian distribution $\mathcal{N}(0, \sigma^2)$ and the system output is $x_n = T(x_{n-1}; x_0) + \varepsilon_n$, without any specific knowledge or assumption about $T$, the quantitative estimation of the noise power $\sigma^2$ is a challenge. Here, we introduce a formal method based on the nonlinear entropy profile to estimate the dynamical noise power $\sigma^2$ without requiring knowledge of the specific $T$ function. We tested the correctness of the proposed method using time series generated from Logistic maps and Pomeau-Manneville systems under different conditions. Our results demonstrate that the proposed estimation algorithm can properly discern different noise levels without any a priori information.**

***Index Terms*—Noise, complex systems, approximate entropy.**

## I. Introduction

The term "noise" typically refers to random or unpredictable fluctuations and disturbances that are not part of a target signal or system. Such noise is called measurement noise $\{\varepsilon_n\}_n$, also known as output noise, which is essentially a perturbation, where $y_n = x_n + \varepsilon_n$, with $y_n$ representing the system output and $x_n$ the system input. Since extensive literature already deals with the identification and eventual removal of output noise $\{\varepsilon_n\}_n$, this letter does not elaborate further on this case.

In dynamical systems, noise can be a peculiar part of the system dynamics; this type of noise, denoted as *dynamical noise*, is given by a sequence $\{\varepsilon_n\}_n$. If we have a sampled signal that represents the output of a dynamical system $T$, that is, a (finite) sequence $\{x_n\}_n$ where its elements satisfy $x_n = T(x_{n-1}; x_0)+\varepsilon_n$, which may depend on $x_{n-1}, x_{n-2}, \ldots, x_0$, where $T$ is a map on a phase space, then the sampled orbit is dependent on the initial (unknown) condition $x_0$, and its dynamical noise can be modeled as a sequence $\{\varepsilon_n\}_n$ of independent and identically distributed (IID) random variables defined on a probability space with a specified statistical distribution. Assuming a Gaussian distribution such that $\{\varepsilon_n\}_n \sim \mathcal{N}(0, \sigma^2)$, to the best of our knowledge, it is not possible to estimate the noise power $\sigma^2$ of dynamical noise $\{\varepsilon_n\}_n$ without any specific knowledge or assumption on $T$.

The main contribution of this letter is to introduce a formal method (theorem and estimation algorithm) to estimate $\sigma^2$ in the non-trivial case of no knowledge of the specific $T$ function, i.e., for any continuous and differentiable $T$ function. Previous attempts at estimating dynamical noise in complex time series have been proposed [1], [2]. These heuristic methods require precise estimation of the system's largest Lyapunov exponent, which is difficult with short and noisy series; they may also be biased by time series behavior and parameter selection and do not detect noise levels when standard deviation exceeds 10% of the signal standard deviation [2]. Other heuristic methods for noise power estimation have been reported in [3], [4], [5], [6], focusing on dynamical noise estimation in chaotic time series. However, these approaches provide only a qualitative characterization of dynamical noise or rely on the knowledge or approximation of the underlying model [7] and on the calculation of correlation dimension, which may not always detect chaos; in fact, one can have a stochastic process with a correlation dimension of 0 [8].

In this letter, we exploit the definition of *Approximate Entropy (ApEn)*, which is a parameter-dependent nonlinear quantifier developed to detect unpredictability over time series [8]. ApEn can be considered an approximation of the Kolmogorov-Sinai entropy for dynamical systems [9], which is actually a theoretical limit value and thus hardly achievable in short-length time series. We focus on how superimposed noise modifies the ApEn profile as a function of its tolerance parameter; while noise increases the maximum of the ApEn profile, it generates a curve with a characteristic behaviour.

## II. Methodology

### A. Approximate Entropy Profile

Given a time series $\{x_n\}_{n=1}^N$ composed by $N$ samples and a positive integer $m$, we form $m$-dimensional vectors $Y_i = (x_i, \ldots, x_{i+m-1})$ with $i = 1, \ldots, N - m + 1$. Let $d$ be a distance in $\mathbb{R}^m$. For a positive value $r$, two vectors $x, z$ in $\mathbb{R}^m$ are *$r$-close* if $d(x, z) < r$.

