

Monopoles, instantons, and the Helmholtz equation

 Guido Franchetti^{a)} and Rafael Maldonado^{b)}
*Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2,
 30167 Hannover, Germany*

(Received 11 April 2016; accepted 24 June 2016; published online 12 July 2016)

In this work we study the dimensional reduction of smooth circle invariant Yang-Mills instantons defined on 4-manifolds which asymptotically become circle fibrations over hyperbolic 3-space. A suitable choice of the 4-manifold metric within a specific conformal class gives rise to singular and smooth hyperbolic monopoles. A large class of monopoles is obtained if the conformal factor satisfies the Helmholtz equation on hyperbolic 3-space. We describe simple configurations and relate our results to the Jackiw-Nohl-Rebbi construction, for which we provide a geometric interpretation. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4955418>]

I. INTRODUCTION

Yang-Mills-Higgs monopoles over Euclidean 3-space have received considerable attention because of their rich mathematical structure⁵ and due to their connections with string theory, see e.g., Ref. 14. Qualitatively similar solutions are obtained by working on hyperbolic space, with the advantage that the monopole equations are then more readily integrated analytically, even for quite complicated configurations.²⁰ The reason for this advantage is that hyperbolic monopoles with specific masses arise as symmetry reductions of smooth circle invariant Yang-Mills instantons on Euclidean 4-space.⁴

Euclidean monopoles with point singularities were first studied by Kronheimer¹⁶ and have been constructed via the Nahm transform.⁸ The analysis was extended to the hyperbolic case by Nash,²⁵ who studied the charge 1 moduli space using twistor techniques.

In this paper we present a technique which allows one to construct both smooth and singular hyperbolic monopoles of mass $m = \frac{1}{2}$. We make use of the fact that hyperbolic monopoles can be obtained from smooth circle invariant instantons via dimensional reduction. This is true not only for smooth monopoles but also for monopoles with singularities. Smooth monopoles come from instantons living on Euclidean 4-space E^4 , which, after removal of a 2-plane, is conformally equivalent to a trivial circle bundle over hyperbolic 3-space H^3 . Singular monopoles, on the other hand, arise from the dimensional reduction of smooth instantons living on spaces which asymptotically become circle fibrations over H^3 . These spaces are the hyperbolic version of Gibbons-Hawking gravitational instantons and we review them in Section II A. Sections II B and II C review some material on singular hyperbolic monopoles and dimensional reduction.

In Section III A we construct monopoles by making use of the fact that the projection of the spin connection of a Riemannian 4-manifold M on the appropriate $\mathfrak{su}(2)$ factor in the Lie algebra decomposition $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is an instanton provided that M is spin, half-conformally flat and scalar flat.⁶ In particular, by rescaling a hyperbolic Gibbons-Hawking (hGH) space via a conformal factor which is circle invariant and satisfies the Helmholtz equation on H^3 , we obtain a large family of solutions of the Bogomolny equations. Some specific examples are studied in Sections III C and III D. In Section III F we remark on how this technique can be modified to obtain the Higgs field of spherically symmetric monopoles with higher mass. A novel family of monopoles is obtained by imposing that the conformally rescaled manifold is Einstein. We study this family in Section IV.

^{a)}franchetti@math.uni-hannover.de

^{b)}rafael.maldonado@itp.uni-hannover.de

Reexamining the case of circle invariant instantons on E^4 from this perspective gives new insight into the construction of hyperbolic monopoles from circle invariant Jackiw-Nohl-Rebbi (JNR) data. The JNR method constructs a class of hyperbolic monopoles starting from circle invariant harmonic functions on E^4 .^{15,9} In Section III E we show that smooth monopoles obtained from solutions of the Helmholtz equation correspond to monopoles coming from JNR data, and we thereby obtain a purely geometrical reformulation of the JNR construction applied to hyperbolic monopoles. Moreover, we identify how the classical JNR data are modified to generate monopole singularities and give a physical interpretation of the JNR poles. We clarify the relation between the two approaches and how to translate from one to the other.

A property of hyperbolic monopoles which is not shared by their Euclidean counterparts is the fact that they are completely determined by the induced asymptotic abelian connection.^{11,22} For monopoles coming from solutions of the Helmholtz equation we show this directly in Section III B by giving an explicit relation between the full monopole solution and its asymptotic data.

II. PRELIMINARIES

It is known from Ref. 16 that a circle invariant instanton on a Gibbons-Hawking space is equivalent to a monopole on E^3 , possibly with singularities, and examples of these monopoles have been constructed.¹² In a similar way, it is possible to obtain a singular hyperbolic monopole starting from a circle invariant instanton on modified Gibbons-Hawking spaces.²⁵ We are now going to review how to generate such instantons.

A. Hyperbolic Gibbons-Hawking spaces

A (Euclidean) Gibbons-Hawking space is a Riemannian 4-manifold M with a metric of the form

$$\mathcal{V}g_{E^3} + \mathcal{V}^{-1}(d\psi + a)^2. \tag{1}$$

Here g_{E^3} is the Euclidean 3-metric and (\mathcal{V}, a) obeys the abelian monopole equation $da = *_E d\mathcal{V}$, hence \mathcal{V} is harmonic on E^3 . Any such metric is hyper-Kähler and therefore Ricci-flat and half-conformally flat. Let $\{p_i\}$ be N distinct points in E^3 and $G_{p_i}^{E^3}(p) = \mu/d^E(p_i, p)$ be Green's function on E^3 centred at p_i . Here d^E is the Euclidean distance in E^3 and $\mu > 0$, a constant related to the range of the angle ψ by $\psi \in [0, 4\pi\mu)$. For \mathcal{V} of the form

$$\mathcal{V} = c + \sum_{i=1}^N G_{p_i}^{E^3} \quad (c = \text{constant}), \tag{2}$$

the metric (1) is known as multi-Eguchi-Hanson if $c = 0$ and as multi-Taub-NUT otherwise. It has a $U(1)$ isometry group generated by the vector field $\partial/\partial\psi$. Away from the NUTs p_i , the fixed points of the $U(1)$ action, M is the total space of a circle fibration over E^3 .

LeBrun¹⁷ obtained a new family of half-conformally flat spaces by replacing the base space E^3 with hyperbolic 3-space H^3 . The metric is now

$$g_0 = Vg_{H^3} + V^{-1}(d\psi + \alpha)^2, \tag{3}$$

where g_{H^3} is the metric on hyperbolic 3-space of sectional curvature -1 and (V, α) is an abelian monopole on H^3 . We will refer to these spaces as hyperbolic Gibbons-Hawking (hGH) spaces and take the orientation specified by the volume form

$$\text{vol}_{hGH} = -V \text{vol}_{H^3} \wedge d\psi, \tag{4}$$

where vol_{H^3} is the volume element on H^3 . Note that (3) is neither hyper-Kähler nor scalar flat. In fact,¹⁸ the scalar curvature s is

$$s = -\frac{6}{V}. \tag{5}$$

We will take V to be of the form

$$V = \frac{2}{\beta} + 2 \sum_{i=1}^N G_{p_i}^{H^3} \quad (\beta = \text{constant}), \tag{6}$$

where $\{p_i\}$ are N distinct points in H^3 and $G_{p_i}^{H^3}$ is the Green's function centred at p_i . If d^H is the distance function on H^3 , then

$$G_{p_i}^{H^3}(p) = \frac{1}{e^{2d^H(p,p_i)} - 1}. \tag{7}$$

With this normalisation of V , the range of ψ in (3) is $\psi \in [0, 4\pi)$.

