

1 **Bezout-like polynomial equations associated with dual**
2 **univariate interpolating subdivision schemes**

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7 **Abstract** The algebraic characterization of dual univariate interpolating subdivi-
8 sion schemes is investigated. Specifically, we provide a constructive approach for
9 finding dual univariate interpolating subdivision schemes based on the solutions
10 of certain associated polynomial equations. The proposed approach also makes it
11 possible to identify conditions for the existence of the sought schemes.

12 **Keywords** Bezout equation; Univariate dual subdivision; Higher arity; Interpo-
13 lation

14 **Mathematics Subject Classification (2000)** 65F05 · 68W30 · 65D05 · 65D17

15 **1 Introduction**

16 Subdivision schemes are useful tools for the fast generation of graphs of functions,
17 smooth curves and surfaces by the application of iterative refinements to an ini-
18 tial set of discrete data. The major fields of application of subdivision schemes
19 are Computer Graphics and Animation, Computer-Aided Geometric Design and
20 Signal/Image Processing, but a further motivation for their study is also their
21 close relation to multiresolution analysis and wavelets. The last connection was
22 especially investigated in the case of interpolating subdivision schemes and it was
23 pointed out that the interpolatory subdivision schemes of Dubuc-Deslauriers [11]
24 are connected to orthonormal wavelets of Daubechies [6, 23]. Interpolating subdivi-
25 sion schemes were also deeply studied, because they are considered to be very
26 efficient in representing smooth curves and surfaces passing through a given set
27 of points. In fact, after five or six subdivision iterations only, they are capable of

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28 providing the refined set of points needed to represent on the screen the desired
 29 smooth limit shape interpolating the given data. The main properties of inter-
 30 polating subdivision schemes were investigated over the past 20 years by several
 31 researchers (see, e.g., [12, 15, 18]) and many approaches were proposed to design
 32 their refinement rules. However, as far as we are aware, before the papers [25, 26],
 33 no one ever tried to construct interpolating subdivision schemes that do not sat-
 34 isfy the stepwise interpolation property and are thus not defined via refinement
 35 rules that at each stage of the iteration leave the previous set of points unchanged.
 36 Stepwise interpolating subdivision schemes - also known as primal interpolating
 37 subdivision schemes [11, 17] - are defined by finite subdivision masks of odd width
 38 that contain as a special submask the sequence $\delta = \{\delta_{0,j}, j \in \mathbb{Z}\}$. Differently, mem-
 39 bers of the most recently introduced class of non-stepwise interpolating subdivision
 40 schemes -also known as dual interpolating subdivision schemes- are characterized
 41 by finite masks with an even number of entries that do not necessarily contain as a
 42 special submask the δ sequence. A first algorithm to construct dual interpolating
 43 quaternary schemes was proposed in [25] and successively extended to arbitrary
 44 arity greater than two in [26]. Precisely, in the latter it was shown that, under
 45 some suitable auxiliary assumptions, the coefficients of the subdivision mask of
 46 a dual interpolating scheme can be (possibly) determined by the solution of an
 47 associated rectangular linear system. This system can be clearly inconsistent for
 48 some choices of input data and/or size (length) of the mask. For a given input
 49 data set the approach taken in [26] consists of an exhaustive analysis of the as-
 50 sociated linear systems of increasing sizes in order to identify possible consistent
 51 configurations.

52 In this paper we pursue a different method for constructing dual interpolating
 53 subdivision schemes based on the reduction of the matrix formulation into a func-
 54 tional setting to solving a certain Bezout-like polynomial equation. The method
 55 makes it possible to address the consistency issues by detecting suitable condi-
 56 tions on the input data which guarantee the existence of a dual interpolating
 57 scheme. Additionally, it yields a full characterization of the set of solutions which
 58 can be exploited to fulfil additional demands and properties of the solution mask.
 59 From the point of view of applications, such a computational approach allows
 60 the user to meet specific requests in terms of polynomial reproduction, support
 61 size and regularity. Even though a general result concerning convergence and/or
 62 smoothness of a dual interpolating subdivision scheme is not yet available, in all
 63 the considered examples the regularity analysis is done via joint spectral radius
 64 techniques (see [4, 20, 22]), rather than by means of the restricted spectral radius
 65 approach (see, e.g., [3]), and the best Hölder exponent for each scheme is computed
 66 up to the 15th decimal digit.

67 2 Background and notation

68 In this section we briefly recall some needed background on subdivision schemes
 69 of arbitrary arity $m \in \mathbb{N}$, $m \geq 2$.

70 Any linear, stationary subdivision scheme is identified by a *refinement mask*
 71 $\mathbf{a} := \{a_i \in \mathbb{R}, i \in \mathbb{Z}\}$ that is usually assumed to have finite support, *i.e.* to satisfy
 72 $a_i = 0$ for $i \notin [-L, L]$ for suitable $L > 0$.

The *subdivision scheme* identified by the mask \mathbf{a} consists of the subsequent application of the *subdivision operator*

$$S\mathbf{a} : \ell(\mathbb{Z}) \rightarrow \ell(\mathbb{Z}), \quad (S\mathbf{a} \mathbf{p})_i := \sum_{j \in \mathbb{Z}} a_{i-mj} p_j, \quad i \in \mathbb{Z},$$

73 which provides the linear rules determining the successive refinements of the ini-
74 tial sequence of discrete data $\mathbf{p} := (p_i \in \mathbb{R}, i \in \mathbb{Z}) \in \ell(\mathbb{Z})$. Introducing the notation
75 $\mathbf{p}^{(0)} := \mathbf{p}$, we can thus describe the subdivision scheme as an iterative method that
76 at the k -th step generates the refined scalar sequence

$$\mathbf{p}^{(k+1)} := S\mathbf{a} \mathbf{p}^{(k)}, \quad k \geq 0. \quad (1)$$

Attaching the data $p_i^{(k)}$ generated at the k -th step to the parameter values $t_i^{(k)}$ with

$$t_i^{(k)} < t_{i+1}^{(k)}, \quad \text{and} \quad t_{i+1}^{(k)} - t_i^{(k)} = m^{-k}, \quad k \geq 0$$

77 (these are usually set as $t_i^{(k)} := m^{-k}i$) we see that the subdivision process generates
78 denser and denser sequences of data so that a notion of convergence can be estab-
79 lished by taking into account the piecewise linear function $P^{(k)}$ that interpolates
80 the data, namely

$$P^{(k)}(t_i^{(k)}) = p_i^{(k)}, \quad P^{(k)}|_{[t_i^{(k)}, t_{i+1}^{(k)}]} \in \Pi_1, \quad i \in \mathbb{Z}, \quad k \geq 0,$$

81 where Π_1 is the space of linear polynomials. If the sequence of the continuous
82 functions $\{P^{(k)}, k \geq 0\}$ converges uniformly, then we denote its limit by

$$f_{\mathbf{p}} := \lim_{k \rightarrow \infty} P^{(k)}$$

and say that $f_{\mathbf{p}}$ is the *limit function* of the subdivision scheme based on the rule (1) for the data \mathbf{p} [2]. When $\mathbf{p} = \boldsymbol{\delta}$, $f_{\boldsymbol{\delta}}$ is called *basic limit function*.

The analysis of convergence of a subdivision scheme can be accomplished by studying the properties of the so-called *symbol* of the subdivision mask [14]. The symbol of a finitely supported sequence \mathbf{a} is defined as the Laurent polynomial

$$a(z) := \sum_{i \in \mathbb{Z}} a_i z^i, \quad z \in \mathbb{C} \setminus \{0\}.$$

83 Besides convergence and smoothness, many other properties of a subdivision scheme,
84 like polynomial generation and reproduction, can be checked by investigating al-
85 gebraic conditions on the subdivision symbol [7]. While the term *polynomial gen-
86 eration* refers to the capability of the subdivision scheme of providing polynomials
87 as limit functions, with *polynomial reproduction* we mean the capability of a subdivi-
88 sion scheme of reproducing in the limit exactly the same polynomial from which
89 the data are sampled. The property of polynomial reproduction is very impor-
90 tant since strictly connected to the approximation order of the subdivision scheme
91 and to its regularity [5, 16]. With respect to the capability of reproducing poly-
92 nomials up to a certain degree, the standard parametrization (corresponding to
93 the choice $t_i^{(k)} := m^{-k}i$, $i \in \mathbb{Z}$) is not always the optimal one. Indeed, the choice
94 $t_i^{(k)} := m^{-k}(i + \sigma/(m-1))$ with $\sigma = a^{(1)}(1)/m$, turns out to be the recommended
95 selection [8]. The subdivision schemes for which $\sigma \in \mathbb{Z}$ are termed *primal*, whereas

the ones for which $\sigma \in (2\mathbb{Z} + 1)/2$ are called *dual*. The target of this work are dual schemes. While dual approximating schemes were investigated extensively (see, e.g., [8, 13] and references therein), to the best of our knowledge dual interpolating schemes were only considered in the recent papers [25, 26]. However, as already acknowledged in [25], the open problem treated in these papers was suggested by Malcolm Sabin, who has the merit of being the first who foresaw the existence of dual m -ary schemes (with $m > 2$) that are capable of interpolating the initial data.

3 The proposed approach

The aim of this section is to investigate the algebraic characterization of univariate dual interpolating subdivision schemes of arity m . According to the results shown in [26], the construction of such schemes requires as input the desired degree of polynomial reproduction (denoted in the following by $d - 1$, $d \in \mathbb{N}$) and some samples of the resulting basic limit function f_δ , i.e.,

$$f_\delta\left(\frac{1}{2} + \ell\right) = \varphi\left(\frac{1}{2} + \ell\right), \quad \forall \ell \in \mathbb{Z}, \quad (2)$$

for a given $\varphi: (2\mathbb{Z} + 1)/2 \rightarrow \mathbb{R}$. A similar procedure was investigated in [9, 10], where the samples of the basic limit function at the integers were required: here instead the samples at the integers are fixed to be the δ sequence and information about the samples at the half-integers are required.

More specifically, in [26] it is seen that taking Fourier transforms on both sides of the refinement equation for the basic limit function f_δ allows one to describe the mask of dual interpolatory schemes in a matrix setting in terms of the solution of certain bi-infinite Toeplitz-like linear systems in banded form. In this paper we exploit the interplay between the functional and the matrix settings into more details. In particular, from the matrix setting we come back to the functional one by relying upon the connection of Toeplitz-like systems with corresponding Bezout-like polynomial equations. This connection yields a constructive approach to determine the associated symbols. Moreover, the proposed approach also makes it possible to identify conditions for the existence of the sought dual interpolatory schemes. In the following, to simplify the presentation, we distinguish between the odd and even arity cases.

