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# A dam management problem with energy production as an optimal switching problem

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**Abstract**

We consider an optimal stochastic control problem for a dam. Electrical power production is operating under an uncertain setting for electricity market prices and water level which has to be kept under control. Indeed, the water level inside the basin cannot exceed a certain threshold for safety reasons, and at the same time cannot decrease below another threshold in order to keep power production active. We model this situation as a mixed control problem with regular and switching controls under constraints. We characterize the value function as solution of an HJB equation and provide some numerical approximating methods. We shall illustrate by numerical examples the main achievements of the present approach.

KEYWORDS

dam management, hydro power production, optimal switching

## 1 | INTRODUCTION

Suitable mathematical models for dam management are of overwhelming importance for both technological and economic reasons. The strong impact of water reservoirs on environment makes even more relevant the need of reliable models for dams and water flows management and control. Strict standard regulations are in force all around the world in order to manage the water flows though dams in a safe and efficient way, and these regulations are usually described via the so-called "Rule Curves."[1] An impressive amount of literature is devoted to description of dam management as an optimal control problem. We refer to the pioneer paper by Faddy[2] for an extensive survey and for an historical perspective on the subject.

Special attention has been devoted to the case of bang-bang policies in which two levels exist. If the water crosses a first level, the decision is to start to release water, the level will then decrease until a second level is reached that is the time to stop the water release, and the cycle is repeated.[3]

In a large amount of papers available in the literature, storage and production models for dams are associated with state-dependent queuing systems. Among these, the paper by Abramov[4] investigates the optimal control problem of a large dam (here large is defined according to the difference between the upper and the lower allowed water levels) by adopting a state-dependent queuing framework and a performance criterion based on minimization of the number of passages across the lower and the upper level. Further contributions in this direction are provided in.[5–8] More recently, McInnes and Miller[9] model the dam dynamics as a controllable time inhomogeneous Markov chain with a compound Poisson process as an input, which approximates the inflow pattern of real dams with seasonal rainfall. Several results related to the Markov chain approach are collected in the PhD Thesis by McInnes.[10]

The optimal control problem arising in the above mentioned framework is often too complicated to be solved in closed form, that implying that suitable numerical methods are required in order to obtain a solution and specific algorithms have been developed to deal with the complexity of the problem involved.[11] In particular, Parallel Computing methods and Genetic Algorithms have been extensively applied in order to solve these optimization problems.[1] More recently, Shardin and Wunderlich[12] consider the valuation problem of an energy storage facility in the presence of stochastic energy prices as it arises in the case of a hydro-electric pump station. This valuation problem is related to the problem of determining the optimal charging/discharging strategy that maximizes the expected value of the resulting discounted cash flows over the lifetime of the storage. By working in a regime switching setting they formulate the problem as a stochastic control problem under partial information in continuous time. Stochastic modeling of energy prices relates dam management to many problems arising in mathematical finance. There is a large literature focusing on money-consuming frictions in economic and/or financial setups, see for instance References 13–20.

In the present paper, we formulate the dam management problem as an optimal mixed regular-switching problem, in which the dam manager has to optimize the revenues from power production and at the same time must keep under control the water level in the basin in order to fulfill safety requirements. We highlight that the key point in hydroelectric production is that the turbine-production system is designed in order to produce electricity at a fixed frequency. This issue is not new but dates back at least to the beginning of the 20th century since the grid and all devices are designed to work at a specific and unified frequency. Industrial plans to change electric frequency exist, for instance in Japan, but it is an exception, especially comparing with devices to change the voltage. The constraint of a fixed frequency implicitly constraint the electric unit production to a on-off strategy and the only way to add intermediate productions is to install multiple units for each dam. The optimal frequency is obtained regulating the water flow using servomechanisms (digital in the modern dam and centrifugal governor in small/old dams) as explained in the following section.

The turbine status (open and closed) and the switching times between status changes are assumed as control variables together with the status of a further depleting device, the spillways, through which the water can flow out without producing energy. The multidimensional setup of nondegenerate mixed switching control problems is a major challenge when one wants to qualitatively describe optimal policies. There are, indeed, no general and easily applicable methods to completely solve this set of switching control problems because each problem is based on specific model features leading in general to a nonstandard HJB equation. The bidimensional setup we consider is a special case of a multidimensional setup where multiple generators are installed as for three gorges dam (China, 34 turbines), Itapu (Brazil, 20 turbines), Hoover (United States, 19 turbines). However, the generalization to multiple-setup is tedious in numerical aspects but relatively easy in mathematical point of view. We will focus on small dams with only one generator. Finally we obtain two optimal threshold endogenously determined related to the optimal turbine and spillover levels.

The paper is organized as follows: in Section 2 we present our mathematical model for the dam dynamics, in Section 3 we formulate the dam management as a stochastic control problem, in Section 4 we discuss existence and uniqueness of a viscosity solution for the variational inequality considered and in Section 5, we provide a numerical illustration of the results obtained. In the final section we shall present some concluding remarks and we shall outline some possible developments of the present approach.

## 2 | MODELING THE WATER DYNAMICS INSIDE THE DAM

Let us consider a dam and denote the water level inside the dam $H_t$ as a function of time. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual hypotheses and equipped with a bi-dimensional Brownian motion denoted by $(W, B)$. The following sources can affect the water level: the water extraction for power production and the external contributions provided by rain and small rivers behind. Moreover, when the water level becomes higher to some critical value, the dam basin can be depleted by opening the spillways, while of course the water flows through the turbine used for power production and also contributes to deplete the basin. When the turbine and the spillways are closed, the water level process $(H_t)_{t\geq 0}$ is given by

$$H_t = X_t^+ \text{ with } X_t = \mu t + \sigma W_t, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. The water supply, originated directly by rain fall or indirectly by rivers entering the basin, is then assumed to vary according to a drifted Brownian motion where negative variations are due to evaporation.[21] If we denote by $Z$ the half-local time of the process $X$ at 0, we deduce from Tanaka's formula that

$$dH_t = \mathbb{1}_{\{H_t>0\}}(\mu dt + \sigma dW_t) + dZ_t.$$

The controlled water level dynamics is then given by the following stochastic differential equation:

$$dH_t = -I_t C(H_t)dt - \nu_t dt + \mathbb{1}_{\{H_t>0\}}(\mu dt + \sigma dW_t) + dZ_t, \tag{2}$$

where $I_t$ denotes the turbine status at time $t$ and it is assumed to be a control variable; it can take values $i = 0, 1$. $i = 0$ describes a closed turbine, while $i = 1$ describes an open turbine. The second control variable $\nu_t$ denotes the spillways status and takes values in $[0, +\infty)$. The deterministic function $C$, defined on $[h_0, +\infty)$ where $h_0$ is the level of the turbine, describes the amount of water extracted by the basin in order to produce power. Notice that, as the zero level of the reference frame is assumed to be the bottom of the reservoir, $h_0$ can be negative, in particular when mountain dams are concerned. The control variable $\nu_t$ denotes the amount of water extracted by the basin without producing energy: it is extracted by opening the auxiliary conducts, the spillways (these are realized just for managing the spillover effects). The optimal level at which the spillways must be opened will be determined as part of the solution to the optimal control problem we are going to introduce. The control variables can then be resumed to be the following: $\{\tau_i\}_{i\in\mathbb{N}^*}$ denoting the switching times for the turbine status and $\nu_t$ denoting the status of the extra flow allowing to deplete the dam in emergency situations. It is important to remark that this kind of flow is not used in order to produce power.

The function $C$ appearing in (2) can be determined by physical arguments: it is given by the product of the open section $Q > 0$ of the penstock connecting the basin with the turbine and the speed $v$ of the water at the entrance of the conduct divided by the surface $S$ of the basin assumed independent of $h$. The speed $v$ can be expressed as a function of $h$ by the energy conservation principle; we can then write:

$$C(h) = Q\frac{v}{S} = Q\frac{\sqrt{2g(h-h_0)}}{S}, \tag{3}$$

where $g$ denotes the gravity constant. By a similar argument the spillover impact $\nu_t$ is proportional to the section of spillover pipe and the square root of the free height $(H_t - h_0)$, where we suppose that the free heigh is the same for spillover and turbine pipes, as in Hoover dam. As a consequence, the spillover impact can be rewritten as $\beta_t\sqrt{2g(H_t - h_0)}$, where $\beta_t$ does not depend of $H_t$ and takes values in the interval $[0, \overline{\beta}]$ where $\overline{\beta} > 0$.