B. Singular hyperbolic monopoles

A hyperbolic monopole (Φ, \mathcal{A}) is a solution of the Bogomolny equations on hyperbolic space,

$$d_{\mathcal{A}}\Phi = -*_{H^3}\mathcal{F}, \tag{8}$$

where \mathcal{A} is a connection on an $SU(2)$ bundle \mathcal{P} over H^3 , \mathcal{F} is its curvature, Φ is a section of the adjoint bundle, and $*_{H^3}$ denotes the Hodge operator on H^3 . On $\mathfrak{su}(2)$ we take the ad-invariant inner product $\langle X, Y \rangle = -\frac{1}{2}\text{Tr}(XY)$.

Let B^{\parallel}, B^{\perp} be the components of \mathcal{A} parallel and orthogonal to the direction of the Higgs field in $\mathfrak{su}(2)$, $B^{\parallel} = \langle \mathcal{A}, \Phi \rangle \Phi / \|\Phi\|^2$, $B^{\perp} = \mathcal{A} - B^{\parallel}$. The monopole is required to satisfy the following asymptotic conditions:^{23,25}

$$(\|\Phi\| - m) \exp(2r) \text{ extends smoothly to } \partial H^3, \text{ for some constant } m > 0, \tag{9}$$

$$B^{\parallel} \text{ extends smoothly to } \partial H^3, \tag{10}$$

$$B^{\perp} \exp(2mr) \text{ extends smoothly to } \partial H^3. \tag{11}$$

Let $\{p_i\}$ be L distinct points in H^3 , $r_i = d^H(p, p_i)$. A singular hyperbolic monopole with singularities at $\{p_i\}$ is a solution of (8) on $H^3 \setminus \{p_i\}$ which satisfies the following conditions:

$$2 \lim_{r_i \rightarrow 0} r_i \|\Phi\| = \ell_i \in \mathbb{Z}_+, \tag{12}$$

$$d(r_i \|\Phi\|) \text{ is bounded in a neighbourhood of } p_i. \tag{13}$$

The quantity

$$\ell = \sum_{i=1}^L \ell_i \tag{14}$$

is called the abelian charge of the monopole.

Because of the decay condition (11) the monopole asymptotically abelianises. The total charge Q of a hyperbolic monopole is the first Chern number associated to the asymptotic abelian fibration. It can be computed as

$$Q = -\frac{1}{2\pi} \int_{\partial H^3} *_H d\|\Phi\|. \tag{15}$$

It follows from (9) that the coefficient of the leading term in an asymptotic expansion of $\|\Phi\|$ is the monopole mass m . If C is the smooth extension of $(\|\Phi\| - m) \exp(2r)$ to ∂H^3 , then

$$Q = \frac{1}{4\pi} \int_{\partial H^3} C \text{vol}_{S^2}. \tag{16}$$

Following Ref. 25, we define the non-abelian charge k to be

$$k = \ell - Q. \tag{17}$$

C. Dimensional reduction

Let P be the total space of an $SU(2)$ principal bundle over an hGH space M . An $SU(2)$ instanton on M is a connection on P having self-dual or anti-self-dual curvature.

Let R be a lift to P of the $U(1)$ action on M generated by the Killing vector field $\partial/\partial\psi$. We say that an instanton ϖ on M is ψ -invariant if $R^*\varpi = \varpi$. For a ψ -invariant instanton ϖ there exists a local section s such that, away from fixed points of the $\partial/\partial\psi$ action, the gauge potential $A = s^*\varpi$ has no explicit ψ dependence.²¹ We call such a choice of local section a circle invariant gauge.

A ψ -invariant self-dual instanton is equivalent to a hyperbolic monopole. In fact,^{16,25} working in a circle invariant gauge s and writing

$$A = s^*\varpi = \mathcal{A} + \frac{\Phi}{V}(d\psi + \alpha), \tag{18}$$

the self-duality equations for A imply the Bogomolny equations (8) for (Φ, \mathcal{A}) .

One could reverse the procedure and define an instanton A on an hGH space M from a singular hyperbolic monopole. Conditions (12) and (13) then ensure that A is globally defined on M as long as the monopole singularities are located at poles of V .^{16,24}

III. THE CONFORMAL RESCALING METHOD

For an oriented spin 4-manifold there is a decomposition of the spin bundle $S(M) = S^+(M) \oplus S^-(M)$ corresponding to the splitting $\mathfrak{so}(4) \simeq \mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$. Denote by $P_{\pm}(\omega)$ the projection of the spin connection ω onto $S^{\pm}(M)$. We will make use of the following result.⁶

Theorem 1 (Atiyah-Hitchin-Singer 1978). *Let M be an oriented Riemannian spin 4-manifold with spin connection ω .*

1. $P_-(\omega)$ is a self-dual $SU(2)$ connection if and only if M is half-conformally flat and scalar flat.
2. $P_+(\omega)$ is a self-dual $SU(2)$ connection if and only if M is Einstein.

Since the self-duality equations are conformally invariant, we can conformally rescale the metric on an hGH space in order to get a metric satisfying either of the above conditions. The appropriate projection of the spin connection is then a self-dual instanton on the hGH space. In the Euclidean case, this method has been used, e.g., in Ref. 12.

A. Hyperbolic monopoles as solutions of the Helmholtz equation

In this section we are going to apply the first method of Theorem 1 to generate self-dual instantons on hGH spaces. We shall see that they can be completely specified by giving a solution of the Helmholtz equation.

Since half-conformal flatness is a conformally invariant condition, we are looking for a function $\Lambda > 0$ for which the metric

$$g = \Lambda^2 g_0 = \Lambda^2 [V g_{H^3} + V^{-1}(d\psi + \alpha)^2] \tag{19}$$

is scalar flat. Under the conformal transformation (19), the scalar curvature s of g_0 transforms as

$$s' = \frac{1}{\Lambda^2} \left(s - 6 \frac{\Delta_{g_0} \Lambda}{\Lambda} \right). \tag{20}$$