3.1 The odd arity case

Now let us consider the solution of the linear system (35) in [26] for the case where m is an odd integer. The system is defined as follows:

$$M\mathbf{a} = \mathbf{c}, \quad M = (\mu_{i,j})_{i,j \in \mathbb{Z}}, \quad \mathbf{c} = (c_i)_{i \in \mathbb{Z}},$$

129 where

$$\mu_{i,j} = \begin{cases} \varphi\left(\frac{i+1}{2} - j\right), & \text{if } i \in 2m\mathbb{Z}, \\ 1, & \text{if } i \in m(2\mathbb{Z} + 1), j = \frac{i+1}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

$$c_i = \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\frac{i}{2m}\right), & \text{if } i \in m(2\mathbb{Z} + 1), \\ 0, & \text{otherwise.} \end{cases}$$

131 By suppressing the zero rows in both M and \mathbf{c} we obtain the equivalent linear
132 system

$$\widehat{M}\mathbf{a} = \widehat{\mathbf{c}}, \quad \widehat{M} = (\widehat{\mu}_{i,j})_{i,j \in \mathbb{Z}}, \quad \widehat{\mathbf{c}} = (\widehat{c}_i)_{i \in \mathbb{Z}}, \quad (3)$$

133 where

$$\widehat{\mu}_{i,j} = \begin{cases} \varphi\left(\frac{im+1}{2} - j\right), & \text{if } \text{mod}(i, 2) = 0, \\ 1, & \text{if } \text{mod}(i, 2) = 1, j = \frac{im+1}{2}, \end{cases}$$

$$\widehat{c}_i = \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\frac{i}{2}\right), & \text{if } \text{mod}(i, 2) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

135 The interplay between computations with polynomials and Toeplitz-like matrices
136 can be exploited to recast the solution of the linear system (3) in terms of solving
137 an associated Bezout-like polynomial equation. Indeed from the proof of Theorem
138 4.1 in [26] one deduces that the entries of the unknown vector \mathbf{a} satisfy

$$\begin{cases} \sum_{\alpha \in m(2\mathbb{Z}+1)} \varphi\left(\frac{\alpha}{2m}\right) z^\alpha = \sum_{\alpha \in m(2\mathbb{Z}+1)} a_{\frac{\alpha+1}{2}} z^\alpha, \\ 1 = \sum_{\alpha \in 2m\mathbb{Z}} \sum_{\beta \in \mathbb{Z}} a_\beta \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^\alpha, \end{cases} \quad (4)$$

139 which implies

$$\begin{cases} a_{mi + \frac{m+1}{2}} = \varphi\left(\frac{2i+1}{2}\right), & i \in \mathbb{Z}, \\ 1 - \sum_{\alpha \in 2m\mathbb{Z}} \sum_{\beta \in m\mathbb{Z} + \frac{m+1}{2}} a_\beta \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^\alpha = \\ \sum_{\alpha \in 2m\mathbb{Z}} \sum_{\substack{\beta \in \mathbb{Z} \\ \text{mod}(m, \beta) \neq \frac{m+1}{2}}} a_\beta \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^\alpha. \end{cases} \quad (5)$$

140 The system (5) can be rewritten into a more compact form by using the decom-
 141 position of $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$ that involves the sub-symbols of the scheme given
 142 by

$$a(z) = \sum_{i=0}^{m-1} a_i (z^m)^i z^i, \quad a_\ell(z) = \sum_{i \in \mathbb{Z}} a_{mi+\ell} z^i, \quad 0 \leq \ell \leq m-1. \quad (6)$$

143 Let us introduce the corresponding decomposition of the Laurent polynomial
 144 $\phi(z) = \sum_{\ell \in \mathbb{Z}} \varphi\left(\frac{1}{2} + \ell\right) z^\ell$ defined by

$$\phi(z) = \sum_{i=0}^{m-1} \phi_i(z^m) z^{-i}, \quad \phi_\ell(z) = \sum_{i \in \mathbb{Z}} \varphi\left(\frac{2mi+1}{2} - \ell\right) z^i, \quad 0 \leq \ell \leq m-1. \quad (7)$$

145 The first equation of (5) determines

$$a_{\frac{m+1}{2}}(z) = \phi(z). \quad (8)$$

146 Then the second equation can be read as follows

$$1 - a_{\frac{m+1}{2}}(z^m) \phi_{\frac{m+1}{2}}(z^m) = \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} a_i(z^m) \phi_i(z^m)$$

147 or, equivalently,

$$1 - \phi(z) \phi_{\frac{m+1}{2}}(z) = \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} a_i(z) \phi_i(z). \quad (9)$$

148 Our computational task is therefore reduced to computing a Laurent polynomial
 149 $a(z)$ defined as in (6) satisfying the Bezout-like polynomial equation (9). It is quite
 150 natural for convergence and reproducibility issues to impose some other constraints
 151 of the form

$$a_i(1) = 1, \quad 0 \leq i \leq m-1, \quad (10)$$

$$a(z) = \left(\frac{1+z+\dots+z^{m-1}}{m} \right)^d b(z),$$

152 for some $b(z) \in \mathbb{R}[z, z^{-1}]$ with $b(\xi_k) \neq 0$, $1 \leq k \leq m-1$, where $\xi_k = e^{2\pi i k/m}$.
 153 Our proposed construction of such a polynomial $a(z)$ works under some additional
 154 assumptions on the input data $\{\varphi((2k+1)/2)\}_{k=-\kappa}^{\kappa-1}$ encoded in the function $\phi(z)$.
 155 More specifically:

156 **ASSUMPTION 1** : We suppose that

$$1 - z\phi(z^2) = (z-1)^d \gamma(z), \quad (11)$$

157 for a certain $\gamma(z) \in \mathbb{R}[z, z^{-1}]$ the ring of Laurent polynomials in z, z^{-1} over \mathbb{R} .

158 **ASSUMPTION 2** : We suppose that

$$g(z) := \gcd\left\{\phi_0(z), \dots, \phi_{\frac{m-1}{2}}(z), \phi_{\frac{m+3}{2}}(z), \dots, \phi_{m-1}(z)\right\},$$

159 divides $1 - \phi(z) \phi_{\frac{m+1}{2}}(z)$ and is such that $g(1) \neq 0$.

160 Assumption 1 is necessary in order to achieve polynomial reproduction of order
 161 d . Indeed, by definition of polynomial reproduction, we should have, for every
 162 polynomial p of degree $d - 1$,

$$p(x) = \sum_{k \in \mathbb{Z}} p(k) f_{\delta}(x - k), \quad \forall x \in \mathbb{R}.$$

163 In particular, taking $x = 1/2 + i$, $i \in \mathbb{Z}$, the (compactly supported) vector $[f_{\delta}(1/2 +$
 164 $\ell) = \varphi(1/2 + \ell)]_{\ell \in \mathbb{Z}}$ defines column-wise a (bandlimited) Toeplitz matrix \mathbf{T} such
 165 that, for every polynomial p of degree $d - 1$,

$$\mathbf{T} [p(k)]_{k \in \mathbb{Z}} = \left[p\left(\frac{1}{2} + i\right) \right]_{i \in \mathbb{Z}}.$$

166 Thus, one can naively define a binary primal interpolating refinement mask as

$$\mathbf{r} = \{ r_i \in \mathbb{R}, i \in \mathbb{Z} \}, \quad \text{with} \quad r_i = \begin{cases} \varphi\left(\frac{1}{2} + \ell\right), & \text{if } i = 2\ell + 1, \\ \delta_{0,\ell}, & \text{if } i = 2\ell, \end{cases}$$

167 which is not guaranteed to be associated with a convergent subdivision scheme,
 168 but it always satisfies

$$\sum_{i \in \mathbb{Z}} r_i z^i = (z + 1)^d \tilde{\gamma}(z), \quad (12)$$

169 for some Laurent polynomial $\tilde{\gamma}(z)$. Now it is easy to check that, replacing z with
 170 $-z$ in (12) and using (7), one indeed obtains Assumption 1, i.e., $1 - z\phi(z^2) =$
 171 $(z - 1)^d \gamma(z)$, with $\gamma(z) = (-1)^d \tilde{\gamma}(-z)$.

172 *Remark 1* The previous observation is also the reason why a suitable way to con-
 173 struct the starting sequence $\{\varphi((2k+1)/2)\}_{k \in \mathbb{Z}}$ is using the mask of a binary primal
 174 interpolating scheme with the desired reproduction properties. In the following Ex-
 175 ample 1 and 2 ((26) and (42) respectively) we choose the mask of the binary 6-point
 176 Dubuc-Deslauriers interpolating scheme [11] since it forms the shortest symmetric
 177 sequence that guarantees polynomial reproduction of order 6.

178 As for Assumption 2, requiring $g(z)$ to divide $1 - \phi(z)\phi_{\frac{m+1}{2}}(z)$ is also necessary
 179 due to equation (9), while asking $g(1) \neq 0$ is only a sufficient condition as it will
 180 be clear in the following. When 1 is a root of $g(z)$, the construction we propose
 181 is still viable but a price has to be paid in terms of polynomial reproduction (see
 182 Remark 2).

183 Under Assumption 1 and Assumption 2 our composite approach for computing
 184 $a(z)$ proceeds by the following steps. The first step consists of determining the
 185 values $a_i^{(s)}(1)$, $0 \leq i \leq m - 1$, $s = 0, \dots, d - 1$. From (10) one gets immediately
 186 $a_i^{(0)}(1) = a_i(1) = 1$, $0 \leq i \leq m - 1$. Due to (8) and Assumption 1, we have that

$$1 - z\phi(z^2) = 1 - za_{\frac{m+1}{2}}(z^2) = (z - 1)^d \gamma(z),$$

187 from which we can compute the values of $a_{\frac{m+1}{2}}(z)$ and its derivatives at $z = 1$.