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If $C_i^m(r) = \text{cardinality}\{Y_j \ \text{r}-\textit{close to } Y_i\}$ and $\Phi^m(r) = (N-m+1)^{-1}\sum_i^{N-m+1}\log C_i^m(r)$, then

$$\text{ApEn}(\{x_n\}_{n=1}^N, m, r) = \Phi^m(r) - \Phi^{m+1}(r).$$

In this context, the $\text{ApEn}(\{x_n\}_{n=1}^N, m, r)$ is non-negative and reflects the likelihood that series patterns $\{Y_j\}_{j=1}^{N-m+1}$ are $r$-close. The lower is the approximate entropy, the higher is the probability of finding similar patterns and hence the more predictable is the time series.

## B. Main Result

*Theorem:* Let $(X, \mu, T)$ be a discrete dynamical system defined by a $C^1$ map $T$, i.e. differentiable, on a compact set $X \subset \mathbb{R}$ and preserving the probability measure $\mu$, whose density $h$ is also $C^1$. Define the noise as the realization of a stochastic process $\{\varepsilon_n\}_n$ of IID random variables following a distribution $q_\xi$, described by an unknown parameter $\xi$. Let $\psi(t)$ denote the probability density function (PDF) of the random variable obtained by taking the difference between any two samples of the noise process. We assume that $\psi(t)$ is continuous in a neighborhood of 0, and differentiable, at most excluding the value 0. For any perturbed orbit of the system $x(\xi) = (x_0, T(x_0) + \varepsilon_1, T(T(x_0) + \varepsilon_1) + \varepsilon_2, \dots)$, it holds

$$\text{ApEn}(\{x_n(\xi)\}_{n=1}^\infty, m, r) \approx -\log\left[2\psi(0)r\right] \qquad (1)$$

for any embedding dimension $m \in \mathbb{N}$ and for the tolerance $r > 0$ small enough.

*Proof:* Without loss of generality, we suppose $X = [0, 1]$ and we consider $|\cdot|$ as the infinity norm.

-*Case* $m = 1$. Observe that for any admissible integer $k$,

$$|x_{n+k+1} - x_{n+1}| = |T(x_{n+k}) - T(x_n) + \varepsilon_{n+k+1} - \varepsilon_{n+1}|,$$

where $\varepsilon_{n+1}$ and $\varepsilon_{n+k+1}$ are independent noise samples.

Let $\psi(t)$ denote the PDF of the random variable $\zeta = \varepsilon_{n+k+1} - \varepsilon_{n+1}$, and let $\mathrm{d}\mu(x) = h(x)\mathrm{d}x$ be a density function that is $C^1$.

We estimate the conditional probability $\mathcal{P}$ of having any two series iterates $x_{n+1}$ and $x_{n+k+1}$ $r$-close, given $x_n$ and $x_{n+k}$ $r$-close for any admissible integer $k$. By setting $z := x_n$ and $y := x_{n+h}$, the conditional probability $P$ can be expressed as follows: $\int \frac{\int_{z-r}^{z+r}\int_{T(z)-T(y)-r}^{T(z)-T(y)+r}\psi(t)\mathrm{d}t\,\mathrm{d}\mu(y)}{\int_{z-r}^{z+r} 1\,\mathrm{d}\mu(y)}\mathrm{d}\mu(z)$. Since the approximate entropy is the negative expectation of the log probability $\mathcal{P}$, we obtain:

$$\text{ApEn}(\{x_n(\xi)\}_{n=1}^\infty, 1, r) = -\int \log \frac{\int_{z-r}^{z+r}\int_{T(z)-T(y)-r}^{T(z)-T(y)+r}\psi(t)\mathrm{d}t\,\mathrm{d}\mu(y)}{\int_{z-r}^{z+r} 1\,\mathrm{d}\mu(y)}\mathrm{d}\mu(z) \qquad (2)$$

Next, we formalize the changes of ApEn when the radius $r$ becomes comparable to noise perturbation effects. Take a fixed value of $z$, and consider $J_z$ to be a small neighborhood centered at 0. Let $N_z(r) = \int_{z-r}^{z+r}\int_{\alpha(y,r)}^{\beta(y,r)}\psi(t)\mathrm{d}t\,\mathrm{d}\mu(y)$ for $r \in J_z$, with $\alpha(y, r) := T(z) - T(y) - r$ and $\beta(y, r) := T(z) - T(y) + r$. We have $N_z(0) = 0$.

