Noise prevents infinite stretching of the passive field in a stochastic vector advection equation

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Abstract

A linear stochastic vector advection equation is considered; the equation may model a passive magnetic field in a random fluid. When the driving velocity field is rough but deterministic, in particular just Hölder continuous and bounded, one can construct examples of infinite stretching of the passive field, arising from smooth initial conditions. The purpose of the paper is to prove that infinite stretching is prevented if the driving velocity field contains in addition a white noise component.

1 Introduction

Consider the linear stochastic vector advection equation in $\mathbb{R}^3$:

$$d\mathbf{B} + \mathrm{curl}(\mathbf{v} \times \mathbf{B})dt + \sigma \sum_{k=1}^{3} \mathrm{curl}(\mathbf{e}_k \times \mathbf{B}) \circ dW^k = 0, \tag{1}$$

where $\mathbf{v}$: $[0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is a given divergence-free vector field, the solution $\mathbf{B}$ is a divergence-free vector field, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the canonical basis of $\mathbb{R}^3$, $\mathbf{W} = \left(W^1, W^2, W^3\right)$ is a Brownian motion in $\mathbb{R}^3$, σ is a real number. The initial condition, at time $t = 0$, will be denoted by $\mathbf{B}_0$. The driving vector field (the velocity field of the fluid, in the usual interpretation) is modeled by the Gaussian field

$$\mathbf{v} + \sigma \sum_{k=1}^{3} \mathbf{e}_k \frac{dW^k}{dt} = \mathbf{v} + \sigma \frac{d\mathbf{W}}{dt}$$

where $\mathbf{v}$ is deterministic, a sort of average or slow-varying component, and $\sigma d\mathbf{W}$ is the fast-varying random component, white noise in time. This equation may model a passive vector field $\mathbf{B}$, like a magnetic field, in a turbulent fluid with a non-trivial average component $\mathbf{v}$. The intensity σ of the noise can be arbitrarily small, in the sequel, to model real situations when the noise (which always exists) is usually neglected in first approximation. However, the trajectories of $\mathbf{W}$ are only Hölder continuous with exponent smaller than $\frac{1}{2}$ and not differentiable at any point, so that the impulses given by the term $\sigma\frac{d\mathbf{W}}{dt}$ are small when cumulated in time ($\sigma\mathbf{W}$) but istantaneously very strong. We aim at studying existence, uniqueness, representation formula and regularity under low regularity assumption on $\mathbf{v}$.

The key point of this work is the fact that the noise prevents blow-up, under assumptions on $\mathbf{v}$ such that blow-up may occur in the deterministic case. When $\sigma = 0$, we give an example of Hölder continuous vector field $\mathbf{v}$ such that infinite values of $\mathbf{B}$ arise in finite time from a bounded continuous initial field $\mathbf{B}_0$; then we prove that Hölder continuity and boundedness of $\mathbf{v}$ is sufficient, in the stochastic case ($\sigma \neq 0$), to prove that continuous initial field $\mathbf{B}_0$ produces continuous fields $\mathbf{B}_t$ for all $t \geq 0$. The singularity in the deterministic case is associated to infinite stretching of $\mathbf{B}$; randomness prevents stretching to blow-up to infinity. Precisely, we prove (see the notations below):

Theorem 1 *i) For $\sigma = 0$, there exists $\mathbf{v}\in C_b^\alpha(\mathbb{R}^3;\mathbb{R}^3)$ and $\mathbf{B}_0 \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\sup_{|x|\leq 1} |\mathbf{B}(t,x)| = +\infty$ for all $t > 0$.*

ii) For $\sigma \neq 0$, for all $\mathbf{v}\in C([0,T];C_b^\alpha(\mathbb{R}^3;\mathbb{R}^3))$ and $\mathbf{B}_0 \in C(\mathbb{R}^3;\mathbb{R}^3)$ one has $\mathbf{B} \in C([0,T] \times \mathbb{R}^3;\mathbb{R}^3)$, with probability one.

Clearly, linear vector advection equation is a very idealized model of fluid dynamics but this result opens the question whether noise may prevent blow-up of the vorticity field of 3D Euler equations. The emergence of singularity seems to require a certain degree of organization of the fluid structures and perhaps this organization is lost, broken, under the influence of randomness. With further degree of speculation, one could even think that a turbulent regime may contain the necessary degree of randomness to prevent blow-up; if so, singularities of the vorticity could more likely be associated to strong transient phases, instead of established turbulent ones.

From the mathematical side, this is not the first result of this nature, see [13], [3], [18], [9], and also [17], [24], [2], [16], [20], [26] for uniqueness of weak solutions due to noise (the other face of the celebrated open problem presented by [14]). However, these papers deal with scalar problems, like linear transport equations, linear continuity equations, vorticity in 2D Euler equations, 1D Vlasov-Poisson equations. The result of the present paper is the first one dealing with vector valued PDEs like 3D Euler equations; the kind of singularity in the vectorial case is different, related to rotations and stretching instead of shocks or mass concentration. Several new technical difficulties arise due to the vectorial nature of the equation (for instance, the proof of uniqueness of non-regular solutions, Lemma 16, usually involving commutator estimates, here is more difficult and is obtained by special cancellations, also inspired to [26]). Let us mention also the improvement of well-posedness due to noise proved for dispersive equations, [10], [5]. In all the works mentioned so far the noise is multiplicative, and often of transport type like in the present paper. The role of additive noise in preventing singularities is more obscure. For uniqueness under poor drift, additive noise is very powerful see [28], [29], [21], [7], [8]; however, its relevance in fluid dynamics is still under investigation. See [6], [19], [15], [27] for partial results.

For additional details on vector advection equations see for instance [4] and references therein. For advanced results on the differentiability of stochastic flow generated by rough drift (key ingredient of the representation formula (5)), see [1], [3], [13], [25]. For a general reference on passive advection driven by random velocity fields see [12], where also the case of a passive magnetic field is discussed; the structure of the noise term in the present work is very simplified with respect to [12] but the point here is to prove that noise has a depleting effect on $\mathbf{B}$ and this fact is true also under this simple noise; generalization to space-homogeneous noise with more complex space structure is possible, if $Q(0)$, the covariance matrix at $x = 0$, is non-degenerate.

The model described here is clearly too idealized for a direct interest in fluid dynamics but once the phenomenon of depletion of stretching is rigorously proved in this particular framework, there is more motivation to investigate generalizations which could become closer to reality. One of them would be the case when $\mathbf{v}$ contains (also just small) high frequency fluctuations, although not being white noise. This extension looks very difficult but potentially not impossible.

1.1 Notations

We denote by $C(\mathbb{R}^3;\mathbb{R}^3)$ (resp. $C^\infty(\mathbb{R}^3;\mathbb{R}^3)$) the space of all continuous (resp. infinitely differentiable) vector fields $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$. We denote by $C_b(\mathbb{R}^3;\mathbb{R}^3)$ the space of all $\mathbf{v} \in C(\mathbb{R}^3;\mathbb{R}^3)$ such that $\|\mathbf{v}\|_0 := \sup_{x\in\mathbb{R}^3} |\mathbf{v}(x)| < \infty$. For any $\alpha \in (0,1)$ we denote by $C_b^\alpha(\mathbb{R}^3;\mathbb{R}^3)$ the space of all $\mathbf{v} \in C_b(\mathbb{R}^3;\mathbb{R}^3)$ such that $[\mathbf{v}]_\alpha := \sup_{x,y\in\mathbb{R}^3, x\neq y} \frac{|\mathbf{v}(x)-\mathbf{v}(y)|}{|x-y|^\alpha} < \infty$; the space $C_b^\alpha(\mathbb{R}^3;\mathbb{R}^3)$ is endowed with the norm $\|\mathbf{v}\|_\alpha = \|\mathbf{v}\|_0 + [\mathbf{v}]_\alpha$. We denote by $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ the space of all $\mathbf{v} \in C^\infty(\mathbb{R}^3;\mathbb{R}^3)$ which have compact support.

For $p \geq 1$, we denote by $L^p_{loc}(\mathbb{R}^3;\mathbb{R}^3)$ the space of measurable vector fields $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\int_{|x|\leq R} |\mathbf{v}(x)|^p \, dx < \infty$ for all $R > 0$; we write $\mathbf{v} \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ when $\int_{\mathbb{R}^3} |\mathbf{v}(x)|^p \, dx < \infty$. The notation $\langle \mathbf{v}, \mathbf{w} \rangle$ stands for $\int_{\mathbb{R}^3} \mathbf{v}(x) \cdot \mathbf{w}(x) \, dx$, when $\mathbf{v}, \mathbf{w} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$.

If $\mathbf{v} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, we usually write $\mathbf{v}(t,x)$, but also $\mathbf{v}_t$ to denote the function $x \mapsto \mathbf{v}(t,x)$ at given $t \in [0,T]$.

If $\mathbf{v} \in \mathbb{R}^3$ we write $\mathbf{v} \cdot \nabla$ for the differential operator $\sum_{i=1}^3 v^i \partial_{x_i}$. If $\mathbf{v}, \mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ the notation $(\mathbf{v} \cdot \nabla) \mathbf{B}$ stands for the vector field with components $(\mathbf{v} \cdot \nabla) B^i$. Similarly, we interpret componentwise operations like $\partial_k \mathbf{B}$, $\Delta \mathbf{B}$.

