Term Structure Models Driven by Hawkes Processes

Guillaume Bernis[*], Matthieu Garcin[†], Simone Scotti[‡] and Carlo Sgarra[§]

March 8, 2022

1 Introduction

The cornerstone paper by Heath, Jarrow and Morton [15] (hereafter HJM model) proposes a general approach which takes the whole yield curve as an input and provides with a dynamic of all forward rates. One of its particular class, proposed by Hull and White [18] offers a Gaussian, Markov version which offers tractable formulae to deal with vanilla interest rates derivatives. The simplicity of this model allows fruitful applications for the valuation and the hedging of some complex derivatives (bermuda swaptions, callable floaters etc.). In particular, efficient numerical methods such as trinomial trees has been developed to deal with these products. See, for instance, Brigo and Mercurio (Brigo07). However, the HJM model does not provide a smile that can be fit to the implied volatilities of a given key rate.

The HJM model exploits a stochastic framework generated by a Brownian motion. Even if rates exhibit low volatility compared to equity, large fluctuations were highlighted in econometric literature. See for instance [5, 6, 8, 17, 19]. A large literature has arised to integrate jumps in HJM approach mainly by exploiting Lévy processes, as, for instance, in [9, 10, 11]. The general law of Lévy processes is useful to fit the data but could be seen also as a drawback. In fact, the market convention is to express the prices of fixed-income derivatives (caps, floors, swaps, etc.) in term of Black/Bachelier volatilities, that is using a log-normal or normal law. It is not surprising that a large part of market models are derived from Black Scholes model,

[*]Natixis Assurances. eMail: guillaume.bernis@natixis.com
[†]Léonard de Vinci Pôle Universitaire, Research Center, 92 916 Paris La Défense, France. eMail: matthieu.garcin@m4x.org
[‡]LPSM, Université Paris Diderot. eMail: simone.scotti@univ-paris-diderot.fr
[§]Politecnico di Milano. eMail: carlo.sgarra@polimi.it

as highlighted in Brigo and Mercurio [4, chapter 10]. When stochastic volatility is added, it is often assumed a zero correlation between rate and its volatility in order to have a conditional log-normal setup. See Renault and Touzi [23] for the theoretical setting and [4, pages 495-496] for the applications in interest rate modelling.

It is easy to remark that the interest in including Lévy processes in fixed income modelling reached his peak in the first decade of 21 century and has faded away during the second decade due to a new phenomenon: the persistence of unusual low interest rates, see [1, 12]. That is, after, or because of, the global financial crisis of 2008 and the European debt crisis of 2010-13 the yield curves have been drawned downward by Central Banks' quantitative easing. For some countries, the curve can be partially (or even totally) negative. Moreover, the volatility of bonds has decreased and the jumps have disappeared. The situation has changed after the corona-virus crisis with a strong recovery and the increase of inflation, particularly in the US.

We summarize the previous empirical results by noticing that jumps cannot be neglected in interest rate modelling, but that their distribution in time is not homogeneous. Jumps are in fact clustered, that is concentrated in relatively small time windows. Hawkes processes [14] can reproduce this behaviour. Our model add to the original HJM set-up a marked Hawkes process. Conditional on the jumps and the intensity process, the bond evolution is log-normal. This feature is particular important for numerical efficiency of swation pricing and is coherent with market convention. In contrast with previous conditional log-normal model, see [4, Chapter 11], the implied volatility is not only smiled but also skewed due to the jumps.

Our model offers a smile that can be fit on the implied volatility of swaptions for a given key rate (tenor). We harness on the log-normality of the model, conditionnaly to jumps, and derive formula to evaluate both caplets/floorlets and swaptions. Our model exhibits negative jumps on the zero-coupon (hence positive on the rates). Therefore, its behaviour is compatible with the situation where globally low interest rates can suddenly show cluster of positive jumps in case of tensions on the market. One of the main difficulties when dealing with Hawkes jumps in the HJM model is to keep a framework that is Markovian. In particular, it is important to preserve the important features of the Hull and White version, especially the reconstruction formula that provides the zero-coupons in terms of the underlying model factors. In our case, this formula is based on two factors: a classical Gaussian one and a pure jump martingale based on Hawkes process.

In a first section, we set-out the stochastic framework and define a general HJM model with Hawkes jumps. Then, we consider a special case of two-factor non-Markovian linear model. In this setting, we derive pricing formula for caplets/floorlets and swaptions. Then, we provide numerical applications. Especially, we fit the model on the implied volatility of swaptions for a given tenor. We also study the influence of the parameters of the model on the form of the smile.

2 General HJM framework with Hawkes processes

Let us set-out the following stochastic framework. Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbb{Q})$ be a filtered probability space, satisfying the usual conditions [20, page 10], equipped with a standard Brownian motion W and a marked Hawkes process, represented by its counting measure $\nu(dt, dz)$, independent from W. The compensator of ν is $\Theta(dz)\lambda_t dt$, where $\Theta(dz)$ is a measure on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$,

Throughout the paper, we assume that $\Theta(dz)$ admits moments of every orders. We will denote by $\widetilde{\nu}(dt, dz)$ the compensated measure of $\nu(dt, dz)$. The intensity of the Hawkes process reads

$$\lambda_t = \lambda_0 + \beta \int_0^t (\lambda_0 - \lambda_s) ds + \int_0^{t^-} \int_{\mathbb{R}^+} z\nu(ds, dz) \tag{1}$$

We introduce the sequence of jumps $\{T_i\}_{i\leq 1}$ and marks $\{Z_i\}_{i\leq 1}$ of the Hawkes process. From now on, we assume the following:

Assumption 1

$$\beta > \int_{\mathbb{R}^+} z\Theta(dz)$$

Under this assumption, the Hawkes process is well defined, mean reverting admits moments of every order and its Laplace Transform is known (see [2], Proposition 7.3, p. 176).