The power produced, in terms of energy units, is given by the following expression:

$$\mathcal{E}_t = Q_t v_t g(H_t - h_0)(1-\chi) = C(H_t)Sg(H_t - h_0)(1-\chi) = \sqrt{2}(1-\chi)Q_t\left[g(H_t - h_0)\right]^{3/2}, \tag{4}$$

where $\chi$ is a dispersion coefficient. The conduct connecting the reservoir with the turbine is designed for the dam and a crucial issue for power production is to keep the frequency of the alternate electric current constant (generally 50 Hz in European Countries, 60 Hz in American Countries). Since this is determined by the rotational speed of the turbine, the open section area will be regulated by servo-mechanism in such a way that water flowing per second $Q_t$ will keep constant the rotational frequency of the turbine. That implies that $\mathcal{E}_t$ could be considered constant as well and in the sequel it will be simply denoted by $\mathcal{E}$.

At the same time we obtain that the volume of water flowing $Q_t$ as inversely proportional to the head of the water power 3/2. We can then write the effect of power production on the water level as follows:

$$C(H_t) = \frac{Q_t\sqrt{2g(H_t - h_0)}}{S} = \frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{H_t - h_0}, \tag{5}$$

That is the impact of the electricity production on the head of water is inversely proportional to itself.

The power produced can be sold at price $P$; this spot price for power is assumed to evolve according to a stochastic dynamics described by the following stochastic differential equation:

$$dP_t = P_t(\lambda dt + \gamma dB_t), \tag{6}$$

where $B$ is a standard Wiener process possibly correlated to $W$, with correlation coefficient $\rho \in (-1, 1)$. The correlation derives from the fact that the unpredictable fluctuations of water heigh are related to rain and the meteorological phenomena influences the other electrical plants either directly (e.g., solar plants) or indirectly (e.g., thermal power plants shall already be cooled down using rivers water, their power is then usually reduced during droughts). We shall assume moreover that whenever the turbine status is changed, a cost in terms of energy must be paid, and we denote by $h_-$ the minimum level under which the turbine cannot operate in order to produce power with the required frequency, and then has to be closed.

As an incentive to keep the water level under control for safety reasons, a penalty is introduced when the water level crosses a critical value $h_+$. The penalty function, defined on $[h_+, +\infty)$ is denoted by $f$ and Lipschitz continuous on $[h_+, +\infty)$.

Finally, we assume that the spillways are at the bottom level of the reservoir, allowing depletion both in case of emergency and for eventual maintenance purposes. The water level at which it is optimal to open the spillways will be determined as a free boundary of the optimal control problem we are going to formulate in next section. The role of the spillways is to allow to deplete the reservoir and we denote by $\beta_t$ the (controlled) flow of water leaving the basin through the spillways at time $t \geq 0$.

## 3 | THE OPTIMAL CONTROL PROBLEM

We formulate our dam management problem as an optimal stochastic control problem. The goal of the dam manager is to maximize the revenues from the power production by keeping the safety level of the dam under control.

Let $(\tau_k)_{k\geq 1}$ be a nondecreasing sequence of $\mathbb{F}$-stopping times. At the controlled time $\tau_k$, the dam manager decide to change the turbine status by switching from state 0 (turbine closed) to state 1 (turbine open) or vice versa. The associated controlled Markov chain taking values in $\{0, 1\}$ will be denoted by $I$ and, for $t \geq 0$, satisfies:

$$I_t := \frac{1}{2}\left[1 - (-1)^{I_0 + N_t}\right], \qquad \text{where } N_t := \sum_{k=1}^{\infty} \mathbb{1}_{\tau_k \leq t}. \tag{7}$$

The second control variable $\beta$ is an $\mathbb{F}$-predictable process, describing the spillways opening, and taking values between 0 and a maximum flow denoted by $\overline{\beta}$. We denote by $\mathcal{B}$ the set of such processes.

The control variables in the present framework are then the sequence of switching times of the turbine status $(\tau_k)_{k\geq 1}$, and the free flow $\beta \in \mathcal{B}$. The problem will be formulated as a stochastic control problem mixing regular and switching controls. We introduce the following set of admissible controls:

$$\mathcal{A} := \left\{\alpha = (\beta, (\tau_k)_{k\geq 1}) : \ \beta \in \mathcal{B} \text{ and } (\tau_k)_{k\geq 1} \text{ is a non decreasing sequence of } \mathbb{F}\text{-stopping times}\right\}. \tag{8}$$

When the water level reaches the value $\overline{h} > h_+$, maximum allowed by safety reasons, the entire problem will end, the dam becomes a nationwide issue and economic questions get irrelevant. We can define then the time horizon $T^\alpha$ of our stochastic control problem as $T^\alpha = \inf\left\{t > 0 \mid H_t^\alpha \geq \overline{h}\right\}$, when no ambiguity can arise, we will denote it by $T$.

The value functions associated to the problem are denoted $v_0$ and $v_1$ and respectively defined on $D^0 := \left[0, \overline{h}\right] \times \mathbb{R}^+$ and $D^1 := \left[h_-, \overline{h}\right] \times \mathbb{R}^+$, by the following expression:

$$v_i(h,p) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{i,h,p,\alpha}\left[\int_0^{T-} e^{-rt}\mathcal{E}\, I_t\, P_t dt - \int_0^{T-} e^{-rt}\kappa P_t dN_t - \int_0^{T-} e^{-rt} P_t f\left((H_t - h_+)^+\right) dt\right], \tag{9}$$

where $r > 0$ is the discounting rate, and $f$ is the penalty function defined on $\mathbb{R}^+$ and creating an incentive to keep the level below $h_+$. $\kappa$ denotes the switching cost proportional to the electricity price. We observe that the first and last integral of (9) equals integrals from 0 to simply $T$ since they are absolute continuous integrals in the Lebesgue sense.

The variational inequality corresponding to the optimal stochastic control problem proposed is the following:

$$\min\left\{ r v_i - \sup_{\beta \in [0,\overline{\beta}]} \mathcal{L}^{(i,\beta)} v_i + p f\left((h - h_+)^+\right) - i\mathcal{E}p, \right.$$
$$\left. (v_i - v_{1-i} - \kappa p)\mathbb{1}_{h \geq h_-} + (1-i)\mathbb{1}_{h < h_-} \right\} = 0, \qquad \text{on } \mathring{D}^i, \tag{10}$$

where the Lagrangian differential operator $\mathcal{L}^{(i,b)}$ is defined as follows:

$$\mathcal{L}^{(i,\beta)}\phi = \left[-i\frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{h-h_0} - \beta\sqrt{2g(h-h_0)^+} + \mu\right]\frac{\partial \phi}{\partial h}$$
$$+ \left(\frac{\sigma^2}{2}\frac{\partial^2 \phi}{\partial h^2} + \sigma\gamma\rho p \frac{\partial^2 \phi}{\partial h \partial p}\right)\mathbb{1}_{h>0} + \lambda p \frac{\partial \phi}{\partial p} + p^2 \frac{\gamma^2}{2}\frac{\partial^2 \phi}{\partial p^2}. \tag{11}$$

For any $p \in \mathbb{R}^+$, we also have the following boundary conditions:

$$v_0(\overline{h}, p) = 0; \qquad \frac{\partial v_0}{\partial h}(0,p) = 0,$$
$$v_1(\overline{h}, p) = 0; \qquad v_1(h_-, p) = v_0(h_-, p) - \kappa p. \tag{12}$$

## 3.1 | Dimensionality reduction

As first result of this subsection we prove that the study of our value function $v_i$ depending on two variables (price $p$ and level $h$) can be reduced to the study of one-single variable function that we will denote by $w_i$ and we will call also *reduced valued function.*

**Proposition 1.** *Let $\mathbb{P}_\star$ be the probability measure equivalent to $\mathbb{P}$ with Radon-Nikodym derivative $Y_t := e^{\rho\gamma W_t - \frac{1}{2}\rho^2\gamma^2 t}$. For $i \in \{0,1\}$, we have $w_i(h) := \frac{v_i(p,h)}{p}$ on $D^i$, where we have set*

$$w_i(h) := \sup_{\alpha \in \mathcal{A}_W} \mathbb{E}_\star^{i,h,\alpha}\left[\int_0^T e^{-(r-\lambda)t}\mathcal{E}\, I_t dt - \int_0^{T-} e^{-(r-\lambda)t}\kappa dN_t\right.$$
$$\left. - \int_0^T e^{-(r-\lambda)t} f\left((H_t - h_+)^+\right) dt\right], \tag{13}$$

*where $\mathcal{A}_W$ is the restriction of the set of admissible control to the stopping times with respect to the filtration, denoted by $\mathbb{F}_W$, generated by the Brownian motion $W$, that is,*

$$\mathcal{A}_W := \left\{\alpha = (\beta, (\hat{\tau}_k)_{k\in\mathbb{N}}) \mid \beta \in \mathcal{B} \text{and } (\hat{\tau}_k)_{k\in\mathbb{N}} \text{ is a nondecreasing sequence of } \mathbb{F}_W\text{-stopping times}\right\}.$$