2 Example of blow-up in the deterministic case

In this section we consider equation (1) in the deterministic case $\sigma = 0$. We give an example of Hölder continuous bounded vector field $\mathbf{v}$ such that $\sup_{|x|\leq 1} |\mathbf{B}(t,x)| = +\infty$ for all $t > 0$, although $\sup_{x\in\mathbb{R}^3} |\mathbf{B}_0(x)| < \infty$ and $\mathbf{B}_0$ is smooth.

Let us also remark that, on the contrary, when $\mathbf{v}$ is of class $C\left([0,T]; C_b^1(\mathbb{R}^3;\mathbb{R}^3)\right)$, for every $\mathbf{B}_0 \in C(\mathbb{R}^3;\mathbb{R}^3)$ there exists a unique continuous weak solution $\mathbf{B}$ (the definition is analogous to Definition 10 below and the proof is similar to the one of Theorem 12); it satisfies identity (5) below where $\Phi_t(x)$ is the deterministic flow given by the equation of characteristics

$$\frac{d}{dt}\Phi_t(x) = \mathbf{v}(t, \Phi_t(x)), \qquad \Phi_0(x) = x.$$

When $\mathbf{v}$ is of class $C\left([0,T]\,; C_b^2(\mathbb{R}^3;\mathbb{R}^3)\right)$ and $\mathbf{B}_0 \in C^1(\mathbb{R}^3;\mathbb{R}^3)$ the solution $\mathbf{B}$ is of class $C\left([0,T]\,; C^1(\mathbb{R}^3;\mathbb{R}^3)\right)$, and so on, from identity (5). The idea of the example of blow-up comes from identity (5): one has to construct a flow $\Phi_t(x)$, corresponding to a vector field $\mathbf{v}$ less regular than $C\left([0,T]\,; C_b^1(\mathbb{R}^3;\mathbb{R}^3)\right)$, such that $D\Phi_t(x)$ blows-up at some point.

2.1 Preliminaries on cylindrical coordinates

Limited to this and next subsection, we denote points of $\mathbb{R}^3$ by (x,y,z) instead of x (and analogous notations for Euclidea coordinates). Let us recall that the material derivative, in cylindrical coordinates, for vectors $\mathbf{A} = \mathbf{A}(r,\theta,z)$, $\mathbf{B} = \mathbf{B}(r,\theta,z)$, $\mathbf{A} = A_r e_r + A_\theta e_\theta + A_z e_z$, $\mathbf{B} = B_r e_r + B_\theta e_\theta + B_z e_z$ (where $e_r = \frac{x}{r}e_x + \frac{y}{r}e_y$, $e_\theta = -\frac{y}{r}e_x + \frac{x}{r}e_y$) are given by the formula

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\,\mathbf{B} = {} & (A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + A_z\frac{\partial B_r}{\partial z} - \frac{A_\theta B_\theta}{r})e_r \\
& + (A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + A_z\frac{\partial B_\theta}{\partial z} + \frac{A_\theta B_r}{r})e_\theta \\
& + (A_r\frac{\partial B_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} + A_z\frac{\partial B_z}{\partial z})e_z
\end{aligned}$$

Consequently,

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\,\mathbf{B} - (\mathbf{B}\cdot\nabla)\,\mathbf{A} = {} & (A_r\frac{\partial B_r}{\partial r} - B_r\frac{\partial A_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} - \frac{B_\theta}{r}\frac{\partial A_r}{\partial \theta} + A_z\frac{\partial B_r}{\partial z} - B_z\frac{\partial A_r}{\partial z})e_r \\
& + (A_r\frac{\partial B_\theta}{\partial r} - B_r\frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} - \frac{B_\theta}{r}\frac{\partial A_\theta}{\partial \theta} + A_z\frac{\partial B_\theta}{\partial z} - B_z\frac{\partial A_\theta}{\partial z} + \frac{A_\theta B_r - B_\theta A_r}{r})e_\theta \\
& + (A_r\frac{\partial B_z}{\partial r} - B_r\frac{\partial A_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} - \frac{B_\theta}{r}\frac{\partial A_z}{\partial \theta} + A_z\frac{\partial B_z}{\partial z} - B_z\frac{\partial A_z}{\partial z})e_z
\end{aligned}$$

With these preliminaries, let us consider a vector field $\mathbf{v}$ of the form

$$\mathbf{v} = v_\theta e_\theta, \quad v_\theta = v_\theta(r)$$

and assume that $\mathbf{B}(t) = B_r(t)e_r + B_\theta(t)e_\theta + B_z(t)e_z, t \geq 0$ is a vector field of class C^1 on $\mathbb{R}^3 \backslash \{0\}$ which satisfies (on $\mathbb{R}^3 \backslash \{0\}$) the equation

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl}(\mathbf{v}\times\mathbf{B}) = 0$$

with divergence-free initial condition $\mathbf{B}_0$. Notice that div$\mathbf{v}$= div$\mathbf{B}$= 0. Indeed, $\mathbf{v}$ is divergence free vector field by definition and $\frac{\partial \operatorname{div}\mathbf{B}}{\partial t} = -\operatorname{div}\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = 0$. Hence we can rewrite equation for $\mathbf{B}$ as follows

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{B} - (\mathbf{B} \cdot \nabla)\,\mathbf{v} = 0.$$

Consequently, in cylindrical coordinates we have

$$\frac{\partial B_r}{\partial t} + \frac{v_\theta}{r}\frac{\partial B_r}{\partial \theta} = 0,$$

$$\frac{\partial B_\theta}{\partial t} = B_r\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\frac{\partial B_\theta}{\partial \theta} - \frac{v_\theta}{r}B_r = -\frac{v_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + B_r(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}),$$

$$\frac{\partial B_z}{\partial t} + \frac{v_\theta}{r}\frac{\partial B_z}{\partial \theta} = 0.$$

2.2 The example

Choose, for some $\alpha \in (0,1)$,

$$v_\theta(r) = r^\alpha, \qquad \text{for } r \in [0,1]$$

and define v_θ for $r > 1$ in a such way that $v_\theta \in C^\infty$, $v_\theta > 0$ and $v_\theta(r) \leq e^{-\gamma r}$, $\gamma > 0$, $r \geq A > 1$ for some γ, A.

Then we have, for $r \in (0,1)$,

$$\frac{\partial B_r}{\partial t} + r^{\alpha-1}\frac{\partial B_r}{\partial \theta} = 0,$$

$$\frac{\partial B_\theta}{\partial t} + r^{\alpha-1}\frac{\partial B_\theta}{\partial \theta} + (1-\alpha)B_r r^{\alpha-1} = 0$$

$$\frac{\partial B_z}{\partial t} + r^{\alpha-1}\frac{\partial B_z}{\partial \theta} = 0.$$

Hence we can deduce that (we write B_r^0, B_z^0, B_θ^0 for the coordinates of $\mathbf{B}_0$), for $r \in (0,1)$,

$$B_r(t,r,\theta,z) = B_r^0(r, \theta - r^{\alpha-1}t, z)$$

$$B_z(t,r,\theta,z) = B_z^0(r, \theta - r^{\alpha-1}t, z)$$

$$B_\theta(t,r,\theta,z) = B_\theta^0(r, \theta - r^{\alpha-1}t, z) - (1-\alpha)\boxed{\mathrm{r}^{\alpha-1}}\, tB_r^0(r, \theta - r^{\alpha-1}t, z).$$

A non-zero radial component B_r^0 of the initial condition near the vertical axis for the origin ($r = 0$) yields a blow-up of the angular component B_θ.

Thus we see that any smooth bounded initial condition $\mathbf{B}_0$, such that $B_r^0(r,\theta,z) > 0$ for all values of the arguments, gives rise to a solution $\mathbf{B}$ such that

$$\lim_{r\to 0} |B_\theta(t,\theta,r,z)| = \infty$$

for any $t > 0$, at any point (θ, z). From this one deduces $\lim_{(x,y,z)\to 0} |\mathbf{B}\,(t,x,y,z)| = +\infty$ for all $t > 0$ (since B_θ is the projection on e_θ at (x,y,z); similarly for B_r, B_z; thus the divergence of B_θ and boundedness of B_r, B_z imply the divergence of $\mathbf{B}$).

Remark 2 *With more work, taking a time-dependent vector field* $\mathbf{v}$ *which is smooth until time* $t_0 > 0$ *when it develops an Hölder singularity of the form above, one can construct an example of solution* $\mathbf{B}$ *which is smooth on* $[0, t_0)$ *but infinite at some point at time* t_0*. Such example would mimic more closely what maybe could happen in a non-passive version of the vector advection equation.*

2.3 The Lagrangian picture

We summarize here the features of this example, with the following items and some pictures (just to give a graphical intuition of what happens).

i) The fluid rotates around the vertical z-axis ζ at the origin; the Lagrangian particles describe circles around ζ, the Cauchy problem

$$\frac{d}{dt} X_t = \mathbf{v}(t, X_t), \qquad X_0 = x \tag{2}$$

is uniquely solvable and generate a continuous flow $\Phi_t(x)$. Figure 1 shows a number of Lagrangian trajectories (solutions of the Cauchy problem (2)) starting on the x-axis, in the regular case $\alpha = 1$, where the velocity produces a rigid motion (no singularity). Figure 2 shows the case $\alpha = 0.2$, where the velocity of rotation near the origin is so large (still infinitesimal, so that the velocity field is Hölder continuous) that very close initial particles are displaced a lot; and the ratio between the displacement at time t and that at time zero diverges when the particles approach zero.