188 **Theorem 1** Under Assumption 1, it holds

$$\begin{cases} a_{\frac{m+1}{2}}(1) = \phi(1) = 1, \\ a_{\frac{m+1}{2}}^{(k)}(1) = \phi^{(k)}(1) = (-1)^k \frac{(2k-1)!!}{2^k}, \quad 1 \leq k \leq d-1. \end{cases}$$

189 *Proof* Substituting $z = \sqrt{w}$ in (11), we get

$$\phi(w) - w^{-1/2} = \frac{(1 - \sqrt{w})^d (-1)^{d+1} \gamma(\sqrt{w})}{\sqrt{w}}.$$

190 The proof easily follows by differentiating this relation at $w = z = 1$. \square

191 The remaining unknowns $a_i^{(s)}(1)$, $0 \leq i \leq m-1$, $i \neq (m+1)/2$, $s = 1, \dots, d-1$,
192 are computed by solving the linear system obtained by differentiation of (10).
193 Specifically, by differentiating s times the expression of $a(z)$ in (6) with respect to
194 the variable z we find that

$$a^{(s)}(z) = \sum_{i=0}^{m-1} \sum_{p=0}^s \frac{a_i^{(p)}(z^m)}{p!} \left(\sum_{j=\max\{s-i, p\}}^s \binom{s}{j} A_{j,p}(z) \frac{i!}{(i-(s-j))!} z^{i-(s-j)} \right), \quad (13)$$

195 where $A_{j,p}(z)$ are polynomials defined by Hoppe's formula (see, e.g., [21]) for the
196 differentiation of composite function according to

$$A_{j,p}(z) = \sum_{\ell=0}^j \binom{p}{\ell} (-f(z))^{p-\ell} \frac{d^j}{dz^j} (f(z))^\ell, \quad f(z) = z^m.$$

197 If $\xi_k = e^{2\pi i k/m}$, $1 \leq k \leq m-1$, are the m -th roots of unity, then from (10) it
198 follows that $a^{(s)}(\xi_k) = 0$, $s = 0, \dots, d-1$, $1 \leq k \leq m-1$. In the view of (13) this
199 implies that the values $a_i^{(s)}(1)$, $0 \leq i \leq m-1$, $i \neq (m+1)/2$, $s = 1, \dots, d-1$, can
200 be computed recursively by solving

$$\sum_{i=0}^{m-1} \sum_{p=0}^s \frac{a_i^{(p)}(1)}{p!} \left(\sum_{j=\max\{s-i, p\}}^s \binom{s}{j} A_{j,p}(\xi_k) \frac{i!}{(i-(s-j))!} \xi_k^{i-(s-j)} \right) = 0,$$

201 with $1 \leq k \leq m-1$. The system can be expressed in matrix form as

$$m^s \mathcal{M} \left[a_0^{(s)}(1), \dots, a_{\frac{m-1}{2}}^{(s)}(1), a_{\frac{m+3}{2}}^{(s)}(1), \dots, a_{m-1}^{(s)}(1) \right]^T = \mathbf{b}_s, \quad (14)$$

202 with

$$\mathcal{M} := \mathcal{D} \left(\xi_1^{(m-1)s}, \dots, \xi_{m-1}^{(m-1)s} \right) \mathcal{V}(\xi_1, \dots, \xi_{m-1}).$$

203 Here $\mathcal{D}(\mathbf{v})$, $\mathbf{v} = [v_1, \dots, v_{m-1}]^T$, is the diagonal matrix with diagonal entries
204 v_k , $1 \leq k \leq m-1$, $\mathcal{V}(\xi_1, \dots, \xi_{m-1})$ is the Vandermonde matrix with nodes ξ_k ,
205 $1 \leq k \leq m-1$, and

$$\begin{aligned} (\mathbf{b}_s)_k &= - \sum_{i=0}^{m-1} \sum_{p=0}^{s-1} \frac{a_i^{(p)}(1)}{p!} \left(\sum_{j=\max\{s-i, p\}}^s \binom{s}{j} A_{j,p}(\xi_k) \frac{i!}{(i-(s-j))!} \xi_k^{i-(s-j)} \right) \\ &= - \frac{a_{\frac{m+1}{2}}^{(s)}(1)}{s!} A_{s,s}(\xi_k) \xi_k^{\frac{m+1}{2}}, \quad 1 \leq k \leq m-1. \end{aligned}$$

206 Since ξ_k , $1 \leq k \leq m-1$, are distinct and non-zero, the coefficient matrix is
 207 nonsingular and $a_i^{(s)}(1)$, $0 \leq i \leq m-1$, $i \neq (m+1)/2$, are uniquely determined.

208 Once the quantities $a_i^{(s)}(1)$, $0 \leq i \leq m-1$, $s = 0, \dots, d-1$, are calculated, then
 209 the sub-symbols $a_i(z)$, $0 \leq i \leq m-1$, $i \neq (m+1)/2$, can be represented as follows

$$a_i(z) = 1 + \sum_{j=1}^{d-1} \frac{a_i^{(j)}(1)}{j!} (z-1)^j + (z-1)^d \hat{a}_i(z) = \check{a}_i(z) + (z-1)^d \hat{a}_i(z), \quad (15)$$

210 for suitable $\hat{a}_i(z) \in \mathbb{R}[z, z^{-1}]$. This representation is exploited in the second step
 211 to find a solution of (9). Combining (9) with (15) we obtain

$$1 - \phi(z)\phi_{\frac{m+1}{2}}(z) - \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z)\phi_i(z) = (z-1)^d \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \hat{a}_i(z)\phi_i(z). \quad (16)$$

212 Thus, setting

$$\theta(z) := 1 - \phi(z)\phi_{\frac{m+1}{2}}(z) - \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z)\phi_i(z), \quad (17)$$

213 the condition

$$\theta^{(s)}(1) = 0, \quad 0 \leq s \leq d-1,$$

214 is needed, but it is always guaranteed by the following result.

215 **Theorem 2** *The function $\theta(z)$ in (17) satisfies $\theta^{(s)}(1) = 0$ for $s = 0, \dots, d-1$.*

216 *Proof* Let us introduce the truncated representation $\check{a}(z)$ of the symbol $a(z)$, that
 217 is,

$$\check{a}(z) = \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z^m)z^i + \phi(z^m)z^{\frac{m+1}{2}},$$

218 and consider the auxiliary function $q(z) = z^{-\frac{m+1}{2}}\check{a}(z^2)z\phi(z^2)$. From (11) it fol-
 219 lows that $q(z) = z^{-\frac{m+1}{2}}\check{a}(z^2) - z^{-\frac{m+1}{2}}\check{a}(z^2)(-1)^d(1-z)^d\gamma(z)$. By construction $\check{a}(z)$
 220 satisfies relations (10). By using the representation of $\check{a}(z)$ provided by (10) this
 221 gives

$$q(z) = z^{-\frac{m+1}{2}}\check{a}(z^2) + \frac{(1-z^m)^d(1+z^m)^d}{(1+z)^d}\hat{\rho}(z)$$

222 with $\hat{\rho}(z) \in \mathbb{R}[z, z^{-1}]$. Observe that

$$z^{-\frac{m+1}{2}}\check{a}(z^2) = z^{\frac{m+1}{2}}\phi(z^{2m}) + \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z^{2m})z^{2i-\frac{m+1}{2}},$$

223 and, hence,

$$q(z) = z^{\frac{m+1}{2}}\phi(z^{2m}) + \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z^{2m})z^{2i-\frac{m+1}{2}} + \frac{(1-z^m)^d(1+z^m)^d}{(1+z)^d}\hat{\rho}(z). \quad (18)$$

224 Moreover it can be easily seen that the two sets $[0, m-1] \cap \mathbb{N}$ and $\{n \in \mathbb{N}: n = 2i -$
 225 $(m+1)/2 \pmod{m}, 0 \leq i \leq m-1\}$ coincide. Besides this, by direct multiplication
 226 of $a(z^2)$ and $\phi(z^2)$, we can write

$$q(z) = z^{\frac{1-m}{2}} \left(\sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z^{2m}) \phi_i(z^{2m}) + \phi(z^{2m}) \phi_{\frac{m+1}{2}}(z^{2m}) \right) + \quad (19)$$

$$+ z^{\frac{1-m}{2}} \sum_{\substack{0 \leq i, j \leq m-1 \\ i \neq j}} z^{2(i-j)} \eta_{i,j}(z^{2m}),$$

227 for suitable Laurent polynomials $\eta_{i,j}(z) \in \mathbb{R}[z, z^{-1}]$. Since $(1-m)/2 \equiv (m+1)/2$
 228 \pmod{m} the class of integers congruent to $(1-m)/2$ modulo m is $\{n \in \mathbb{Z}: n =$
 229 $(1-m)/2 + \ell m, \ell \in \mathbb{Z}\}$. It follows that $n = (1-m)/2 + 2(i-j)$, $i \neq j$, $0 \leq i, j \leq m-1$,
 230 is such that $n \not\equiv (1-m)/2 \pmod{m}$. Hence, by comparison of classes mod m in
 231 (18) and (19), we obtain that

$$z^m \phi(z^{2m}) = \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z^{2m}) \phi_i(z^{2m}) + \phi(z^{2m}) \phi_{\frac{m+1}{2}}(z^{2m}) + (1-z^m)^d \tilde{\rho}(z),$$

232 for some $\tilde{\rho}(z) \in \mathbb{R}[z, z^{-1}]$. From (11) this implies that

$$\sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z^2) \phi_i(z^2) + \phi(z^2) \phi_{\frac{m+1}{2}}(z^2) = 1 + (1-z)^d \rho(z), \quad \rho(z) \in \mathbb{R}[z, z^{-1}],$$

233 which concludes the proof. \square

234 Theorem 2, along with Assumption 2, guarantees the existence of $\hat{\theta}(z) \in \mathbb{R}[z, z^{-1}]$
 235 such that

$$\theta(z) = (z-1)^d g(z) \hat{\theta}(z). \quad (20)$$

236 Thus, due to (16), the polynomial corrections $\hat{a}_i(z)$, $0 \leq i \leq m-1$, $i \neq (m+1)/2$,
 237 must satisfy the Bezout equation

$$\hat{\theta}(z) = \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \hat{a}_i(z) \frac{\phi_i(z)}{g(z)}. \quad (21)$$

238 Under Assumption 2 the Laurent polynomials $\phi_i(z)/g(z)$, $i \neq (m+1)/2$, are rela-
 239 tively prime and thus equation (21) is solvable. In particular, following [19] every
 240 solution of (21) can be written as

$$\hat{a}_i(z) = \tilde{a}_i(z) + \sum_{\substack{j=i+1 \\ j \neq \frac{m+1}{2}}}^{m-1} H_{i,j}(z) \frac{\phi_j(z)}{g(z)} - \sum_{\substack{j=0 \\ j \neq \frac{m+1}{2}}}^{i-1} H_{j,i}(z) \frac{\phi_j(z)}{g(z)}, \quad (22)$$

241 where $\{\tilde{a}_i(z), i \neq (m+1)/2\}$ is a particular solution of (21) and $H_{i,j}(z)$ is any
 242 element of $\mathbb{R}[z, z^{-1}]$. Upper bounds for the minimal length of the coefficient
 243 vectors associated to the solution of (21) are known a priori [19]. Using these
 244 bounds the computation of a particular solution $\tilde{a}_i(z)$, $0 \leq i \leq m-1$, $i \neq \frac{m+1}{2}$,
 245 reduces to solving a square linear system.