*Proof.* We first notice that

$$P_t = P_0 e^{\left(\lambda - \frac{1}{2}\gamma^2\right)t + \gamma B_t} = P_0 \exp\left\{\lambda t + \gamma\rho W_t - \frac{\rho^2\gamma^2}{2}t + \gamma\sqrt{1-\rho^2}\hat{B}_t - \frac{1-\rho^2}{2}\gamma^2 t\right\},$$

where we have introduced $\hat{B} := (B - \rho W)/\sqrt{1-\rho^2}$, that is a Brownian motion independent with respect to $W$. So previous term can be rewritten as follows

$$P_t = P_0 e^{\lambda t} \underbrace{\exp\left\{\gamma\rho W_t - \frac{\rho^2\gamma^2}{2}t\right\}}_{=Y_t} \underbrace{\exp\left\{\gamma\sqrt{1-\rho^2}\hat{B}_t - \frac{1-\rho^2}{2}\gamma^2 t\right\}}_{:=M_t}.$$

We remark that the last factor define a martingale, denoted hereafter $M_t$ independent with respect to the filtration generated by $W$, the integrability condition is guaranteed since $P$ is integrable. We also recognize $Y_t$, that is, the change of probability from $\mathbb{P}$ to $\mathbb{P}_\star$ and, finally, we recall that $W_t - \gamma\rho t$ is a $\mathbb{P}_\star$-Brownian motion.

Putting it into the definition of value function (9), we obtain

$$\begin{aligned}
\frac{1}{p}v_i(h,p) &= \frac{1}{p}\sup_{\alpha\in\mathcal{A}} \mathbb{E}^{i,h,p,\alpha}\left[\int_0^T e^{-rt}\mathcal{E}\, I_t\, p e^{\lambda t} Y_t M_t dt\right. \\
&\quad \left. - \int_0^T e^{-rt} p e^{\lambda t} Y_t M_t f\left((H_t - h_+)^+\right)dt - \int_0^{T-} e^{-rt}\kappa p e^{\lambda t} Y_t M_t dN_t\right] \\
&= \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{i,h,\alpha}\left[\int_0^{T-} e^{-(r-\lambda)t}\mathcal{E}\, I_t\, Y_t dt\right. \\
&\quad \left. - \int_0^{T-} e^{-(r-\lambda)t} Y_t f\left((H_t - h_+)^+\right)dt - \int_0^{T-} e^{-(r-\lambda)t}\kappa Y_t dN_t\right] \\
&= \sup_{\alpha\in\mathcal{A}_W} \mathbb{E}_\star^{i,h,\alpha}\left[\int_0^T e^{-(r-\lambda)t}\mathcal{E}\, I_t dt\right. \\
&\quad \left. - \int_0^T e^{-(r-\lambda)t} f\left((H_t - h_+)^+\right)dt - \int_0^{T-} e^{-(r-\lambda)t}\kappa dN_t\right],
\end{aligned}$$

where at the second step we have used that $M_t$ is a $\mathbb{P}$-martingale independent of the filtration generated by the Brownian motion $W$, at the third step, we have used the Radon–Nykodym derivative of $\mathbb{P}_\star$ with respect to $\mathbb{P}$, that is, $Y$. It follows that there exists a couple of functions $w_i$ defined on $\{0,1\} \times D^i$ such that $w_i(h) = p^{-1}v_i(h,p)$ and the optimal switching times

$$\begin{aligned}
\tau_\star^{i,h,p} &= \inf\left\{t < T | v_i(H_t^{i,h}, P_t^p) = v_{1-i}(H_t^{i,h}, P_t^p) - \kappa P_t^p\right\} \\
&= \inf\left\{t < T | w_i(H_t^{i,h}) = w_{1-i}(H_t^{i,h}) - \kappa\right\} := \tau_\star^{i,h},
\end{aligned}$$

belong to the set stopping times with respect to $\mathbb{F}_W$. ∎

Under $\mathbb{P}_\star$ the evolution of the head of water reads

$$dH_t = -I_t \frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{H_t - h_0}dt - \beta_t\sqrt{2g(H_t - h_0)}dt + \mathbb{1}_{\{H_t>0\}}((\mu + \sigma\gamma\rho)dt + \sigma dW_t) + dZ_t, \tag{14}$$

and the Dynkin operator associated to the controlled diffusion reads

$$\mathcal{L}_\star^{(i,\beta)}\phi = \left[\mu + \sigma\gamma\rho\mathbb{1}_{h>0} - i\frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{h-h_0} - \beta_t\sqrt{2g(h-h_0)}\right]\frac{\partial\phi}{\partial h} + \frac{\sigma^2}{2}\mathbb{1}_{h>0}\frac{\partial^2\phi}{\partial h^2}. \tag{15}$$

The HJB associated to the problem reads

$$\min\left\{(r-\lambda)w_i - \sup_{\beta\in\mathcal{B}}\mathcal{L}_\star^{(i,\beta)}w_i + f\big((h-h_+)^+\big) - i\mathcal{E},\right.$$
$$\left.(w_i - w_{1-i} - \kappa)\mathbb{1}_{h\geq h_-} + (1-i)\mathbb{1}_{h<h_-}\right\} = 0, \qquad \text{on } \mathring{\tilde{D}}^i, \tag{16}$$

where $\tilde{D}^0 = [0, \overline{h}]$ and $\tilde{D}^1 = [h_-, \overline{h}]$. For sake of readability, we will denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}_\star$ in the rest of the paper instead $\mathbb{E}_\star$.

**Proposition 2.** *Assume $r > \lambda$, then the reduced value function $w_i$ is finite and we have $w_i \leq \mathcal{E}/(r-\lambda)$.*

*Proof.* Following Duffie and Zariphopoulou,[22] we exploit an argument of "fictitious" control problem. We consider the same problem under the hypotheses $f \equiv 0$, $h_- = 0$ and $\overline{h} = \infty$, that is the dam has an infinite height and no cost for operating exists. The associated value function is denoted $\widehat{w}_i$. Consider now the control policy to switch immediately the production state from 0 to 1, if this is not the case, and then wait. It is easy to see that the production is always positive and the ending time $T$ will never take place. It is then easy to see that the reward function reads $\int_0^\infty e^{-(r-\lambda)t}\mathcal{E}dt - \kappa(1-i) = \mathcal{E}/(r-\lambda) - \kappa(1-i)$ and that is optimal, i.e. $\widehat{w}_i = \kappa(i-1) + \mathcal{E}/(r-\lambda)$. It is also easy to check that the "fictitious" control problem is an upper bound for the true one, since $f$ is non-negative and $\overline{h} < \infty$. Then we obtain the second part of the Proposition statement. ∎

**Proposition 3.** *For $i \in \{0, 1\}$, $w_i$ is Lipschitz.*

*Proof.* Let $h, y \in D_i$, we want to prove $|w_i(y) - w_i(h)| \leq C_w|y-h|$ for some positive constant $C_w$. Without lost of generality, assume $h < y$, let $\alpha_h$ be an admissible control for the initial state $H_0 = h$. Let $\theta_{h,y}^{\alpha_h} := \inf\left\{t \mid H_t^{h,\alpha_h} \leq 0 \text{ or } T_y^{\alpha_h} \leq t\right\}$, that is the first time when the process $H$ starting at $h$ and following $\alpha_h$ reaches 0 or the process $H$ starting at $y$ and following $\alpha_h$ reaches $\overline{h}$. It is easy to check that up to $\theta_{h,y}^{\alpha_h}$ the SDE (14) has Lipschitz coefficients and then the uniqueness of the solution guarantees that $H_t^{h,\alpha_h} < H_t^{y,\alpha_h}$. We could deduce that $\alpha_h$ is an admissible control for the initial state $H_0 = y$. In the sequel, for notational simplicity, we will omit the dependency with respect to the control $\alpha_h$. Using the SDE (14), we have, introducing $Y_t := H_t^y - H_t^h$, that $Y_0 = y - h$ and

$$dY_t = \left[\frac{\mathcal{E}\,I_t^i}{Sg(1-\chi)}\left(\frac{1}{H_t^h - h_0} - \frac{1}{H_t^y - h_0}\right) + \beta_t\sqrt{2g}\left(\sqrt{H_t^h - h_0} - \sqrt{H_t^y - h_0}\right)\right]dt$$
$$= Y_t\left[\frac{\mathcal{E}}{Sg(1-\chi)}\frac{I_t^i}{(H_t^y - h_0)(H_t^h - h_0)} - \frac{\beta_t\sqrt{2g}}{\sqrt{H_t^h - h_0} + \sqrt{H_t^y - h_0}}\right]dt\ .$$