We use the notation Δ_g to denote the Laplace-Beltrami operator with respect to the metric g . Let us assume that $\partial_\psi \Lambda = 0$. Then $\Delta_{g_0} = V^{-1} \Delta_{H^3}$ and, using (5), imposing $s' = 0$ gives the Helmholtz equation on hyperbolic space,

$$\Delta_{H^3} \Lambda + \Lambda = 0. \tag{21}$$

As we are going to show below, in terms of Λ , the instanton obtained from the spin connection ω on the conformally hGH space with metric (19) is given by

$$A = \mathcal{A} + \frac{\Phi}{V}(d\psi + \alpha), \tag{22}$$

with (Φ, \mathcal{A}) the hyperbolic monopole

$$\Phi = \frac{1}{2} \left[\frac{\epsilon_1(\Lambda)}{\Lambda} \mathbf{i} + \frac{\epsilon_2(\Lambda)}{\Lambda} \mathbf{j} + \frac{\epsilon_3(\Lambda)}{\Lambda} \mathbf{k} \right], \tag{23}$$

$$\begin{aligned} \mathcal{A} = & \frac{1}{4} [(C_{3k2} + C_{k23} - C_{23k}) \mathbf{i} + (C_{1k3} + C_{k31} - C_{31k}) \mathbf{j} + (C_{2k1} + C_{k12} - C_{12k}) \mathbf{k}] \epsilon^k + \tag{24} \\ & + \frac{1}{2\Lambda} [(\epsilon_2(\Lambda)\epsilon^3 - \epsilon_3(\Lambda)\epsilon^2) \mathbf{i} + (\epsilon_3(\Lambda)\epsilon^1 - \epsilon_1(\Lambda)\epsilon^3) \mathbf{j} + (\epsilon_1(\Lambda)\epsilon^2 - \epsilon_2(\Lambda)\epsilon^1) \mathbf{k}]. \end{aligned}$$

Here $\{\epsilon_i\}$ is an orthonormal frame on H^3 , $\{\epsilon^i\}$ is the dual coframe, $[\epsilon_i, \epsilon_j] = C_{ij}{}^k \epsilon_k$, and $C_{ijk} = C_{ij}{}^m (g_{H^3})_{mk}$. Latin indices range from 1 to 3.

To prove (23) and (24) take the g -orthonormal coframe $\{e^i, e^4\}$ and dual frame $\{e_i, e_4\}$,

$$\begin{aligned} e^i &= \Lambda \sqrt{V} \epsilon^i, & e^4 &= -\frac{\Lambda(d\psi + \alpha)}{\sqrt{V}}, \tag{25} \\ e_i &= \frac{\epsilon_i}{\Lambda \sqrt{V}}, & e_4 &= -\frac{\sqrt{V}}{\Lambda} d\psi. \end{aligned}$$

Let $[e_\alpha, e_\beta] = C_{\alpha\beta}{}^\gamma e_\gamma$, $C_{\alpha\beta\gamma} = g_{\mu\gamma} C_{\alpha\beta}{}^\mu$, with Greek indices ranging from 1 to 4. The spin connection coefficients can be computed making use of the equation

$$\omega_{\alpha\beta} = \frac{1}{2} (C_{\alpha\beta\mu} - C_{\beta\mu\alpha} - C_{\mu\alpha\beta}) e^\mu. \tag{26}$$

Since

$$\begin{aligned} C_{i4}{}^4 &= \frac{dV(\epsilon_i)}{2\Lambda V^{3/2}} - \frac{d\Lambda(\epsilon_i)}{\Lambda^2 V^{1/2}}, \\ C_{ij}{}^4 &= \frac{{}^*H^3 dV(\epsilon_i, \epsilon_j)}{\Lambda V^{3/2}}, \\ C_{i4}{}^j &= 0, \\ C_{ij}{}^k &= \frac{1}{\Lambda \sqrt{V}} (C_{ij}{}^k + \epsilon_j(\sqrt{V}\Lambda) \delta_i^k - \epsilon_i(\sqrt{V}\Lambda) \delta_j^k), \end{aligned} \tag{27}$$

we have

$$\begin{aligned} \omega_{ij} &= \frac{1}{2} (C_{ijk} - C_{jki} - C_{kij}) \epsilon^k + \frac{1}{\Lambda} [\epsilon_j(\Lambda) \epsilon^i - \epsilon_i(\Lambda) \epsilon^j] + \frac{1}{2V} [\epsilon_j(V) \epsilon^i - \epsilon_i(V) \epsilon^j] \\ &\quad - \frac{{}^*H^3 dV(\epsilon_i, \epsilon_j)}{2V} \left(\frac{d\psi + \alpha}{V} \right), \tag{28} \\ \omega_{i4} &= \left(\frac{d\Lambda(\epsilon_i)}{\Lambda} - \frac{dV(\epsilon_i)}{2V} \right) \left(\frac{d\psi + \alpha}{V} \right) + \frac{{}^*H^3 dV(\epsilon_i, \epsilon_j)}{2V} \epsilon^j. \end{aligned}$$

The projection of ω onto $S^-(M)$ is given by

$$P_-(\omega) = \frac{1}{4} (\bar{\eta}_a)_{ij} \omega_{ij} \bar{\eta}_a, \tag{29}$$

with $\{\bar{\eta}_i\}$ the anti-self-dual 't Hooft matrices

$$\bar{\eta}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \bar{\eta}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \bar{\eta}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

Identifying

$$\bar{\eta}_1 = -\mathbf{k}, \quad \bar{\eta}_2 = -\mathbf{j}, \quad \bar{\eta}_3 = -\mathbf{i}, \tag{31}$$

we get

$$A = P_-(\omega) = \frac{1}{2} [(\omega_{14} - \omega_{23})\mathbf{i} + (\omega_{24} + \omega_{13})\mathbf{j} + (\omega_{34} - \omega_{12})\mathbf{k}]. \tag{32}$$

Using

$$*_H^3 dV(\epsilon_i, \epsilon_j) = \epsilon_{ijk} dV(\epsilon_k), \tag{33}$$

with ϵ_{ijk} the Levi-Civita symbol in 3 dimensions, $\epsilon_{123} = 1$, we have

$$\begin{aligned} \omega_{ij} - \epsilon_{ijk} \omega_{k4} &= \frac{1}{2} (C_{ijk} - C_{jki} - C_{kij}) \epsilon^k + \frac{1}{\Lambda} [\epsilon_j(\Lambda) \epsilon^i - \epsilon_i(\Lambda) \epsilon^j] \\ &+ \epsilon_{ijk} \frac{d\Lambda(\epsilon_k)}{\Lambda} \left(\frac{d\psi + \alpha}{V} \right). \end{aligned} \tag{34}$$

By comparing (32) with (22), Equations (23) and (24) follow.