A zero-coupon is a risk-free instrument which pays 1 at a given maturity $T \geq 0$. Its price at any time $0 \leq t \leq T$ is denoted by $B_t(T)$, and, of course satisfies $B_T(T) = 1$.

We now turn our attention on the dynamics of the zero-coupons bonds. Let $T_\star$ be the finite maximal horizon of the set-up.

Definition 1 *[Zero-coupons Dynamics]. For any* $0 \leq t \leq T \leq T_\star$,

$$\frac{dB_t(T)}{B_{t^-}(T)} = r_t dt + \Gamma(t, T) dW_t + \int_{\mathbb{R}^+} J(t, T, z)\widetilde{\nu}(dz, dt) \tag{2}$$

under the initial condition $B_0(t)$ *where* Γ *and* J *are deterministic functions and satisfy the following conditions, for any* $0 \leq t \leq T \leq T_\star$*:*

Well-posedness and integrability for Brownian part *For* $t < T$, $\Gamma(t,T) > 0$ *and* $\int_0^T \Gamma(t,T)^2 dt < +\infty$

Well-posedness and integrability for jump part $J(t,T,z) > -1$

Closure $\Gamma(T,\cdot,T) = 0$ *and* $J(T,T,z) = 0$.

These assumptions are quite classical at this stage.

The stochastic process r denotes the *risk-free short-time rate*. We define also the *money market account* as follows:

$$Q(t) := \exp\left(-\int_0^t r(u)du\right). \tag{3}$$

In this framework, the probability $\mathbb{Q}$ represents the *spot risk-neutral* probability, i.e. the probability with the cash as *numéraire*.

A direct integration of (2) yields

$$B_t(T) = B_0(T)\mathcal{E}\left(\int_0^\cdot r(s)ds + \int_0^\cdot \Gamma(s,T)dW_s + \int_0^\cdot \int_{\mathbb{R}^+} J(s,T,z)\widetilde{\nu}(ds,dz)\right)_t \tag{4}$$

where $\mathcal{E}\,()$ denotes the Doléans-Dade exponential.

Set $T \geq 0$ and define $C_t(T) = Q(t)\frac{B_t(T)}{B(0,T)}$. From Equation (4), we have that the process C can be decomposed into two terms, $C_t(T) = M_t(T)\,N_t(T)$ with

$$M_t(T) = \mathcal{E}\left(\int_0^\cdot \Gamma(s,\lambda_s,T)dW_s\right)_t \tag{5}$$

$$N_t(T) = \mathcal{E}\left(\int_0^\cdot \int_{\mathbb{R}^+} J(s,T,z)\widetilde{\nu}(ds,dz)\right)_t \tag{6}$$

The process $\{M_t(T)\}_{0\leq t\leq T}$ has continuous paths whereas $\{N_t(T)\}_{0\leq t\leq T}$ has finite variation and captures the jumps.

Remark 1 *We have the following formulation of the Doléans-Dade martingale,* $\{N_t(T)\}_{0\leq t\leq T}$*:*

$$N_t(T) = \prod_{T_i\leq t}\left(1 + J(T_i,T,Z_i)\right)\exp\left(-\int_0^t \int_{\mathbb{R}^+} J(s,T,z)\Theta(dz)\lambda_s ds\right)$$

see [22, Theorem 36, p.77].

From now on, set $\overline{J}(t,T) = \int_{\mathbb{R}^+} J(t,T,z)\Theta(dz)$.

Proposition 1 *Assume that, for any $z \geq 0$, $0 \leq S \leq t \leq T$,*

$$\ln\left(1 + J(S,T,z)\right) - z\int_S^t e^{-\beta(s-S)}\overline{J}(s,t)ds \leq 0$$

Then, $\{C_t(T)\}_{0\leq t\leq T}$ is a martingale with moments of every order.

Proof. The process C is the product of two local martingales, as Doléans-Dade exponentials of martingales. According to the square integrability of Γ, see Definition 1, M is a true martingale with log-normal law and then it admits moment of every order.

We will, now, focus on N. First, recall that the integrated form of λ is given by

$$\lambda_t = \lambda_0 + \sum_{T_i<t} Z_i e^{-\beta(t-T_i)}.$$

Using this expression together with the form of $N_t(T)$ given in Remark 1, we have the following majoration

$$N_t(T) \leq K \prod_{T_i\leq t} \exp\left[\ln\left(1 + J(T_i,T,Z_i)\right) - Z_i\int_{T_i}^t \overline{J}(s,T)e^{-\beta(s-T_i)}ds\right]$$

where $K := \sup_{0\leq t\leq T} \exp\left(-\lambda_0\int_0^t \overline{J}(s,T)ds\right)$. Under the assumption that prevails, the local martingale $\{N_t(T)\}_{0\leq t\leq T}$ is a true martingale because $\sup_{0\leq t\leq T} N_t(T)$ is integrable, and it also has moments of every orders. Then, the process $\{C_t(T)\}_{0\leq t\leq T}$ is a martingale with moments of every order as the product of two independent martingales with moments of every order. $\square$

One of the key elements when dealing with interest rates derivatives is the expression of forward zero-coupons.