We could solve the differential equation and deduce

$$Y_t = (y-h)\exp\left\{\int_0^t \frac{\mathcal{E}}{Sg(1-\chi)}\frac{I_s^i}{(H_s^y - h_0)(H_s^h - h_0)} - \frac{\beta_s\sqrt{2g}}{\sqrt{H_s^h - h_0} + \sqrt{H_s^y - h_0}}ds\right\}$$
$$\leq (y-h)\exp\left\{\int_0^t \frac{\mathcal{E}}{Sg(1-\chi)}\frac{I_s^i}{(H_s^y - h_0)(H_s^h - h_0)}ds\right\}$$
$$\leq (y-h)\exp\left\{\int_0^t \frac{\mathcal{E}}{Sg(1-\chi)}\frac{\mathbb{1}_{H_s^h\geq h_-}}{(h_- - h_0)^2}ds\right\} \leq (y-h)e^{\frac{E}{Sg(1-\chi)(h_- - h_0)^2}t},$$

where we have used that $\beta$ is nonnegative and when $H_s^h < h_-$ the state has to be 0 due to physical constraints. ∎

### 3.2 | Dynamic programming principle

For $\theta$ any $\mathbb{F}$-stopping time,

$$\begin{aligned} w_i(h) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{i,h,\alpha}\bigg[ &\int_0^{(\theta\wedge T)} e^{-(r-\lambda)t}\mathcal{E}\, I_t dt - \int_0^{(\theta\wedge T)} e^{-(r-\lambda)t} f\big((H_t-h_+)^+\big)dt \\ &- \int_0^{(\theta\wedge T)-} e^{-(r-\lambda)t}\kappa dN_t + e^{-(r-\lambda)(\theta\wedge T)} w_{I_{\theta\wedge T}}(H_{\theta\wedge T})\bigg], \end{aligned} \tag{17}$$

## 4 | CHARACTERIZATION AS VISCOSITY SOLUTION

In the present section we want to characterize the solution of the problem formulated in the previous section as the unique viscosity solution of the following HJB equation

$$\begin{cases} 0 = \min\Big\{(r-\lambda)w_0 - \Big[(\mu+\sigma\rho\gamma)w_0' + \overline{\beta}\ \sqrt{2g(h-h_0)}\big(-w_0'\big)^+ + \frac{\sigma^2}{2}w_0''\Big] \\ \qquad + f\big((h-h_+)^+\big)\ ;\ (w_0 - w_1 + \kappa)\mathbb{1}_{(h\geq h_-)} + \mathbb{1}_{(h<h_-)}\Big\}, \quad on\ (0,\overline{h}), \\ 0 = w_0'(0), \\ 0 = w_0(\overline{h}), \end{cases} \tag{18}$$

$$\begin{cases} 0 = \min\Big\{(r-\lambda)w_1 - \Big[\Big(\mu+\sigma\rho\gamma - \frac{\mathcal{E}}{Sg(1-\chi)(h-h_0)}\Big)w_1' + \overline{\beta}\ \sqrt{2g(h-h_0)}\big(-w_1'\big)^+ + \frac{\sigma^2}{2}w_1''\Big] \\ \qquad - \mathcal{E} + f\big((h-h_+)^+\big), w_1 - w_0 + \kappa\Big\}, \quad \text{on } (h_-,\overline{h}) \\ 0 = w_1\big(\overline{h}\big) \\ \kappa = w_0(h_-) - w_1(h_-) \end{cases} \quad .. \tag{19}$$

We now state a standard first result for this system of PDE.

**Proposition 4.** *Let $\{\varphi_i\}_{i=0;1} \in C^2$ on $[0,\overline{h}]$ such that $\varphi(\overline{h}) \geq 0$ and*

$$\begin{aligned} 0 \leq \min\Big\{&(r-\lambda)\varphi_0 - \Big[(\mu+\sigma\rho\gamma)\varphi_0' + \overline{\beta}\ \sqrt{2g(h-h_0)}\big(-\varphi_0'\big)^+ + \frac{\sigma^2}{2}\varphi_0''\Big] \\ &+ f\big((h-h_+)^+\big)\ ;\ (\varphi_0-\varphi_1+\kappa)\mathbb{1}_{(h\geq h_-)} + \mathbb{1}_{(h<h_-)}\Big\}, \quad on\ (0,\overline{h}), \\ 0 \leq \min\Big\{&(r-\lambda)\varphi_1 - \Big[\Big(\mu+\sigma\rho\gamma + \frac{\mathcal{E}}{Sg(1-\chi)(h-h_0)}\Big)\varphi_1' + \overline{\beta}\sqrt{2g(h-h_0)}(-\varphi_1')^+ + \frac{\sigma^2}{2}\varphi_1''\Big] \\ &- \mathcal{E} + f\big((h-h_+)^+\big), \varphi_1 - \varphi_0 + \kappa\Big\}, \quad on\ (h_-,\overline{h}), \end{aligned}$$

*then we have $w_i(h) \leq \varphi_i(h)$, for all $h \in (0,\overline{h})$ and for $i = 0, 1$.*

*Proof.* Given an initial state-regime value $(h;i) \in (0,\overline{h}) \times \{0;1\}$ take an arbitrary control $\{\{\beta_t\}_{t\geq 0}; \{\tau_n\}_{n\in\mathbb{N}}\}$, we recall that $I_t = i$ if $t \in [\tau_{2k};\tau_{2k+1})$ and $I_t = 1-i$ if $t \in [\tau_{2k+1};\tau_{2k+2})$ since only two states are allowed. We apply then Ito formula to $e^{-(r-\lambda)t}\varphi_{I_t}\Big(H_t^{(h;i)}\Big)$ between $T\wedge\tau_n$ and $T\wedge\tau_{n+1}$.

$$\begin{aligned} e^{-(r-\lambda)(T\wedge\tau_{n+1})}\varphi_{I_{T\wedge\tau_{n+1}^-}}\Big(H^{(h;i)}_{T\wedge\tau_{n+1}^-}\Big) = {}& e^{-(r-\lambda)(T\wedge\tau_n)}\varphi_{I_{T\wedge\tau_n}}\Big(H^{(h;i)}_{T\wedge\tau_n}\Big) \\ &+ \int_{T\wedge\tau_n}^{T\wedge\tau_{n+1}} e^{-(r-\lambda)t}\Big[\mathcal{L}_\star^{(i,\beta)}\varphi_{I_t} - (r-\lambda)\varphi_{I_t}\Big]\Big(H_t^{(h;i)}\Big)dt \\ &+ \int_{T\wedge\tau_n}^{T\wedge\tau_{n+1}} e^{-(r-\lambda)t}\sigma\mathbb{1}_{H_t^{(h;i)}>0}\varphi_{I_t}'\Big(H_t^{(h;i)}\Big)dW_t, \end{aligned}$$

taking the expectation we obtain

$$\mathbb{E}\left[e^{-(r-\lambda)(T\wedge\tau_{n+1})}\varphi_{I_{T\wedge\tau_{n+1}^-}}\left(H^{(h;i)}_{T\wedge\tau_{n+1}^-}\right)\right]=\mathbb{E}\left[e^{-(r-\lambda)(T\wedge\tau_n)}\varphi_{I_{T\wedge\tau_n}}\left(H^{(h;i)}_{T\wedge\tau_n}\right)\right]$$
$$+\mathbb{E}\left[\int_{T\wedge\tau_n}^{T\wedge\tau_{n+1}}e^{-(r-\lambda)t}\left[\mathcal{L}_\star^{(i,\beta)}\varphi_{I_t}-(r-\lambda)\varphi_{I_t}\right]\left(H^{(h;i)}_t\right)dt\right],$$

where the stochastic integral has disappeared since $\varphi\in C^2$ and then its derivative is bounded on the compact $[0,\overline{h}]$ and then the stochastic integral is a true martingale. Now exploiting the first term in supersolution inequality in the statement of the proposition, we obtain

$$\mathbb{E}\left[e^{-(r-\lambda)(T\wedge\tau_{n+1})}\varphi_{I_{T\wedge\tau_{n+1}^-}}\left(H^{(h;i)}_{T\wedge\tau_{n+1}^-}\right)\right]\le\mathbb{E}\left[e^{-(r-\lambda)(T\wedge\tau_n)}\varphi_{I_{T\wedge\tau_n}}\left(H^{(h;i)}_{T\wedge\tau_n}\right)\right]$$
$$-\mathbb{E}\left[\int_{T\wedge\tau_n}^{T\wedge\tau_{n+1}}e^{-(r-\lambda)t}\left\{\mathcal{E}I_t-f\left(\left(H^{(h;i)}_t-h_+\right)^+\right)\right\}dt\right].$$