246 *Remark 2* If $g(z) = (z-1)^q \hat{g}(z)$ for some $q \in \mathbb{N}$, $q < d$, and $\hat{g}(z) \in \mathbb{R}[z, z^{-1}]$, with
 247 $\hat{g}(1) \neq 0$, then the result of Theorem 2 is unchanged but, differently from (20), we
 248 can only factorize $\theta(z)$ as

$$\theta(z) = (z-1)^{d-q} g(z) \hat{\theta}(z).$$

249 Thus, in this case, one should consider

$$a_i(z) = 1 + \sum_{j=1}^{d-q-1} \frac{a_i^{(j)}(1)}{j!} (z-1)^j + (z-1)^{d-q} \hat{a}_i(z) = \check{a}_i(z) + (z-1)^{d-q} \hat{a}_i(z), \quad (23)$$

250 instead of (15), and the illustrated procedure will lead to a symbol $a(z)$ of the
 251 form

$$a(z) = \left(\frac{1+z+\dots+z^{m-1}}{m} \right)^{d-q} b(z) \quad (24)$$

252 rather than (10). This means that the scheme associated to $a(z)$ would reproduce
 253 only polynomials up to degree $d-q-1$.

254 *Remark 3* Combining (6), (15) and (22), we get

$$\begin{aligned} a(z) = & \sum_{i=0}^{m-1} \check{a}_i(z^m) z^i + (z^m - 1)^d \sum_{i=0}^{m-1} \tilde{a}_i(z^m) z^i + \\ & + (z^m - 1)^d \sum_{i=0}^{m-1} \left(\sum_{\substack{j=i+1 \\ j \neq \frac{m+1}{2}}}^{m-1} H_{i,j}(z^m) \frac{\phi_j(z^m)}{g(z^m)} - \sum_{\substack{j=0 \\ j \neq \frac{m+1}{2}}}^{i-1} H_{j,i}(z^m) \frac{\phi_j(z^m)}{g(z^m)} \right) z^i, \end{aligned}$$

255 where on the right-hand-side the unique unknowns are the coefficients of the Lau-
 256 rent polynomials $H_{i,j}(z)$, $i, j = 0, \dots, m-1$. Knowing the first and the last non-zero
 257 coefficients of

$$\sum_{i=0}^{m-1} \check{a}_i(z^m) z^i + (z^m - 1)^d \sum_{i=0}^{m-1} \tilde{a}_i(z^m) z^i, \quad (25)$$

258 it is possible to establish the indices of the first and the last non-zero coefficients
 259 of each $H_{i,j}(z)$, so that the range of the powers in (25) and in

$$(z^m - 1)^d \sum_{i=0}^{m-1} \left(\sum_{\substack{j=i+1 \\ j \neq \frac{m+1}{2}}}^{m-1} H_{i,j}(z^m) \frac{\phi_j(z^m)}{g(z^m)} - \sum_{\substack{j=0 \\ j \neq \frac{m+1}{2}}}^{i-1} H_{j,i}(z^m) \frac{\phi_j(z^m)}{g(z^m)} \right) z^i,$$

260 are the same. After that one can start imposing the first (or the last) coefficient
 261 of $a(z)$ to be 0, which is a linear condition with respect to the coefficients of all
 262 $H_{i,j}(z)$. It is possible then to add linear constraints in the same unknowns in
 263 order to annihilate additional coefficients of $a(z)$ as long as the new added linear
 264 condition is compatible with the previous ones. Since (22) encodes all possible
 265 solutions of (21), when there are no more compatible conditions to be added, the
 266 mask with minimal support has been reached. A naive implementation of this
 267 strategy has been used in our experiments to produce the interpolatory mask of
 268 minimal support. A more general algorithmic description of this strategy should
 269 incorporate some preprocessing algebraic computations such as the reduction of
 270 the polynomials in reduced form as described in [1].

271 *Remark 4* Of great importance for applications is the case of symbols $a(z)$ that
 272 satisfy the symmetry condition $a(z) = za(z^{-1})$. The existence of such a sym-
 273 metric symbol can be proven under the auxiliary assumption that $\varphi(1/2 + \ell) =$
 274 $\varphi(-1/2 - \ell)$, $\ell \in \mathbb{N} \cup \{0\}$. Under this assumption, we obtain that the coefficients
 275 of $a(z)$ satisfy (4) if and only if the coefficients of $za(z^{-1})$ also satisfy (4). By
 276 linearity this implies that the coefficients of $(a(z) + za(z^{-1}))/2$ satisfy (4) too, with
 277 $(a(z) + za(z^{-1}))/2$ fulfilling the symmetry condition.

278 The presented procedure for the odd arity case can be summarized as in Algo-
 279 rithm 1, at the end of which Remark 3 and Remark 4 can be exploited to reduce the
 280 support of the resulting mask and/or to obtain a symmetric mask. The following
 281 example is used to illustrate our composite approach for the odd arity case. Here
 282 we construct the dual ternary interpolating scheme, reproducing quintic polyno-
 283 mials, sharing with the primal binary Dubuc-Deslauriers 6-point scheme the same
 284 samples at the half integers, and having symmetric mask with shortest support.

285 *Example 1* We choose $m = 3$, $d = 6$ and (see Remark 1)

$$\varphi\left(\frac{1}{2} + \ell\right) = \begin{cases} \frac{3}{256}, & \text{if } \ell \in \{-3, 2\}, \\ -\frac{25}{256}, & \text{if } \ell \in \{-2, 1\}, \\ \frac{75}{128}, & \text{if } \ell \in \{-1, 0\}, \\ 0, & \text{otherwise.} \end{cases} \quad (26)$$

286 These values are taken from the mask of the primal binary 6-point interpolat-
 287 ing scheme which reproduces quintic polynomials and it has a basic limit func-
 288 tion supported in $[-5, 5]$ with best Hölder exponent 2.830074998557687. Its pri-
 289 mal ternary counterpart (see, e.g., [24]) reproduces quintic polynomials as well,
 290 but it has a basic limit function supported in $[-4, 4]$ with best Hölder exponent
 291 2.319856140753624. According to (7), we have

$$\begin{aligned} \phi(z) &= \frac{3}{256z^3} - \frac{25}{256z^2} + \frac{75}{128z} + \frac{75}{128} - \frac{25z}{256} + \frac{3z^2}{256} \\ &= \phi_0(z^3) + \phi_1(z^3)z^{-1} + \phi_2(z^3)z^{-2}, \end{aligned}$$

292 with

$$\phi_0(z) = \frac{3}{256z} + \frac{75}{128}, \quad \phi_1(z) = \frac{75}{128} + \frac{3z}{256}, \quad \phi_2(z) = -\frac{25}{256} - \frac{25z}{256}.$$

293 In particular, we observe that

$$1 - z\phi(z^2) = -(z-1)^6 \frac{3z^4 + 18z^3 + 38z^2 + 18z + 3}{256z^5} \quad (27)$$

294 and

$$g(z) = \gcd\{\phi_0(z), \phi_1(z)\} = 1.$$

295 Thus, Assumption 1 and Assumption 2 are satisfied. After solving the linear
 296 system (14), we have from (15)

$$a_0(z) = \check{a}_0(z) + (z-1)^6 \hat{a}_0(z), \quad a_1(z) = \check{a}_1(z) + (z-1)^6 \hat{a}_1(z)$$

Algorithm 1 [odd arity case]

Input: $m \in 2\mathbb{N} + 1$ and a compactly supported sequence $\{\varphi\left(\frac{2k+1}{2}\right) \in \mathbb{R}\}_{k \in \mathbb{Z}}$ such that the Laurent polynomials

$$\begin{aligned}\phi_\ell(z) &= \sum_{i \in \mathbb{Z}} \varphi\left(\frac{2mi+1}{2} - \ell\right) z^i, \quad \ell \in \{0, \dots, m-1\}, \\ g(z) &= \gcd\left\{\phi_0(z), \dots, \phi_{\frac{m-1}{2}}(z), \phi_{\frac{m+3}{2}}(z), \dots, \phi_{m-1}(z)\right\}, \\ \phi(z) &= \sum_{i=0}^{m-1} \phi_i(z^m) z^{-i},\end{aligned}$$

satisfy

- (a) $1 - z\phi(z^2) = (z-1)^d \gamma(z)$ for some $d \in \mathbb{N}$, $\gamma \in \mathbb{R}[z, z^{-1}]$;
- (b) $g(z)$ divides $1 - \phi(z)\phi_{\frac{m+1}{2}}(z)$;
- (c) $g(z) = (z-1)^q \hat{g}(z)$ for some $0 \leq q < d$, and $\hat{g}(z) \in \mathbb{R}[z, z^{-1}]$ with $\hat{g}(1) \neq 0$.