As the limit bounds of integration and integrand functions in $N_z(r)$ are differentiable by hypothesis, we can apply Leibniz's rule to obtain the following result:

$$N_z'(r) = \int_{z-r}^{z+r}[\psi(\beta(y,r)) + \psi(\alpha(y,r))]h(y)\mathrm{d}y + \int_{\alpha(z+r,r)}^{\beta(z+r,r)} h(z+r)\psi(t)\mathrm{d}t + \int_{\alpha(z-r,r)}^{\beta(z-r,r)} h(z-r)\psi(t)\mathrm{d}t.$$

$N_z'(r)$ is continuous in $J_z$, and we still have $N'(0) = 0$.

If $\psi(t)$ is differentiable, then integrand functions and bounds of integration in $N'(r)$ are differentiable: Leibniz rule yields

$$\begin{aligned} N_z''(r) &= h(z+r)(1 - T'(z+r))(\psi(\beta(z+r,r))) \\ &\quad + h(z+r)(1 + T'(z+r))(\psi(\alpha(z+r,r))) \\ &\quad + \int_{\alpha(z+r,r)}^{\beta(z+r,r)} h'(z+r)\psi(t)\mathrm{d}t - \int_{\alpha(z-r,r)}^{\beta(z-r,r)} h'(z-r)\psi(t)\mathrm{d}t \\ &\quad + h(z-r)(1 + T'(z-r))(\psi(\beta(z-r,r))) \\ &\quad - h(z-r)(-1 + T'(z-r))(\psi(\alpha(z-r,r))) \\ &\quad + h(z+r)(\psi(\beta(z+r,r)) + \psi(\alpha(z+r,r))) \\ &\quad + h(z-r)(\psi(\beta(z-r,r)) + \psi(\alpha(z-r,r))) \\ &\quad + \int_{z-r}^{z+r} h(y)[\psi'(\beta(y,r)) - \psi'(\alpha(y,r))]\mathrm{d}y. \end{aligned}$$

Since $N_z''(0) = 8h(z)\psi(0)$ is defined and $N'(r)$ is $C^1$, we can use the Taylor polynomial expansion $N_z(r) = 4h(z)\psi(0)r^2 + o(r^2)$ in $J_z$.

If $\psi(t)$ is not differentiable at $t = 0$, we can still obtain $N_z''(0) = 8h(z)\psi(0)$ by using the smoothing of $\psi$ through mollifiers, which is a sequence of $C^\infty$ functions that uniformly converges to $\psi$ in any small neighborhood of 0.

Moreover, $\int_{z-r}^{z+r} 1\mathrm{d}\mu(y) \approx 2rh(z)$ in $J_z$.

By substituting the previous polynomial expansions into (2), we can obtain the desired result since, for $r > 0$:

$$\begin{aligned} \text{ApEn}(\{x_n(\xi)\}_{n=1}^\infty, 1, r) &\approx -\int \log\frac{4h(z)\psi(0)r^2}{2rh(z)}\mathrm{d}\mu(z) \\ &= -\int \log\left[2\psi(0)r\right]\mathrm{d}\mu(z) = -\log 2\psi(0)r \end{aligned}$$

-*Case* $m > 1$. We embed the series $\{x_n(\xi)\}_{n=1}^\infty$ in a $m$-dimensional space by forming vectors $\mathbf{X}_j = (x_{j-(m-1)}, \dots, x_j)$ for all admissible $j$. Analogously, the $\text{ApEn}(\{x_n(\xi)\}_{n=1}^\infty, m, r)$ is minus the expectation of the log probability $\mathcal{P}$ of having two iterates $x_{n+k+1}$ and $x_{n+1}$ $r$-close, being the previous $m$ points $\mathbf{X}_{n+k}$ $\mathbf{X}_n$ $r$-close for any admissible integer $k$.

The proof directly retraces the case $m = 1$: the only exception is of considering the invariant measure $\mu$ defined on a $m$-dimensional space. □

*Remark:* The previous result can be weakened to maps $T$ and density functions $h$ that are piece-wise differentiable. This is because the argument used before is not valid for a set of values of $z$ with measure 0.

*Corollary:* If the noise samples are IID Gaussian distributed random variables $\sim \mathcal{N}(0, \sigma^2)$, by setting $\xi = \sigma$, relation (1) becomes

$$\text{ApEn}(\{x_n(\sigma)\}_{n=1}^{\infty}, m, r) \approx -\log\left[r/(\sigma\sqrt{\pi})\right]$$

and therefore:

$$\log(\sigma) \approx \text{ApEn}(\{x_n(\sigma)\}_{n=1}^{\infty}, m, r) + \log(r\sqrt{\pi})$$

when $r < \sigma$ and $r \to 0^+$.

