A NOTE ON BV AND 1-SOBOLEV FUNCTIONS ON THE WEIGHTED EUCLIDEAN SPACE

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ABSTRACT. In the setting of the Euclidean space equipped with an arbitrary Radon measure, we prove the equivalence between several notions of function of bounded variation present in the literature. We also study the relation between various definitions of 1-Sobolev function.

1. Introduction

In the setting of the Euclidean space $\mathbb{R}^d$ equipped with an arbitrary Radon measure $\mu \geq 0$ (hereafter referred to as the *weighted Euclidean space*) the first notion of function of bounded variation (BV, for short) was introduced in the late nineties, proposed by Bellettini, Bouchitté, and Fragalà in [2]. The approach in there follows the ideas developed in [6], where the Sobolev space $W^{1,p}$ with $p > 1$ has been introduced. It is based on a notion of *space tangent to the measure* μ and the related concept of *μ-tangential gradient*, that we shall discuss below. The study of functional spaces in the weighted Euclidean space setting is motivated by numerous applications in different kinds of variational problems; e.g. shape optimization [5, 3, 4], optimal transport problems with gradient penalization [15], homogenization [21, 13].

In the last twenty years, both Sobolev and BV calculus have been extensively studied also in a more general setting, that of *metric measure spaces* (namely, complete and separable metric spaces endowed with a boundedly-finite Borel measure), see for example [17, 1, 8, 14]. The first instance of the definition of BV function appeared in [17] by Miranda, where a relaxation-type approach has been adopted. Ten years later it was followed by the definitions by Ambrosio and Di Marino in [1] and by Di Marino in [9], where a thorough study of all the approaches has been performed and where it was also proven that all of them are equivalent. All the results from [1] and [9] are collected in Di Marino's PhD thesis [8], to which we will often refer to.

In the main result of this paper (given in Theorem 5.7) we will prove that the notion of BV function proposed in [2], that is tailored for the Euclidean setting, coincides with several (equivalent) notions of BV function coming from the framework of metric measure spaces [8]. This note comes as a natural follow-up to the paper [16], where the equivalence between different notions of Sobolev spaces $W^{1,p}$ with $p > 1$ has been proven.

Let us now briefly explain the main ideas that lie behind the definition of BV and $W^{1,1}$ functions proposed in [2]. The objects that play a key role in this approach are bounded vector fields having bounded distributional divergence. In the sequel, the space of such vector fields will be denoted by $\mathrm{D}_\infty(\mathrm{div}_\mu)$. Their role is, in a sense, two-fold: the space $\mathrm{BV}(\mathbb{R}^d, \mu)$ of functions of bounded variation is defined as the space of those 1-integrable (with respect to μ) functions $f \in L^1_\mu(\mathbb{R}^d)$

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such that the quantity

$$\|D_\mu f\| := \sup\left\{\int_{\mathbb{R}^d} f\,\mathrm{div}_\mu(v)\,\mathrm{d}\mu :\ v\in \mathrm{D}_\infty(\mathrm{div}_\mu), |v|\le 1\ \mu\text{-a.e.}\right\},$$

referred to as the *total variation* of f, is finite. On the other hand, one can show that there exists a unique (up to μ-a.e. equality) minimal subbundle of $\mathbb{R}^d$ – denoted $\{T_\mu(x)\}_{x\in\mathbb{R}^d}$ – such that for every $v\in \mathrm{D}_\infty(\mathrm{div}_\mu)$ it holds $v(x)\in T_\mu(x)$ for μ-a.e. $x\in\mathbb{R}^d$. It then permits to give the notion of *tangential gradient* $\nabla_\mu f$ of a compactly-supported smooth function, by simply setting

$$\nabla_\mu f(x) := \mathrm{pr}_{T_\mu(x)}(\nabla f(x)),\quad f\in C_c^\infty(\mathbb{R}^d).$$

It has been proven in [2] that the space $\mathrm{BV}(\mathbb{R}^d,\mu)$ can be equivalently characterized as the domain of finiteness of the relaxation (in the strong $L^1_\mu(\mathbb{R}^d)$-topology) of the functional associating to every $f\in C_c^\infty(\mathbb{R}^d)$ the quantity $\int_{\mathbb{R}^d}|\nabla_\mu f|\,\mathrm{d}\mu$ and set to be $+\infty$ elsewhere in $L^1_\mu(\mathbb{R}^d)$. Moreover, it holds that

$$\|D_\mu f\| = \inf \varliminf_{n\to\infty}\int_{\mathbb{R}^d}|\nabla_\mu f_n|\,\mathrm{d}\mu,\quad f\in \mathrm{BV}(\mathbb{R}^d,\mu), \tag{1.1}$$

where the infimum is taken among all $(f_n)_n\subseteq C_c^\infty(\mathbb{R}^d)$ converging strongly in $L^1_\mu(\mathbb{R}^d)$ to f.

The Sobolev space $W^{1,1}(\mathbb{R}^d,\mu)$ is defined (as in the case $p>1$ in [6]) as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{W^{1,1}(\mathbb{R}^d,\mu)} = \|f\|_{L^1_\mu(\mathbb{R}^d)} + \|\nabla_\mu f\|_{L^1_\mu(\mathbb{R}^d;\mathbb{R}^d)}.$$

Such defined space is indeed a space of functions (not just an abstract Banach space), due to the closability property of the tangential gradient operator, which therefore extends to the whole space $W^{1,1}(\mathbb{R}^d,\mu)$. These properties of ∇_μ, that have been stated in [2], are proven in Lemma 4.4, Corollary 4.5, and Proposition 4.6 for the sake of completeness.