Recalling that $H$ has continuous paths and exploiting the second term in supersolution inequality, we obtain $\varphi_{I_{T\wedge\tau_{n+1}^-}}\left(H^{(h;i)}_{T\wedge\tau_{n+1}^-}\right)=\varphi_{I_{T\wedge\tau_{n+1}^-}}\left(H^{(h;i)}_{T\wedge\tau_{n+1}}\right)\ge\varphi_{I_{T\wedge\tau_{n+1}}}\left(H^{(h;i)}_{T\wedge\tau_{n+1}}\right)-\kappa$ and then

$$\mathbb{E}\left[e^{-(r-\lambda)(T\wedge\tau_n)}\varphi_{I_{T\wedge\tau_n}}\left(H^{(h;i)}_{T\wedge\tau_n}\right)\right]\ge$$
$$\mathbb{E}\left[e^{-(r-\lambda)(T\wedge\tau_{n+1})}\left(\varphi_{I_{T\wedge\tau_{n+1}}}\left(H^{(h;i)}_{T\wedge\tau_{n+1}}\right)-\kappa\right)+\int_{T\wedge\tau_n}^{T\wedge\tau_{n+1}}e^{-(r-\lambda)t}\left\{\mathcal{E}I_t-f\left(\left(H^{(h;i)}_t-h_+\right)^+\right)\right\}dt\right].$$

By iterating the previous inequality for all $n$ up to $T$ and recalling that $H^{(h;i)}_T=\overline{h}$ we then obtain

$$\varphi_i(h)\ge\mathbb{E}\left[e^{-(r-\lambda)T}\varphi_{I_T}\left(H^{(h;i)}_T\right)-\kappa\sum_{\tau_n\le T}e^{-(r-\lambda)(T\wedge\tau_n)}+\int_0^T e^{-(r-\lambda)t}\left\{\mathcal{E}I_t-f\left(\left(H^{(h;i)}_t-h_+\right)^+\right)\right\}dt\right]$$
$$\ge\mathbb{E}\left[\int_0^T e^{-(r-\lambda)t}\left\{\mathcal{E}I_t-f\left(\left(H^{(h;i)}_t-h_+\right)^+\right)\right\}dt-\int_0^{T-}e^{-(r-\lambda)t}\kappa dN_t\right],$$

and we obtain the required result from the arbitrariness of the control $\{\{\beta_t\}_{t\ge0};\{\tau_n\}_{n\in\mathbb{N}}\}$. ■

## 4.1 | Existence and uniqueness of solutions

We then have the PDE characterization of the value functions $w_i(h)$.

**Theorem 1.** *The value functions $w_i(h)$, $i=0,1$, are continuous on $(0,\overline{h})$, and are the unique viscosity solutions on $(0,\overline{h})$ with boundary data $w_i(\overline{h})=0$, $w_0'(0)=0$, to the system of variational inequalities (18) and (19).*

*Proof. of supersolution property*: Let $\hat{h}\in(0,\overline{h})$ and $\phi_i\in C^2((0,\overline{h}))$ such that $\phi_i(\hat{h})=w_i(\hat{h})$ and $\phi_i\le w_i$ on $(0,\overline{h})$. We have to prove that

$$0\le\min\left\{(r-\lambda)\phi_0(\hat{h})-\left[(\mu+\sigma\rho\gamma)\phi_0'+\overline{\beta}\sqrt{2g(\hat{h}-h_0)}\left(-\phi_0'(\hat{h})\right)^++\frac{\sigma^2}{2}\phi_0''(\hat{h})\right]\right.$$
$$\left.+f\left((\hat{h}-h_+)^+\right);\ (\phi_0(\hat{h})-\phi_1(\hat{h})+\kappa)\mathbb{1}_{(\hat{h}\ge h_-)}+\mathbb{1}_{(\hat{h}<h_-)}\right\}$$
$$0\le\min\left\{(r-\lambda)\phi_1(\hat{h})-\left[\left(\mu+\sigma\rho\gamma-\frac{\mathcal{E}}{Sg(1-\chi)(\hat{h}-h_0)}\right)\phi_1'(\hat{h})+\overline{\beta}\sqrt{2g(\hat{h}-h_0)}\left(-\phi_0'(h_0)\right)^++\frac{\sigma^2}{2}\phi_1''(\hat{h})\right]\right.$$
$$\left.-\mathcal{E}+f\left((\hat{h}-h_+)^+\right);\phi_1(\hat{h})-\phi_0(\hat{h})+\kappa\right\},$$

It is easy to remark that $(\phi_0(\hat{h}) - \phi_1(\hat{h}) + \kappa)\mathbb{1}_{(\hat{h}\geq h_-)} + \mathbb{1}_{(\hat{h}<h_-)} \geq 0$ and $\phi_1(\hat{h}) - \phi_0(\hat{h}) + \kappa \geq 0$ since $\phi_i(\hat{h}) = w_i(\hat{h})$ and the properties of value function. We have then to show the two other inequalities.

Given an initial regime value $i \in \{0;1\}$ consider the initial state-regime $(i, \hat{h})$ take an arbitrary control $\{\{\beta_t\}_{t\geq 0}; \{\tau_n\}_{n\in\mathbb{N}}\}$, we could now adapt the argument of Proposition 4 replacing $\varphi$ with $\phi$. More precisely, let $\mathcal{B}_\delta := (\hat{h}-\delta, \hat{h}+\delta)$ for all $\delta > 0$, we define $\theta_\delta := \inf\{t \mid H_t \notin \mathcal{B}_\delta\}$, that is, the exit time of the process $H_t$ from the ball $\mathcal{B}_\delta$. Without loss of generality we could assume that $\mathcal{B}_\delta \subset (0, \bar{h})$, that is, $\hat{h} + \delta < \bar{h}$ and then $\theta_\delta < T$ We apply the dynamic programming principle up to $\theta_\delta \wedge \tau_1 \wedge t$ where $t > 0$ and we obtain

$$\begin{aligned}\phi_i(\hat{h}) &= w_i(\hat{h}) \\ &\geq \mathbb{E}^{i,\hat{h},\beta}\Bigg[\int_0^{\theta_\delta\wedge\tau_1\wedge t} e^{-(r-\lambda)s}\mathcal{E}\, I_s ds - \int_0^{\theta_\delta\wedge\tau_1\wedge t} e^{-(r-\lambda)s} f\big((H_s - h_+)^+\big)ds \\ &\quad - \int_0^{(\theta_\delta\wedge\tau_1\wedge t)^-} e^{-(r-\lambda)s}\kappa dN_s + e^{-(r-\lambda)(\theta_\delta\wedge\tau_1\wedge t)^-} w_{I_{(\theta_\delta\wedge\tau_1\wedge t)^-}}(H_{(\theta_\delta\wedge\tau_1\wedge t)^-})\Bigg] \\ &\geq \mathbb{E}^{i,\hat{h},\beta}\Bigg[\int_0^{\theta_\delta\wedge\tau_1\wedge t} e^{-(r-\lambda)s}\mathcal{E}\, i ds - \int_0^{\theta_\delta\wedge\tau_1\wedge t} e^{-(r-\lambda)s} f\big((H_s - h_+)^+\big)ds \\ &\quad + e^{-(r-\lambda)(\theta_\delta\wedge\tau_1\wedge t)^-} w_i(H_{\theta_\delta\wedge\tau_1\wedge t})\Bigg],\end{aligned}$$

where we have used the definition of $\tau_1$, the path-continuity of the SDE satisfied by $H$ and the fact that $\theta_\delta < T$. A direct application of Ito-formula between 0 and $\theta_\delta \wedge \tau_1 \wedge t$ to $e^{-(r-\lambda)s}\phi_i(H_s)$ gives

$$\mathbb{E}^{i,\hat{h}}\left[e^{-(r-\lambda)(\theta_\delta\wedge\tau_1\wedge t)^-}\phi_i(H_{(\theta_\delta\wedge\tau_1\wedge t)^-})\right] = \phi_i(\hat{h}) + \mathbb{E}^{i,\hat{h}}\left[\int_0^{\theta_\delta\wedge\tau_1\wedge t} e^{-(r-\lambda)s}\left(\mathcal{L}_\star^{(i,\beta)} - r + \lambda\right)\phi_i(H_s)ds\right].$$

Then, combining with the previous inequality, we obtain

$$\begin{aligned}&\mathbb{E}^{i,\hat{h}}\left[\int_0^{\theta_\delta\wedge\tau_1\wedge t} e^{-(r-\lambda)s}\left\{(r-\lambda)\phi_i(H_s) - \mathcal{L}_\star^{(i,\beta)}\phi_i(H_s) - \mathcal{E}i + f\big((H_s-h_+)^+\big)\right\}ds\right] \geq \\ &\mathbb{E}^{i,\hat{h}}\left[e^{-(r-\lambda)(\theta_\delta\wedge\tau_1\wedge t)^-}(w_i - \phi_i)(H_{(\theta_\delta\wedge\tau_1\wedge t)^-})\right] \geq 0.\end{aligned} \tag{20}$$

From the definition of $\theta_\delta$, we readily see that the integrand part of (20) is bounded. Dividing the previous inequality by $t$ and taking $t$ to 0, we may apply the dominated convergence theorem and obtain

$$(r-\lambda)\phi_i(\hat{h}) - \mathcal{L}_\star^{(i,\beta)}\phi_i(\hat{h}) - \mathcal{E}i + f\left(\left(\hat{h} - h_+\right)^+\right) \geq 0,$$

for all control $\beta \in \mathcal{B}$ leading us to the two required supersolution inequalities by taking the supremum over $\beta$.