In this case, the noise standard deviation (std) $\sigma$ can be approximated by the tolerance value $r$ for which the functions $z \to \text{ApEn}(\{x_n(\sigma)\}_{n=1}^{\infty}, m, z)$ and $z \to -\log z$ show the most similar differential behavior.

[*Proof of the Corollary*] If the noise samples are IID Gaussian distributed random variables of mean 0 and variance $\sigma^2$, then the density function $\psi \sim \mathcal{N}(0, 2\sigma^2)$.

In this context,

$$\psi(0) = \frac{1}{2\sigma\sqrt{\pi}},$$

and hence relation (1) becomes

$$\text{ApEn}(\{x_n(\sigma)\}_{n=1}^{\infty}, m, r) \approx -\log\left[r/(\sigma\sqrt{\pi})\right]. \qquad \square$$

### C. Assumptions

Unlike Markovian formalism, where time series points are seen as a sequence of random variables governed by transition probabilities between consecutive states, in our framework we make the following assumptions:

- The dynamics is deterministic (it may be chaotic) and ruled by a differentiable map $T$;
- Noise is modeled as a sequence of IID Gaussian random variables with $\mathcal{N}(0, \sigma^2)$;
- The system dynamics is perturbed by dynamical noise $\{\varepsilon_n\}_n$ in $\mathcal{N}(0, \sigma^2)$; any orbit of the dynamical systems is of the form $x_n = T(x_{n-1}, \ldots, x_0) + \varepsilon_n$.

In terms of time series analysis, the properties of the map $T$ appear the most general and suitable to describe and model a wide class of observable phenomena.

### D. Noise Estimation Algorithm

The MATLAB source code for executing the proposed noise power estimation is available at https://github.com/AndScar/noise_estimation.

Based on the assumptions made previously, we propose a method for estimating the dynamical noise power $\sigma^2$ from a noisy series $X$ of $N$ samples, using an embedding dimension $m$ and a step $\Delta r$. The estimation method is presented in Algorithm 1 and its graphical representation is shown in Fig. 1. In summary, the method first calculates the approximate entropy profile of $X$ as a function of the tolerance $r$. Then, based on the main result above (see Section II-B), it obtains an initial estimate of the noise power $\bar{\sigma}$ by identifying the value $\bar{r}$ where $\text{ApEn}(X, m, r)$ and $-\log r$ have the most similar slope. Finally, the Corollary is used to improve $\bar{\sigma}$ by finding the best fit $\sigma$ for the function $s \to \text{ApEn}(X, m, r) + \log\left[r/(s\sqrt{\pi})\right]$ in a neighborhood of $\bar{r}$.

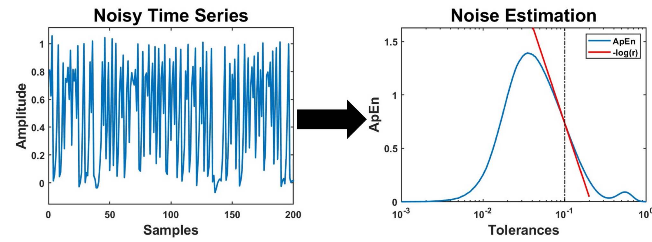


Fig. 1. Dynamical noise estimation procedure. Take a noisy series as input (left panel). First, the function $r \to \text{ApEn}(X, m, r)$ is estimated (right panel, blue line)- $r$ ranging between 0 and the series amplitude with a $\Delta r$ step. A first noise power estimate is obtained by searching for the point in which the ApEn profile and the curve $-\log r$ (red line) have the same slope. An improved standard deviation estimation is obtained by a successive curve fitting procedure.

**Algorithm 1:** Noise Estimation Algorithm.

**Require** $\Delta r > 0$; a noisy series $X$; $m > 0$.
  **for** $r = 0 : \Delta r : (\max(X) - \min(X))$ **do**
    Evaluate $\text{ApEn}(X, m, r)$;
  **end for**
  $\text{argmin}_r \text{ApEn}(X, m, r) + \log(r) \leftarrow \bar{r}$;
  $\bar{r} \leftarrow \sigma$; ▷ Rough Noise Estimation
  $\text{argmax}_r \text{ApEn}(X, m, r) \leftarrow r_{\max}$;
  $I(\bar{r}) := [r_{\max}, \bar{r}]$; ▷ Define a neighborhood of $\bar{r}$
  In $I(\bar{r})$, evaluate $\sigma$ as the best fit of $s \to \text{ApEn}(X, m, r)$ $+ \log[r/(s\sqrt{\pi})]$;
**return** $\sigma$