Looking from the point of view of the metric measure space theory, there are two approaches that will be relevant for the purposes of the present paper.

(i) The first one is a variant of the relaxation-type approach from [17], given in [8]. This approach involves, due to the lack (in general) of a smooth structure of the underlying space, locally Lipschitz functions. The role of $|\nabla_\mu f|$ in the relaxation argument above is here played by the asymptotic Lipschitz constant (denoted hereafter by $\mathrm{lip}_a(f)$ for any f Lipschitz; see (3.2) for its definition). Hence, when we stick to the specific case of the weighted Euclidean space, the space of BV functions $\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ is defined as the set of those $f\in L^1_\mu(\mathbb{R}^d)$ for which the quantity

$$\|D_\mu f\|_{\mathrm{Lip}} := \inf \varliminf_{n\to\infty}\int_{\mathbb{R}^d}\mathrm{lip}_a(f_n)\,\mathrm{d}\mu$$

is finite. The infimum above is taken among all sequences $(f_n)_n$ of locally Lipschitz functions converging to f strongly in $L^1_\mu(\mathbb{R}^d)$. Localizing the above procedure, one can associate to each $f\in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ its *total variation measure* $|D_\mu f|_{\mathrm{Lip}}$, which on any open set $\Omega\subseteq\mathbb{R}^d$ reads as

$$|D_\mu f|_{\mathrm{Lip}}(\Omega) := \inf \varliminf_{n\to\infty}\int_\Omega \mathrm{lip}_a(f_n)\,\mathrm{d}\mu.$$

The corresponding definition of the Sobolev space $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ is as follows: $f\in L^1_\mu(\mathbb{R}^d)$ belongs to $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ if there exists a sequence $(f_n)_n$ of compactly-supported Lipschitz functions converging to f strongly in $L^1_\mu(\mathbb{R}^d)$ and such that $(\mathrm{lip}_a(f_n))_n$ is weakly convergent in $L^1_\mu(\mathbb{R}^d)$.

(ii) Another definition of BV function we will consider was proposed in [9] and is based on the notion of *bounded derivation* $\mathbf{b}$ admitting *bounded divergence* $\mathrm{div}(\mathbf{b})$. Such a derivation can be thought of as a linear map acting on boundedly-supported Lipschitz functions and having values in the space of essentially bounded functions. Also, this derivation enjoys a suitable Leibniz rule and a locality property. We refer to Subsection 3.3 for the definition, in the specific case of the weighted Euclidean space, of the above-described space of derivations, that we denote by $\mathrm{Der}_b(\mathbb{R}^d,\mu)$. With this notion at disposal, the space of BV functions $\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)$ is defined as the space of those $f\in L^1_\mu(\mathbb{R}^d)$ for which there exists a continuous, linear (also with respect to the multiplication by Lipschitz functions) operator $L_f\colon \mathrm{Der}_b(\mathbb{R}^d,\mu)\to\mathscr{M}(\mathbb{R}^d)$, such that

$$L_f(\mathbf{b})(\mathbb{R}^d)=-\int_{\mathbb{R}^d} f\,\mathrm{div}(\mathbf{b})\,\mathrm{d}\mu \quad \text{for all } \mathbf{b}\in\mathrm{Der}_b(\mathbb{R}^d,\mu).$$

Here we denote by $\mathscr{M}(\mathbb{R}^d)$ the space of all finite signed Borel measures on $\mathbb{R}^d$. The total variation associated to a BV function $f\in\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)$ is given by the quantity

$$\|D_\mu f\|_{\mathrm{Der}}\coloneqq\sup\left\{\int_{\mathbb{R}^d} f\,\mathrm{div}(\mathbf{b})\,\mathrm{d}\mu:\ \mathbf{b}\in\mathrm{Der}_b(\mathbb{R}^d,\mu),|\mathbf{b}|\le 1\ \mu\text{-a.e.}\right\}.$$

Similarly, the total variation measure associated with f is given on any $\Omega\subseteq\mathbb{R}^d$ open by

$$|D_\mu f|_{\mathrm{Der}}(\Omega)\coloneqq\sup\left\{\int_{\Omega} f\,\mathrm{div}(\mathbf{b})\,\mathrm{d}\mu:\ \mathbf{b}\in\mathrm{Der}_b(\mathbb{R}^d,\mu),|\mathbf{b}|\le 1\ \mu\text{-a.e. and }\mathrm{supp}(\mathbf{b})\Subset\Omega\right\}.$$

It follows from [8] that

$$\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)=\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu),\quad |D_\mu f|_{\mathrm{Lip}}=|D_\mu f|_{\mathrm{Der}}\quad \text{for every } f\in\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu). \tag{1.2}$$

The present paper provides the following results:

(I) $\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)=\mathrm{BV}_{C^\infty}(\mathbb{R}^d,\mu)$: As one might expect, we show that in the setting of the weighted Euclidean space, smooth functions are enough for the approximation in the relaxation process that leads to the definition of the space $\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ described in (i). The same holds also for the total variation measure, namely $|D_\mu f|_{\mathrm{Lip}}=|D_\mu f|_{C^\infty}$ (see Theorem 5.3). Recalling that for every $f\in C^\infty(\mathbb{R}^d)$ it holds that $\mathrm{lip}_a(f)=|\nabla f|$, note that the quantity $\|D_\mu f\|_{\mathrm{Lip}}$, a priori, might differ from the quantity $\|D_\mu f\|$ given in (1.1).