To see what constraints on Λ result from the asymptotic conditions (9)–(11), let us take the hyperboloid model of H^3 with metric

$$g_{H^3} = dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2). \tag{35}$$

The orthonormal coframe,

$$\epsilon^1 = dr, \quad \epsilon^2 = \sinh r d\theta, \quad \epsilon^3 = \sinh r \sin \theta d\phi, \tag{36}$$

has non-trivial commutation coefficients,

$$C_{122} = C_{133} = -\coth r, \quad C_{233} = -\frac{\cot \theta}{\sinh r}. \tag{37}$$

If Λ satisfies the Helmholtz Equation (21), which in these coordinates reads

$$\sinh^2 r (\partial_r^2 \Lambda + 2 \coth r \partial_r \Lambda + \Lambda) + \partial_\theta^2 \Lambda + \cot \theta \partial_\theta \Lambda + \frac{1}{\sin^2 \theta} \partial_\phi^2 \Lambda = 0, \tag{38}$$

Equations (23) and (24) give the following solution of the Bogomolny equations:

$$\Phi = \frac{1}{2\Lambda} \left(\partial_r \Lambda \mathbf{i} + \frac{\partial_\theta \Lambda}{\sinh r} \mathbf{j} + \frac{\partial_\phi \Lambda}{\sinh r \sin \theta} \mathbf{k} \right), \tag{39}$$

$$\begin{aligned} \mathcal{A} &= \frac{\mathbf{i}}{2} \left[\left(\cos \theta + \sin \theta \frac{\partial_\theta \Lambda}{\Lambda} \right) d\phi - \frac{\partial_\phi \Lambda}{\Lambda \sin \theta} d\theta \right] \\ &- \frac{\mathbf{j}}{2} \left[\left(\cosh r + \sinh r \frac{\partial_r \Lambda}{\Lambda} \right) \sin \theta d\phi - \frac{\partial_\phi \Lambda}{\Lambda \sinh r \sin \theta} dr \right] \\ &+ \frac{\mathbf{k}}{2} \left[\left(\cosh r + \sinh r \frac{\partial_r \Lambda}{\Lambda} \right) d\theta - \frac{\partial_\theta \Lambda}{\Lambda \sinh r} dr \right]. \end{aligned} \tag{40}$$

Note that

$$\|\Phi\|^2 = -\frac{1}{4} (1 + \Delta_{H^3} \log \Lambda) = \frac{1}{4} |d \log \Lambda|_{H^3}^2. \tag{41}$$

From the boundary conditions (9)–(11) we find $m = \frac{1}{2}$ and

$$\log \Lambda = -r + f_0(\theta, \phi) + f_2(\theta, \phi) e^{-2r} + \mathcal{O}(e^{-4r}). \tag{42}$$

It follows that

$$\|\Phi\| = \frac{1}{2} - e^{-2r} (\Delta_{S^2} f_0 - 1) + \mathcal{O}(e^{-4r}). \tag{43}$$

The total charge (15) corresponding to (43) is then

$$Q = 1 - \frac{1}{4\pi} \int_{S^2} \Delta_{S^2} f_0 \sin \theta d\theta \wedge d\phi = 1 - \frac{1}{4\pi} \int_{S^2} d^*_{S^2} d f_0. \tag{44}$$

In order for Q to be an integer, we need

$$f_0(\theta, \phi) = -n \log(\sin \theta) + \phi_0(\theta, \phi), \tag{45}$$

with $n \in \mathbb{Z}$ and ϕ_0 such that $d *_S d\phi_0$ is the zero element in $H^2_{\text{dR}}(S^2)$, the second de Rham cohomology group of S^2 . Hence $d *_S d\phi_0 = d\beta$, $\beta \in \Omega^1(S^2)$. Note that generally $\beta \neq *_S d\phi_0$ as $*_S d\phi_0$ is not necessarily a globally defined 1-form on S^2 . For f_0 given by (45) we have $Q = 1 - n$, while the monopole moduli are encoded in ϕ_0 .

We end this section with some remarks on the monopole boundary conditions at infinity and at the singularities.

Conditions (12) and (13) at the singularities imply that, for r_i the hyperbolic distance from a singularity at p_i ,

$$\Lambda = r_i^{-\ell_i} + \mathcal{O}(r_i^{1-\ell_i}). \tag{46}$$

The Helmholtz equation requires us to take $\ell_i = 1$, hence the total abelian charge ℓ defined in (14) is equal to the number L of singularities.

Conditions (12) and (13) also allow one to reverse the construction and obtain a smooth instanton on an hGH space starting from a singular hyperbolic monopole. Let (Φ, \mathcal{A}) be a singular monopole on H^3 with singularities at $\{p_i\}$, $i = 1, \dots, L$, and construct an hGH space M as follows.¹⁷ For $V = 2/\beta + 2 \sum_{i=1}^L G_{p_i}^{H^3}$ take the circle fibration \widehat{M} over $H^3 \setminus \{p_i\}$ with connection 1-form $d\psi + \alpha$, where ψ is the angular coordinate along the fibre and $d\alpha = *_3 dV$ and equip it with the metric

$$Vg_{H^3} + V^{-1}(d\psi + \alpha)^2. \tag{47}$$

Adjoin a point \hat{p}_i above each point $p_i \in H^3$ and set $M = \widehat{M} \cup \{\hat{p}_i\}$. The metric (3) on \widehat{M} extends smoothly to M and the points $\{\hat{p}_i\}$ are fixed points of the circle action on M . Equations (12) and (13) guarantee that the instanton gauge potential (22), *a priori* only defined on \widehat{M} , extends smoothly to M .^{16,25,1} Elsewhere in this paper, in order not to burden notation, we have identified each point $p_i \in H^3$ with its corresponding point $\hat{p}_i \in M$.

Finally, the asymptotic condition (42) on Λ ensures that the field strength F of (22) satisfies $\|F\|^2 = \mathcal{O}(\exp(-2r))$ for large r , hence the instanton action on M is finite.

B. The boundary data of a hyperbolic monopole

Hyperbolic monopoles are completely determined by the connection \mathcal{A}_∞ on the boundary of H^3 .^{11,22} We can show this explicitly for solutions of the Helmholtz equation, which can be written in the form

$$\Lambda(r, \theta, \phi) = \sum_{k=0}^{\infty} c_k(\theta, \phi) e^{-(2k+1)r}. \tag{48}$$

According to (40) and (42), the boundary connection is

$$\mathcal{A}_\infty = \frac{\mathbf{i}}{2} (\cos \theta d\phi + *_S d f_0) \tag{49}$$

and will completely determine the monopole as long as the fields Φ and \mathcal{A} depend only on the derivatives of f_0 . Imposing the Helmholtz equation order by order in (48) results in the recursion relation

$$c_{k+1} = c_k - \frac{1}{(k+1)^2} \sum_{j=0}^k \Delta_{S^2} c_j, \tag{50}$$

with $c_0 = e^{f_0}$. Thus $\log \Lambda$ can be written in the form (42) with f_k , $k \geq 1$, depending only on derivatives of f_0 , and the result follows.

The asymptotic expansion of the Higgs field of our hyperbolic monopoles thus only contains terms which decay exponentially with respect to hyperbolic distance. This is in contrast with what

happens for the Higgs field of a Euclidean monopole, which has both an algebraic tail and an exponential falloff near the core.

Taking $c_0 = \text{const}$, we obtain the solution $\Lambda \propto (\sinh r)^{-1}$, which we will discuss in Subsection III C. For a typical solution of the form (45), the expansion (48) breaks down at $\theta = 0, \pi$, where a more careful analysis is required. However, for the simple solution $f_0 = -2 \log(\sin \theta)$ we can restrict to the plane $\theta = \pi/2$ to find $\Lambda_{\theta=\pi/2} = (2 \cosh r)^{-1}$ and $(\partial_\theta \Lambda)_{\theta=\pi/2} = 0$, which gives the Higgs field of the smooth charge 1 monopole, $\|\Phi\| = \frac{1}{2} \tanh r$.