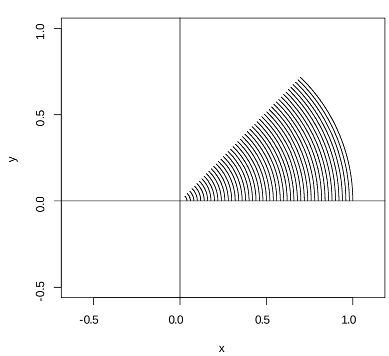

Fig.1. Lagrangian trajectories for $\alpha = 1$.

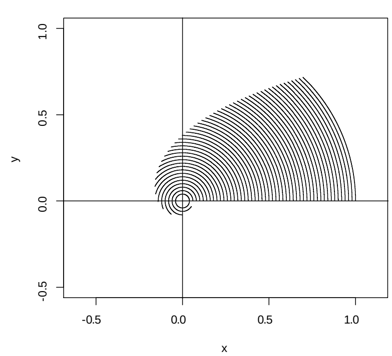

Fig.2. Lagrangian trajectories for $\alpha = 0.3$.

ii) The flow $\Phi_t(x)$ is however not differentiable at the vertical axis ζ (it is smooth outside ζ), as it may be guesses from Figure 2; ideal lines of Lagrangian points in a plane orthogonal to ζ are stretched near ζ and the stretching becomes infinite at ζ, see Figure 3 below.

iii) The passive field $\mathbf{B}$ is also stretched by the fluid and the stretching blows-up at ζ.

With this picture in mind, we may anticipate the behavior when we add noise. As we shall see below, the transport type noise, in Stratonovich form, introduced in equation (1), corresponds at the Lagrangian level to the addition of a random shift to all Lagrangian particles (see equation (4)). Figures 3 and 4 below are obtained simulating the time evolution of 400 points initially on the x-axis, as they were a line; the pictures show the line at time $t = 1$. Again $\alpha = 0.3$. In the deterministic case (Figure 3) this line is infinitely stretched near the origin. In the stochastic case (Figure 4), even with very small noise intensity ($\sigma = 0.1$), the line is shifted by noise a little bit in all possible directions and thus it passes through the origin only for a negligible amount of time. Stretching still occurs but not with infinite strength and the visible result is that the line at time $t = 1$ looks still relatively regular, although strongly curved.

Stretching still exists but it is smeared-out, distributed among different portions of fluid; the deterministic concentration of stretching at ζ is broken.

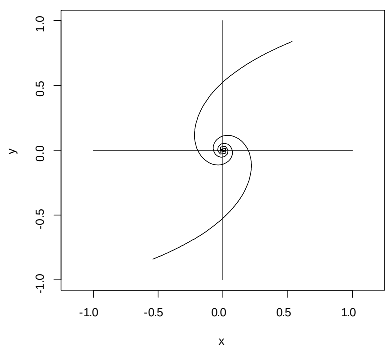

Fig.3. Ideal lines evolution, no noise.

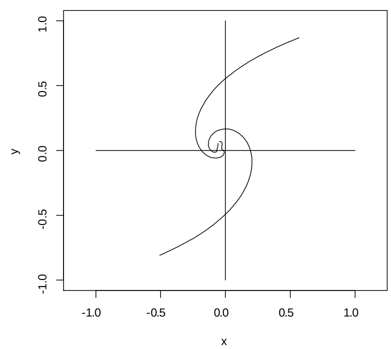

Fig.4. Ideal lines evolution, noise with $\sigma = 0.1$.

3 The stochastic case: absence of blow-up

3.1 The regular case

In this section we study the regular case. Let $\mathbf{W} = \left(W^1, W^2, W^3\right)$ be a 3-dimensional Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$ and let $(\mathcal{F}_t)_t$ be its natural completed filtration. Let $\mathbf{v}$ be a divergence-free vector field in $C^1([0,T]; C_c^\infty(\mathbb{R}^3; \mathbb{R}^3))$ and $\mathbf{B}_0$ be a divergence-free vector field in $C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$. For a divergence-free solution $\mathbf{B}$, equation (1) reads formally

$$d\mathbf{B}+\left(\left(\mathbf{v}\cdot\nabla\right)\mathbf{B}-\left(\mathbf{B}\cdot\nabla\right)\mathbf{v}\right)dt+\sigma\sum_{k}\partial_{k}\mathbf{B}\circ dW^{k}=0. \tag{3}$$

We will always use (1) in this form.

Remark 3 *The Stratonovich operation $\partial_k\mathbf{B}\circ dW^k$ is the natural one from the physical viewpoint, because of Wong-Zakai principle, see the Appendix of [17] for an example, and because of the formal validity of conservation laws. More rigorously, it is responsible for the validity of relation (5) between $\mathbf{B}$ and the Lagrangian motion, relation which extends to the stochastic case a well know deterministic relation.*

For mathematical convenience, we translate Stratonovich in Itô form. Formally, the martingale part of $\partial_k\mathbf{B}$ is (from equation (3) itself) equal to $\sigma\sum_j\partial_k\partial_j\mathbf{B}dW^j$ and thus the quadratic variation $d\left[\partial_k\mathbf{B},W^k\right]$ is equal to $\sigma\partial_k\partial_k\mathbf{B}$; therefore

$$\sigma\sum_{k}\partial_{k}\mathbf{B}\circ dW^{k}=\sigma\sum_{k}\partial_{k}\mathbf{B}dW^{k}+\frac{\sigma^{2}}{2}\Delta\mathbf{B}.$$

This is the heuristic justification of the following rigorous definition.

Definition 4 *A regular solution to (1) is a vector field $\mathbf{B}:[0,T]\times\mathbb{R}^3\times\Omega\to\mathbb{R}^3$ such that*

i) $\mathbf{B}(t,x)$ and its derivatives in x up to fourth order are continuous in (t,x)

ii) for every $i,j=1,...,d$ and $x\in\mathbb{R}^3$, $\mathbf{B}(t,x)$, $\partial_{x_i}\mathbf{B}(t,x)$, $\partial_{x_j}\partial_{x_i}\mathbf{B}(t,x)$ are adapted processes

ii) for every (t,x), $\operatorname{div}\mathbf{B}(t,x)=0$ and

$$\begin{aligned}\mathbf{B}(t,x)=\mathbf{B}_0(x)+&\int_0^t\left[\left(\mathbf{B}(r,x)\cdot\nabla\right)\mathbf{v}(r,x)-\left(\mathbf{v}(r,x)\cdot\nabla\right)\mathbf{B}(r,x)\right]dr\\&-\sigma\sum_{k=1}^{3}\int_0^t\partial_k\mathbf{B}(r,x)dW_r^k+\frac{\sigma^2}{2}\int_0^t\Delta\mathbf{B}(r,x)dr.\end{aligned}$$

Remark 5 *In order to give a meaning to the equation it is not necessary to ask C^4 regularity in x in point (i); the requirement is imposed to apply Itô-Kunita-Wentzell formula (Theorem 3.3.1 in [23]) below.*

Remark 6 *For the purpose of this paper, one can simplify and ask that $\mathbf{B}$ is C^∞ in x, with all derivatives continuous in (t,x); the results below remain true.*

Consider now the SDE on $\mathbb{R}^3$

$$dX_t = \mathbf{v}(t, X_t)dt + \sigma d\mathbf{W}_t, \qquad X_0 = x. \tag{4}$$

It is a classical result (see [23]) that there exists a stochastic flow Φ of C^∞ diffeomorphisms (see Definition 13 in Section 3.2) solving the above SDE. Since $\mathbf{v}$ is divergence-free, Φ_t and Φ_t^{-1} are also measure-preserving for every t, i.e. $det(D\Phi_t) = 1$.

We can now prove the representation formula for the regular solution to equation (1), which will be the key ingredient of our work.