(II) $\mathrm{BV}(\mathbb{R}^d,\mu)=\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)$: In order to prove it, we first show in Section 5.2 that there exists an isometric isomorphism between $\mathrm{D}_\infty(\mathrm{div}_\mu)$ and $\mathrm{Der}_b(\mathbb{R}^d,\mu)$. Due to this fact, we have that $\|D_\mu f\|=\|D_\mu f\|_{\mathrm{Der}}$ for every $f\in L^1_\mu(\mathbb{R}^d)$. This immediately implies $\mathrm{BV}(\mathbb{R}^d,\mu)\supseteq\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)$. To get the opposite inclusion, we use the equivalent characterization of $\|D_\mu f\|$ given in (1.1) in order to construct the operator L_f as in point (ii) above, associated with $f\in\mathrm{BV}(\mathbb{R}^d,\mu)$ (see Theorem 5.7). Taking into account (I) and (1.2) we finally get

$$\mathrm{BV}(\mathbb{R}^d,\mu)=\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)=\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)=\mathrm{BV}_{C^\infty}(\mathbb{R}^d,\mu),$$

and, moreover, that $|D_\mu f|_{\mathrm{Der}}=|D_\mu f|_{\mathrm{Lip}}=|D_\mu f|_{C^\infty}$ as measures, for every $f\in\mathrm{BV}(\mathbb{R}^d,\mu)$.

(III) $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)\subseteq W^{1,1}(\mathbb{R}^d,\mu)$: In the case of $W^{1,1}$ spaces, there are not many instances where the various notions provided in [8] do coincide. Also in this case we obtain (in Theorem 5.8) only the above inclusion: to do so, we need to perform a careful study of the tangential gradient operator and its behaviour on compactly-supported Lipschitz functions. The entire Subsection 4.1 is devoted to this. In particular, it allows us to give an equivalent characterization of the space $W^{1,1}(\mathbb{R}^d,\mu)$ (see Theorem 4.13), which turns out to be more suitable for showing that $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)\subseteq W^{1,1}(\mathbb{R}^d,\mu)$.

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2. Preliminaries

In this paper we are going to work in the setting of the weighted Euclidean space, namely, in the space $\mathbb{R}^d$ equipped with the Euclidean distance $\mathsf{d}_{\rm Eucl}(x,y) \coloneqq |x-y|$ and an arbitrary non-negative Radon measure μ. The space $(\mathbb{R}^d, \mathsf{d}_{\rm Eucl}, \mu)$ will be fixed to the end of the paper. Let us recall some basic notions that will be used throughout.

We denote by $\mathrm{LIP}(\mathbb{R}^d)$ the space of all real-valued Lipschitz functions on $\mathbb{R}^d$, whereas $\mathrm{LIP}_c(\mathbb{R}^d)$ stands for the family of all elements of $\mathrm{LIP}(\mathbb{R}^d)$ having compact support. The Lipschitz constant of the restriction of a function $f \in \mathrm{LIP}(\mathbb{R}^d)$ to a set $E \subseteq \mathbb{R}^d$ will be denoted by $\mathrm{Lip}(f;E) \in [0,+\infty)$, while the global Lipschitz constant of f will be denoted by $\mathrm{Lip}(f) \coloneqq \mathrm{Lip}(f;\mathbb{R}^d)$ for brevity.

Given any $p \in [1,\infty)$, we denote by $L^p_\mu(\mathbb{R}^d;\mathbb{R}^k)$ the space of p-integrable (with respect to μ) $\mathbb{R}^k$-valued maps on $\mathbb{R}^d$, while $L^\infty_\mu(\mathbb{R}^d;\mathbb{R}^k)$ stands for the space of μ-essentially bounded $\mathbb{R}^k$-valued maps on $\mathbb{R}^d$, in both cases considered up to μ-a.e. equality. By $L^0_\mu(\mathbb{R}^d)$ we shall denote the space of all μ-measurable functions on $\mathbb{R}^d$, again considered up to μ-a.e. equality. It is well-known that the space $L^p_\mu(\mathbb{R}^d;\mathbb{R}^k)$ is a Banach space for any $p \in [1,\infty]$, with respect to the norm

$$\|v\|_{L^p_\mu(\mathbb{R}^d;\mathbb{R}^k)} \coloneqq \||v|\|_{L^p_\mu(\mathbb{R}^d)}, \qquad \text{for every } v \in L^p_\mu(\mathbb{R}^d;\mathbb{R}^k).$$

A mollification argument will be often useful. Hence, we fix once and for all a kernel of mollification ρ on $\mathbb{R}^d$, *i.e.*, a smooth, symmetric function $\rho \in C^\infty_c(\mathbb{R}^d)$ such that $\rho \geq 0$, $\mathrm{supp}(\rho) \subseteq B_1(0)$, and $\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}\mathcal{L}^n(x) = 1$. Given any $\varepsilon > 0$, we define $\rho_\varepsilon \in C^\infty_c(\mathbb{R}^d)$ as

$$\rho_\varepsilon(x) \coloneqq \varepsilon^n \rho(x/\varepsilon), \qquad \text{for every } x \in \mathbb{R}^d.$$