C. Superposing singular monopoles

The simplest solution of (38) with the asymptotic behaviour specified by (42) is

$$\Lambda = \frac{1}{\sinh r}. \tag{51}$$

It corresponds to the abelian monopole

$$\mathcal{A} = \frac{i}{2} \cos \theta d\phi, \quad \Phi = -\frac{i}{2} \coth r, \tag{52}$$

which has a singularity at $r = 0$ with abelian charge $\ell = 1$ and total charge $Q = 1$.

Further solutions of (38) are obtained by superposing Green’s function (51) at distinct points $\{p_i\}$,

$$\Lambda = \sum_{i=1}^L \frac{1}{\sinh(d^H(p, p_i))}. \tag{53}$$

The resulting monopole configuration has an abelian charge, $\ell = L$, and a total charge, $Q = 1$. From (17) it has a non-abelian charge, $k = L - 1$. The non-Abelian charge is reflected in the number of zeros of the Higgs field $\|\Phi\|$. For example, the Higgs field for $L = 2$ has two poles with a zero at the midpoint, see Figure 1. Notice that the configuration in which all the poles and zeros are placed on the $\theta = 0$ axis is axially symmetric. This should be contrasted with the case of strings of non-abelian monopoles, for which there is a breaking of axial symmetry.¹⁹

D. Generating solutions of the Helmholtz equation

Let (M, g) be a scalar flat conformally hGH space,

$$g = \Omega^2 g_0 = \Omega^2 [Vg_{H^3} + V^{-1}(d\psi + \alpha)^2]. \tag{54}$$

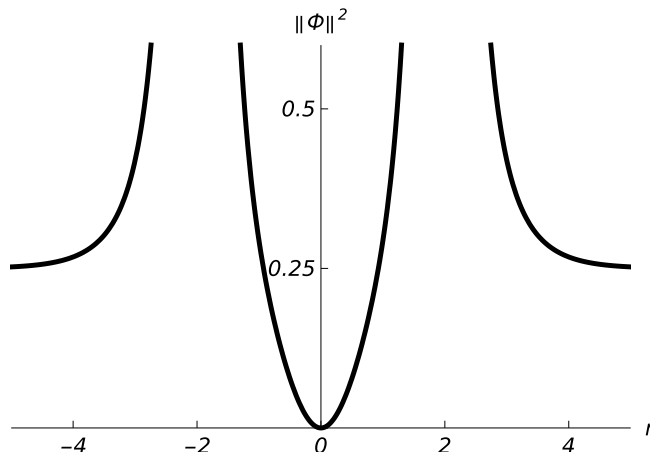


FIG. 1. The Higgs field along the polar axis ($\theta = 0, \pi$) of a monopole constructed by superposing poles of Λ at $(r, \theta) = (2, 0)$ and at $(r, \theta) = (2, \pi)$.

As shown in Section III A, g is conformally flat if Ω satisfies the Helmholtz equation,

$$\Delta_{H^3}\Omega + \Omega = 0. \tag{55}$$

The Laplacian of a ψ -independent function h on M is

$$\Delta_M h = \frac{1}{\Omega^2 V} \left(\Delta_{H^3} h + \frac{2}{\Omega} \langle dh, d\Omega \rangle_{H^3} \right), \tag{56}$$

where $\langle dh, d\Omega \rangle_{H^3} = *_H^3(dh \wedge *_H^3 d\Omega)$. To find new monopole solutions we replace Ω in (54) by $\Lambda = \rho \Omega$ and impose that $\Lambda^2 g_0$ is scalar flat, $\Delta_{H^3} \Lambda + \Lambda = 0$. Using (55) one can then see that $\Delta_M \rho = 0$. This procedure allows us to obtain a new solution of the Helmholtz equation starting from a simpler solution together with a harmonic function.

We illustrate the procedure by constructing a smooth non-abelian $k = 1$ hyperbolic monopole starting from a Yang-Mills instanton on Eguchi-Hanson (EH) gravitational instanton. The metric on EH space is often expressed in Euclidean Gibbons-Hawking form,

$$g_{EH} = \mathcal{V} (du^2 + u^2 (d\chi^2 + \sin^2 \chi d\psi^2)) + \mathcal{V}^{-1} (2 d\phi + a)^2, \tag{57}$$

where $d\mathcal{V} = *_E^3 da$. Here $u \in [0, \infty)$, $\chi \in [0, \pi]$, $\psi \in [0, 2\pi)$, $\phi \in [0, 2\pi)$. The abelian monopole (\mathcal{V}, a) has two poles,

$$\mathcal{V} = \frac{1}{u_+} + \frac{1}{u_-}, \quad u_{\pm}^2 = u^2 + u_0^2 \pm 2uu_0 \cos \chi, \tag{58}$$

where u_0 is an arbitrary constant.

It is a remarkable fact that EH space is conformal to a hyperbolic Gibbons-Hawking metric.⁷ This is achieved by the following coordinate transformation (originating from²⁷):

$$u^2 = u_0^2 \left(\frac{1}{\sinh^2 r} + \cos^2 \theta \right), \quad \tan \chi = \frac{\tan \theta}{\cosh r}. \tag{59}$$

In these coordinates the EH metric reads

$$g_{EH} = \frac{2u_0}{\sinh^2 r} \left[V (dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2)) + V^{-1} (d\psi + \cos \theta d\phi)^2 \right], \tag{60}$$

with $V = (\tanh r)^{-1}$. We remark that when transforming between (57) and (60), the roles of ϕ and ψ are interchanged. The metric inside square brackets has the form (3) with V given by (6), a single pole at the origin and $\beta = 2$. However, the range of ψ in (60) is $[0, 2\pi)$, while the metric (3) of an hGH space has $\psi \in [0, 4\pi)$. Therefore, a $\beta = 2$ single pole hGH space is conformally equivalent to a branched double cover of the EH space, in which large r hypersurfaces have the topology of $SU(2)$ rather than that of $SO(3)$.

Let us now look for monopole solutions. As EH space is scalar flat, we immediately recover the abelian monopole (52) from the conformal factor $\Omega \propto (\sinh r)^{-1}$. In order to find other monopoles we need harmonic functions on EH space. For this purpose it is convenient to work with the metric in the form (57). A ϕ -independent function ρ on EH space satisfies $\Delta_{EH} \rho = \mathcal{V}^{-1} \Delta_{E^3} \rho$, therefore a ϕ -independent harmonic function on EH space is simply a harmonic function on E^3 . In particular, $\rho = \mathcal{V}$ is a suitable function.

By expressing \mathcal{V} in (r, θ) coordinates we get a new solution of the Helmholtz equation,

$$\Lambda = \mathcal{V} \Omega = \frac{4 \cosh r}{1 + \sinh^2 r \sin^2 \theta}. \tag{61}$$

Equation (39) shows that this is the smooth spherically symmetric $k = 1$ non-abelian monopole, for which

$$\|\Phi\| = \frac{1}{2} \tanh r. \tag{62}$$

Furthermore, the manifold with the conformally related metric $(\Lambda/\Omega)^2 g_{EH}$ has a finite volume.