Proposition 7 *Suppose* $\mathbf{B}_0 \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ *and* $\mathbf{v} \in C^1([0,T];C_c^\infty(\mathbb{R}^3;\mathbb{R}^3))$*, both divergence free. Then equation (1) admits a unique regular solution, satisfying the identity*

$$\mathbf{B}(t, \Phi_t(x)) = D\Phi_t(x)\mathbf{B}_0(x). \tag{5}$$

Remark 8 *Notice that* $D\Phi_t(\Phi_t^{-1}(x)) = (D\Phi_t^{-1}(x))^{-1}$*. This inverse matrix is the transpose of the cofactor matrix of* $D\Phi_t^{-1}$*, multiplied by the inverse of the determinant of* $D\Phi_t^{-1}(x)$*, which is* 1 *since* Φ_t *is measure-preserving; the cofactor matrix of a given* 3×3 *matrix* A *is a polynomial function* $H(A)^T$*, of degree* 2*, of* A*. So we have* $D\Phi_t(\Phi_t^{-1}(x)) = H(D\Phi_t^{-1}(x))$ *and formula (5) also reads*

$$\mathbf{B}(t,x) = H(D\Phi_t^{-1}(x))\mathbf{B}_0(\Phi_t^{-1}(x)). \tag{6}$$

Proof. Step 1 (chain rule). Let us recall the so called Itô-Kunita-Wentzell formula (Theorem 8.1 in [22]; see also Theorem 3.3.1 in [23] for a variant). We state it with the notations of interest for us. Assume that $F(t,x)$, $t\in[0,T]$, $x\in\mathbb{R}^d$, is a continuous random field, twice differentiable in x with second derivatives continuous in (t,x), of the form

$$F(t,x) = F_0(x) + \int_0^t f_0(s,x)\,ds + \sum_{k=1}^n \int_0^t f_k(s,x)\,dW_s^k$$

where W^k, $k=1,...,n$ are independent Brownian motions and f_k, $k=0,1,...,n$ are twice differentiable in x, continuous in (t,x) with their second space derivatives, and for each x the processes $t\mapsto f_k(t,x)$ are adapted. Let X_t be a continuous semimartingale in $\mathbb{R}^d$. Then

$$\begin{aligned} F(t,X_t) &= F_0(X_0) + \int_0^t f_0(s,X_s)\,ds + \sum_{k=1}^n \int_0^t f_k(s,X_s)\,dW_s^k \\ &+ \int_0^t \nabla F(t,X_s)\cdot dX_s + \frac{1}{2}\sum_{k,h=1}^n \int_0^t \partial_{x_k}\partial_{x_h} F(t,X_s)\,d\left[X^h,X^k\right]_s \\ &+ \sum_{k=1}^n\sum_{i=1}^d \int_0^t \partial_{x_i} f_k(s,X_s)\,d\left[X^i,W^k\right]_s \end{aligned}$$

where $\left[X^h, X^k\right]_t$ denotes the quadratic mutual variation between the components of X and similarly for $\left[X^h, W^k\right]_t$.

Step 2 (uniqueness). Fix x in $\mathbb{R}^3$; observe that $D\Phi_t(x)\mathbf{B}_0(x)$ is the unique solution to

$$\frac{d\mathbf{Z}_t}{dt} = (\mathbf{Z}_t \cdot \nabla)\, \mathbf{v}(t, \Phi_t(x)). \tag{7}$$

with $\mathbf{Z}_0 = \mathbf{B}_0(x)$ (uniqueness follows from the fact the the stochastic drift for this ODE, namely $(t, y) \to D\mathbf{v}(t, \Phi_t(x))y$ is in C_b^1). Thus, in order to get uniqueness for equation (1) and prove formula (5), it is enough to prove that, for any regular solution $\mathbf{B}$ to (1), $\mathbf{B}(t, \Phi_t(x))$ satisfies equation (7). For this purpose, we use the chain rule of Step 1 (the assumptions in the definition of regular solution above are imposed precisely in order to apply this result). For each component $j = 1, 2, 3$ we apply the formula with

$$F = B^j, f_0 = ((\mathbf{B} \cdot \nabla)\, \mathbf{v} - (\mathbf{v} \cdot \nabla)\, \mathbf{B})^j + \frac{\sigma^2}{2} \Delta B^j, f_k = -\sigma \partial_k B^j, X_t = \Phi_t(x).$$

The result, rewritten in vector form, is

$$\begin{aligned} d[\mathbf{B}(t, \Phi_t(x))] \;&=\; (d\mathbf{B})\,(t, \Phi_t(x)) \\ &\quad + \sum_{i=1}^{3} \partial_{x_i} \mathbf{B}(t, \Phi_t(x)) d\Phi_t^i(x) \\ &\quad + \frac{\sigma^2}{2} \Delta \mathbf{B}(t, \Phi_t(x))dt - \sigma^2 \Delta \mathbf{B}(t, \Phi_t(x))dt \end{aligned}$$

because

$$\sum_{k=1}^{3} \sum_{i=1}^{3} \int_0^t \partial_{x_i} f_k d\left[X^i, W^k\right]_s = -\sigma \sum_{k=1}^{3} \sum_{i=1}^{3} \int_0^t \partial_{x_i} \partial_k B^j \sigma \delta_{ik} ds = -\sigma^2 \int_0^t \Delta B^j ds$$

(since $d\left[X^i, W^k\right]_s = \sigma \delta_{ik} ds$). Therefore

$$\begin{aligned} d[\mathbf{B}(t, \Phi_t(x))] \;&=\; ((\mathbf{B} \cdot \nabla)\, \mathbf{v} - (\mathbf{v} \cdot \nabla)\, \mathbf{B})\, dt + \frac{\sigma^2}{2} \Delta \mathbf{B} dt - \sigma \sum_{k=1}^{3} \partial_k \mathbf{B} dW_t^k \\ &\quad + (\mathbf{v} \cdot \nabla)\, \mathbf{B} dt + \sigma \sum_{k=1}^{3} \partial_k \mathbf{B} dW_t^k \\ &\quad - \frac{\sigma^2}{2} \Delta \mathbf{B}(t, \Phi_t(x))dt \\ &= ((\mathbf{B} \cdot \nabla)\, \mathbf{v})\,(t, \Phi_t(x))dt. \end{aligned}$$

Therefore $\mathbf{B}(t, \Phi_t(x))$ satisfies (7). Uniqueness and formula (5) are proved.

Step 3 (existence). Conversely, given $\mathbf{B}$ defined by (5), let us prove that it is a regular solution to equation (1). Properties (i)-(ii) of the definition of regular solution are obvious from (6) (indeed $\mathbf{B}$ is C^∞ in x). It also follows, from Itô-Kunita-Wentzell formula, that $\mathbf{B}(t,x)$ has the form

$$d\mathbf{B}(t,x) = \mathbf{A}(t,x)\,dt + \sum_{k=1}^{3} \mathbf{S}_k(t,x)\,dW_t^k \tag{8}$$

where $\mathbf{B}(t,x), \mathbf{A}(t,x), \mathbf{S}_k(t,x)$ are continuous in (t,x) with their second space derivatives, and are adapted in t for every x. Notice that we do not need to compute explicitly $\mathbf{A}(t,x)$ and $\mathbf{S}_k(t,x)$ (by Itô-Kunita-Wentzell formula) from the identity (6) (this would involve too complex expressions with derivatives of the flow). We just need to realize that Itô-Kunita-Wentzell formula can be applied and gives a decomposition of the form (8) with $\mathbf{B}(t,x), \mathbf{A}(t,x), \mathbf{S}_k(t,x)$ having the regularity stated above.

Thanks to this regularity, we may apply Itô-Kunita-Wentzell formula to $\mathbf{B}(t,\Phi_t(x))$, where now we only know that identities (8) and (5) are satisfied by $\mathbf{B}$. On one side, from (5) and the fact that $D\Phi_t(x)\mathbf{B}_0(x)$ is the unique solution to (7) we get

$$d[\mathbf{B}(t,\Phi_t(x))] = (\mathbf{B}(t,\Phi_t(x))\cdot\nabla)\,\mathbf{v}(t,\Phi_t(x))dt.$$

On the other side, similarly to the computation of Step 2, from Itô-Kunita-Wentzell formula applied to the function

$$F = B^j, f_0 = A^j, f_k = S_k^j, X_t = \Phi_t(x)$$

we get

$$\begin{aligned} d[\mathbf{B}(t,\Phi_t(x))] &= (d\mathbf{B})(t,\Phi_t(x)) \\ &\quad + \sum_{i=1}^{3} \partial_{x_i}\mathbf{B}(t,\Phi_t(x))d\Phi_t^i(x) \\ &\quad + \frac{\sigma^2}{2}\Delta\mathbf{B}(t,\Phi_t(x))dt + \sigma\sum_{k=1}^{3}\partial_{x_k}\mathbf{S}_k(t,\Phi_t(x))dt \end{aligned}$$

because now

$$\sum_{k=1}^{3}\sum_{i=1}^{3}\int_0^t \partial_{x_i} f_k d\left[X^i, W^k\right]_s = \sum_{k=1}^{3}\sum_{i=1}^{3}\int_0^t \partial_{x_i} S_k^j \sigma\delta_{ik} ds = \sigma\sum_{k=1}^{3}\int_0^t \partial_{x_k} S_k^j ds.$$

Therefore

$$\begin{aligned} d[\mathbf{B}(t,\Phi_t(x))] &= \mathbf{A}(t,\Phi_t(x))dt + \sum_{k=1}^{3}\mathbf{S}_k(t,\Phi_t(x))dW_t^k \\ &\quad + ((\mathbf{v}\cdot\nabla)\mathbf{B})(t,\Phi_t(x))dt + \sigma\sum_{k=1}^{3}\partial_k\mathbf{B}(t,\Phi_t(x))dW_t^k \\ &\quad + \frac{\sigma^2}{2}\Delta\mathbf{B}(t,\Phi_t(x))dt + \sigma\sum_{k=1}^{3}\partial_{x_k}\mathbf{S}_k(t,\Phi_t(x))dt. \end{aligned}$$

Equating the two identities satisfied by $d[\mathbf{B}(t,\Phi_t(x))]$, and using the invertibility of Φ_t, we get

$$\begin{aligned} \mathbf{S}_k &= -\sigma\partial_k\mathbf{B} \\ (\mathbf{B}\cdot\nabla)\mathbf{v} &= \mathbf{A} + (\mathbf{v}\cdot\nabla)\mathbf{B} + \frac{\sigma^2}{2}\Delta\mathbf{B} + \sigma\sum_{k=1}^{3}\partial_{x_k}\mathbf{S}_k. \end{aligned}$$

Thus

$$\mathbf{A} = (\mathbf{B}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{B} + \frac{\sigma^2}{2}\Delta\mathbf{B}$$

which completes the proof that $\mathbf{B}$ satisfies the SPDE.