*Proof of subsolution*: We prove the subsolution property by contradiction. Suppose that the claim is not true. Then there exist $\hat{h} \in (0, \bar{h})$ and $\hat{i} \in \{0, 1\}$, a neighborhood of $\hat{h}$ denoted by $\mathcal{B}_\delta := (\hat{h} - \delta, \hat{h} + \delta)$ with $\delta > 0$ small enough, a couple of functions $\phi_i$ with $\phi_i \geq w_i$ on $\mathcal{B}_\delta$ and $\phi_i(\hat{h}) = w_i(\hat{h})$, a control $\hat{\alpha} := (\hat{\beta}, \{\hat{\tau}_k\}_{k\in\mathbb{N}}) \in \mathcal{A}_W$ and $\eta > 0$ such that for all $h \in \mathcal{B}_\delta$ the two following inequalities hold.

$$\begin{aligned}\eta &< \min\Bigg\{(r-\lambda)\phi_0(\hat{h}) - \left[(\mu + \sigma\rho\gamma)\phi_0' + \bar{\beta}\sqrt{2g(\hat{h}-h_0)(-\phi_0'(\hat{h}))^+} + \frac{\sigma^2}{2}\phi_0''(\hat{h})\right] \\ &\quad + f\left((\hat{h}-h_+)^+\right);\ (\phi_0(\hat{h}) - \phi_1(\hat{h}) + \kappa)\mathbb{1}_{(\hat{h}\geq h_-)} + \mathbb{1}_{(\hat{h}<h_-)}\Bigg\} \\ \eta &< \min\Bigg\{(r-\lambda)\phi_1(\hat{h}) - \left[\left(\mu + \sigma\rho\gamma - \frac{\mathcal{E}}{Sg(1-\chi)(\hat{h}-h_0)}\right)\phi_1'(\hat{h}) + \bar{\beta}\sqrt{2g(\hat{h}-h_0)(-\phi_1'(\hat{h}))^+} + \frac{\sigma^2}{2}\phi_1''(\hat{h})\right] \\ &\quad - \mathcal{E} + f\left((\hat{h}-h_+)^+\right); \phi_1(\hat{h}) - \phi_0(\hat{h}) + \kappa\Bigg\},\end{aligned} \tag{21}$$

We consider the exit time $\theta_\delta := \inf\{t \mid H_t \notin \mathcal{B}_\delta\}$. Let $\hat{\gamma}$ be a stopping time, and apply Itô's Formula to $e^{-(r-\lambda)t}\phi$ between 0 and $\gamma_\delta := \theta_\delta \wedge \hat{\gamma} \wedge \hat{\tau}_1^-$. Taking the expectation, we obtain

$$\mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\gamma_\delta}\phi_{I_{\gamma_\delta}}(H_{\gamma_\delta})\right] = \phi_{\hat{i}}\left(\hat{h}\right) + \mathbb{E}^{(\hat{i},\hat{h})}\left[\int_0^{\gamma_\delta} e^{-(r-\lambda)t}\left\{\mathcal{L}_\star^{(\hat{i},\hat{\beta})}\phi_{\hat{i}}(H_t) - (r-\lambda)\phi_{\hat{i}}(H_t)\right\}dt\right].$$

From relation (21), the above inequality becomes

$$\mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\gamma_\delta}\phi_{I_{\gamma_\delta}}(H_{\gamma_\delta})\right] \le \phi_{\hat{i}}\left(\hat{h}\right) + \mathbb{E}^{(\hat{i},\hat{h})}\left[\int_0^{\gamma_\delta} e^{-(r-\lambda)t}\{\eta + i\mathcal{E} - f\left((H_t - h_+)^+\right)\}dt\right],$$

by splitting the stopping time $\gamma_\delta$ into its components, we have

$$\begin{aligned}\phi_{\hat{i}}\left(\hat{h}\right) \ge& -\eta\mathbb{E}^{(\hat{i},\hat{h})}\left[\frac{1-e^{-(r-\lambda)\gamma_\delta}}{(r-\lambda)}\right] + \mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\hat{\tau}_1}\phi_i(H_{\hat{\tau}_1^-})\mathbb{1}_{\hat{\tau}_1\le\theta_\delta\wedge\hat{\gamma}}\right]\\ &+\mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\theta_\delta}\phi_i(H_{\theta_\delta})\mathbb{1}_{\theta_\delta<\hat{\tau}_1\wedge\hat{\gamma}}\right] + \mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\hat{\gamma}}\phi_i(H_{\hat{\gamma}})\mathbb{1}_{\hat{\gamma}<\hat{\tau}_1\wedge\theta_\delta}\right]\\ &-\mathbb{E}^{(\hat{i},\hat{h})}\left[\int_0^{\gamma_\delta}\{i\mathcal{E} - f\left((H_t-h_+)^+\right)\}dt\right].\end{aligned}$$

Using relation (21) in order to obtain un upper bound for the second expectation and recalling that $\phi_i \ge w_i$ on $\mathcal{B}_\delta$, we obtain for all stopping time $\hat{\gamma}$

$$\begin{aligned}\phi_{\hat{i}}\left(\hat{h}\right) \ge& -\eta\mathbb{E}^{(\hat{i},\hat{h})}\left[\frac{1-e^{-(r-\lambda)\gamma_\delta}}{(r-\lambda)}\right] + \mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\hat{\tau}_1}\left(w_{1-i}(H_{\hat{\tau}_1^-}) + \eta\right)\mathbb{1}_{\hat{\tau}_1\le\theta_\delta\wedge\hat{\gamma}}\right]\\ &+\mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\theta_\delta}w_i(H_{\theta_\delta})\mathbb{1}_{\theta_\delta<\hat{\tau}_1\wedge\hat{\gamma}}\right] + \mathbb{E}^{(\hat{i},\hat{h})}\left[e^{-(r-\lambda)\hat{\gamma}}w_i(H_{\hat{\gamma}})\mathbb{1}_{\hat{\gamma}<\hat{\tau}_1\wedge\theta_\delta}\right] - \mathbb{E}^{(\hat{i},\hat{h})}\left[\int_0^{\gamma_\delta}\{i\mathcal{E} - f\left((H_t-h_+)^+\right)\}dt\right].\end{aligned}$$

We then identify easily the three cases specified by the Dynamic Programming Principle (17) and then we have $\phi_{\hat{i}}\left(\hat{h}\right) \ge w_{\hat{i}}\left(\hat{h}\right) + \eta\mathbb{E}^{(\hat{i},\hat{h})}\left[\frac{1-e^{-(r-\lambda)\gamma_\delta}}{(r-\lambda)}\right]$. We obtain the contradiction by noticing that $\mathbb{E}^{(\hat{i},\hat{h})}\left[\frac{1-e^{-(r-\lambda)\gamma_\delta}}{(r-\lambda)}\right] > 0$. ■

## 5 | APPROXIMATION OF SOLUTIONS

To solve the HJB equation (16) arising from the reduced stochastic control problem (13), we choose to use a deterministic approach based on a finite difference scheme which leads to the resolution of a controlled Markov chain problem, see for instance Kushner and Dupuis.[23] The convergence of the solution of the numerical scheme toward the solution of the HJB equation, when the space step goes to zero, can be shown using standard arguments, that is, it satisfies monotonicity, consistency and stability properties. Similar numerical schemes, involving a controlled Markov chain problem, are exploited in operational research, see for instance References 24–27.

We first reformulate the problem on a discretized grid. Let $N \in \mathbb{N}^*$ be the number of discretization points in our grid and $\delta > 0$ be the discretization step along the direction $h$. Of course we need a relation between $\bar{h}$ and $h_-$ such that the discretized grid make sense.* We define the space grid as $\mathcal{G}_\delta := \{0, \delta, 2\delta, \dots, \bar{h}-\delta, \bar{h}\}$.