Procedure:

- (i) set $a_{\frac{m+1}{2}}(z) = \phi(z)$;
- (ii) for $s \in \{1, \dots, d-q-1\}$, solve linear system (14) for $\{a_i^{(s)}(1)\}_{i \in \{0, \dots, m-1\} \setminus \{\frac{m+1}{2}\}}$;
- (iii) for $i \in \{0, \dots, m-1\} \setminus \{\frac{m+1}{2}\}$, define

$$\check{a}_i(z) = 1 + \sum_{s=1}^{d-q-1} \frac{a_i^{(s)}(1)}{s!} (z-1)^s;$$

- (iv) compute

$$\hat{\theta}(z) = \frac{(z-1)^{q-d}}{g(z)} \left(1 - \phi(z)\phi_{\frac{m+1}{2}}(z) - \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \check{a}_i(z)\phi_i(z) \right);$$

- (v) follow the strategy in [19] to compute Laurent polynomials $\{\hat{a}_i(z)\}_{i \in \{0, \dots, m-1\} \setminus \{\frac{m+1}{2}\}}$ such that

$$\hat{\theta}(z) = \sum_{\substack{i=0 \\ i \neq \frac{m+1}{2}}}^{m-1} \hat{a}_i(z) \frac{\phi_i(z)}{g(z)};$$

Output: the symbol

$$a(z) = \sum_{i=0}^{m-1} a_i(z^m) z^i \quad \text{with} \quad a_i(z) = \check{a}_i(z) + \hat{a}_i(z)(z-1)^{d-q}, \quad i \neq \frac{m+1}{2},$$

of an m -ary dual interpolating subdivision scheme reproducing polynomials up to degree $d-q-1$ and having basic limit function with the given samples at $\mathbb{Z}/2$.

with

$$\begin{aligned}\check{a}_0(z) &= 1 + \frac{(z-1)}{6} - \frac{5(z-1)^2}{72} + \frac{55(z-1)^3}{1296} - \frac{935(z-1)^4}{31104} + \frac{4301(z-1)^5}{186624}, \\ \check{a}_1(z) &= 1 - \frac{(z-1)}{6} + \frac{7(z-1)^2}{72} - \frac{91(z-1)^3}{1296} + \frac{1729(z-1)^4}{31104} - \frac{8645(z-1)^5}{186624},\end{aligned}$$

and

$$a_2(z) = \phi(z).$$

To search for compatible $\hat{a}_0(z)$ and $\hat{a}_1(z)$, we first compute

$$\hat{\theta}(z) = \frac{8645z^3 + 215471z^2 - 24300z + 18225}{15925248z^3}$$

in such a way that (20) holds, i.e.,

$$(z-1)^6 \hat{\theta}(z) = 1 - a_2(z) \phi_2(z) - \sum_{i=0}^1 \check{a}_i(z) \phi_i(z).$$

Then we look for particular solutions $\tilde{a}_0(z)$ and $\tilde{a}_1(z)$ such that

$$\hat{\theta}(z) = \tilde{a}_0(z) \phi_0(z) + \tilde{a}_1(z) \phi_1(z).$$

A possible choice is

$$\begin{aligned}\tilde{a}_0(z) &= -\frac{9903400z - 45544275}{466373376z^2}, \\ \tilde{a}_1(z) &= \frac{21603855z - 46560721}{466373376z^2}.\end{aligned}$$

To obtain a shorter mask, according to Remark 3, we search for a suitable $H_{0,1}(z)$ so that replacing

$$\begin{aligned}\hat{a}_0(z) &= \tilde{a}_0(z) + H_{0,1}(z) \phi_1(z), \\ \hat{a}_1(z) &= \tilde{a}_1(z) - H_{0,1}(z) \phi_0(z),\end{aligned}$$

in the previous expressions of $a_0(z)$ and $a_1(z)$, leads to a symbol

$$a(z) = a_0(z^3) + a_1(z^3)z + a_2(z^3)z^2$$

with a shorter associated mask. The choice of $H_{0,1}(z)$ that leads to the shortest mask is

$$H_{0,1}(z) = -\frac{844799}{5465313z^2},$$

and, after symmetrization (see Remark 4), the resulting symmetric mask \mathbf{a} is such that $a_i = 0$ for $i \notin [-14, 15]$, with the first half of its entries being

$$\left\{ \frac{16567}{466373376}, 0, -\frac{414175}{233186688}, \frac{224821}{66624768}, \frac{3}{256}, \frac{589847}{33312384}, \right. \\ \left. -\frac{83995}{2776032}, -\frac{25}{256}, -\frac{2042857}{22208256}, \frac{1290971}{8328096}, \frac{75}{128}, \frac{63152905}{66624768} \right\}. \quad (28)$$

The basic limit function f_{δ} related to the mask in (28) is shown in Figure 1, and two examples of interpolating curves can be found in Figure 2. We have that $\text{supp}(f_{\delta}) = [-23/4, 23/4]$ and $f_{\delta} \in C^{\omega}(\mathbb{R})$ with the best Hölder exponent ω being 3.006664260760692. By construction the corresponding subdivision scheme reproduces polynomials of degree 5.

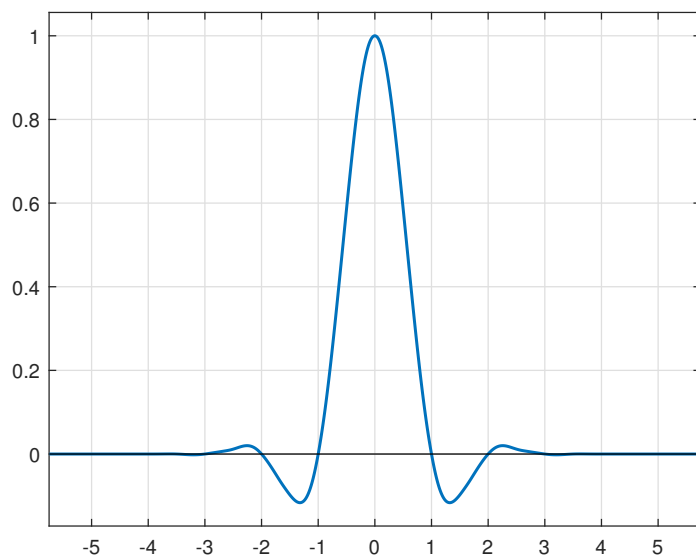


Fig. 1: The graph of the basic limit function f_δ related to the mask in (28).

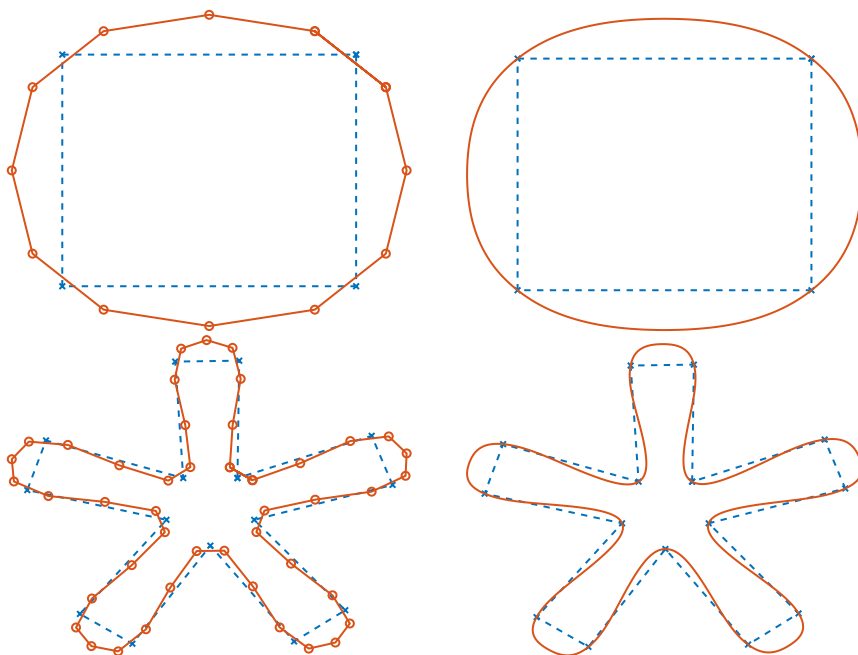


Fig. 2: Two examples of interpolating curves given by the subdivision scheme associated to the mask in (28). On the left, the first level of subdivision starting with the dotted control polygons; on the right, the corresponding interpolating limit curves.

318 3.2 The even arity case

319 Let us now consider the solution of the linear system (35) in [26] for the case where
 320 m is an even integer. The system is defined as follows:

$$M\mathbf{a} = \mathbf{c}, \quad M = (\mu_{i,j})_{i,j \in \mathbb{Z}}, \quad \mathbf{c} = (c_i)_{i \in \mathbb{Z}} \quad (29)$$

321 where

$$\mu_{i,j} = \begin{cases} \varphi\left(\frac{i+1}{2} - j\right), & \text{if } i \in m\mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases} \quad c_i = \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\frac{i}{2m}\right), & \text{if } i \in m(2\mathbb{Z} + 1), \\ 0, & \text{otherwise.} \end{cases}$$

322 By suppressing the zero rows in both M and \mathbf{c} we obtain the equivalent linear
 323 system

$$\widehat{M}\mathbf{a} = \widehat{\mathbf{c}}, \quad \widehat{M} = (\widehat{\mu}_{i,j})_{i,j \in \mathbb{Z}}, \quad \widehat{\mathbf{c}} = (\widehat{c}_i)_{i \in \mathbb{Z}} \quad (30)$$

324 where

$$\widehat{\mu}_{i,j} = \varphi\left(\frac{im+1}{2} - j\right), \quad i, j \in \mathbb{Z}, \quad \widehat{c}_i = \begin{cases} 1, & \text{if } i = 0, \\ \varphi\left(\frac{i}{2}\right), & \text{if } \text{mod}(i, 2) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

325 According to [26], (29) and (30) can be expressed in functional form as

$$\sum_{\alpha \in m\mathbb{Z}} \sum_{\beta \in \mathbb{Z}} a_{\beta} \varphi\left(\frac{\alpha+1}{2} - \beta\right) z^{\alpha} = 1 + \sum_{\alpha \in m(2\mathbb{Z}+1)} \varphi\left(\frac{\alpha}{2m}\right) z^{\alpha}$$

326 which can be rewritten as

$$\begin{aligned} \sum_{\ell \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}} a_{\beta} \varphi\left(\frac{m\ell+1}{2} - \beta\right) z^{\ell} &= 1 + \sum_{\ell \in \mathbb{Z}} \varphi\left(\frac{2\ell+1}{2}\right) z^{2\ell+1} \\ &= 1 + \sum_{\ell \in \mathbb{Z}} \varphi\left(\ell + \frac{1}{2}\right) z^{2\ell+1}. \end{aligned} \quad (31)$$

327 Supposing that, as in the odd arity case, Assumption 1 holds for

$$\phi(z) = \sum_{\ell \in \mathbb{Z}} \varphi\left(\frac{1}{2} + \ell\right) z^{\ell},$$

328 the right-hand side of (31) satisfies

$$\begin{aligned} 1 + \sum_{\ell \in \mathbb{Z}} \varphi\left(\ell + \frac{1}{2}\right) z^{2\ell+1} &= 1 + z\phi(z^2) \\ &= (z+1)^d (-1)^d \gamma(-z) \\ &= (z+1)^d \widetilde{\gamma}(z), \quad \widetilde{\gamma}(z) \in \mathbb{R}[z, z^{-1}]. \end{aligned}$$