It remains to prove the divergence-free property. For this, since $\mathbf{B}$ is regular, it is enough to show that, for every fixed t, for a.e. ω, $\mathrm{div}\mathbf{B}(t,\cdot,\omega)$ is 0 in the sense of distributions. For this, take φ in $C_c^\infty(\mathbb{R}^3)$; then, using integration by parts (notice that also $\mathbf{B}(\omega)$ has compact support and remember that Φ_t is measure-preserving)

$$\int_{\mathbb{R}^3}\mathbf{B}_t\cdot\nabla\varphi dx = \int_{\mathbb{R}^3}D\Phi_t\mathbf{B}_0\cdot\nabla\varphi(\Phi_t)dx$$

$$= \int_{\mathbb{R}^3}\mathbf{B}_0\cdot(D\Phi_t)^T\nabla\varphi(\Phi_t)dx = \int_{\mathbb{R}^3}\mathbf{B}_0\cdot\nabla[\varphi(\Phi_t)]dx = 0$$

since $\mathbf{B}_0$ is divergence-free. The proof is complete. ■

3.2 The case when v is only Hölder continuous and bounded

In this section we shall always assume $\sigma \neq 0$ and the following condition.

Condition 9 *The vector field* $\mathbf{v}$ *is in* $C([0,T];C_b^\alpha(\mathbb{R}^3;\mathbb{R}^3))$ *for some* $\alpha \in (0,1)$ *and it is divergence-free.*

Definition 10 *Let $\mathbf{B}_0$ be divergence-free and in $C(\mathbb{R}^3;\mathbb{R}^3)$. A continuous weak solution to equation (1) is a vector field $\mathbf{B} : [0,T] \times \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$, with a.e. path in $C([0,T] \times \mathbb{R}^3;\mathbb{R}^3)$, weakly adapted to $(\mathcal{F}_t)_t$ (namely such that $\langle \mathbf{B}, \varphi \rangle$ is adapted for all φ in $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$) such that:*

i) for every φ in $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, the continuous adapted process $\langle \mathbf{B}, \varphi \rangle$ satisfies

$$\langle \mathbf{B}_t, \varphi \rangle = \langle \mathbf{B}_0, \varphi \rangle + \int_0^t \langle (D\varphi)_r^A \mathbf{v}, \mathbf{B}_r \rangle dr + \sigma \sum_{k=1}^d \int_0^t \langle (D\varphi)\mathbf{e}_k, \mathbf{B}_r \rangle dW_r^k + \frac{\sigma^2}{2} \int_0^t \langle \Delta\varphi, \mathbf{B}_r \rangle dr, \tag{9}$$

where $((D\varphi)(x))^A = D\varphi(x) - (D\varphi(x))^T$ is the antisymmetric part of the matrix $D\varphi(x)$;

ii) $\mathbf{B}_t$ is divergence-free, in the sense that $P\{\mathrm{div}\mathbf{B}_t = 0,\ \forall t \in [0,T]\} = 1$.

Notice that the Itô integrals are well defined since the processes $\langle (D\varphi)\mathbf{e}_k, \mathbf{B}_r \rangle$ are continuous and adapted.

Remark 11 *One can define a similar notion of L^p weak solution and, at least for $p > 1$, existence and uniqueness should remain true with a more elaborated proof. We restrict ourselves to continuous solution to emphasize the no blow-up result.*

The aim of this section is to prove the following main result.

Theorem 12 *Assume that $\sigma \neq 0$. Let $\mathbf{B}_0$ be divergence-free in $C(\mathbb{R}^3;\mathbb{R}^3)$ and suppose Condition 9. Then there exists a unique continuous weak solution $\mathbf{B}$ to equation (1), starting from $\mathbf{B}_0$. In particular no blow-up occurs.*

Let us recall the notion of stochastic flow of $C^{1,\beta}$ diffeomorphisms, limited to the properties of interest to us.

Definition 13 *A stochastic flow Φ of $C^{1,\beta}$ diffeomorphisms (on $\mathbb{R}^3$), $\beta \in (0,1)$, is a map $[0,T] \times \mathbb{R}^3 \times \Omega \to \mathbb{R}^3$ such that*

- *for every x in $\mathbb{R}^3$, $\Phi(x)$ is adapted to $(\mathcal{F}_t)_t$;*
- *for a.e. ω in Ω, $\Phi(\omega)$ is a flow of $C^{1,\beta}$ diffeomorphisms, i.e. $\Phi_0(\omega) = id$, for every t, $\Phi_t(\omega)$ is a diffeomorphism, Φ_t, Φ_t^{-1}, $D\Phi_t$ and $D\Phi_t^{-1}$ are jointly continuous on $[0,T] \times \mathbb{R}^3$, β-Hölder continuous in space uniformly in time.*

In the definition of flows we did not mention the cocycle property, since it is not useful for our purposes.

We need the following result (valid more in general in $\mathbb{R}^d$), see [17], Theorem 5.

Theorem 14 *Assume that $\sigma \neq 0$. Let $\mathbf{v}$ satisfy Condition 9 and consider the SDE (4) on $\mathbb{R}^3$.*

1. *For every x in $\mathbb{R}^3$, there exists a unique strong solution to the SDE (4) starting from x. There exists a stochastic flow of $C^{1,\alpha'}$ diffeomorphisms, for every $\alpha' < \alpha$, solving the SDE and belonging to $L^\infty_{loc}([0,T] \times \mathbb{R}^3; L^m(\Omega))$ for every finite m.*

2. *Let $(\mathbf{v}^\epsilon)_{\epsilon>0}$ be a family of divergence-free vector fields in $C([0,T]; C^\alpha_b(\mathbb{R}^3;\mathbb{R}^3))$ converging to $\mathbf{v}$ in this space, as $\epsilon \to 0$. For every $\epsilon > 0$, let Φ^ϵ be the stochastic flow of diffeomorphisms solving (4) with drift $\mathbf{v}^\epsilon$. Then, for every $R > 0$ and every $m \geq 1$, the following results hold:*

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T]} \sup_{|x|\leq R} E[|\Phi^\epsilon_t(x) - \Phi_t(x)|^m] = 0, \tag{10}$$

$$\lim_{\epsilon\to 0} \sup_{t\in[0,T]} \sup_{|x|\leq R} E[|D\Phi^\epsilon_t(x) - D\Phi_t(x)|^m] = 0 \tag{11}$$

and the same for the inverse flow Φ_t^{-1} and its derivative in space.

3. *for every t, Φ_t is measure-preserving, i.e. $det(D\Phi_t(x)) = 1$ for every x in $\mathbb{R}^3$.*

We split the proof of Theorem 12 in two lemmata, one of existence and the other of uniqueness.

Lemma 15 *Let $\mathbf{B}_0$ be divergence-free and in $C(\mathbb{R}^3;\mathbb{R}^3)$ and suppose Condition 9 hold; let Φ be the flow of diffeomorphisms solving the SDE (4) (as given in Theorem 14). Define the random vector field $\mathbf{B}$ as*

$$\mathbf{B}(t,x) = D\Phi_t(\Phi_t^{-1}(x))\mathbf{B}_0(\Phi_t^{-1}(x)). \tag{12}$$

Then $\mathbf{B}$ is a continuous weak solution to equation (1).

Proof. Step 1 (regularity). By definition (12), the assumption on $\mathbf{B}_0$ and the continuity properties in (t,x) of $\Phi_t^{-1}(x)$ and $D\Phi_t(x)$ it follows that $\mathbf{B} \in C([0,T] \times \mathbb{R}^3;\mathbb{R}^3)$ with probability one; since $\Phi_t^{-1}(x)$ and $D\Phi_t(x)$ are $\mathcal{F}_t$ measurable, for every x the process $\mathbf{B}(t,x)$ is adapted to $(\mathcal{F}_t)_t$, hence also weakly adapted. It remains to prove properties (i) and (ii) of Definition 10.