## III. Numerical Analysis

We tested the proposed estimation method on the following analytical benchmarks: *Logistic maps:* a family of maps described by the expression

$$f_\lambda : [0, 1] \to [0, 1], \quad f_\lambda(x) = \lambda x(1 - x)$$

where λ is a real-valued parameter in the interval [0, 4]. Simulations were performed with $\lambda = 3.5$, in the periodic regime, and with $\lambda = 3.5699456\ldots$, in the onset of chaos, i.e., in the chaotic regime but with null Kolmogorov-Sinai entropy. *Pomeau-Manneville maps:* a family of chaotic maps described by the function

$$T_\alpha : [0, 1] \to [0, 1], \quad T_\alpha(x) = \{x + x^\alpha\}$$

where $\alpha$ is a real number greater than 1, and the braces denote the reduction modulo 1.

For each $\alpha$, there exists only one $T_\alpha$-invariant measure $\mu_\alpha$. In addition, the measure $\mu_\alpha([0, 1])$ is finite for $1 < \alpha < 2$, and infinite for $\alpha \geq 2$ [10]. We consider a limit situation for Pomeau-Manneville maps with $\alpha = 2.05$, since for $\alpha \geq 2$ they do not preserve an absolutely continuous probability measure, leading to a not straightforward application of the Theorem. These maps are studied by the results of *infinite ergodic theory*, and it is known that the statistical properties of such maps are peculiar [11], [12].

We generate the perturbed series by fixing an initial condition $x_0$, which we exploit to produce a noise-free series of $N = 20000$ samples. Then, we consider a realization of a white Gaussian process (20000 samples) with a standard deviation equal to a percentage of the amplitude of the noise-free series. Finally, restarting from $x_0$, we construct the perturbed series

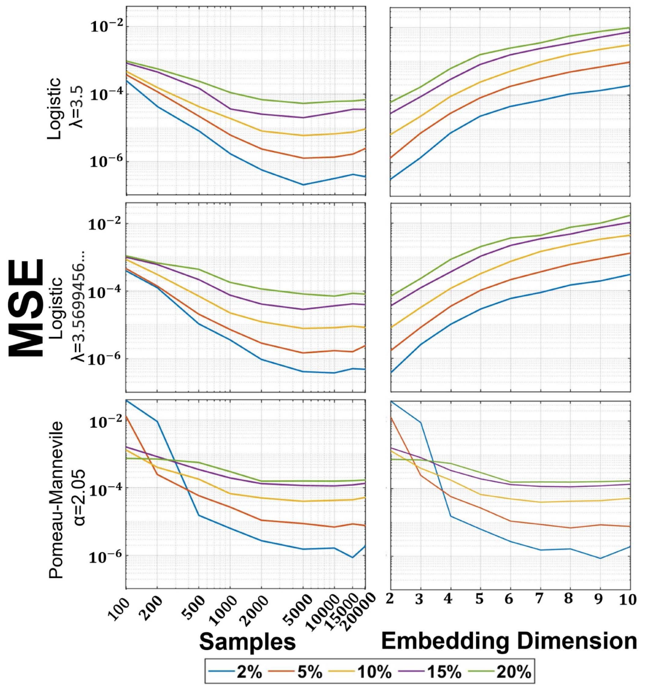


Fig. 2. Mean Squared Error (MSE) among 100 realizations at different levels of noise and series length in Logistic and Pomeau-Manneville (P-M) maps. MSE is computed between the estimated std noise and the actual superimposed noise for 5 noise levels (2%,5%,10%,15% and 20% with respect to the noise-free series amplitude). (left panels) MSE at 9 different series lengths when embedding dimension is fixed ($m = 2$). (right panels) MSE at 9 embedding dimensions when series length is fixed at $N = 10000$ samples.