Notice that $\mathrm{supp}(\rho_\varepsilon) \subseteq B_\varepsilon(0)$ and $\int_{\mathbb{R}^d} \rho_\varepsilon(x)\,\mathrm{d}\mathcal{L}^n(x) = 1$. Given a locally integrable, Borel function $f\colon \mathbb{R}^d \to \mathbb{R}$, we define its ε-mollification (with kernel ρ) as the convolution between ρ_ε and f, *i.e.*,

$$(\rho_\varepsilon * f)(x) \coloneqq \int_{\mathbb{R}^d} \rho_\varepsilon(x-y) f(y)\,\mathrm{d}\mathcal{L}^n(y) = \int_{\mathbb{R}^d} \rho_\varepsilon(y) f(x+y)\,\mathrm{d}\mathcal{L}^n(y), \qquad \text{for every } x \in \mathbb{R}^d.$$

In the following result we collect the main well-known properties of the mollification:

Lemma 2.1 (Approximation of compactly-supported Lipschitz functions)**.** *Let $f \in \mathrm{LIP}_c(\mathbb{R}^d)$ be given. Then for every $\varepsilon > 0$ the ε-mollification $f_\varepsilon \coloneqq \rho_\varepsilon * f \in C^\infty_c(\mathbb{R}^d)$ satisfies*

$$\mathrm{supp}(f_\varepsilon) \subseteq B_\varepsilon(\mathrm{supp}(f)), \tag{2.1a}$$

$$\big|f_\varepsilon(x) - f(x)\big| \leq \mathrm{Lip}(f)\varepsilon, \quad \textit{for every } x \in \mathbb{R}^d, \tag{2.1b}$$

$$|\nabla f_\varepsilon(x)| \leq \mathrm{Lip}\big(f; B_{2\varepsilon}(x)\big), \quad \textit{for every } x \in \mathbb{R}^d. \tag{2.1c}$$

Moreover, it holds $f = \lim_{\varepsilon \searrow 0} f_\varepsilon$ strongly in $L^p_\mu(\mathbb{R}^d)$ for any $p \in [1,\infty)$ and weakly* in $L^\infty_\mu(\mathbb{R}^d)$.

Proof. The fact that $f_\varepsilon \in C^\infty(\mathbb{R}^d)$ is well-known. Given any $x \in \mathbb{R}^d \setminus B_\varepsilon(\mathrm{supp}(f))$, we have that $x + y \notin \mathrm{supp}(f)$ for every $y \in B_\varepsilon(0)$, thus $f_\varepsilon(x) = \int_{B_\varepsilon(0)} \rho_\varepsilon(y) f(x+y)\,\mathrm{d}\mathcal{L}^n(y) = 0$, getting (2.1a) and in particular that $f_\varepsilon \in C^\infty_c(\mathbb{R}^d)$. Now observe that for any $x \in \mathbb{R}^d$ we may estimate

$$\big|f_\varepsilon(x) - f(x)\big| \leq \int_{\mathbb{R}^d} \big|f(x+y) - f(x)\big| \rho_\varepsilon(y)\,\mathrm{d}\mathcal{L}^n(y) \leq \mathrm{Lip}(f) \int_{B_\varepsilon(0)} |y| \rho_\varepsilon(y)\,\mathrm{d}\mathcal{L}^n(y) \leq \mathrm{Lip}(f)\varepsilon,$$

which proves (2.1b). To verify (2.1c), take $y \in B_\varepsilon(x)$ with $y \neq x$. Then it holds that

$$\big|f_\varepsilon(y) - f_\varepsilon(x)\big| \leq \int_{B_\varepsilon(0)} \big|f(y+v) - f(x+v)\big| \rho_\varepsilon(v)\,\mathrm{d}\mathcal{L}^n(v) \leq \mathrm{Lip}\big(f; B_{2\varepsilon}(x)\big)|y-x|.$$

By dividing the above inequality by $|y-x|$ and passing to the limit as $y \to x$, we get that $|\nabla f_\varepsilon(x)| \le \mathrm{Lip}\big(f; B_{2\varepsilon}(x)\big)$, proving (2.1c). Finally, for any $\varepsilon \in (0,1)$ and $x \in \mathbb{R}^d$ we have that

$$|f_\varepsilon(x)| \le \int_{\mathbb{R}^d} |f(x+y)|\rho_\varepsilon(y)\,\mathrm{d}\mathcal{L}^n(y) \le \sup_{\mathbb{R}^d}|f| \int_{\mathbb{R}^d} \rho_\varepsilon\,\mathrm{d}\mathcal{L}^n = \sup_{\mathbb{R}^d}|f|,$$

which together with (2.1a) grant that $|f_\varepsilon| \le \chi_K \|f\|_{L^\infty_\mu(\mathbb{R}^d)} \in L^1_\mu(\mathbb{R}^d) \cap L^\infty_\mu(\mathbb{R}^d)$ for all $\varepsilon \in (0,1)$, where K stands for the closed 1-neighbourhood of $\mathrm{supp}(f)$. We know from (2.1b) that f_ε pointwise converges to f as $\varepsilon \searrow 0$, so applying the dominated convergence theorem we get $f = \lim_{\varepsilon\searrow 0} f_\varepsilon$ strongly in $L^p_\mu(\mathbb{R}^d)$ for any $p \in [1,\infty)$. For any $h \in L^1_\mu(\mathbb{R}^d)$ we have that $|hf_\varepsilon| \le \chi_K |h| \|f\|_{L^\infty_\mu(\mathbb{R}^d)}$ holds μ-a.e. for every $\varepsilon \in (0,1)$ and $(hf)(x) = \lim_{\varepsilon\searrow 0}(hf_\varepsilon)(x)$ for μ-a.e. $x \in \mathbb{R}^d$, thus by applying again the dominated convergence theorem we conclude that $\int_{\mathbb{R}^d} hf\,\mathrm{d}\mu = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d} hf_\varepsilon\,\mathrm{d}\mu$. Thanks to the arbitrariness of $h \in L^1_\mu(\mathbb{R}^d)$, we conclude that $f = \lim_{\varepsilon\searrow 0} f_\varepsilon$ weakly* in $L^\infty_\mu(\mathbb{R}^d)$. □

For the reader's usefulness, in the following statement we collect some well-known fundamental results in functional analysis, which will be used several times in the sequel.