Step 2 (property (i)). Let $(\mathbf{v}^\epsilon)_{\epsilon>0}$ be a family of $C^1([0,T]; C^\infty_c(\mathbb{R}^3;\mathbb{R}^3))$ divergence-free vector fields, approximating $\mathbf{v}$ in $C([0,T]; C^\alpha_b(\mathbb{R}^3;\mathbb{R}^3))$; let $(\mathbf{B}_0^\epsilon)_\epsilon$ be a family of $C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)$ divergence-free vector fields, approximating $\mathbf{B}_0$ in $C_b(\mathbb{R}^3;\mathbb{R}^3)$. We know from Lemma 7 that, for every $\epsilon > 0$,

$$\mathbf{B}^\epsilon(t,x) = D\Phi^\epsilon_t((\Phi^\epsilon_t)^{-1}(x))\mathbf{B}^\epsilon_0((\Phi^\epsilon_t)^{-1}(x)) \tag{13}$$

solves equation (1), where Φ^ϵ is the regular stochastic flow solving the SDE (4) with drift $\mathbf{v}^\epsilon$. Let us first show that for every (t,x), $(\mathbf{B}^\epsilon(t,x))_\epsilon$ converges to $\mathbf{B}(t,x)$, defined by (12), in $L^m(\Omega;\mathbb{R}^3)$, for every finite m.

Fix (t,x) and $m \geq 1$. Using Remark 8 (which also applies to Φ, since $\det(D\Phi_t)=1$), We have

$$\begin{aligned} & |\mathbf{B}^\epsilon(t,x)-\mathbf{B}(t,x)| \\ \leq \ & |\mathbf{B}_0^\epsilon((\Phi_t^\epsilon)^{-1}(x))||H((D\Phi_t^\epsilon)^{-1}(x))-H(D\Phi_t^{-1}(x))| \\ & +|H((D\Phi_t^\epsilon)^{-1}(x))||\mathbf{B}_0^\epsilon((\Phi_t^\epsilon)^{-1}(x))-\mathbf{B}_0((\Phi_t^\epsilon)^{-1}(x))| \\ & +|H((D\Phi_t^\epsilon)^{-1}(x))||\mathbf{B}_0((\Phi_t^\epsilon)^{-1}(x))-\mathbf{B}_0(\Phi_t^{-1}(x))| \end{aligned}$$

so, by Hölder inequality, we get

$$E[|\mathbf{B}^\epsilon(t,x)-\mathbf{B}(t,x)|^m] \tag{14}$$

$$\begin{aligned} \leq \ & C\|\mathbf{B}_0^\epsilon\|_0 E[|H((D\Phi_t^\epsilon)^{-1}(x))-H(D\Phi_t^{-1}(x))|^m] \\ & +CE[|H(D\Phi_t^{-1}(x))|^m]\|\mathbf{B}_0^\epsilon-\mathbf{B}_0\|_0 \\ & +CE[|H(D\Phi_t^{-1}(x))|^{2m}]^{1/2}E[|\mathbf{B}_0((\Phi_t^\epsilon)^{-1}(x))-\mathbf{B}_0(\Phi_t^{-1}(x))|^{2m}]^{1/2}. \end{aligned}$$

We will prove that every term on the right-hand-side of (14) tends to 0. First notice that $\|\mathbf{B}_0^\epsilon\|_0$ and $E[|H(D\Phi_t^{-1}(x)|^m]$ are bounded uniformly in ϵ, for every m, since H is a polynomial function. The convergence of $\|\mathbf{B}_0^\epsilon-\mathbf{B}_0\|_0$ is ensured by our assumptions, that of $|\mathbf{B}_0((\Phi_t^\epsilon)^{-1}(x))-\mathbf{B}_0(\Phi_t^{-1}(x))|$ by Theorem 14 and dominated convergence theorem ($\mathbf{B}_0$ is bounded). Also the convergence of $|H(D(\Phi_t^\epsilon)^{-1}(x))-H(D\Phi_t^{-1}(x))|$ in $L^m(\Omega)$ is a consequence of Theorem 14 and the fact that H is a polynomial. To see this in detail, we can write this term as

$$\begin{aligned} & H(D(\Phi_t^\epsilon)^{-1}(x))-H(D\Phi_t^{-1}(x)) \\ & =\sum_{i,j=1}^3\left(\int_0^1\frac{\partial H}{\partial x_{ij}}(\xi D(\Phi_t^\epsilon)^{-1}(x)+(1-\xi)D\Phi_t^{-1}(x))d\xi\right)\left(D(\Phi_t^\epsilon)^{-1}(x)-D\Phi_t^{-1}(x)\right)_{ij}. \end{aligned}$$

The function H' is linear (because H is quadratic), so we can use Hölder inequality and get

$$\begin{aligned} & E[|H((D\Phi_t^\epsilon)^{-1}(x))-H(D\Phi_t^{-1}(x))|^m] \\ & \leq CE[|(D\Phi_t^\epsilon)^{-1}(x)|^{2m}+|D\Phi_t^{-1}(x)|^{2m}]^{1/2}E[|(D\Phi_t^\epsilon)^{-1}(x)-D\Phi_t^{-1}(x)|^{2m}]^{1/2}. \end{aligned}$$

The term $E[|(D\Phi_t^\epsilon)^{-1}(x)|^{2m}+|D\Phi_t^{-1}(x)|^{2m}]$ is uniformly bounded in ϵ, thanks to Theorem 14, and the term $E[|(D\Phi_t^\epsilon)^{-1}(x)-D\Phi_t^{-1}(x)|^{2m}]$ tends to 0 by Theorem 14. Putting all together, we get convergence of $\mathbf{B}^\epsilon(t,x)$ to $\mathbf{B}(t,x)$ in $L^m(\Omega)$.

Now, with the help of this convergence, we may prove that $\mathbf{B}$ solves equation (9). We know that (9) is satisfied by $\mathbf{B}^\epsilon$, pointwise and thus in the distributional form (formula (9)), by integration by parts. Let us prove that, for every φ in $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, for every t, every term of (9) for $\mathbf{B}^\epsilon$ converges in $L^m(\Omega)$,for any fixed finite m, to the corresponding term for $\mathbf{B}$. We will use the previous convergence result and the uniform estimate

$$\sup_\epsilon \sup_{t\in[0,T]} \sup_{|x|\le R} E[|\mathbf{B}^\epsilon(t,x)|^m] < +\infty, \quad \sup_{t\in[0,T]} \sup_{|x|\le R} E[|\mathbf{B}(t,x)|^m] < +\infty \tag{15}$$

which again follows from Theorem 14. Take the term $\langle \mathbf{B}_t, \varphi\rangle$. Since

$$E[|\langle \mathbf{B}^\epsilon_t - \mathbf{B}_t, \varphi\rangle|^m] \le C\langle E[|\mathbf{B}^\epsilon_t - \mathbf{B}_t|^m], |\varphi|^m\rangle, \tag{16}$$

(in the last term $\langle .,.\rangle$ denotes the scalar product in $L^2(\mathbb{R}^3)$ between real-valued functions, not vector fields as usual), $\mathbf{B}^\epsilon(t,x)$ tends to $\mathbf{B}(t,x)$ in $L^m(\Omega)$ for every x and $E[|\mathbf{B}^\epsilon_t|^m + |\mathbf{B}_t|^m]$ is bounded uniformly in ϵ and x, the convergence of this term follows from dominated convergence theorem. Similarly one can prove the convergence of the terms $\int_0^t \langle \mathbf{B}_r, \Delta\varphi\rangle dr$ and $\int_0^t \langle (D\varphi)e_k, \mathbf{B}_r\rangle dW^k_r$, $k = 1, 2, 3$, the last ones using Burkholder inequality

$$E\left[\left|\int_0^t \langle (D\varphi)e_k, \mathbf{B}^\epsilon_r - \mathbf{B}_r\rangle dW^k_r\right|^m\right] \le C\int_0^t \langle |(D\varphi)e_k|^m, E[|\mathbf{B}^\epsilon_r - \mathbf{B}_r|^m]\rangle dr. \tag{17}$$

For the last term, $\int_0^t \langle (D\varphi)^A \mathbf{v}_r, \mathbf{B}_r\rangle dr$, we have

$$\begin{aligned} &E\left[\left|\int_0^t \left\langle (D\varphi)^A \mathbf{v}^\epsilon_r, \mathbf{B}^\epsilon_r\right\rangle dr - \int_0^t \left\langle (D\varphi)^A \mathbf{v}_r, \mathbf{B}_r\right\rangle dr\right|^m\right] \\ &\le C\int_0^t \langle |D\varphi|^m |\mathbf{v}^\epsilon_r - \mathbf{v}_r|^m, E[|\mathbf{B}^\epsilon_r|^m]\rangle dr + C\int_0^t \langle |D\varphi|^m |\mathbf{v}_r|^m, E[|\mathbf{B}^\epsilon_r - \mathbf{B}_r|^m]\rangle dr. \end{aligned}$$

Both the two addends in the right-hand-side of this equation tend to 0 by dominated convergence theorem, because $\mathbf{v}^\epsilon \to \mathbf{v}$ and $E[|\mathbf{B}^\epsilon_r - \mathbf{B}_r|^m] \to 0$ for every (t,x) and $|\mathbf{v}^\epsilon| + |\mathbf{v}|$, $E[|\mathbf{B}^\epsilon_t|^m + |\mathbf{B}_t|^m]$ are uniformly bounded. Since all the terms of (9) converge, (9) holds for $\mathbf{B}$. Thus $\mathbf{B}$ solves (1) in the sense of distributions.