Proposition 2 *For any $0 \leq s \leq t \leq T$, we obtain*

$$\frac{B_s(T)}{B_s(t)} = \frac{B_0(T)}{B_0(t)} M_s(t,T) N_s(t,T)$$

with

$$\begin{aligned} M_s(t,T) &= \mathcal{E}\left(\int_0^{\cdot} \left[\Gamma(u,T) - \Gamma(u,t)\right]\left[dW_u - \Gamma(u,t)du\right]\right)_s \\ N_s(t,T) &= \mathcal{E}\left(\int_0^{\cdot}\int_{\mathbb{R}+} \frac{J(u,T,z) - J(u,t,z)}{1 + J(u,t,z)}\widehat{\nu}^t(du,dz)\right)_s \\ \widehat{\nu}_t(du,dz) &= \nu(du,dz) - (1 + J(u,t,z))\Theta(dz)\lambda_u du \end{aligned} \tag{7}$$

Let $C_t(T)$ as defined in Proposition 1. Then, $\frac{d\mathbb{Q}^t}{d\mathbb{Q}}(s) = C_s(t)$ defines a probability, known as the t-forward probability*. Under $\mathbb{Q}^t$, $\{\frac{B_s(T)}{B_s(t)}\}_{0\leq s\leq T}$ is a martingale, moreover, $W_u - \Gamma(u,t)du$ is a standard Brownian motion and ν admits for compensator $(1 + J(u,t,z))\Theta(dz)\lambda_u du$.*

Proof. The expressions of the forward zero-coupon is direct from (4) and the reorganisation of both Brownian and jump parts. The existence of probability $\mathbb{Q}^t$ is a consequence of 1. We have, for any $0 \leq u \leq s \leq t$,

$$C_u(t)\mathbb{E}^{\mathbb{Q}^t}\left\{\frac{B_s(T)}{B_s(t)} \mid \mathcal{F}_u\right\} = \mathbb{E}^{\mathbb{Q}}\left\{C_s(t)\frac{B_s(T)}{B_s(t)} \mid \mathcal{F}_u\right\}$$

Injecting the expression of $C_s(t)$ is the right-hand member, we obtain, after simplification,

$$B_u(t)\mathbb{E}^{\mathbb{Q}^t}\left\{\frac{B_s(T)}{B_s(t)} \mid \mathcal{F}_u\right\} = \mathbb{E}^{\mathbb{Q}}\left\{exp\left(-\int_u^s r(v)dv\right)B_s(T) \mid \mathcal{F}_u\right\}$$

By Equation (4), and proposition 1, we can simplify the right-hand member into

$$\mathbb{E}^{\mathbb{Q}^t}\left\{\frac{B_s(T)}{B_s(t)} \mid \mathcal{F}_u\right\} = \frac{B_u(T)}{B_u(t)}$$

It shows that $\frac{B_s(T)}{B_s(t)}$ is a t-forward martingale.

The last assertions is twofold. For the Brownian part, the result is well known: see [20, Theorem 5.1, p 191]. The form of M given by (5) gives the form of the density of Girsanov Theorem. For the pure jump part, we can refer to [21, Theorem 10.2.6, p. 339]. In this case, Equation (6) is the expression of the density of the change of probability from compensator $\Theta(dz)\lambda_t dt$ to $(1 + J(u,t,z))\Theta(dz)\lambda_u du$. □

3 Two-Factor Non-Markovian Linear Model

In order to carry on formal computation of zero-coupons and short-term rate, we will need to specify the structure of volatility and jump effect. This is the purpose of the following assumption.

Assumption 2 *For any* $0 \leq s \leq t$, $\Gamma(s,t) = \sigma(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}$ *and, for a.s any* $z \in \mathbb{R}^+$, $J(s,t,z) = exp\left(zj(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}\right) - 1$, *where* $\sigma > 0$ *and* $j \leq 0$

Assumption 2 implies that the amortizing factor of both volatility and jump factor is the same. It will play a crucial part in order to obtain a tractable form for the short-rate. Besides, $j \leq 0$ implies that $J(s,t,Z) \leq 0$, hence, the jumps of the zero-coupons are negative. It implies that the jumps of the short-rates are positive. It is clear that both Γ and J satisfy the requirements of Definition 1.

The volatility chosen takes the form of the Hull and White model and a similar feature is taken for the jumps. The Hull and White form of the volatility is very close to the general form required to have a Markovian model, as shown in [24]. So it is not a strong requirement on the Brownian part if we want to keep tractable pricing formulae. Even with this assumption, the model with jumps is not Markovian anymore, since the expression of the zero-coupon will depend on the whole integrated path of the intensity λ multiplied by a deterministic mapping.

When the jumps are set to 0, i.e. $J \equiv 0$, we find back a version of the Hull and White model which is Linear, Gaussian and Markovian.