Another singular monopole is obtained by adding a constant λ to the harmonic function \mathcal{V} . The resulting configuration has $\ell = 1$, $Q = -1$, hence $k = 2$. See Figure 2 for a plot of the squared norm of the Higgs field. The singularity is located at $r = 0$ and aligned with two equally spaced zeros whose separation depends on the value of λ .

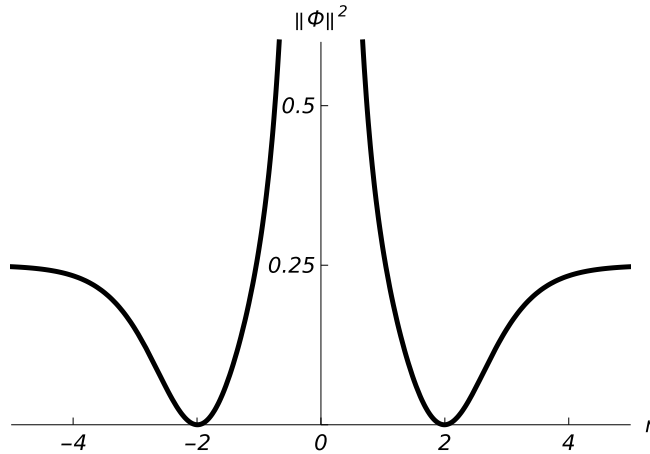


FIG. 2. The Higgs field along the polar axis ($\theta=0,\pi$) of a monopole constructed from the harmonic function $\rho = \lambda + \mathcal{V}$, with \mathcal{V} given by (58). Here $\lambda = 200$.

E. JNR equivalence

A large family of Yang-Mills instantons on E^4 can be constructed from the JNR ansatz.¹⁵ The prescription takes a harmonic function ρ on E^4 of the form

$$\rho = \sum_{i=0}^N \lambda_i G_{p_i}^{E^4}, \tag{63}$$

where $G_{p_i}^{E^4}(p) = |p - p_i|^{-2}$, with $|\cdot|$ the Euclidean distance. A charge \mathcal{N} instanton is then constructed as

$$A = -\frac{1}{2} \sigma_{\mu\nu} \partial^\nu \log \rho \, dx^\mu, \tag{64}$$

where the tensor σ is given in terms of the unit quaternions and the anti-self-dual 't Hooft matrices (30) by $\sigma = \mathbf{i}\bar{\eta}_1 + \mathbf{j}\bar{\eta}_2 + \mathbf{k}\bar{\eta}_3$.

In order to obtain a hyperbolic monopole, one makes use of the conformal equivalence between $E^4 \setminus E^2$ and $H^3 \times S^1$,

$$g_{E^4} = z^2 [g_{H^3} + d\psi^2] = z^2 \left(\frac{dx^2 + dy^2 + dz^2}{z^2} + d\psi^2 \right). \tag{65}$$

Circle invariance is ensured by placing all the poles p_i of ρ on the fixed plane of the $\partial/\partial\psi$ action, a 2-plane in E^4 which, under the conformal equivalence, is mapped to the boundary of H^3 . In a circle invariant gauge write $A = \mathcal{A} + \Phi d\psi$. Then (Φ, \mathcal{A}) is a hyperbolic monopole. The Higgs field has squared norm⁹

$$\|\Phi\|^2 = \frac{z^2}{4\rho^2} \left((\partial_x \rho)^2 + (\partial_y \rho)^2 + \left(\frac{\rho}{z} + \partial_z \rho \right)^2 \right) = -\frac{1}{4} (1 + \Delta_{H^3} \log(z\rho)), \tag{66}$$

where for the second equality we have made use of the relations

$$\Delta_{H^3} \rho = z^2 (\partial_x^2 + \partial_y^2 + \partial_z^2) \rho - z \partial_z \rho, \tag{67}$$

$$z^2 \Delta_{E^4} \rho = z^2 (\partial_x^2 + \partial_y^2 + \partial_z^2) \rho + z \partial_z \rho = 0. \tag{68}$$

Note that $H^3 \times S^1$ is a special case of an hGH space with $V = 1$, $\alpha = 0$. Since E^4 is scalar flat, we recover the JNR construction of monopoles by applying the method of Section III D: take the conformal factor $\Omega = z$ and a harmonic function ρ on E^4 , then $\Lambda = z\rho$ generates a hyperbolic monopole via Equations (23) and (24).²

We expect that all the solutions of the Helmholtz equation giving smooth hyperbolic monopoles correspond to harmonic functions on E^4 which are of JNR type. To translate from JNR data to

solutions of the hyperbolic Helmholtz equation in hyperboloid model coordinates we make use of the coordinate transformation

$$\begin{aligned} x &= \frac{\sinh r \sin \theta \cos \phi}{\cosh r - \sinh r \cos \theta}, & \tan \phi &= \frac{y}{x}, \\ y &= \frac{\sinh r \sin \theta \sin \phi}{\cosh r - \sinh r \cos \theta}, & \tan^2 \theta &= \frac{4(x^2 + y^2)}{(x^2 + y^2 + z^2 - 1)^2}, \\ z &= \frac{1}{\cosh r - \sinh r \cos \theta}, & \cosh r &= \frac{x^2 + y^2 + z^2 + 1}{2z}. \end{aligned} \tag{69}$$

Singular hyperbolic monopoles can also be described in terms of JNR data. Using (69) we find that the JNR function corresponding to the singular monopole with $\Lambda = (2 \sinh r)^{-1}$ is

$$\rho = \frac{\Lambda}{z} = \frac{1}{\sqrt{(1 + x^2 + y^2 + z^2)^2 - 4z^2}}. \tag{70}$$

Note that the function ρ is singular at the monopole location, $(x, y, z) = (0, 0, 1)$. A monopole singularity may thus be interpreted as a JNR pole which has been moved from the boundary of hyperbolic space to the interior. This interpretation of singular monopoles explains our observation in Section III C that by superposing two abelian monopoles the Higgs field acquires a zero. Separating the poles causes the profile of the Higgs field near the zero to approach that of the smooth unit charge non-abelian monopole. In general, a JNR configuration consisting of $\mathcal{N} > 0$ poles on the boundary of H^3 and $L \geq 0$ poles in the interior has an abelian charge, $\ell = L$, a non-Abelian charge, $k = \mathcal{N} + \ell - 1$, and a total charge, $Q = 1 - \mathcal{N}$, independent of ℓ .