Proposition 2.2. *The following properties are verified:*

i) *Let $(v_n)_n \subseteq L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ and $v \in L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ be such that $v_n \to v$ strongly in $L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$. Then some subsequence $(v_{n_i})_i$ of $(v_n)_n$ is* dominated*, i.e. there exists $g \in L^1_\mu(\mathbb{R}^d)$ such that $|v_{n_i}| \le g$ holds μ-a.e. for every $i \in \mathbb{N}$. Moreover, we can further require that $v_{n_i}(x) \to v(x)$ as $i \to \infty$ for μ-a.e. $x \in \mathbb{R}^d$.*

ii) *Let $(v_n)_n \subseteq L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ and $v \in L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ be such that $v_n \rightharpoonup v$ weakly in $L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$. Then for any $n \in \mathbb{N}$ there exist coefficients $(\alpha^n_i)_{i=n}^{N_n} \subseteq [0,1]$, for some $N_n \in \mathbb{N}$ with $N_n \ge n$, such that $\sum_{i=n}^{N_n}\alpha^n_i = 1$ and $\sum_{i=n}^{N_n}\alpha^n_i v_i \to v$ strongly in $L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ as $n \to \infty$.*

iii) *Let $(v_n)_n \subseteq L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ be a dominated sequence. Then there exist $v \in L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ and a subsequence $(v_{n_i})_i$ of $(v_n)_n$ such that $v_{n_i} \rightharpoonup v$ weakly in $L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ as $i \to \infty$.*

Proof. Observe that, writing $v = (v^1,\dots,v^k) \in L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$, we may estimate

$$\max_{i=1,\dots,k}|v^i| \le |v| = \big(|v^1|^2 + \dots + |v^k|^2\big)^{1/2} \le \sqrt{k}\max_{i=1,\dots,k}|v^i|, \quad \text{in the } \mu\text{-a.e. sense.}$$

In particular, a sequence $(v_n)_n \subseteq L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ is dominated if and only if $(v^i_n)_n \subseteq L^1_\mu(\mathbb{R}^d)$ is dominated for every $i = 1,\dots,k$. Moreover, it is easy to check that v_n converges strongly (resp. weakly) in $L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ to some vector field $v = (v^1,\dots,v^k) \in L^1_\mu(\mathbb{R}^d;\mathbb{R}^k)$ if and only if v^i_n converges strongly (resp. weakly) in $L^1_\mu(\mathbb{R}^d)$ to v^i for every $i = 1,\dots,k$. Thanks to these observations, we can prove the statement by arguing componentwise, i.e. it suffices to deal with the case $k = 1$.

i) Fix any $(f_n)_n \subseteq L^1_\mu(\mathbb{R}^d)$ and $f \in L^1_\mu(\mathbb{R}^d)$ such that $f_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$. Then we can find a subsequence $(n_i)_i$ satisfying $\|f_{n_i} - f_{n_{i+1}}\|_{L^1_\mu(\mathbb{R}^d)} \le 1/2^i$ for every $i \in \mathbb{N}$. Now let us define

$$g(x) \coloneqq |f_{n_1}|(x) + \sum_{i=1}^\infty |f_{n_i} - f_{n_{i+1}}|(x), \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

The μ-a.e. defined functions $g_j \coloneqq |f_{n_1}| + \sum_{i=1}^j |f_{n_i} - f_{n_{i+1}}|$ satisfy $g_j \nearrow g$ in the μ-a.e. sense and

$$\int g_j\,\mathrm{d}\mu \le \int |f_{n_1}|\,\mathrm{d}\mu + \sum_{i=1}^j \int |f_{n_i} - f_{n_{i+1}}|\,\mathrm{d}\mu \le \|f_{n_1}\|_{L^1_\mu(\mathbb{R}^d)} + \sum_{i=1}^j \frac{1}{2^i} \le \|f_{n_1}\|_{L^1_\mu(\mathbb{R}^d)} + 1.$$

By using the monotone convergence theorem we get that $\int g\,\mathrm{d}\mu = \lim_j \int g_j\,\mathrm{d}\mu \le \|f_{n_1}\|_{L^1_\mu(\mathbb{R}^d)} + 1$, thus in particular $g \in L^1_\mu(\mathbb{R}^d)$. Notice also that for any $j \in \mathbb{N}$ with $j \ge 2$ we have that

$$|f_{n_j}| = \left|f_{n_1} + \sum_{i=2}^j (f_{n_i} - f_{n_{i-1}})\right| \le |f_{n_1}| + \sum_{i=2}^j |f_{n_i} - f_{n_{i-1}}| \le g, \quad \text{in the } \mu\text{-a.e. sense.}$$