Proposition 3 *Assume that the jumps are exponentially distributed, i.e.* $\Theta(dz) = \alpha e^{-\alpha z}\mathbb{I}_{z\geq 0}dz$. *Assume that* $\beta \leq \gamma$ *and* $j < -1$, *the process* $\{N_t(T)\}_{0\leq t\leq T}$ *is a martingale with moments of every order.*

Proof. The idea of the proof is to apply Proposition 1. In this case, direct calculation shows that

$$N_t \leq K \prod_{T_i \leq t} e^{Z_i f(t,T_i,T)}$$

where $K > 0$ defined in proof of Proposition 1 and

$$\begin{aligned} f(t,T_i,T) :=& j(T_i)A_\gamma(T-T_i) - \int_{T_i}^t J(s,T)e^{-\beta(s-T_i)}ds \\ =& j(T_i)A_\gamma(T-T_i) + A_\beta(t-T_i) - \int_{T_i}^t \frac{\alpha}{\alpha - j(s)A_\gamma(T-s)} e^{-\beta(z-T_i)}ds \end{aligned}$$

with $A_h(u) = \frac{1-e^{-hu}}{h}$ for any $u \geq 0$ and $h > 0$. It is clear that $\sup_{0\leq t\leq T} f(t,T_i,T) = f(T,T_i,T)$. With $\beta \geq \gamma$, we have $A_\beta(s) \geq A_\gamma(s)$ for any $s \geq 0$. By the following inequality, we can conclude that $f(T, T_i, T) \leq 0$:

$$\forall\, 0 \leq u \leq T,\ f(T, u, T) \leq j(u) A_\gamma(T-u) + A_\beta(u)$$

Hence, the end of proof. $\square$

3.1 Short rate dynamics

First, let us define the initial (deterministic) forward rate curve $f(0, \dot{)}$ by $B_0(t) := \exp(-\int_0^t f(0,u)du)$,

Let us set $\hat{r}(\cdot) := r(\cdot) - f(0, \cdot)$. Take $t = T$ in (4). We obtain:

$$\begin{aligned}
-\int_0^t \hat{r}(u)du &= \int_0^t \sigma(s) \frac{1-e^{-\gamma(t-s)}}{\gamma} dW_s - \frac{1}{2}\int_0^t \sigma^2(u) \left(\frac{1-e^{-\gamma(t-u)}}{\gamma}\right)^2 du \\
&+ \int_0^t \int_{\mathbb{R}^+} zj(s) \frac{1-e^{-\gamma(t-s)}}{\gamma} \nu(ds, dz) - \int_0^t \int_{\mathbb{R}^+} J(u,t,z)\Theta(dz)\nu(du,dz)
\end{aligned}$$

Next step consists in inverting the stochastic integration and the integration with respect to time, in order to obtain on both left and right members an integral with respect to time. It yields

$$\begin{aligned}
-\int_0^t \hat{r}(u)du &= \int_0^t \int_0^u \sigma(s) e^{-\gamma(u-s)} dW_s du - \frac{1}{2}\int_0^t \sigma^2(u) \left(\frac{1-e^{-\gamma(t-u)}}{\gamma}\right)^2 du \\
&+ \int_0^t \int_0^u j(s) e^{-\gamma(u-s)} \int_{\mathbb{R}^+} z\nu(ds, dz) - \int_0^t \int_{\mathbb{R}^+} J(u,t,z)\Theta(dz)\nu(du,dz)du
\end{aligned}$$

In order to extract $\hat{r}$, we need to differentiate with respect to t. For notational convenience, set

$$\Phi(t) := \int_0^t \sigma^2(s) e^{-\gamma(t-s)} ds$$

$$h(s,t) := \int_{\mathbb{R}^+} \frac{\partial J(s,t,z)}{\partial x_2} \Theta(dz) = j(s) e^{-\gamma(t-s)} \int_{\mathbb{R}^+} z e^{zj(s)\frac{1-e^{-\gamma(t-s)}}{\gamma}} \Theta(dz)$$

We obtain the following result:

Proposition 4 *Under Assumption 2, the dynamic of the short rate is given by*

$$
\begin{aligned}
r(t) = f(0,t) &- \int_0^t \sigma(s) e^{-\gamma(t-s)} dW_s + \int_0^t e^{-\gamma(t-s)} \Phi(s) ds \\
&- \int_0^t j(s) e^{-\gamma(t-s)} \int_{\mathbb{R}+} z \nu(ds, dz) + \int_0^t h(s,t) \lambda_s ds
\end{aligned}
$$

Now, let us provide the *reconstruction formula* which enables to express the forward zero-coupons at time t, in terms of $r(t)$ and of $\lambda(\cdot)$.

Proposition 5 *[Reconstruction Formula] For any* $0 \leq t \leq T$,

$$
B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left(A_\gamma(T-t) r(t) + \int_0^t b(u,t,T) \lambda_u du + C(t,T) \right)
$$

with

$$
\begin{aligned}
b(s,t,T) &= A_\gamma(T-t) h(s,t) - \int_t^T h(s,v) dv \\
C(t,T) &= A_\gamma(T-t) \int_0^t \frac{\sigma^2(s)}{\gamma^2} \left[e^{-\gamma(t-u)} - e^{-2\gamma(t-u)} \right] du \\
&\quad + A_\gamma(T-t) \int_0^t e^{-\gamma(t-s)} \Phi(s) ds \\
&\quad + A_\gamma(T-t) f(0,t)
\end{aligned}
$$

3.2 Caplets/floorlets pricing

A caplet (respectively, a floorlet) is a call option on a so-called “Ibor” rate (Euribor, Libor USD, Libor GBP...), paid at the “end date” of the interest period which defines the rates. Indeed, this Ibor rate is caracterized by 3 dates $T^f \leq T^s < T^e$:

- The date T^f is the fixing date, where the rate is known
- The date T^s is the start date of the interest period of the rate
- The date T^e is the end date of the interest period of the rate.