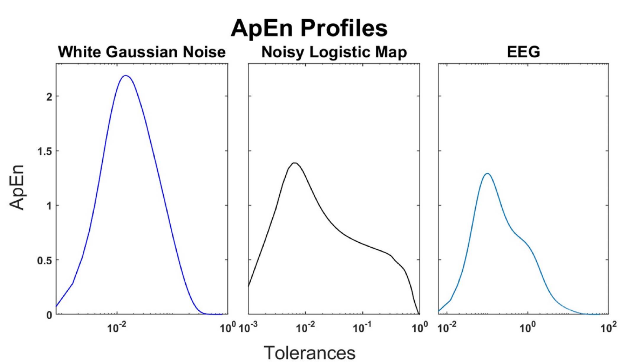


Fig. 3. Approximate Entropy (ApEn) Profiles of noisy maps. The shapes of a realization of a WGN process (first panel), of a noisy chaotic logistic map (second panel) - $\lambda = 4$, 5% of noise level - and for a channel EEG recording (third panel) are depicted for embedding dimension $m = 2$. Noise presence in time series breaks the dynamical *plateau* by generating an evident peak.

by adding a sample of the noise realization at each step of the map equation. We consider noise standard deviation percentages of $2\%, 5\%, 10\%, 15\%, 20\%$ with respect to the noise-free series amplitude. For each noise level and parameter, $M = 100$ perturbed series are obtained. Furthermore, for Logistic maps, we also consider a *bounce* effect: each time noise moves a series point out of [0, 1], we put the sample back into the box [0, 1] through reduction modulo 1. For these bounce effects, the actual noise standard deviation may result in a lower theoretical imposed one. Such an effect is automatically verified by the reduction modulo 1 in the Pomeau-Manneville maps definition.

For any series, noise estimation is performed with a resolution step of $\Delta r = \{\text{series amplitude}\}/1000$.

Results are expressed in terms of mean squared error (MSE) and illustrated in Fig. 2. Numerical results confirm that the method properly estimates the different levels of superimposed dynamical noise with a minimal variance across repetitive estimates. For a fixed embedding dimension, the higher the series length, the more accurate is the estimation. On the other hand, increasing embedding dimension leads to worse noise estimation, for a fixed number of samples. If the embedding dimension $m$ is appropriate, the method performs a reliable noise power estimation starting from $250-500$ samples.

## IV. Discussion and Conclusion

We propose a formal method to estimate the noise power, $\sigma^2$, of a noise sequence, $\{\varepsilon_n\}_n$, that dynamically influences an unknown dynamical system, $T$, whose elements satisfy $x_n = T(x_{n-1}; x_0)+\varepsilon_n$, possibly including maps of the form $x_n = T(x_{n-1}, x_{n-2}, \ldots, x_0)+\varepsilon_n$. To test the correctness of the method, we perturb the Logistic and Pomeau-Manneville maps in periodic as well as chaotic regimes with finite and infinite probability measures by imposing different levels of dynamical white Gaussian noise. Numerical results confirm that the method properly estimates the different levels of superimposed noise with minimal variance across repetitive estimates, even in the case of noise disturbance as high as $\%20$. Despite the sample dependence of the ApEn quantifier [13], and the validity of the theorem for infinite-length series for any embedding dimension, the proposed method seems quite robust to the cardinality of the series, even if a small $m$ is suggested. Numerical results show that a higher MSE may be associated with a higher $m$, which would require a larger number of sample points to properly estimate conditional probabilities. The method applies to any continuous and differentiable $T$ map, and no further *a priori* information on $T$ is required. Note that the theorem above applies for any IID random sequence; thus, its validity stands for noise with uniform distribution, for example. Furthermore, the proposed method is effective for the estimation of additive (output) noise, too. While the method does not provide information on the nature of the noise (output or dynamical) or its statistical properties, such an estimation may have practical advantages in real applications. In fact, no *a priori* information or hypotheses are necessary to perform the proposed noise power estimation.

The fact that the proposed method is applicable to both discrete-time and continuous-time dynamical systems deserves attention. We consider the noise effects on dynamics between $t_1$ and $t_2$ as the amount of noise carried by a continuous-time dynamical system at the instant $t_2$. We then replace this sum of noise effects with a discrete-time noise deviation, which can be numerically handled in observable, sampled (and hence discrete-time) time series. Therefore, we can assume that the observable series are the output of a discrete-time dynamical system. This is particularly relevant for applications involving, e.g., financial, natural, and complex physiological systems, since their recordings are discrete-time signals by definition, being the result of sampling. To illustrate, Fig. 3 shows the ApEn profile in the case of noise-free series, noisy logistic map series, and a real electroencephalographic (EEG) series gathered from a healthy subject in a resting condition. The profile suggests that dynamical noise is an inherent part of neural dynamics.

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