All in all, we have proved that $(f_{n_i})_i$ is dominated by g. Finally, from the fact that $g(x) < +\infty$ for μ-a.e. $x \in \mathbb{R}^d$ we deduce that $\big(f_{n_i}(x)\big)_i \subseteq \mathbb{R}$ is a Cauchy sequence for μ-a.e. $x \in \mathbb{R}^d$. Given that $f_{n_i} \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$, we thus conclude that $f_{n_i}(x) \to f(x)$ for μ-a.e. $x \in \mathbb{R}^d$.
ii) Immediate consequence of Mazur Lemma, applied to the Banach space $L^1_\mu(\mathbb{R}^d)$.
iii) It follows from Dunford–Pettis Theorem. For a more direct proof, see [12, Lemma 1.3.22]. □

Remark 2.3. With $(\mathscr{M}(\mathbb{R}^d), \|\cdot\|_{\mathsf{TV}})$ we denote the space of finite, signed Borel measures on $\mathbb{R}^d$. Endowed with the total variation norm, denoted above by $\|\cdot\|_{\mathsf{TV}}$, it results in a Banach space. We recall that we can identify $(\mathscr{M}(\mathbb{R}^d), \|\cdot\|_{\mathsf{TV}})$ with the dual of the Banach space $C_0(\mathbb{R}^d) \coloneqq \mathrm{cl}_{C_b(\mathbb{R}^d)}(C_c(\mathbb{R}^d))$. Here, $C_b(\mathbb{R}^d)$ (resp. $C_c(\mathbb{R}^d)$) stands for the space of bounded (resp. compactly-supported) continuous, real-valued functions on $\mathbb{R}^d$. Recall also that $C_b(\mathbb{R}^d)$ is a Banach space when endowed with the supremum norm $\|f\|_{C_b(\mathbb{R}^d)} \coloneqq \sup_{x\in\mathbb{R}^d} |f(x)|$. ■

3. Different notions of BV space

3.1. **BV space via vector fields.** The first attempt to the definition of BV functions in the setting of the weighted Euclidean spaces has been done in [2]. It is based on the notion of μ-divergence of a vector field which we are going to recall below.

A vector field $v \in L^\infty_\mu(\mathbb{R}^d;\mathbb{R}^d)$ is said to be a vector field with bounded μ-divergence (in a distributional sense) if there exists a function $\mathrm{div}_\mu(v) \in L^\infty_\mu(\mathbb{R}^d)$ such that the following integration-by-parts formula holds:

$$\int_{\mathbb{R}^d} \nabla f \cdot v \, \mathrm{d}\mu = -\int_{\mathbb{R}^d} f \, \mathrm{div}_\mu(v) \, \mathrm{d}\mu, \qquad \text{for every } f \in C^\infty_c(\mathbb{R}^d).$$

Whenever it exists, $\mathrm{div}_\mu(v)$ is uniquely determined. Let us define the space

$$\mathrm{D}_\infty(\mathrm{div}_\mu) \coloneqq \big\{ v \in L^\infty_\mu(\mathbb{R}^d;\mathbb{R}^d) : \, v \text{ has bounded } \mu\text{-divergence} \big\}.$$

The following definition of BV function has been proposed in [2]:

Definition 3.1 (BV space via vector fields)**.** *We say that a function $f \in L^1_\mu(\mathbb{R}^d)$ belongs to the space $\mathrm{BV}(\mathbb{R}^d,\mu)$ if the quantity*

$$\|D_\mu f\| \coloneqq \sup\left\{ \int_{\mathbb{R}^d} f \, \mathrm{div}_\mu(v) \, \mathrm{d}\mu : \, v \in \mathrm{D}_\infty(\mathrm{div}_\mu), \, |v| \leq 1 \, \mu\text{-a.e.} \right\} \tag{3.1}$$

is finite.

The total variation defined in (3.1) can be localized on open sets as follows: given any function $f \in \mathrm{BV}(\mathbb{R}^d,\mu)$ and any open set $\Omega \subseteq \mathbb{R}^d$, we set

$$|D_\mu f|(\Omega) \coloneqq \sup\left\{ \int_\Omega f \, \mathrm{div}_\mu(v) \, \mathrm{d}\mu : \, v \in \mathrm{D}_\infty(\mathrm{div}_\mu), \, \mathrm{supp}(v) \Subset \Omega, \, |v| \leq 1 \, \mu\text{-a.e.} \right\}.$$

Notice that $|D_\mu f|(\mathbb{R}^d) = \|D_\mu f\|$ by definition. The function $|D_\mu f|$ can be extended to all Borel sets $B \subseteq \mathbb{R}^d$ via Carathéodory construction, as follows:

$$|D_\mu f|(B) \coloneqq \inf\big\{ |D_\mu f|(\Omega) : \, \Omega \subseteq \mathbb{R}^d \text{ open}, \, B \subseteq \Omega \big\}.$$

It turns out that $|D_\mu f|$ is a finite Borel measure on $\mathbb{R}^d$. However, we do not verify it right now; we will obtain it as a consequence of Theorem 5.7.

3.2. **BV space via relaxation.** The relaxation-type approach to the definition of BV space has been firstly introduced in [17] in the setting of metric measure spaces. We shall present here a slight variant of it, which has been proposed in [8].