The interest period $[T^s, T^e]$ defines the so-called “frequency” of the rate (3, 6, 12 months for the most common frequencies). It also defined, with a specific day-count fraction, the coverage δ, which is the year fraction between T^s and T^e. The link between the Ibor rate and the zero-coupons is given by:

$$L(T^f, T^s, T^e) = \frac{1}{\delta}\left(\frac{B(T^f, T^s)}{B(T^f, T^e)} - 1\right)$$

The caplet price of strike K at time $t = 0$ writes

$$C(K) = \delta \times \mathbb{E}\left\{Q(T^e)\left(L(T^f, T^z, T^e) - K\right)_+\right\}$$

By injecting the expression of the Ibor rate in terms of zero-coupons, and then, switching to the T^s-forward probability, we obtain

$$C(K) = B(0, T^s) \times \mathbb{E}^{T^s}\left\{\left(1 - (1 + \delta K)\frac{B(T^f, T^e)}{B(T^f, T^s)}\right)_+\right\}$$

i.e the caplet is a put on the forward zero-coupon.

According to Proposition 2, $\left\{\frac{B(t,T^e)}{B(t,T^s)}\right\}_{0\leq t\leq T}$ is a $\mathbb{Q}^{T^s}$-martingale. This leads us to the following result:

Proposition 6 *The price at time $t = 0$ of the caplet of strike K written on the Ibor rate $L(T^f, T^s, T^e)$ is given by*

$$C(K) = B(0, T^s)\mathbb{E}^{T^s}\left\{BS_p\left((1 + \delta K)\frac{B(0, T^e)}{B(0, T^s)}N_{T^f}(T^s, T^e), 1, T^f, \sigma^{LGM}(T^f, T^s, T^e)\right)\right\}$$

where $BS_p(f, k, t, v)$ denotes the price in the Black model of a put option with forward f, strike k, time-to expiry t and annualized volatility v. Besides, $\sigma^{LGM}(T^f, T^s, T^e)$ is the volatility of the caplet under the Linear, Gaussian Markov version (i.e. the model with no jumps). It is simply given by

$$\left(\sigma^{LGM}(T^f, T^s, T^e)\right)^2 T^f = A_\gamma(T^e - T^s)^2 \int_0^{T^f} \sigma^2(u)e^{-2\gamma(T^s - u)}du$$

Proof: The Brownian motion and the Hawkes process being independent, let us consider the price of the instrument conditional to the jumps. The formula is similar to the one in the Hull and White model (cf. [4]). The results follows. □

3.3 Swaption pricing

A swaption is an option to enter into an interest rate swap at a given date, t^e, called the expiry (at least the physically settled version that we are dealing with). When the swap pays (respectively, receives) the fixed rate, the swaption is called a payer (respectively, receiver) swaption and happens to be a call (respectively, a put) on the swap rate. The swap is characterized mainly by the dates of the fixed leg. Let t_0 be the start date of the swap (typically, 2 business days after the expiry t^e for Euro swaps). The payment dates of the fixed leg are given by the schedule $\mathcal{T} := \{t_1, \ldots, t_M\}$, with $t_1 < t_2 < \cdots < t_M$. The distance $t_M - t_0$, expressed as a number of years, is called the tenor of the swap. The associated payment coverages (representing the year fractions of each interest periods are given by the $(\delta_i)_{1\leq i\leq M}$. We define the associated level (or annuity) by

$$\mathrm{LVL}(t, [T_0, \mathcal{T}]) := \sum_{i=1}^{M} \delta_i B(t, t_i)$$

The floating leg represents the sequence of consecutive Ibor rates. When neglecting the difference between the start date of a rate and the end date of the previous one (typically, when we do not use business days), the value of the floating leg at time t^e writes (with the convention that $s_0 = t_0$ and $s_N = t_M$)

$$\sum_{j=1}^{N} \delta \times B(T^e, s_j) \times \frac{1}{\delta}\left(\frac{B(t^e, s_{j-1})}{B(t^e, s_j)} - 1\right) = B(t^e, t_0) - B(t^e, t_M)$$

The price of a payer swaption at time $t = 0$, with strike κ, writes

$$\begin{aligned}
\mathrm{Swpn}_p(0, [t_0, \mathcal{T}], \kappa) &= \mathbb{E}\left\{Q(t^e)\left[B(t^e, t_0) - B(t^e, t_M) - \kappa \times \mathrm{LVL}(t^e, [t_0, \mathcal{T}])\right]_+\right\} \\
&= B(0, t_0)\mathbb{E}^{t_0}\left\{\left[1 - \sum_{i=1}^{M} c_i \frac{B(t^e, t_i)}{B(t^e, t_0)}\right]_+\right\}
\end{aligned}$$

where

$$c_i := \begin{cases} 1 + \kappa \times \delta_M, & \text{if } i = M \\ \kappa \times \delta_i, & \text{elsewhere} \end{cases}$$

As in the context of Proposition 6, let us work conditionally to the jumps and marks. In this case, according to Proposition 2, under the T_0-forward probability any of the $\frac{B(t^e,t_i)}{B(t^e,t_0)}$, $1 \leq i \leq M$, is a log-normal random variable with volatility ν_i, defined by

$$\nu_i^2 := (A_\gamma(t_i - t_0))^2 \times \int_0^{t^e} \sigma^2(u) e^{-2\gamma(t_0-u)} du$$