329 Concerning the representation of the left-hand side of (31), let us introduce the
330 modified subsymbols defined by

$$\widehat{\phi}_\ell(z) = \sum_{i \in \mathbb{Z}} \varphi \left(\frac{mi+1}{2} - \ell \right) z^i, \quad 0 \leq \ell \leq m-1. \quad (32)$$

331 Notice that if $\phi_\ell(z)$, $0 \leq \ell \leq m/2 - 1$, denote the subsymbols of the mask of arity
332 $m/2$, then we have

$$\widehat{\phi}_\ell(z) = \phi_\ell(z), \quad \widehat{\phi}_{\ell+m/2}(z) = z\widehat{\phi}_\ell(z), \quad 0 \leq \ell \leq m/2 - 1. \quad (33)$$

333 In particular this implies that

$$\widehat{\phi}_{\ell+m/2}(-1) = -\widehat{\phi}_\ell(-1), \quad \widehat{\phi}_{\ell+m/2}(1) = \widehat{\phi}_\ell(1), \quad 0 \leq \ell \leq m/2 - 1.$$

334 Moreover, from $1 + z\phi(z^2) = (z+1)^d \widetilde{\gamma}(z)$ and $1 - z\phi(z^2) = (z-1)^d \gamma(z)$ one deduces
335 that

$$(z+1)^d \widetilde{\gamma}(z) = 2 - (z-1)^d \gamma(z). \quad (34)$$

336 Then for the left-hand side of (31) it holds

$$\sum_{\ell \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}} a_{\beta\varphi} \left(\frac{m\ell+1}{2} - \beta \right) z^\ell = a_0(z^2) \widehat{\phi}_0(z) + \dots + a_{m-1}(z^2) \widehat{\phi}_{m-1}(z).$$

337 Hence, it follows that relation (31) can be reformulated as the Bezout-like poly-
338 nomial equation

$$a_0(z^2) \widehat{\phi}_0(z) + \dots + a_{m-1}(z^2) \widehat{\phi}_{m-1}(z) = (z+1)^d \widetilde{\gamma}(z). \quad (35)$$

339 From (33) it follows that equation (35) can be equivalently rewritten as

$$\left(a_0(z^2) + za_{m/2}(z^2) \right) \phi_0(z) + \dots + \left(a_{m/2-1}(z^2) + za_{m-1}(z^2) \right) \phi_{m/2-1}(z) = (z+1)^d \widetilde{\gamma}(z). \quad (36)$$

340 To proceed we consider the following assumption that plays the same role as
341 Assumption 2 in the odd arity case.

342 **ASSUMPTION 3** : We suppose that

$$g(z) := \gcd \left\{ \phi_0(z), \dots, \phi_{m/2-1}(z) \right\},$$

343 divides $(z+1)^d \widetilde{\gamma}(z)$ and is such that $g(\pm 1) \neq 0$.

344 Requiring $g(z)$ to divide $(z+1)^d \widetilde{\gamma}(z)$ and satisfy $g(1) \neq 0$ is clearly a necessary
345 condition because of (36) and Assumption 1. Condition $g(-1) \neq 0$ however is only
346 sufficient to construct $a(z)$ as in (10) and, when it is not satisfied, a price has to
347 be paid in terms of polynomial reproduction (see Remark 5).

348 Under Assumption 3 the solution to equation (35) can be found similarly to the
349 odd arity case. Specifically, at the first step the unknowns $a_i^{(s)}(1)$, $0 \leq i \leq m-1$,
350 $s = 1, \dots, d-1$, are computed by solving a Vandermonde linear system. The system
351 is formed as follows. The first $m-1$ equations are obtained by differentiation of
352 (10) complemented with relation (35). The last equation is found by imposing the

property (34) on the left hand-side of (35). If $\xi_k = e^{2\pi ik/m}$, $1 \leq k \leq m$, denote the m -th roots of unity, then the system is of the form

$$m^s \mathcal{D}(\xi_1^{(m-1)s}, \dots, \xi_{m-1}^{(m-1)s}, (2/m)^s) \mathcal{V}(\xi_1, \dots, \xi_{m-1}, \phi) [a_0^{(s)}(1), \dots, a_{m-1}^{(s)}(1)]^T = \mathbf{b}_s, \quad (37)$$

where $\mathcal{D}(\mathbf{v})$ is the diagonal matrix with diagonal entries v_k and $\mathcal{V}(\xi_1, \dots, \xi_{m-1}, \phi)$ is a Vandermonde-like matrix with nodes ξ_k of the form

$$\mathcal{V}(\xi_1, \dots, \xi_{m-1}, \phi) = \begin{bmatrix} \xi_1^0 & \dots & \xi_1^{m-1} \\ \vdots & \vdots & \vdots \\ \xi_{m-1}^0 & \dots & \xi_{m-1}^{m-1} \\ (\phi)_1 & \dots & (\phi)_m \end{bmatrix} \quad \text{with} \quad (\phi)_j = \hat{\phi}_{j-1}(1), \quad 1 \leq j \leq m.$$

The solvability of the systems (37) follows from the next lemma.

Lemma 1 For any $\mathbf{v} = [v_1, \dots, v_{m/2}] \in \mathbb{R}^{m/2}$ and $\mathbf{w} = [\mathbf{v}, \mathbf{v}] \in \mathbb{R}^m$, it holds

$$\det \mathcal{V}(\xi_1, \dots, \xi_{m-1}, \mathbf{w}) = \frac{2}{m} \prod_{1 \leq i, j \leq m} (\xi_i - \xi_j) \sum_{i=1}^{m/2} v_i.$$

Proof By Laplace's rule we find that $\det \mathcal{V}(\xi_1, \dots, \xi_{m-1}, \mathbf{w})$ is linear in $v_1, \dots, v_{m/2}$. If $\sum_{i=1}^{m/2} v_i = 0$, then $\det \mathcal{V}(\xi_1, \dots, \xi_{m-1}, \mathbf{w}) = 0$ since the all-ones vector $\mathbf{1}$ belongs to the kernel of the matrix. This implies that $\det \mathcal{V}(\xi_1, \dots, \xi_{m-1}, \mathbf{w}) = \gamma \sum_{i=1}^{m/2} v_i$ for a suitable γ . The value of γ can be determined by setting $\mathbf{w} = \mathbf{1}$ which amounts to consider the customary Vandermonde matrix. \square

As a consequence of Assumption 1 with $z = 1$, we have that

$$\sum_{i=1}^{m/2} \hat{\phi}_{i-1}(1) = \sum_{i=1}^{m/2} \phi_{i-1}(1) = 1.$$

Therefore, by Lemma 1, the coefficient matrix in (37) is non-singular and $a_i^{(s)}(1)$, $0 \leq i \leq m-1$, are uniquely determined. Once these quantities are computed, the sub-symbols can be represented as follows

$$a_i(z) = \check{a}_i(z) + (z-1)^d \hat{a}_i(z), \quad 0 \leq i \leq m-1, \quad (38)$$

$$\check{a}_i(z) = 1 + \sum_{j=1}^{d-1} \frac{a_i^{(j)}(1)}{j!} (z-1)^j,$$

for suitable $\hat{a}_i(z) \in \mathbb{R}[z, z^{-1}]$. This representation is exploited in the second step to find a solution of (35). If we set

$$\theta(z) := (z+1)^d \check{\gamma}(z) - \sum_{j=0}^{m-1} \check{a}_j(z^2) \hat{\phi}_j(z), \quad (39)$$

by using similar arguments as in the proof of Theorem 2, together with Assumption 3, it is shown that there exists $\hat{\theta}(z) \in \mathbb{R}[z, z^{-1}]$ such that

$$\theta(z) = (z^2 - 1)^d g(z) \hat{\theta}(z). \quad (40)$$

372 In this way equation (35) can be simplified as follows

$$\hat{a}_0(z^2) \frac{\hat{\phi}_0(z)}{g(z)} + \dots + \hat{a}_{m-1}(z^2) \frac{\hat{\phi}_{m-1}(z)}{g(z)} = \hat{\theta}(z),$$

373 which yields to its reduced analogue

$$\left(\hat{a}_0(z^2) + z\hat{a}_{m/2}(z^2)\right) \frac{\phi_0(z)}{g(z)} + \dots + \left(\hat{a}_{m/2-1}(z^2) + z\hat{a}_{m-1}(z^2)\right) \frac{\phi_{m/2-1}(z)}{g(z)} = \hat{\theta}(z).$$

374 By setting $\tilde{a}_i(z) = \hat{a}_i(z^2) + z\hat{a}_{i+m/2}(z^2)$, $0 \leq i \leq m/2 - 1$, thanks to Assumption 3
375 we deduce that the equation

$$\tilde{a}_0(z) \frac{\phi_0(z)}{g(z)} + \dots + \tilde{a}_{m/2-1}(z) \frac{\phi_{m/2-1}(z)}{g(z)} = \hat{\theta}(z) \quad (41)$$

376 is solvable and every solution can be written as

$$\bar{a}_i(z) = \tilde{a}_i(z) + \sum_{j=i+1}^{m/2-1} H_{i,j}(z) \frac{\phi_j(z)}{g(z)} - \sum_{j=0}^{i-1} H_{j,i}(z) \frac{\phi_j(z)}{g(z)},$$

377 where $\tilde{a}_i(z)$ is a particular solution of (41) and $H_{i,j}(z)$ is any element of $\mathbb{R}[z, z^{-1}]$.

378 *Remark 5* If $g(z) = (z+1)^q \hat{g}(z)$ for some $q \in \mathbb{N}$, $q < d$, and $\hat{g}(z) \in \mathbb{R}[z, z^{-1}]$, with
379 $\hat{g}(-1) \neq 0$, then one can only factorize $\theta(z)$ as

$$\theta(z) = (z^2 - 1)^{d-q} (z - 1)^q g(z) \hat{\theta}(z),$$

380 rather than (40). In this case, similarly to what was done in Remark 2, one should
381 consider (23) instead of (38) and the illustrated procedure will lead to a symbol
382 $a(z)$ of the form (24) instead of (10). This means that the subdivision scheme
383 associated to $a(z)$ would reproduce only polynomials up to degree $d - q - 1$.

384 *Remark 6* For $m = 2$ equation (35) becomes

$$(a_0(z^2) + za_1(z^2))\phi(z) = 1 + z\phi(z^2)$$

385 which implies that the first and the last non-zero elements of $a(z)$ must be equal
386 to 1. It follows that the associated subdivision scheme cannot be convergent [26].