Step 3 (property (ii)). Concerning property (ii) of Definition 10, we will prove that, for every t, for every φ in $C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)$,

$$E\left[\int_{\mathbb{R}^3} \mathbf{B}_t \cdot \nabla\varphi dx\right] = 0. \tag{18}$$

Since, for a.e. ω, $\mathbf{B}$ is continuous in (t,x) and since $C^\infty_c(\mathbb{R}^3;\mathbb{R}^3)$ is separable, (18) implies that, outside a negligible set in Ω, B_t is divergence-free for every t. We know that (18) is satisfied for $\mathbf{B}^\epsilon$ and that $\mathbf{B}^\epsilon$ tends to $\mathbf{B}$ in $L^m(\Omega)$, for every m, with L^m-norm bounded uniformly in x (in a ball). Then, applying dominated convergence theorem as before, we get (18). The proof is complete. ■

Finally, let us prove that the solution given by the previous theorem is unique.

Lemma 16 *Let $\mathbf{B}_0$ be divergence-free and in $C(\mathbb{R}^3;\mathbb{R}^3)$ and suppose Condition 9 hold. The there is at most one continuous weak solution to equation (1), given by formula (12).*

Proof. Step 1 (origin of the proof). Since the equation is linear, it is sufficient to consider the case $\mathbf{B}_0 = 0$ and prove that, if $\mathbf{B}$ is a continuous weak solution to equation (1) with $\mathbf{B}_0 = 0$, then $\mathbf{B} = 0$.

In Proposition 7 we proved, by Itô-Kunita-Wentzell formula, that a regular solution $\mathbf{B}$ satisfies the identity

$$\frac{d}{dt}[\mathbf{B}(t,\Phi_t(x))] = (\mathbf{B}(t,\Phi_t(x)) \cdot \nabla)\, \mathbf{v}(t,\Phi_t(x))$$

and thus, by uniqueness for equation (7), we got $\mathbf{B}(t,\Phi_t(x)) = D\Phi_t(x)\mathbf{B}_0(x)$, namely $\mathbf{B}(t,\Phi_t(x)) = 0$ in the present case (hence $\mathbf{B} = 0$). We may also go further and drop the step involving equation (7): it is sufficient to differentiate $(D\Phi_t(x))^{-1}\,\mathbf{B}(t,\Phi_t(x))$:

$$\frac{d}{dt}\left[(D\Phi_t(x))^{-1}\,\mathbf{B}(t,\Phi_t(x))\right] = 0$$

which readily implies $(D\Phi_t(x))^{-1}\,\mathbf{B}(t,\Phi_t(x)) = \mathbf{B}_0(x) = 0$, hence $\mathbf{B}(t,\Phi_t(x)) = 0$ and thus $\mathbf{B} = 0$. We have used the fact that

$$\frac{d}{dt}(D\Phi_t(x))^{-1} = -(D\Phi_t(x))^{-1}\, D\mathbf{v}(t,\Phi_t(x))$$

which comes from the computation (in the regular case)

$$\begin{aligned}
\frac{d}{dt}(D\Phi_t(x))^{-1} &= \lim_{h\to 0}\frac{(D\Phi_{t+h}(x))^{-1} - (D\Phi_t(x))^{-1}}{h} \\
&= \lim_{h\to 0}(D\Phi_{t+h}(x))^{-1}\frac{(D\Phi_t(x) - D\Phi_{t+h}(x))}{h}(D\Phi_t(x))^{-1} \\
&= -(D\Phi_t(x))^{-1}\frac{d}{dt}D\Phi_t(x)\,(D\Phi_t(x))^{-1} \\
&= -(D\Phi_t(x))^{-1}\, D\mathbf{v}(t,\Phi_t(x)).
\end{aligned}$$

These are proofs of uniqueness for regular solutions. If $\mathbf{B}$ is only a continuous weak solution, Itô-Kunita-Wentzell formula cannot be applied. Moreover, $D\mathbf{v}$ is a distribution, hence everywhere it enters the computations it may cause troubles (for instance, the meaning of equation (7) is less clear; although in mild form it is meaningful because $D\Phi_t(x)$, which exists also in the non-regular case, is formally its fundamental solution).

Thus we regularize both $\mathbf{B}$ and the flow $\Phi_t(x)$. Usually, with this procedure, the regularized field $\mathbf{B}^\epsilon$ satisfies an equation similar to (7) but with a remainder, a commutator; this has been a successful procedure for linear transport equations with non-smooth coefficients, see [11]; in the stochastic case one has a commutator composed with the flow and the approach works again well due to variants of the commutator lemma, see [17]. The commutator estimates are the central tool in this approach, both deterministic and stochastic. When special cancellations apply, in particular due to divergence free conditions, it is possible to follow an interesting variant of this approach, not based on commutator estimates, developed by [26]. We follow this approach and exploit special cancellations; in absence of them, the vectorial case proper of this paper could not be treated (see below the argument about second space derivatives of the flow).

Step 2 (approximation). Let ρ be a C^∞ compactly supported even function on $\mathbb{R}^3$ and define the approximations of identity as $\rho_\epsilon(x) := \epsilon^{-3}\rho(\epsilon^{-1}x)$, for $\epsilon > 0$. Call $\mathbf{B}^\epsilon = \mathbf{B} * \rho_\epsilon$, $\mathbf{v}^\epsilon = \mathbf{v} * \rho_\epsilon$ (and similarly for other fields). Then, using ρ^ϵ as test function, we get the following equation for $\mathbf{B}^\epsilon$, satisfied pointwise (actually, for a.e. ω, for every (t,x), up to a suitable modification):

$$\begin{aligned}\mathbf{B}^\epsilon(t,x) &= \int_0^t [((\mathbf{B}\cdot\nabla)\mathbf{v})^\epsilon(r,x) - ((\mathbf{v}\cdot\nabla)\mathbf{B})^\epsilon(r,x)]dr \\ &\quad -\sigma\sum_{k=1}^3\int_0^t \partial_k\mathbf{B}^\epsilon(r,x)dW_r^k + \frac{\sigma^2}{2}\int_0^t \Delta\mathbf{B}^\epsilon(r,x)dr\end{aligned}$$

where we have used the fact that $\mathbf{B}_0^\epsilon = 0$. Let $\Phi_t^\epsilon(x)$ be the regular flow associated to $\mathbf{v}^\epsilon$. Since $\mathbf{B}^\epsilon$ is regular, we can now apply Itô-Kunita-Wentzell formula to $\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))$ and get (as in the proof of Proposition 7):

$$\begin{aligned}d[\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))] &= (d\mathbf{B}^\epsilon)(t,\Phi_t^\epsilon(x)) \\ &\quad + \sum_{i=1}^3 \partial_{x_i}\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))d(\Phi_t^\epsilon)^i(x) \\ &\quad + \frac{\sigma^2}{2}\Delta\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dt - \sigma^2\Delta\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dt\end{aligned}$$

$$\begin{aligned}&= [((\mathbf{B}\cdot\nabla)\mathbf{v})^\epsilon - ((\mathbf{v}\cdot\nabla)\mathbf{B})^\epsilon(t,\Phi_t^\epsilon(x))]dt + \frac{\sigma^2}{2}\Delta\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dt - \sigma\sum_{k=1}^3\partial_k\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dW_t^k \\ &\quad + (\mathbf{v}^\epsilon\cdot\nabla)\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dt + \sigma\sum_{k=1}^3\partial_k\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dW_t^k \\ &\quad - \frac{\sigma^2}{2}\Delta\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dt \\ &= (((\mathbf{B}\cdot\nabla)\mathbf{v})^\epsilon - ((\mathbf{v}\cdot\nabla)\mathbf{B})^\epsilon)(t,\Phi_t^\epsilon(x))dt + (\mathbf{v}^\epsilon\cdot\nabla)\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))dt.\end{aligned}$$

Since $\frac{d}{dt}(D\Phi_t^\epsilon(x))^{-1} = -(D\Phi_t^\epsilon(x))^{-1}D\mathbf{v}^\epsilon(t,\Phi_t^\epsilon(x))$, we get

$$\begin{aligned}&\frac{d}{dt}\left[(D\Phi_t^\epsilon(x))^{-1}\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x))\right] \\ &= (D\Phi_t^\epsilon(x))^{-1}[((\mathbf{B}\cdot\nabla)\mathbf{v})^\epsilon - ((\mathbf{v}\cdot\nabla)\mathbf{B})^\epsilon + (\mathbf{v}^\epsilon\cdot\nabla)\mathbf{B}^\epsilon - (\mathbf{B}^\epsilon\cdot\nabla)\mathbf{v}^\epsilon](t,\Phi_t^\epsilon(x)).\end{aligned}$$

Fix φ in $C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$. We multiply the previous formula by φ, integrate in space and change variable $x = \Phi_t^\epsilon(x')$ recalling that Φ_t^ϵ is measure preserving:

$$\int_{\mathbb{R}^3} \left(D\Phi_t^\epsilon(x)\right)^{-1} \mathbf{B}^\epsilon(t, \Phi_t^\epsilon(x))\varphi\left(x\right) dx$$

$$= \int_0^t \int_{\mathbb{R}^3} \left[\left(\left(\mathbf{B}\cdot\nabla\right)\mathbf{v}\right)^\epsilon - \left(\left(\mathbf{v}\cdot\nabla\right)\mathbf{B}\right)^\epsilon + \left(\mathbf{v}^\epsilon\cdot\nabla\right)\mathbf{B}^\epsilon - \left(\mathbf{B}^\epsilon\cdot\nabla\right)\mathbf{v}^\epsilon\right](s,x)\psi^\epsilon\left(s,x\right) dxds$$

where we have introduced the random field

$$\psi^\epsilon\left(s,x\right) := \left(D\Phi_s^\epsilon((\Phi_s^\epsilon)^{-1}\left(x\right))\right)^{-1}\varphi\left((\Phi_s^\epsilon)^{-1}\left(x\right)\right).$$