Morevover, with the separability of Γ, their underlying normal laws are correlated to 1. Thus, we can write

$$\begin{aligned}
&\mathrm{Swpn}_p(0, [t_0, \mathcal{T}], \kappa) \\
=&B(0,t_0)\mathbb{E}^{t_0}\left\{\int_{-\infty}^{x^*}\left(1 - \sum_{i=1}^{M} c_i \frac{B(0,t_i)}{B(0,t_0)} N_{t^e}(t_0,t_i) e^{\nu_i x - \frac{\nu_i^2}{2}}\right)\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right\}
\end{aligned} \tag{8}$$

where x^* (depending on the jumps) is the solution of

$$\sum_{i=1}^{M} c_i \frac{B(0,t_i)}{B(0,t_0)} N_{t^e}(t_0,t_i) e^{\nu_i x^* - \frac{\nu_i^2}{2}} = 1 \tag{9}$$

This solution always exists as the ν_i are positive as well as the zero-coupons and the $N_{t^e}(t_0, t_i)$. With Equations (8) and (9) at hand, a simple integration with respect to the Gaussian density yields

$$\begin{aligned}
&\mathrm{Swpn}_p(0, [t_0, \mathcal{T}], \kappa) \\
=&\mathbb{E}^{t_0}\left\{B(0,t_0)\mathcal{N}(x^*) - \sum_{i=1}^{M} c_i B(0,t_i) N_{t^e}(t_0,t_i)\mathcal{N}(x^* - \nu_i)\right\} \\
=&\mathbb{E}\left\{N_{t^e}(t_0)\left[B(0,t_0)\mathcal{N}(x^*) - \sum_{i=1}^{M} c_i B(0,t_i) N_{t^e}(t_0,t_i)\mathcal{N}(x^* - \nu_i)\right]\right\}
\end{aligned} \tag{10}$$

In Equation (10), the only random parts of the terms in the expectation operator are $N_{t^e}(t_0)$ and x^*. They both depend of the jumps occuring on $]0, t^e]$.

4 Numerical application

In this section we conduct a numerical application to show the relevance of the introduced model in order to price swaptions. More precisely, we show that the model, with a well-chosen set of parameters, is able to reproduce stylized facts of this asset class. We work with a set of swaption contracts of start date t_0 in 5 years, tenor $t_M - t_0$ equal to 10 years, and annual payments. The strike of these swaptions is in the interval $[\kappa^{ATM} - 4\%, \kappa^{ATM} + 4\%]$, where κ^{ATM} is the strike at the money, defined by the forward swap rate

$$\kappa^{ATM} = \frac{B(0,t_0) - B(0,t_M)}{\mathrm{LVL}(0,[t_0,\mathcal{T}])}.$$

In our dataset, which corresponds to the observation of the Euro zero-coupon rate curve as of 14^{th} June 2021, we have κ^{ATM} equal to 0.678%. We also observe the prices of these swaptions at the same date, from which we determine the annual implied volatility under the Bachelier model, which provides us with the call price of forward f, strike k, maturity t, and volatility v,

$$C_B(f,k,t,v) = (f-k) \times \mathcal{N}\left(\frac{f-k}{v\sqrt{t}}\right) + v\sqrt{t} \times g\left(\frac{f-k}{v\sqrt{t}}\right),$$

where $\mathcal{N}$ and g are respectively the Gaussian cdf and the Gaussian pdf. In the case of a swaption of strike κ and market price $\mathrm{Swpn}^{\mathrm{market}}$, the forward rate is κ^{ATM} and the implied volatility $\sigma^{\mathrm{Bachelier}}$ is solution of the equation

$$C_B(\kappa^{ATM}, \kappa, T_0, \sigma^{\mathrm{Bachelier}}) = \mathrm{Swpn}^{\mathrm{market}}.$$

Solving this equation numerically with the Newton-Raphson algorithm, we represent the smile of implied volatilities in Figure 1.

Remark 2 *The use of Bachelier model, i.e. normal volatility, to quote the swaptions has been generalized to the Euro rates, which have been negative for many years.*

Following the Hawkes-HJM approach, we can generate swaption prices with equation (10), which defines the price as an expectation of a given transformation of the Hawkes process. We evaluate the expectation thanks to a Monte Carlo method. We thus have to simulate Hawkes processes in the time interval $[0, T_M]$. To this end, we use the exact simulation method of Dassios and Zhao [7]. This method provides us with the quadruplet $(n_{t_M}, \{T_i\}_{i=1}^{n_{t_M}}, \{\lambda_{T_i}\}_{i=1}^{n_{t_M}}, \{Z_i\}_{i=1}^{n_{t_M}})$, where n_{t_M} is the number of jumps simulated in the interval $[0, t_M]$, T_i is the i^{th} jump time, λ_{T_i} is the intensity at this time, and Z_i is the mark of the i-th jump.

In the general framework of the Hawkes-HJM model, we focus on a particular specification. Indeed, we assume that the distribution Θ is of exponential type:

$\Theta(dz) = \alpha \exp(-\alpha z)dz$. We also assume that the functions σ and j, introduced in Assumption 2, are constant. Finally, our model has globally six parameters: σ, j, γ, β, λ_0, and α.