For sake of readability we introduce the following quantities:

$$\begin{aligned}&h_1 := h + \delta, \quad h_2 := h - \delta,\\ &\mu_h^i := \mu + \sigma\gamma\rho - i\frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{h-h_0} - \beta\sqrt{2g(h-h_0)}.\end{aligned}$$

*The relation is that $\delta = \frac{q}{N}MCD\left(\bar{h}, h_-\right)$ for some positive integer $q$ In our example, see Table 1, we fix $N = 1000$, $\bar{h} = 100$ and $h_- = 50$. The discretization step is $\delta := \bar{h}/N = 1/10$, that is, $q = 2$; $N_+ = 0.8 * N = 800$ and $N_- = 0.5 * N = 500$; $h_+ = N_+ * \delta = 80$ and $h_- = N_- * \delta = 50$.

For $h$ in the space grid $\mathcal{G}_\delta$ we consider approximations of the following form:

$$\frac{\partial w}{\partial h}(h) \approx \frac{w(h+\delta)-w(h)}{\delta}\mathbb{1}_{\mu_h^i \geq 0} - \frac{w(h-\delta)-w(h)}{\delta}\mathbb{1}_{\mu_h^i < 0},$$
$$\frac{\partial^2 w}{\partial h^2}(h) \approx \frac{w(h+\delta)+w(h-\delta)-2w(h)}{\delta^2}.$$

Thus, using the above notations and applying a finite difference scheme, the HJB equation (16) can be formulated, for $i \in \{0,1\}$, as the following:

$$w_i(h) = \max\left\{\max_{\beta\in\{0,\overline{\beta}\}} \frac{\sum_{j=1}^{2} p_j w_i(h_j) + G_h^i \Delta t^\delta}{1+(\lambda-r)\Delta t^\delta}, w_{1-i}-\kappa\right\}, \quad \text{on} \quad \mathcal{G}_\delta \setminus \{0,\delta,\ \dots\ ,h_- - \delta, \overline{h}\}, \tag{22}$$

$$w_0(h) = \max_{\beta\in\{0,\overline{\beta}\}} \frac{\sum_{j=1}^{2} p_j w_0(h_j) + G_h^0 \Delta t^\delta}{1+(\lambda-r)\Delta t^\delta}, \quad \text{on} \quad \{\delta, 2\delta,\ \dots\ ,h_- - \delta\}, \tag{23}$$

where

$$p_1 := \frac{\sigma^2/2 + \delta(\mu_h^i)^+}{Q^\delta}, \quad p_2 := \frac{\sigma^2/2 + \delta(\mu_h^i)^-}{Q^\delta},$$
$$\Delta t^\delta := \frac{\delta^2}{Q^\delta}, \quad G_h^i := i\mathcal{E} - f\big((h-h_+)^+\big),$$
$$Q^\delta := |\mu_h^i|\delta + \sigma^2,$$

with the notation $(.)^+$ (resp. $(.)^-$) representing the positive (resp. negative) part of a given function. To compute explicitly the approximated solution of the latter discrete problem we use, for $n \in \mathbb{N}$, the following iterative scheme:

$$w_i^{(n+1)}(h) = \max\left\{\max_{\beta\in\{0,\overline{\beta}\}} \frac{\sum_{j=1}^{2} p_j w_i^{(n)}(h_j) + G_h^i \Delta t^\delta}{1+(\lambda-r)\Delta t^\delta}, w_{1-i}^{(n)}-\kappa\right\}, \text{ on } \mathcal{G}_\delta \setminus \{0,\delta,\ \dots\ ,h_- - \delta, \overline{h}\},$$
$$w_0^{(n+1)}(h) = \max_{\beta\in\{0,\overline{\beta}\}} \frac{\sum_{j=1}^{2} p_j w_0^{(n)}(h_j) + G_h^0 \Delta t^\delta}{1+(\lambda-r)\Delta t^\delta}, \quad \text{on} \quad \{\delta, 2\delta,\ \dots\ ,h_- - \delta\},$$
$$w_i^{(0)} \equiv 0.$$

Recalling the following boundary conditions:

$$w_0^{(n)}(\overline{h}) = 0; \qquad \frac{\partial w_0^{(n)}}{\partial h}(0) = 0,$$
$$w_1^{(n)}(\overline{h}) = 0; \qquad w_1^{(n)}(h_-) = v_0(h_-) - \kappa, \tag{24}$$

for $i \in \{0,1\}$ and $n \in \mathbb{N}$, the above iterative scheme is explicit and fully implementable on the enlarged grid $\mathcal{G}_\delta^+ := \{-\delta, 0,\ \dots\ , \overline{h}\}$.

Using the parameters defined in Table 1 (see the following section), about 7 s are necessary to obtain the approximated value functions and policy using Intel$^{\text{TM}}$Core i7 at 2.70 Ghz CPU with 8 Go of RAM.

## 6 | NUMERICAL RESULTS

We focus on a dam, which height with respect to the river level is lower than 100 m, that is the typical height. It is important also to remark that higher dam height are generally associated with small basins due to the mountain relief and also to nonsmooth fluctuation in the water level due to storms. To compare very large dams have an height less than

**TABLE 1** Central values for the parameters used for the numerical tests.

| Parameter | Meaning | Value |
|---|---|---|
| $\overline{h}$ | Maximal water level inside the dam | 100 |
| $h_+$ | Critical water level | 80 |
| $h_-$ | Minimal water level to produce electricity | 50 |
| $\overline{\beta}$ | Maximum flow of the spillover | 0.01 |
| $\mathcal{E}$ | Energy production normalized for unity of surface | 50 |
| $h_0$ | Turbine position with respect to the dam | −1 |
| $\chi$ | Standard waste for a turbine of type *Francis* | 0.05 |
| $\kappa$ | Switch cost | 0.1 |
| $\mu$ | Expected river flow level | 0.1 |
| $\sigma$ | Volatility for the river flow level | 0.05 |
| $\lambda$ | Return of standard price model | 0.1 |
| $\gamma$ | Volatility of standard price model | 0.05 |
| $r$ | Discount rate | 0.2 |
| $\rho$ | Correlation between Brownian motions | 0.04 |
| $g$ | Gravity acceleration | 9.806 |
| $S$ | Renormalized surface | 1 |

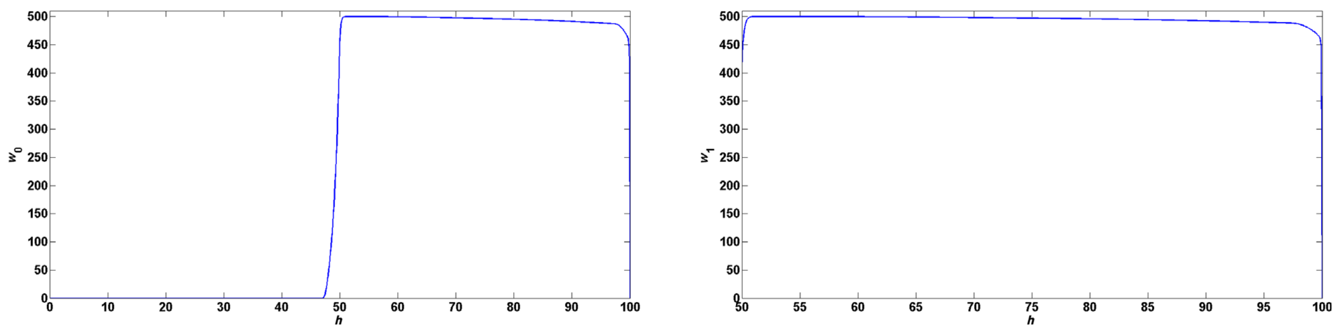


**FIGURE 1** Value functions $w_0$ and $w_1$.

200 m, for instance Three Gorges (181 m), Itaipu (196 m) and Hoover (180 m) and the tallest dam in the world is Jinping-I with 305-m height.

Hydraulic head, meaning the use-full height for electric production, is around 50% of the dam height, for instance Three Gorges (80.6 m) and Itaipu (118 m). The electricity production essentially depends on the water volume collected on the artificial lake and then on the geography of the dam site. For sake of readability, we renormalize the surface $S$ to the unit, that is the value function is expressed for unit of squared meter of the basin surface. The penalization function is assumed quadratic with a coefficient $10^{-3}$. The rest of the parameters are resumed in Table 1 where they are split according with their kind, that is, dam construction, hydro production setup, water evolution, electricity price model and physical constant. Values are chosen in a synthetic way such that we capture every aspect of the dynamics and are perfectly reasonable. Table resumes all involved parameters, their meaning and numerical value normalized for a unitary surface $S$. Value $k = 0.1$ means that the switching cost is equal to 10% of the price, $\mu = 0.1$ and $\sigma = 0.05$ means that after every unit of time, we expect an increase of the water level by 0.1 m with an error of 0.05. For instance if we consider a day to be our unit of time, after 10 days the water height in the dam will increase by 1 m. Numerical values for $\lambda$ and $\gamma$ are standards when dealing with Black–Scholes model for prices in (6), $\rho$ is the Brownian motion correlation, $r$ the discount rate and as usual $r$ has to be bigger than $\lambda$; $g$ is a physical constant of gravitational acceleration $g = 9.8067\ m/s^2$.