Given any open set $\Omega \subseteq \mathbb{R}^d$, we denote by $\mathrm{LIP}_{loc}(\Omega)$ the family of all locally Lipschitz functions on Ω, i.e. those functions $f\colon \Omega \to \mathbb{R}$ satisfying the following: for every $x \in \Omega$ there exists $r_x > 0$ such that $f|_{B_{r_x}(x)}$ is Lipschitz. Given any $f \in \mathrm{LIP}_{loc}(\Omega)$, we shall denote by $\mathrm{lip}_a(f)\colon \Omega \to [0,+\infty)$ its *asymptotic Lipschitz constant*, which is defined as

$$\mathrm{lip}_a(f)(x) \coloneqq \lim_{r\to 0} \mathrm{Lip}\big(f; B_r(x)\big) = \varlimsup_{\substack{y\neq z\\ y,z\to x}} \frac{|f(y)-f(z)|}{\mathsf{d}(y,z)}, \tag{3.2}$$

if $x \in \Omega$ is an accumulation point and $\mathrm{lip}_a(f)(x) \coloneqq 0$ otherwise.

Taking into account [8, Theorem 4.5.3], we have the following definition of BV space:

Definition 3.2 (BV space via relaxation)**.** *We say that a function $f \in L^1_\mu(\mathbb{R}^d)$ belongs to the space $\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$, if one of the following equivalent conditions is satisfied:*

1) *There exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_{loc}(\mathbb{R}^d) \cap L^1_\mu(\mathbb{R}^d)$ such that*

$$f_n \to f \ \text{ in } L^1_\mu(\mathbb{R}^d) \quad \text{ and } \quad \sup_n \int_{\mathbb{R}^d} \mathrm{lip}_a(f_n)\,\mathrm{d}\mu < +\infty.$$

2) *There exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ such that*

$$f_n \to f \ \text{ in } L^1_\mu(\mathbb{R}^d) \quad \text{ and } \quad \sup_n \int_{\mathbb{R}^d} \mathrm{lip}_a(f_n)\,\mathrm{d}\mu < +\infty.$$

Given any $f \in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$, the *total variation measure* $|D_\mu f|_{\mathrm{Lip}}$ associated with f is defined in [8] as

$$|D_\mu f|_{\mathrm{Lip}}(B) \coloneqq \inf\big\{|D_\mu f|_{\mathrm{Lip}}(\Omega) : \Omega \subseteq \mathbb{R}^d \text{ open},\ B \subseteq \Omega\big\}, \qquad \text{for every } B \subseteq \mathbb{R}^d \text{ Borel},$$

where for any open set $\Omega \subseteq \mathbb{R}^d$ we set

$$|D_\mu f|_{\mathrm{Lip}}(\Omega) \coloneqq \inf\bigg\{ \varliminf_{n\to\infty} \int_\Omega \mathrm{lip}_a(f_n)\,\mathrm{d}\mu : (f_n)_n \subseteq \mathrm{LIP}_{loc}(\Omega) \cap L^1_\mu(\Omega),\ f_n \to f \text{ in } L^1_\mu(\Omega)\bigg\}. \tag{3.3}$$

The *total variation* of f, *i.e.*, the total variation measure evaluated at the entire space, can be recovered by using only compactly-supported Lipschitz functions (cf. [8, Theorem 4.5.3]), namely:

$$|D_\mu f|_{\mathrm{Lip}}(\mathbb{R}^d) = \inf\bigg\{ \varliminf_{n\to\infty} \int_{\mathbb{R}^d} \mathrm{lip}_a(f_n)\,\mathrm{d}\mu : (f_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d),\ f_n \to f \text{ in } L^1_\mu(\mathbb{R}^d)\bigg\}. \tag{3.4}$$

As observed in the example preceding [8, Proposition 4.4.1], the formulation in (3.4), using compactly-supported Lipschitz functions instead of locally Lipschitz ones, cannot be used (in general) to compute the quantity $|D_\mu f|_{\mathrm{Lip}}(\Omega)$ for any open set $\Omega \subseteq \mathbb{R}^d$.

Remark 3.3. While Definition 3.1 is tailored to the weighted Euclidean space setting, the concept in Definition 3.2 (as well as the one in Definition 3.5) actually makes sense on any metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, *i.e.*, (X,d) is a complete separable metric space and $\mathfrak{m}$ is a boundedly-finite Borel measure on X; we refer to [8] for the details. This remark will play a role in Section 5.1. ■

3.3. **BV space via derivations.** In this subsection we report the definition of BV space via derivations proposed in [8]. We start by recalling the definition of derivation introduced in [8] and point out some of its basic properties.

Definition 3.4. *By a* derivation *on $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$ we mean any linear map $\mathbf{b}\colon \mathrm{LIP}_c(\mathbb{R}^d) \to L^0_\mu(\mathbb{R}^d)$ satisfying the following two properties:*

1) Leibniz rule: *For every $f, g \in \mathrm{LIP}_c(\mathbb{R}^d)$, it holds that*

$$\mathbf{b}(fg) = \mathbf{b}(f)g + f\mathbf{b}(g).$$

2) Weak locality: *There exists a non-negative function $G \in L^0_\mu(\mathbb{R}^d)$ such that*

$$|\mathbf{b}(f)| \leq G \,\mathrm{lip}_a(f) \quad \text{holds } \mu\text{-a.e., for every } f \in \mathrm{LIP}_c(\mathbb{R}^d).$$

The least function G as above will be denoted by $|\mathbf{b}|$.