For a given set of these six parameters, we calculate the price of eleven swaptions of various strikes with the Hawkes-HJM model. Then, we translate these model prices in implied volatilities under the Bachelier model. The results is displayed in Figure 1, in which each curve represents a particular set of parameters, namely $\gamma = 0.1$, $\beta = 0.05$, $\lambda_0 = 2$, and $\alpha = 100$, with the parameters σ and j fixed so as to get an implied volatility equal to the true implied volatility for the at-the-money swaption.

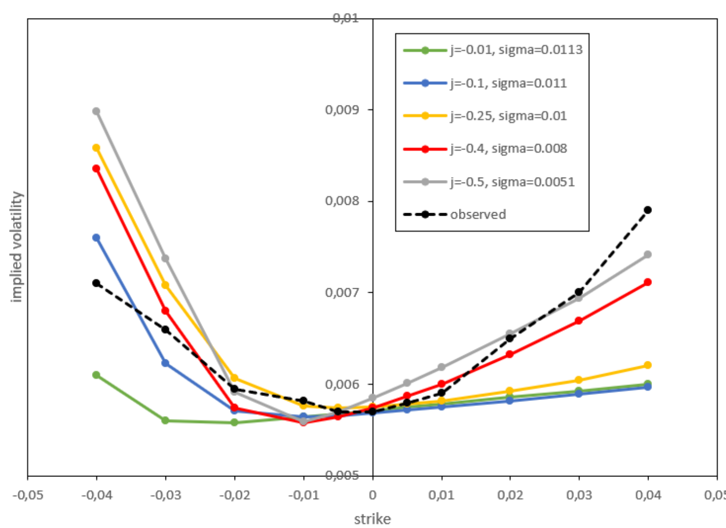

Figure 1: Implied volatility of the swaption contracts under the Bachelier model for the market prices (dotted line) and for prices obtained by simulations with the Hawkes-HJM model (continuous lines), as a function of $\kappa - \kappa^{ATM}$.

We observe that the Hawkes-HJM model reproduces properly the smile of the true implied volatilities of swaptions. This smile is even more pronounced for larger jumps, that is for a larger $|j|$. On the contrary, values of j close to 0 lead to an almost flat curve, indicating the relevance of the jumps in the rate dynamic.

For each set of parameters, all the prices are computed with the same series of pseudo-random numbers, in order to avoid that the differences of prices for swaptions of different strikes comes from an inaccuracy of the Monte Carlo method. In this numerical application, we have considered 1,000 simulations in the Monte Carlo method, which leads to a satisfying accuracy, as reported in Figure 2, in which we display the standard deviation of the output of our method as a function of the number of Monte Carlo simulations.

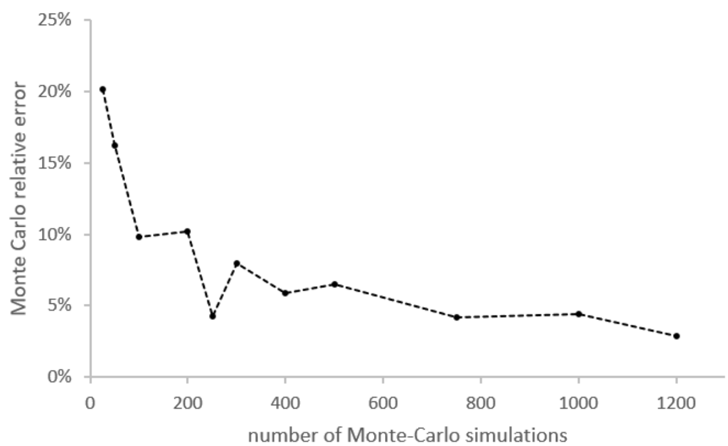

Figure 2: Monte Carlo error, defined as the standard deviation of the implied volatility (obtained from the model prices evaluated with a given number of simulations in the Monte Carlo method), relatively to the average implied volatility (obtained with the same method). Parameters of the model are $\sigma = 0.0057$, $j = -1$, and the same values as in Figure 1 for the other parameters. The swaption considered is at the money.

Finally, we want to get some insight on the role of each parameter of the Hawkes-HJM model on the shape of the volatility smile. We thus study the sensitivity of the smile to a change in only one parameter. Results are gathered in Figure 3. A higher γ, that is a smaller impact of jumps and volatility on rates, tends to move globally the smile downwards but may also reinforce the skew, with still a strong volatility for in-the-money swaptions. A higher value of λ_0, that is the more frequent occurrence of jumps, tends to translate the smile upwards without any obvious impact on the skew. A higher value of α, that is the occurrence of smaller jumps, moves the smile downwards and tends to flatten the right-hand side of the curve. Regarding β, the strength of the mean reversion of the intensity process, the sensitivity study has not led to any obvious interpretation, since the smile does not change significantly when increasing or decreasing, even to a large extent, the value of this parameter.