F. Higher mass monopoles

Dimensional reduction of instantons also gives monopoles of mass $m > \frac{1}{2}$. Take the axially symmetric instantons on Eguchi-Hanson space described by Boutaleb-Joutei *et al.*¹⁰ In the coordinates (60) their solution reads

$$A_r = 0, \quad A_\theta = -D \frac{\mathbf{k}}{2}, \quad A_\phi = G \cos \theta \frac{\mathbf{i}}{2} - D \sin \theta \frac{\mathbf{j}}{2}, \quad A_\psi = (G - 1) \frac{\mathbf{i}}{2}, \tag{71}$$

where

$$D = \frac{\alpha \sinh r}{\sinh(\alpha r)}, \quad G = \frac{\alpha \tanh r}{\tanh(\alpha r)}. \tag{72}$$

This instanton is manifestly circle invariant so we can reduce it by a circle action of weight $\frac{1}{2}$ to get a hyperbolic monopole. As far as we know this has not been noticed before. The monopole Higgs field $\Phi = VA_\psi$, where $V = \coth r$, is given by

$$\|\Phi\| = \frac{1}{2}(\alpha \coth(\alpha r) - \coth r) \tag{73}$$

and has mass $\frac{1}{2}(\alpha - 1)$. This monopole arises either as a higher weight reduction of an axially symmetric Euclidean instanton²⁴ or, as we have just shown, as a weight $\frac{1}{2}$ reduction of an instanton on Eguchi-Hanson space.

The modified Helmholtz equation

$$\Delta_{H^3} \Lambda + K \Lambda = 0, \quad K < 1 \tag{74}$$

has solutions of the form

$$\Lambda = \frac{\sinh(\sqrt{1 - K} r)}{\sinh r}. \tag{75}$$

It is remarkable that if we use (41) to compute $\|\Phi\|$ we recover (73) with $\alpha = \sqrt{1 - K}$. While the pair (Φ, \mathcal{A}) obtained using (39), (40) is not a solution of the Bogomolny equations, the fact that we can reproduce (73) suggests that our construction can be extended to $m > \frac{1}{2}$.

IV. A FAMILY OF CONFORMALLY EINSTEIN MANIFOLDS

We still have not made use of the second method of Theorem 1 which will give us a family of monopoles which, as far as we know, has not been discussed before. It has been proved²⁶ that an hGH space is conformally Einstein if and only if V is spherically symmetric. Therefore we take

$$V = \frac{2}{\beta} + \frac{2}{e^{2r} - 1}, \tag{76}$$

so that $\alpha = \cos \theta d\phi$.³ The metric

$$g = \frac{2}{\beta((2 - \beta) \cosh r + \beta \sinh r)^2} (V g_{H^3} + V^{-1}(d\psi + \alpha)^2), \quad \beta \in (0, 2] \tag{77}$$

is Einstein with constant $\frac{3}{2}\beta^2(2 - \beta)$.⁷ The case $\beta = 2$ corresponds to (a branched double cover of) the Eguchi-Hanson space discussed previously. For $\beta = 1$ one obtains the Fubini-Study metric on $\mathbb{C}P^2$. With the rescaling $r \rightarrow r/\beta$ the pointwise limit for $\beta \rightarrow 0$ of (77) is the Taub-NUT metric.

To get a self-dual instanton we need to project the spin connection ω onto $S^+(M)$,

$$P_+(\omega) = \frac{1}{4}(\eta_a)_{ij} \omega_{ij} \eta_a = \frac{1}{2} [(\omega_{34} + \omega_{12})\mathbf{i} + (\omega_{24} - \omega_{13})\mathbf{j} + (\omega_{14} + \omega_{23})\mathbf{k}], \tag{78}$$

where $\{\eta_i\}$ are the self-dual 't Hooft matrices

$$\eta_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{79}$$

and we have identified $\eta_1 = -\mathbf{k}, \eta_2 = -\mathbf{j}, \eta_3 = -\mathbf{i}$.

The corresponding instanton has gauge potential $A = \mathcal{A} + \Phi(d\psi + \cos \theta d\phi)/V$. The fields Φ, \mathcal{A} satisfy the Bogomolny equations (8) and are given by

$$\Phi = \frac{\mathbf{i}}{4}(1 - \coth r) \left(\frac{(\beta - 1)^2(1 - 3e^{2r}) - e^{6r}(1 - 3e^{-2r})}{e^{4r} - (\beta - 1)^2} \right), \tag{80}$$

$$\mathcal{A} = \frac{\mathbf{i}}{2} \cos \theta d\phi + \frac{2(\beta - 1)e^{2r} \sinh r}{e^{4r} - (\beta - 1)^2} (\mathbf{j} \sin \theta d\phi - \mathbf{k} d\theta). \tag{81}$$

Note that Φ and \mathcal{A} are invariant under the change $\beta \rightarrow 2 - \beta$, coupled to a gauge transformation by \mathbf{i} . The monopole has a mass $\frac{1}{2}$ and a total charge, $Q = -1$. The value $\beta = 2$ gives the smooth spherically symmetric $k = 1$ monopole with $\|\Phi\| = \frac{1}{2} \tanh r$. For $\beta \neq 2$ there is a singularity at $r = 0$ with abelian weight $\ell = 1$ and a 2-sphere worth of zeros given by $r = r_0 \in (0, \frac{1}{2} \log(3))$ as $2 - \beta$ varies in $(0, 1]$. The Higgs field profile is shown in Figure 3. Note that Φ and \mathcal{A} abelianise for

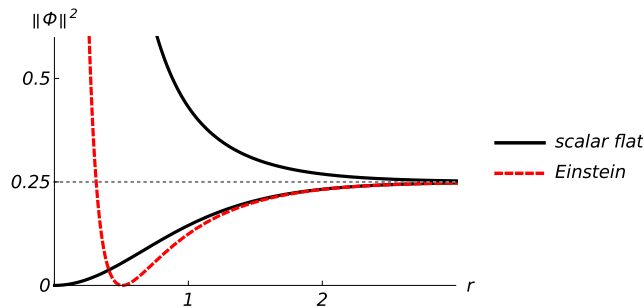


FIG. 3. Solid curves show the Higgs field profile of the smooth monopole (62) and singular monopole (52) constructed by conformally rescaling (3) to be scalar flat. The dashed curve shows the Higgs field (80) for $\beta = 1.5$ obtained by rescaling (3) to be Einstein. The sign of Q depends on whether the asymptotic value $m^2 = \frac{1}{4}$ is approached from above ($Q > 0$) or from below ($Q < 0$).

both small and large values of r . Plotting the energy density shows no special features at $r = r_0$. In the limiting case $\beta = 1$ the fields abelianise for all values of r and we have $\|\Phi\| = \frac{1}{2}|\coth r - 2|$. This family has charge Q opposite to that of the singular monopole $\|\Phi\| = \frac{1}{2}\coth r$ we encountered before. From (17), we expect the 2-sphere of zeros to contribute a non-abelian charge $k = 2$.

V. CONCLUSIONS

In this work we have shown how a class of smooth and singular hyperbolic monopoles of mass $m = \frac{1}{2}$ can be expressed in terms of solutions of the hyperbolic Helmholtz equation. All the solutions we found have an equivalent description in terms of JNR data. However, thinking in terms of the Helmholtz equation presents a number of advantages.

First, our construction is entirely coordinate free, see Equations (23) and (24), while the JNR construction is adapted to the upper half-space model of hyperbolic space.