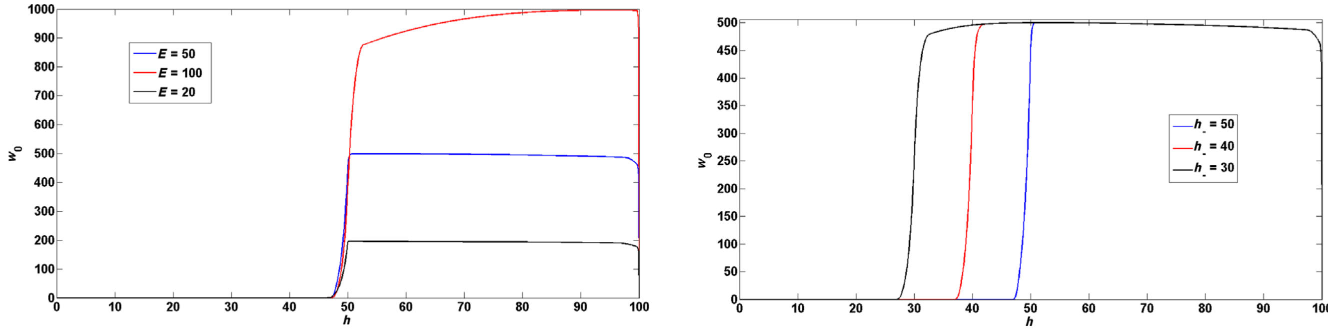


FIGURE 2 Value function $w_0$ for different values of the electricity cost $\mathcal{E}$ and the minimal height to produce electricity $h_{-}$ respectively.

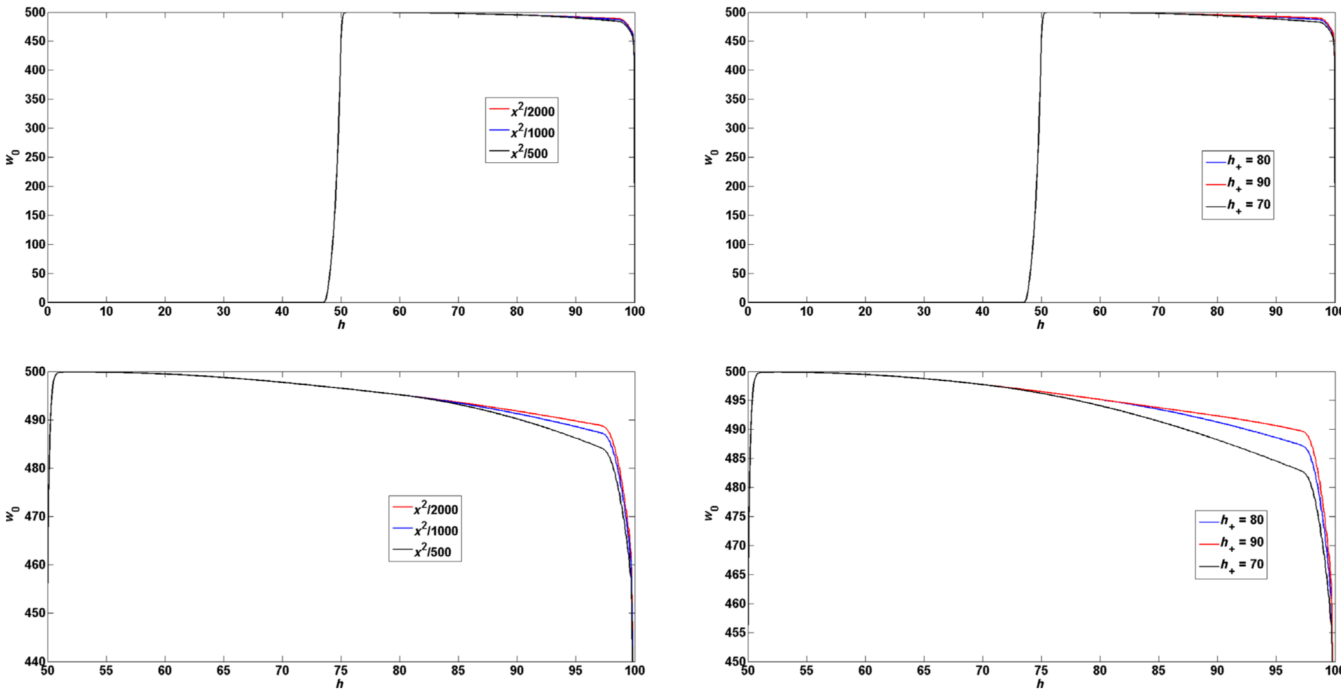


FIGURE 3 Value function $w_0$ for different values of the electricity cost $E$ and the minimal height to produce electricity $h_{-}$ respectively.

$\chi$ represents the waste level in a hydroelectric turbines,† $\mathcal{E}$ and $h_0$ are also physically reasonable and we chose in order to have a drift of an equivalent size with $\mu$.

Figure 1 shows the value function $w_0$ and $w_1$ for the two states, that is, closed and open turbines, as function of the water height in the reservoir. We remark that the value function $w_0$ increases abruptly near the minimal level to produce electricity $h_{-}$. After this level, the value function is relatively flat and increasing showing that the dam works in optimal condition up to a level where the combination of the quadratic penalization and the risk of dam failure force the opening of spillover and the value function will fall dramatically.

Figure 2 shows the behavior of value function for different values of the electricity revenue $E$ and the minimal high to produce electricity $h_{-}$. We remark that when the electricity increases, the value function increases and when the electricity revenue rises to 100, the value function is not longer flat but remains increasing almost up to the maximal level $\overline{h}$ due to the high potential energy of a reservoir with high free water. As a consequence, the manager opens the spillovers only

†Hydroelectric turbines are of three possible type: Francis, Pelton, and Kaplan which refers to three different ways to exploit the power of water. The Francis turbine was the first modern hydropower turbine and was invented by British-American engineer James Francis in 1849. Hydroelectric turbine runners can be categorized in various ways. The three common hydroelectric turbine designs are the Kaplan, Pelton, and Francis designs. There are *reaction* (Kaplan and Francis) and *impulse* (Pelton) turbines, which are pressure and pressureless, respectively.

when the water almost reaches the dam head and the risk of the dam failure is really high. This result highlights the importance for the regulator to define a fair penalization in order to incentive the manager to keep the water level in a reasonable interval. This is the case when the value function is relatively flat for a large interval inside $[h_-, h^+]$ and far from $\overline{h}$. The impact of the minimal height to produce electricity $h_-$ is quite standard pushing right the value function since it is impossible to switch on the turbine before $h_-$. There is no impact on the value function far from $h_-$ showing that this level is a consequence of the physical/geographical setup of the dam.

We now focus on the penalization term. In our setup, it is quadratic and it is then controlled by two parameters, the coefficient of the quadratic term and the minimal height $h_+$ at which the penalization is implemented. Figure 3 shows that the impact on the value function is relatively small an concentrated on the last interval $[h_+, \overline{h}]$. In particular, the second line of Figure 3 is a zoom of the value function in order to evaluate the distance between the value function with the three different parameters. We easily remark that a change on the minimal height $h_+$ has a larger impact compared with the quadratic term.

## 7 | CONCLUDING REMARKS

Dam management plays a crucial role for electricity production. Hydropower production importance increases due to the necessity to reduce coal/oil power for ecological reasons and for the instability of solar/wind power production. The electricity price has huge impacts on economics.

This paper mainly contributes to dam management addressing the issue of evaluate the optimal policy of open and close the turbine to produce electricity. In particular, we focus on physical aspects of hydro-power generation combining with regulation/control constraints. Up to our knowledge, this is the first work highlighting the physical constraints of dam management. Under mild hypotheses, we showed that the stochastic fluctuation of electricity prices has no impact on the optimal policy that depends mainly on the heigh of the water inside the dam.

Our setup gives birth to a switching stochastic control problem that has shown a large and increasing financial interest mainly to describe productivity/debt management, see References 14–16, 18, and 19 and also for other common resources see References 25 and 26. Our analysis evaluates the sensitivity of electric interest with respect to the main physical and regulation parameters.

Our work, rather than herald the end of the debate about the optimal management of dam for hydro-power production, opens different questions about the evolution of water inside the dam due to the regulation constraints. Moreover the climate change will dramatically modify the evolution of the water inside the artificial lakes and the request of agriculture is expected to increase. We can then forecast an increasing role of government regulation into the dam management. We postpone the modeling and the analysis of the impact of climate change in a future paper.

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### DATA AVAILABILITY STATEMENT
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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