387 The presented procedure for the even arity case can be summarized as in
388 Algorithm 2, after which similar arguments as in Remark 3 and Remark 4 can
389 be exploited, to reduce the support of the resulting mask and/or to obtain a
390 symmetric one. Next, we conclude with the illustration of our composite approach,
391 in the even arity case, by means of a computational example where we construct the
392 dual quaternary interpolating scheme, reproducing quintic polynomials, sharing
393 with the primal binary Dubuc-Deslauriers 6-point scheme the same samples at the
394 half integers, and having symmetric mask with shortest support.

Algorithm 2 [even arity case]

Input: $m \in 2\mathbb{N} \setminus \{2\}$ and a compactly supported sequence $\{\varphi\left(\frac{2k+1}{2}\right) \in \mathbb{R}\}_{k \in \mathbb{Z}}$ such that the Laurent polynomials

$$\phi_\ell(z) = \sum_{i \in \mathbb{Z}} \varphi\left(\frac{mi+1}{2} - \ell\right) z^i, \quad \ell \in \{0, \dots, m/2 - 1\},$$

$$g(z) = \gcd\{\phi_0(z), \dots, \phi_{m/2-1}(z)\},$$

$$\phi(z) = \sum_{i=0}^{m/2-1} \phi_i(z^m) z^{-i},$$

satisfy

- (a) $1 - z\phi(z^2) = (z-1)^d \gamma(z)$ for some $d \in \mathbb{N}$, $\gamma \in \mathbb{R}[z, z^{-1}]$;
- (b) $g(z)$ divides $1 + z\phi(z^2)$;
- (c) $g(z) = (z+1)^q \hat{g}(z)$ for some $0 \leq q < d$, and $\hat{g}(z) \in \mathbb{R}[z, z^{-1}]$ with $\hat{g}(-1) \neq 0$.

Procedure:

- (i) for $s \in \{1, \dots, d-q-1\}$, solve linear system (37) for $\{a_i^{(s)}(1)\}_{i=0, \dots, m-1}$;
- (ii) for $i \in \{0, \dots, m-1\}$, define

$$\check{a}_i(z) = 1 + \sum_{s=1}^{d-q-1} \frac{a_i^{(s)}(1)}{s!} (z-1)^s;$$

- (iii) compute

$$\hat{\theta}(z) = \frac{(z^2-1)^{q-d}}{(z-1)^q g(z)} \left(1 + z\phi(z^2) - \sum_{i=0}^{m/2-1} (\check{a}_i(z^2) + z\check{a}_{m/2+i}(z^2)) \phi_i(z) \right);$$

- (iv) follow the strategy in [19] to compute Laurent polynomials $\{\tilde{a}_i(z)\}_{i=0, \dots, m/2-1}$ such that

$$\hat{\theta}(z) = \sum_{i=0}^{m/2-1} \tilde{a}_i(z) \frac{\phi_i(z)}{g(z)};$$

- (v) for $i \in \{0, \dots, m/2-1\}$, compute the Laurent polynomials $\hat{a}_i(z)$ and $\hat{a}_{m/2+i}(z)$ uniquely defined by the relation

$$\tilde{a}_i(z) = \hat{a}_i(z^2) + z\hat{a}_{m/2+i}(z^2).$$

Output: the symbol

$$a(z) = \sum_{i=0}^{m-1} a_i(z^m) z^i \quad \text{with} \quad a_i(z) = \check{a}_i(z) + \hat{a}_i(z)(z-1)^{d-q},$$

of an m -ary dual interpolating subdivision scheme reproducing polynomials up to degree $d-q-1$ and having basic limit function with the given samples at $\mathbb{Z}/2$.

395 *Example 2* We choose $m = 4$, $d = 6$ and (see Remark 1)

$$\varphi\left(\frac{1}{2} + \ell\right) = \begin{cases} \frac{3}{256}, & \text{if } \ell \in \{-3, 2\}, \\ -\frac{25}{256}, & \text{if } \ell \in \{-2, 1\}, \\ \frac{75}{128}, & \text{if } \ell \in \{-1, 0\}, \\ 0, & \text{otherwise.} \end{cases} \quad (42)$$

396 These values are again taken from the mask of the primal binary 6-point inter-
 397 interpolating scheme which reproduces quintic polynomials and it has a basic limit
 398 function supported in $[-5, 5]$ with best Hölder exponent 2.830074998557687. Its
 399 primal quaternary counterpart (see, e.g., [24]) reproduces quintic polynomials as
 400 well, but it has a basic limit function supported in $[-11/3, 11/3]$ with best Hölder
 401 exponent 2.099550050039848. In view of (32) and (33), we have

$$\hat{\phi}_0(z) = \phi_0(z) = -\frac{25}{256}z + \frac{75}{128} + \frac{3z}{256},$$

$$\hat{\phi}_1(z) = \phi_1(z) = \frac{3}{256}z + \frac{75}{128} - \frac{25z}{256},$$

$$\hat{\phi}_2(z) = z\phi_0(z) = -\frac{25}{256} + \frac{75z}{128} + \frac{3z^2}{256},$$

$$\hat{\phi}_3(z) = z\phi_1(z) = \frac{3}{256} + \frac{75z}{128} - \frac{25z^2}{256}.$$

405 Assumption 1 is satisfied since $\phi(z)$ is the same as in Example 1 (27), while As-
 406 sumption 3 holds because

$$\phi_1(z) = \phi_0(1/z) \implies \gcd\{\phi_0(z), \phi_1(z)\} = 1.$$

407 After solving the linear system (37), from (38) we obtain

$$a_i(z) = \check{a}_i(z) + (z-1)^d \hat{a}_i(z), \quad 0 \leq i \leq 3,$$

408 with

$$\check{a}_0(z) = 1 + \frac{(z-1)}{8} - \frac{7(z-1)^2}{128} + \frac{35(z-1)^3}{1024} - \frac{805(z-1)^4}{32768} + \frac{4991(z-1)^5}{262144},$$

$$\check{a}_1(z) = 1 - \frac{(z-1)}{8} + \frac{9(z-1)^2}{128} - \frac{51(z-1)^3}{1024} + \frac{1275(z-1)^4}{32768} - \frac{8415(z-1)^5}{262144},$$

$$\check{a}_2(z) = 1 - \frac{3(z-1)}{8} + \frac{33(z-1)^2}{128} - \frac{209(z-1)^3}{1024} + \frac{5643(z-1)^4}{32768} - \frac{39501(z-1)^5}{262144},$$

$$\check{a}_3(z) = 1 - \frac{5(z-1)}{8} + \frac{65(z-1)^2}{128} - \frac{455(z-1)^3}{1024} + \frac{13195(z-1)^4}{32768} - \frac{97643(z-1)^5}{262144}.$$

412 To search for compatible $\hat{a}_0(z)$, $\hat{a}_1(z)$, $\hat{a}_2(z)$ and $\hat{a}_3(z)$, we first compute

$$\hat{\theta}(z) = \frac{3}{256z^5} - \frac{7}{256z^3} + \frac{5086563}{16777216z} - \frac{580643}{16777216}$$

such that, according to (39) and (40),

$$(z^2 - 1)^6 \hat{\theta}(z) = (z + 1)^6 \tilde{\gamma}(z) - \sum_{i=0}^3 \tilde{a}_i(z^2) \hat{\phi}_i(z),$$

with

$$\tilde{\gamma}(z) = \frac{3}{256 z^5} - \frac{9}{128 z^4} + \frac{19}{128 z^3} - \frac{9}{128 z^2} + \frac{3}{256 z},$$

due to (34). Then we search for $\tilde{a}_0(z)$ and $\tilde{a}_1(z)$ that solve the reduced Bezout equation in (41), i.e.,

$$\hat{\theta}(z) = \tilde{a}_0(z) \phi_0(z) + \tilde{a}_1(z) \phi_1(z). \quad (43)$$

A possible choice is

$$\tilde{a}_0(z) = \frac{2126507351527}{157810688 z} - \frac{176620228675}{78905344},$$

$$\tilde{a}_1(z) = \frac{1}{z^4} - \frac{50}{z^3} + \frac{2506}{z^2} - \frac{2118539063675}{157810688 z} - \frac{21194427441}{78905344}.$$

Once we have a solution of (43), we search for

$$\bar{a}_0(z) = \tilde{a}_0(z) + H_{0,1}(z) \phi_1(z),$$

$$\bar{a}_1(z) = \tilde{a}_1(z) - H_{0,1}(z) \phi_0(z),$$

so that $\{\hat{a}_k(z)\}_{k=0,\dots,3}$ fulfilling

$$\bar{a}_i(z) = \hat{a}_i(z^2) + z \hat{a}_{i+2}(z^2), \quad i \in \{0, 1\},$$

lead to a symbol $a(z)$ satisfying $a(z) = za(z^{-1})$. For example, the choice

$$\begin{aligned} H_{0,1}(z) = & -\frac{7064809147}{308224 z} + \frac{281633113}{616448 z^2} - \frac{2817667}{308224 z^3} + \frac{119853}{616448 z^4} \\ & + \frac{7302199}{596413440 z^5} - \frac{3127}{1232896 z^6} + \frac{947}{1331280 z^7} \end{aligned}$$

leads to

$$\begin{aligned} \bar{a}_0(z) = & \frac{39501}{262144 z} - \frac{4991}{262144 z^2} - \frac{5643}{262144 z^3} + \frac{24415849}{4362338304 z^4} + \frac{394938757}{40715157504 z^5} \\ & - \frac{61600783}{43623383040 z^6} + \frac{15760091}{40715157504 z^7} + \frac{947}{113602560 z^8} \end{aligned}$$

$$\begin{aligned} \bar{a}_1(z) = & \frac{97643}{262144 z} + \frac{8415}{262144 z^2} - \frac{7917}{262144 z^3} - \frac{49446367}{7270563840 z^4} + \frac{482174039}{40715157504 z^5} \\ & + \frac{116624327}{43623383040 z^6} - \frac{27054815}{40715157504 z^7} + \frac{4735}{68161536 z^8}, \end{aligned}$$

and so

$$\hat{a}_0(z) = -\frac{4991}{262144 z} + \frac{24415849}{4362338304 z^2} - \frac{61600783}{43623383040 z^3} + \frac{947}{113602560 z^4},$$

$$\hat{a}_1(z) = \frac{8415}{262144 z} - \frac{49446367}{7270563840 z^2} + \frac{116624327}{43623383040 z^3} + \frac{4735}{68161536 z^4},$$

$$\begin{aligned} \hat{a}_2(z) &= \frac{39501}{262144z} - \frac{5643}{262144z^2} + \frac{394938757}{40715157504z^3} + \frac{15760091}{40715157504z^4}, \\ \hat{a}_3(z) &= \frac{97643}{262144z} - \frac{7917}{262144z^2} + \frac{482174039}{40715157504z^3} - \frac{27054815}{40715157504z^4}. \end{aligned}$$