By integration by parts we get:

$$\int_{\mathbb{R}^3} \left(D\Phi_t^\epsilon(x)\right)^{-1} \mathbf{B}^\epsilon(t, \Phi_t^\epsilon(x))\varphi\left(x\right) dx = -\sum_{i,j=1}^{3}\int_0^t \int_{\mathbb{R}^3} \left[(v^j B^i)^\epsilon - v^{\epsilon,j}B^{\epsilon,i}\right](s,x)(D\psi^\epsilon)_{ij}^A(s,x)dxds. \tag{19}$$

Step 3 (support and convergence of ψ^ϵ). In the next step we need a technical fact about the support of $x \mapsto \psi^\epsilon\left(t,x,\omega\right)$. Let $R' > 0$ be such that the support of φ is contained in $B\left(0,R'\right)$. Define $R^\epsilon\left(\omega\right)$ as

$$R_t^\epsilon\left(\omega\right) = \max_{x\in B(0,R')}\left|\Phi_t^\epsilon(x,\omega)\right|.$$

Then the support of $x \mapsto \psi^\epsilon\left(t,x,\omega\right)$ is contained in $\overline{B\left(0,R_t^\epsilon\left(\omega\right)\right)}$. We have

$$\Phi_t^\epsilon(x,\omega) = x + \int_0^t \mathbf{v}^\epsilon\left(s,\Phi_s^\epsilon(x,\omega)\right)ds + \sigma\mathbf{W}_t\left(\omega\right)$$

and there is a constant $C > 0$ such that $\left|\mathbf{v}^\epsilon\left(s,\Phi_s^\epsilon((\Phi_t^\epsilon)^{-1}\left(x,\omega\right),\omega)\right)\right| \leq C$; thus

$$\left|\Phi_t^\epsilon(x,\omega)\right| \leq |x| + Ct + \sigma\max_{t\in[0,T]}\left|\mathbf{W}_t\left(\omega\right)\right|.$$

It implies that

$$R_t^\epsilon\left(\omega\right) \leq \overline{R}\left(\omega\right) := R' + CT + \sigma\max_{t\in[0,T]}\left|\mathbf{W}_t\left(\omega\right)\right|$$

for all $\epsilon > 0$, $t \in [0,T]$. The r.v. $\overline{R}\left(\omega\right)$ is finite a.s. and thus we have proved that the function $x \mapsto \psi^\epsilon\left(t,x,\omega\right)$ has a random support which is contained in $B\left(0,\overline{R}\left(\omega\right)\right)$ for all $\epsilon > 0$, $t \in [0,T]$, with probability one. The same result is true replacing $\Phi_t^\epsilon(x,\omega)$ with $\Phi_t(x,\omega)$.

About the convergence of ψ^ϵ, we shall use the following fact: for a.e. ω, possibly passing to a subsequence, $\psi^\epsilon\left(t,\cdot,\omega\right)$ tends to $\psi\left(t,\cdot,\omega\right)$ in $L_{loc}^m(\mathbb{R}^3)$ and $\psi^\epsilon\left(\cdot,\cdot,\omega\right)$ tends to $\psi\left(\cdot,\cdot,\omega\right)$

in $L^m_{loc}([0,T]\times\mathbb{R}^3)$, for every finite m. Indeed, first notice that $\left(D\Phi_s^\epsilon((\Phi_s^\epsilon)^{-1}(x))\right)^{-1} = D(\Phi_s^\epsilon)^{-1}(x)$, so that

$$\psi^\epsilon(s,x) = D(\Phi_s^\epsilon)^{-1}(x)\,\varphi\left((\Phi_s^\epsilon)^{-1}(x)\right).$$

By Theorem 14 and standard arguments like in the proof of Lemma 15, Step 2, $\psi^\epsilon(t,\cdot,\cdot)$ converges in $L^m(B_R\times\Omega)$ for every finite m and every $R>0$; this implies that, for a.e. ω, possibly passing to a subsequence, for a.e. ω it converges in $L^m(B_R)$ for every finite m and every $R>0$; by a diagonal procedure we can choose this subsequence independently of m and R. The proof of the convergence of $\psi^\epsilon(\cdot,\cdot,\omega)$ in $L^m_{loc}([0,T]\times\mathbb{R}^3)$ is similar.

Step 4 (passage to the limit). Now we fix $t>0$ and let ϵ go to 0 in formula (19). We will prove we obtain in the limit

$$\int_{\mathbb{R}^3}(D\Phi_t(x))^{-1}\,\mathbf{B}(t,\Phi_t(x))\varphi(x)\,dx = 0 \tag{20}$$

which implies $\mathbf{B}=0$ as already explained above.

The term on the left-hand-side of (19) converges, possibly up to subsequences, to the one on the left-hand-side of (20). Indeed, by the change variable $x=\Phi_t^\epsilon(x')$ and the support result of the previous step we have (recall that $\overline{R}$ is random but independent of $\epsilon>0$)

$$\begin{aligned}\int_{\mathbb{R}^3}\left(D\Phi_t^\epsilon(x')\right)^{-1}\mathbf{B}^\epsilon(t,\Phi_t^\epsilon(x'))\varphi\left(x'\right)dx' &= \int_{\mathbb{R}^3}\mathbf{B}^\epsilon(t,x)\psi^\epsilon(t,x)\,dx\\ &= \int_{B\left(0,\overline{R}\right)}\mathbf{B}^\epsilon(t,x)\psi^\epsilon(t,x)\,dx\end{aligned}$$

With probability one, for every $R>0$ the function $\mathbf{B}^\epsilon(t,x)$ converges to $\mathbf{B}(t,x)$ uniformly on $[0,T]\times B(0,R)$, by classical mollifiers arguments. We have seen in Step 3 that, for a.e. ω, possibly passing to a subsequence, $\psi^\epsilon(t,\cdot,\omega)$ tends to $\psi(t,\cdot,\omega)$ in $L^1_{loc}(\mathbb{R}^3)$. Hence we may pass to the limit in $\int_{B\left(0,\overline{R}\right)}\mathbf{B}^\epsilon(t,x)\psi^\epsilon(t,x)\,dx$, for a.e. ω; the limit is $\int_{B\left(0,\overline{R}\right)}\mathbf{B}(t,x)\psi(t,x)\,dx$ which gives the left-hand-side of (20) by going backwards with the same computations.

Let us consider now the term on the right-hand-side of (19); we want to prove that it converges to zero. It is not difficult to show that, for a.e. ω, both $(v^jB^i)^\epsilon$ and $v^{\epsilon,j}B^{\epsilon,i}$ converge to v^jB^i in $C([0,T]\times\mathbb{R}^3)$ (namely, uniformly on compact sets) so $(v^jB^i)^\epsilon - v^{\epsilon,j}B^{\epsilon,i}$ tends to 0 in that space. The term $(D\psi^\epsilon)^A_{ij}(s,x)$ could look problematic at a first view, since it seems to involve the second derivatives of the flow Φ^ϵ, which are not under control. But this is not the case, because we only need the antisymmetric part of the derivative. Indeed, differentiating $\psi^\epsilon = (D((\Phi^\epsilon)^{-1}))^T\varphi((\Phi^\epsilon)^{-1})$, we get

$$(D\psi^\epsilon)_{ij} = \sum_{k=1}^3 \partial_j\partial_i((\Phi^\epsilon)^{-1})^k\varphi_k((\Phi^\epsilon)^{-1}) + \sum_{k=1}^3 \partial_i((\Phi^\epsilon)^{-1})^k\partial_i[\varphi_k((\Phi^\epsilon)^{-1})].$$

The possible problem is only with the first addend. Its antisymmetric part however is

$$\sum_{k=1}^{3} \partial_j \partial_i ((\Phi^\epsilon)^{-1})^k \varphi_k((\Phi^\epsilon)^{-1}) - \sum_{k=1}^{3} \partial_i \partial_j ((\Phi^\epsilon)^{-1})^k \varphi_k((\Phi^\epsilon)^{-1}) = 0.$$

So $(D\psi^\epsilon)^A$ involves only powers of first derivatives of Φ^ϵ. Hence, using again arguments like in proof of Lemma 15, up to subsequences, $(D\psi^\epsilon)^A$ converges to $(D\psi)^A$ in $L^1_{loc}([0,T] \times B_R)$, with probability one. Using again the uniform random support of ψ^ϵ we see that the term on the right-hand-side of (19), equal to

$$-\sum_{i,j=1}^{3} \int_0^t \int_{B(0,\overline{R})} \left[(v^j B^i)^\epsilon - v^{\epsilon,j} B^{\epsilon,i} \right] (s,x) (D\psi^\epsilon)^A_{ij}(s,x) dx ds$$

converges to zero, with probability one. Then (20) is proved and the proof is complete. ■

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