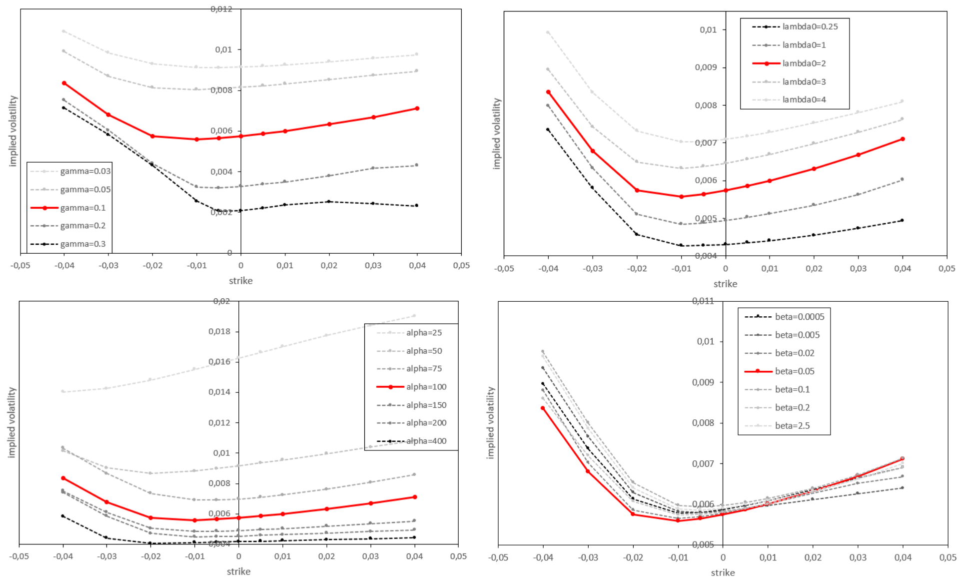

Figure 3: Volatility smiles when changing only one parameter: γ, λ_0, α, and β. The reference set of parameters (red curve) corresponds to $(\sigma, j, \gamma, \beta, \lambda_0, \alpha) = (0.008,\ -0.4,\ 0.1,\ 0.05,\ 2,\ 100)$.

5 Conclusion

References

[1] Bean, C. R., Broda, C. M., Ito, T., and Kroszner, R. (2015). Low for long?: Causes and consequences of persistently low interest rates. ICMB, International Center for Monetary and Banking Studies.

[2] Bernis, G. and Scotti, S.: Clustering Effects via Hawkes Processes, in *From Probability to Finance*, Jiao Editor (2020). Mathematics and Financial Economics (2021)

[3] Bernis, G. Brignone, R., Scotti, S. and Sgarra, C.: A Gamma Ornstein–Uhlenbeck model driven by a Hawkes process. Mathematics and Financial Economics (2021)

[4] Brigo, D., and Mercurio, F. (2007). Interest rate models-theory and practice: with smile, inflation and credit. Springer Science and Business Media.

[5] Chacko, G., and Das, S. (2002). Pricing interest rate derivatives: a general approach. The review of financial studies, 15(1), 195-241.

[6] Das, S. R. (2002). The surprise element: jumps in interest rates. Journal of Econometrics, 106(1), 27-65.

[7] Dassios, A. and Zhao, H. (2013). Exact simulation of Hawkes process with exponentially decaying intensity. Electronic Communications in Probability, 18, 1-13.

[8] Dungey, M., McKenzie, M., and Smith, L. V. (2009). Empirical evidence on jumps in the term structure of the US Treasury market. Journal of Empirical Finance, 16(3), 430-445.

[9] Eberlein, E., Raible, S.: Term Structure Models Driven by General Lévy Processes . Finance and Stoch., **9**, 31-53 (1999)

[10] Eberlein, E., Jacod, J. and Raible, S.: The Lévy LIBOR Model. Finance and Stoch., **9**, 67-88 (2005)

[11] Eberlein, E., Özkan, F.: Term Structure Models Driven by General Lévy Processes . Finance and Stoch., **9**, 327-348 (2005)

[12] García-Schmidt, M., and Woodford, M. (2019). Are low interest rates deflationary? A paradox of perfect-foresight analysis. American Economic Review, 109(1), 86-120.

[13] Hainault, D.: A model for interest rates with clustering effects. Quantitative Finance, **16**, 1203-1218 (2016)

[14] Hawkes, A. G. (1971). Spectra of some self-exciting and mutually exciting point processes. Biometrika, 58(1), 83-90.

[15] Heath, D., Jarrow, R., and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. Econometrica: Journal of the Econometric Society, 77-105

[16] Ho, T. S., and Lee, S. B. (1986). Term structure movements and pricing interest rate contingent claims. the Journal of Finance, 41(5), 1011-1029.

[17] Hong, Y., and Li, H. (2005). Nonparametric specification testing for continuous-time models with applications to term structure of interest rates. The Review of Financial Studies, 18(1), 37-84.

[18] Hull, J., and White, A. (1990). Pricing interest-rate-derivative securities. The review of financial studies, 3(4), 573-592.

[19] Johannes, M. (2004). The statistical and economic role of jumps in continuous-time interest rate models. The Journal of Finance, 59(1), 227-260.

[20] Karatzas, I. and Shreve, S.: Brownian Motion and Stochastic Calculus (2^{nd} edition). Springer, Heidelberg (1991)

[21] Last, G. and Brandt, A.: Marked Point Processes on the Real Line - The Dynamic Approach. Springer, Heidelberg (1995)

[22] Protter, Ph.: Stochastic Integration and Differential Equations. Springer, Heidelberg (1992)

[23] Renault, E., and Touzi, N. (1996). Option hedging and implied volatilities in a stochastic volatility model 1. Mathematical Finance, 6(3), 279-302.

[24] Ritchken, P., and Sankarasubramanian, L. (1995). Volatility Structures of Forward Rates and the Dynamics of the Term Structure 1. Mathematical finance, 5(1), 55-72.