Replacing the previous expressions in the above equations of $a_0(z)$, $a_1(z)$, $a_2(z)$ and $a_3(z)$ and using

$$a(z) = a_0(z^4) + a_1(z^4)z + a_2(z^4)z^2 + a_3(z^4)z^3,$$

the first half of the resulting symmetric mask \mathbf{a} is

$$\left\{ \begin{aligned} &\frac{947}{113602560}, \frac{4735}{68161536}, \frac{15760091}{40715157504}, -\frac{27054815}{40715157504}, -\frac{63782671}{43623383040}, \\ &\frac{98441927}{43623383040}, \frac{42911173}{5816451072}, \frac{92071847}{5816451072}, \frac{154804477}{10905845760}, -\frac{79247347}{3635281920}, \\ &-\frac{143318065}{1938817024}, -\frac{215643011}{1938817024}, -\frac{71706399}{969408512}, \frac{4869166267}{43623383040}, \frac{2428957997}{5816451072}, \\ &\frac{4331006815}{5816451072}, \frac{528433771}{545292288} \end{aligned} \right\}. \quad (44)$$

The basic limit function f_δ related to this mask is shown in Figure 3, and two examples of interpolating curves can be found in Figure 4. We have that $\text{supp}(f_\delta) = [-11/2, 11/2]$ and, via joint spectral radius techniques, one can prove that $f_\delta \in \mathcal{C}^\omega(\mathbb{R})$ with the best Hölder exponent ω being 3.050871089158321. By construction the corresponding subdivision scheme reproduces polynomials of degree 5.

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References

1. Michael Alekhovich. Linear Diophantine equations over polynomials and soft decoding of Reed-Solomon codes. *IEEE Trans. Inform. Theory*, 51(7):2257–2265, 2005.
2. Alfred S. Cavaretta, Wolfgang Dahmen, and Charles A. Micchelli. Stationary subdivision. *Mem. Amer. Math. Soc.*, 93(453), 1991.
3. Maria Charina, Costanza Conti, and Tomas Sauer. Regularity of multivariate vector subdivision schemes. *Numer. Algorithms*, 39(1-3):97–113, 2005.
4. Maria Charina and Thomas Mejsirik. Multiple multivariate subdivision schemes: matrix and operator approaches. *J. Comput. Appl. Math.*, 349:279–291, 2019.

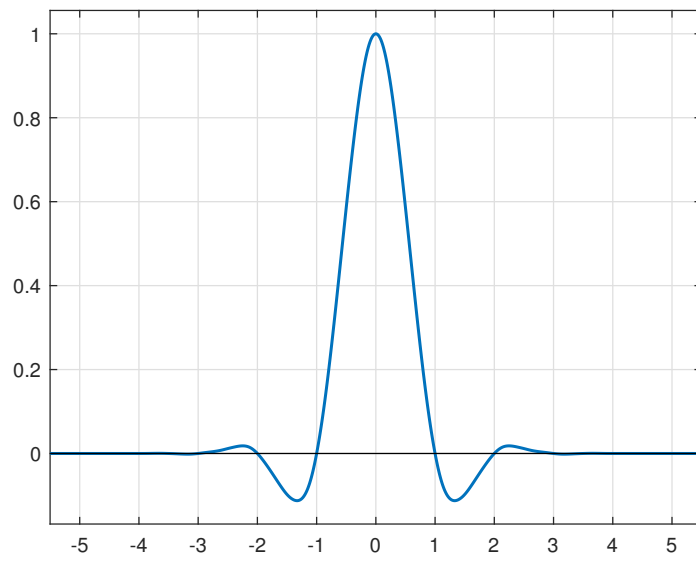


Fig. 3: The graph of the basic limit function f_δ related to the mask in (44).

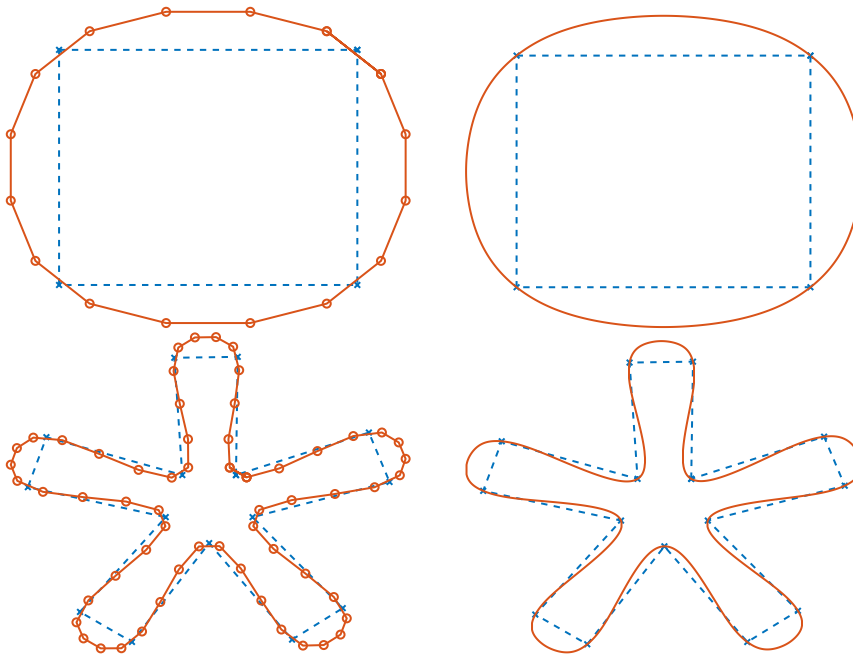


Fig. 4: Two examples of interpolating curves given by the subdivision scheme associated to the mask in (44). On the left, the first level of subdivision starting with the dotted control polygons; on the right, the corresponding interpolating limit curves.

- 455 5. Sung Woo Choi, Byung-Gook Lee, Yeon Ju Lee, and Jungho Yoon. Stationary subdivision
456 schemes reproducing polynomials. *Comput. Aided Geom. Design*, 23(4):351 – 360, 2006.
- 457 6. Charles Chui and Johan M. de Villiers. *Wavelet Subdivision Methods: GEMS for Render-*
458 *ing Curves and Surfaces*. CRC Press Boca Raton, FL, 2010.
- 459 7. Costanza Conti and Kai Hormann. Polynomial reproduction for univariate subdivision
460 schemes of any arity. *J. Approx. Theory*, 163(4):413–437, 2011.
- 461 8. Costanza Conti and Lucia Romani. Dual univariate m -ary subdivision schemes of de
462 Rham-type. *J. Math. Anal. Appl.*, 407(2):443–456, 2013.
- 463 9. Johan de Villiers and Mpfareleni Rejoyce Gavhi. Local interpolation with optimal poly-
464 nomial exactness in refinement spaces. *Math. Comp.*, 85(298):759–782, 2016.
- 465 10. Johan M. de Villiers, Charles A. Micchelli, and Tomas Sauer. Building refinable functions
466 from their values at integers. *Calcolo*, 37(3):139–158, 2000.
- 467 11. Gilles Deslauriers and Serge Dubuc. Symmetric iterative interpolation processes. *Constr.*
468 *Approx.*, 5(1):49–68, 1989.
- 469 12. Serge Dubuc. Interpolation through an iterative scheme. *J. Math. Anal. Appl.*, 114(1):185
470 – 204, 1986.
- 471 13. Serge Dubuc. de Rham transforms for subdivision schemes. *J. Approx. Theory*, 163(8):966
472 – 987, 2011.
- 473 14. Nira Dyn. Subdivision schemes in computer-aided geometric design. In *Advances in*
474 *Numerical Analysis, Vol. II (Lancaster, 1990)*, Oxford Sci. Publ., pages 36–104. Oxford
475 Univ. Press, New York, 1992.
- 476 15. Nira Dyn. Interpolatory subdivision schemes. In Floater M.S. Iske A., Quak E., edi-
477 tor, *Tutorials on Multiresolution in Geometric Modelling*, pages 25–50. Springer, Berlin,
478 Heidelberg, 2002.
- 479 16. Nira Dyn, Kai Hormann, Malcolm A. Sabin, and Zuowei Shen. Polynomial reproduction
480 by symmetric subdivision schemes. *J. Approx. Theory*, 155(1):28–42, 2008.
- 481 17. Nira Dyn and David Levin. Subdivision schemes in geometric modelling. *Acta Numer.*,
482 11:73–144, 2002.
- 483 18. Mpfareleni Rejoyce Gavhi-Molefe and Johan de Villiers. On interpolatory subdivision
484 symbol formulation and parameter convergence intervals. *J. Comput. Appl. Math.*, 349:354
485 – 365, 2019.
- 486 19. Graziano Gentili and Daniele C. Struppa. Minimal degree solutions of polynomial equa-
487 tions. *Kybernetika*, 23(1):44–53, 1987.
- 488 20. Nicola Guglielmi and Vladimir Protasov. Exact computation of joint spectral character-
489 istics of linear operators. *Found. Comput. Math.*, 13(1):37–97, 2013.
- 490 21. Warren P. Johnson. The curious history of Faà di Bruno’s formula. *Amer. Math. Monthly*,
491 109(3):217–234, 2002.
- 492 22. Thomas Mejstrik. Algorithm 1011: improved invariant polytope algorithm and applica-
493 tions. *ACM Trans. Math. Software*, 46(3):Art. 29, 26, 2020.
- 494 23. Charles A. Micchelli. Interpolatory subdivision schemes and wavelets. *J. Approx. Theory*,
495 86:41–71, 1996.
- 496 24. Georg Muntingh. Symbols and exact regularity of symmetric pseudo-splines of any arity.
497 *BIT*, 57(3):867–900, 2017.
- 498 25. Lucia Romani. Interpolating m -refinable functions with compact support: The second
499 generation class. *Appl. Math. Comput.*, 361:735–746, 2019.
- 500 26. Lucia Romani and Alberto Viscardi. Dual univariate interpolatory subdivision of every
501 arity: Algebraic characterization and construction. *J. Math. Anal. Appl.*, 484(1):123713,
502 2020.