Second, we relate *singular* hyperbolic monopoles to *smooth* instantons on a scalar flat 4-manifold which is conformally equivalent to a hyperbolic Gibbons-Hawking space. The JNR data giving the same monopoles describes instantons on E^4 which are singular along circles. In this sense we have illustrated how a conformally Gibbons-Hawking geometry, which is commonly seen as encoding an abelian monopole, also encodes non-abelian monopoles.

Third, our approach shows in a very explicit fashion, albeit in a special case, how a hyperbolic monopole can be reconstructed from its asymptotic data.

Fourth, we provide a physical interpretation of the poles in the JNR ansatz as singular hyperbolic monopoles. Separating two such poles gives rise to a non-abelian monopole between them. This should be contrasted with the related Euclidean instanton, for which there is no direct physical interpretation of the JNR poles.

Interestingly, the Helmholtz equation also arises in Prasad's generalisation of the Atiyah-Ward ansatz for Euclidean monopoles.²⁸

The manifold $\mathbb{C}P^2$ is conformally hGH and Einstein, so it is natural to ask whether circle invariant instantons on this space can be reduced to hyperbolic monopoles. Instantons with instanton number 1 are studied in Ref. 13 and there is a 3-parameter family which is invariant under a circle action. However, this is not the circle action which reduces $\mathbb{C}P^2$ to a conformally hGH space, therefore these instantons do not descend to hyperbolic monopoles.

It is natural to ask if our construction can be generalised to $m > \frac{1}{2}$. The results described in Section III F suggest that, at least for the spherically symmetric case, some more general construction exists. However such an extension is not straightforward and we leave it for future work.

ACKNOWLEDGMENTS

We wish to thank Olaf Lechtenfeld for useful discussions. G.F. was supported by the DFG Research Training Group No. 1463.

¹ A smooth instanton would also result if we were to construct M by adjoining points $\{\hat{p}_i\}$, $i = 1, \dots, N$; $N > L$, to a circle fibration over $H^3 \setminus \{p_i\}$, where $\{p_1, \dots, p_L\}$ are the singular points of the monopole and $\{p_{L+1}, \dots, p_N\}$ are arbitrary distinct points in H^3 . All the points $\{\hat{p}_i\}$, $i = 1, \dots, N$, would then be fixed points of the circle action on M .

² Of course, since z satisfies the hyperbolic Helmholtz equation it also gives rise to a hyperbolic monopole. It is easy to check that this is the vacuum solution $\Phi = \mathbf{i}/2$, \mathcal{A} gauge equivalent to zero.

³ Note that with our conventions V in (76) is twice that appearing in Ref. 7. Consequently, our conformal factor is one half of theirs.

⁴ Atiyah, M. F., *Magnetic Monopoles in Hyperbolic Spaces*, M. Atiyah: Collected Works Vol. 5 (Oxford University Press, 1988).

⁵ Atiyah, M. and Hitchin, N. J., *The Geometry and Dynamics of Magnetic Monopoles* (Princeton University Press, 1988).

⁶ Atiyah, M. F., Hitchin, N. J., and Singer, I. M., "Self-duality in four-dimensional Riemannian geometry," *Proc. R. Soc. London A* **362**, 425 (1978).

⁷ Atiyah, M. and LeBrun, C., "Curvature, cones and characteristic numbers," *Math. Proc. Cambridge Philos. Soc.* **155**, 13 (2013).

⁸ Blair, C. D. A. and Cherkis, S. A., "Singular monopoles from Cheshire bows," *Nucl. Phys. B* **845**, 140 (2011).

⁹ Bolognesi, S., Cockburn, A., and Sutcliffe, P., "Hyperbolic monopoles, JNR data and spectral curves," *Nonlinearity* **28**, 211 (2015).

- ¹⁰ Boutaleb-Joutei, H., Chakrabarti, A., and Comtet, A., "Gauge field configurations in curved space-times. III. Self-dual SU(2) fields in Eguchi-Hanson space," *Phys. Rev. D* **21**, 979 (1980).
- ¹¹ Braam, P. J. and Austin, D. M., "Boundary values of hyperbolic monopoles," *Nonlinearity* **3**, 809 (1990).
- ¹² Etesi, G. and Hausel, T., "On Yang-Mills instantons over multi-centered gravitational instantons," *Commun. Math. Phys.* **235**, 275 (2003).
- ¹³ Groisser, D., "The geometry of the moduli space of $\mathbb{C}P^2$ instantons," *Invent. Math.* **99**, 393 (1990).
- ¹⁴ Hanany, A. and Witten, E., "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys. B* **492**, 152 (1997).
- ¹⁵ Jackiw, R., Nohl, C., and Rebbi, C., "Conformal properties of pseudoparticle configurations," *Phys. Rev. D* **15**, 162 (1977).
- ¹⁶ Kronheimer, P. B., "Monopoles and Taub-NUT metrics," M.S. dissertation, University of Oxford, Oxford, 1985.
- ¹⁷ LeBrun, C., "Explicit self-dual metrics on $\mathbb{C}P^2 \# \dots \# \mathbb{C}P^2$," *J. Differ. Geom.* **34**, 223 (1991).
- ¹⁸ LeBrun, C., Nayatani, S., and Nitta, T., "Self-dual manifolds with positive Ricci curvature," *Math. Z.* **224**, 49 (1997).
- ¹⁹ Maldonado, R., "Hyperbolic monopoles from hyperbolic vortices," e-print [arXiv:1508.07304](https://arxiv.org/abs/1508.07304).
- ²⁰ Manton, N. S. and Sutcliffe, P. M., "Platonic hyperbolic monopoles," *Commun. Math. Phys.* **325**, 821 (2014).
- ²¹ Mason, L. J. and Woodhouse, N. M. J., *Integrability, Self-Duality, and Twistor Theory* (Oxford University Press, 1996).
- ²² Murray, M. K., Norbury, P., and Singer, M. A., "Hyperbolic monopoles and holomorphic spheres," *Ann. Global Anal. Geom.* **23**, 101 (2003).
- ²³ Murray, M. K. and Singer, M. A., "On the complete integrability of the discrete Nahm equations," *Commun. Math. Phys.* **210**, 497 (2000).
- ²⁴ Nash, C., "Geometry of hyperbolic monopoles," *J. Math. Phys.* **27**, 2160 (1986).
- ²⁵ Nash, O., "Singular hyperbolic monopoles," *Commun. Math. Phys.* **277**, 161 (2008).
- ²⁶ Pedersen, H. and Tod, P., "Einstein metric and hyperbolic monopoles," *Classical Quantum Gravity* **8**, 751 (1991).
- ²⁷ Prasad, M. K., "Equivalence of Eguchi-Hanson metric to two center Gibbons-Hawking metric," *Phys. Lett. B* **83**, 310 (1979).
- ²⁸ Prasad, M. K., "Yang-Mills-Higgs monopole solutions of arbitrary topological charge," *Commun. Math. Phys.* **80**, 137 (1981).