We recall from [10] that for a given derivation $\mathbf{b}$, one has the following formula for $|\mathbf{b}|$:

$$|\mathbf{b}| = \operatorname{ess\,sup} \big\{ \mathbf{b}(f) : \, f \in \mathrm{LIP}_c(\mathbb{R}^d), \, \mathrm{Lip}(f) \leq 1 \big\}. \tag{3.5}$$

By the *support* of a derivation $\mathbf{b}$ we mean the support of the associated function $|\mathbf{b}|$, and we denote it by $\mathrm{supp}(\mathbf{b})$.

We denote by $\mathrm{Der}(\mathbb{R}^d, \mu)$ the space of all derivations on $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$, while $\mathrm{Der}_\infty(\mathbb{R}^d, \mu)$ stands for the space of all *bounded derivations*, *i.e.*, $\mathrm{Der}_\infty(\mathbb{R}^d, \mu) \coloneqq \big\{ \mathbf{b} \in \mathrm{Der}(\mathbb{R}^d, \mu) : \, |\mathbf{b}| \in L^\infty_\mu(\mathbb{R}^d) \big\}$. We have that $\big(\mathrm{Der}_\infty(\mathbb{R}^d, \mu), \| \cdot \|_b\big)$ is a Banach space, where we set

$$\|\mathbf{b}\|_b \coloneqq \| |\mathbf{b}| \|_{L^\infty_\mu(\mathbb{R}^d)}, \quad \text{for every } \mathbf{b} \in \mathrm{Der}_\infty(\mathbb{R}^d, \mu).$$

In what follows we will be concentrated on those elements $\mathbf{b} \in \mathrm{Der}_\infty(\mathbb{R}^d, \mu)$ admitting *bounded divergence*, *i.e.*, for which there is a (uniquely determined) function $\mathrm{div}(\mathbf{b}) \in L^\infty_\mu(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} \mathbf{b}(f) \, \mathrm{d}\mu = - \int_{\mathbb{R}^d} f \, \mathrm{div}(\mathbf{b}) \, \mathrm{d}\mu, \quad \text{for every } f \in \mathrm{LIP}_c(\mathbb{R}^d).$$

The space of all bounded derivations with bounded divergence will be denoted by $\mathrm{Der}_b(\mathbb{R}^d, \mu)$, which is a Banach space when endowed with the norm

$$\|\mathbf{b}\|_{b,b} \coloneqq \|\mathbf{b}\|_b + \|\mathrm{div}(\mathbf{b})\|_{L^\infty_\mu(\mathbb{R}^d)}.$$

Let us recall the Leibniz rule for the divergence: given any $h \in \mathrm{LIP}_c(\mathbb{R}^d)$ and $\mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu)$, it holds that $h\mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu)$ and

$$\mathrm{div}(h\mathbf{b}) = \mathbf{b}(h) + h \, \mathrm{div}(\mathbf{b}). \tag{3.6}$$

Definition 3.5. *We say that a function $f \in L^1_\mu(\mathbb{R}^d)$ belongs to the space $\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu)$ if there exists a $\mathrm{LIP}_c(\mathbb{R}^d)$-linear and $\| \cdot \|_b$-continuous map $Df \colon \mathrm{Der}_b(\mathbb{R}^d, \mu) \to \mathscr{M}(\mathbb{R}^d)$ satisfying*

$$\int_{\mathbb{R}^d} \mathrm{d}Df(\mathbf{b}) = - \int_{\mathbb{R}^d} f \, \mathrm{div}(\mathbf{b}) \, \mathrm{d}\mu, \quad \textit{for every } \mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu).$$

In this case, the map Df is uniquely determined.

Given any $f \in \mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu)$, the *total variation measure* associated with f is defined as the unique finite Borel measure $|D_\mu f|_{\mathrm{Der}}$ on $\mathbb{R}^d$ that for each open set $\Omega \subseteq \mathbb{R}^d$ satisfies

$$|D_\mu f|_{\mathrm{Der}}(\Omega) = \sup \left\{ \int_\Omega f \, \mathrm{div}(\mathbf{b}) \, \mathrm{d}\mu : \, \mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu), \, \mathrm{supp}(\mathbf{b}) \Subset \Omega, \, |\mathbf{b}| \leq 1 \ \mu\text{-a.e.} \right\}.$$

4. Different notions of $W^{1,1}$ space

4.1. **$W^{1,1}$ space via vector fields.** An approach based on a notion of a 'space tangent to a measure' (and in turn on the properties of vector fields with divergence) has been used in the pioneering work [6] to propose a concept of Sobolev space $W^{1,p}$ in the case $p \in (1, \infty)$. As observed in [2], the very same technique may be applied in the case $p = 1$. Below, we recall the definition of $W^{1,1}$ space from [2] and study more in details the properties of the 'tangential gradient operator' which plays a crucial role in its definition.

First of all, let us recall the definition of a (measurable) bundle in $\mathbb{R}^d$, following quite closely the presentation in [16]. Let V be a map assigning to any point $x \in \mathbb{R}^d$ a vector subspace $V(x)$ of $\mathbb{R}^d$. Then we say that V is a (measurable) bundle in $\mathbb{R}^d$ provided $\mathbb{R}^d \ni x \mapsto \mathsf{d}_{\rm Eucl}(y, V(x)) \in \mathbb{R}$ is Borel measurable for any $y \in \mathbb{R}^d$. A partial order (depending on μ) on the family of all bundles in $\mathbb{R}^d$ is given as follows: if V and W are bundles in $\mathbb{R}^d$, we declare that $V \preceq W$ provided $V(x) \subseteq W(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Given an exponent $p \in [1, \infty]$ and a bundle V in $\mathbb{R}^d$, we denote by $\Gamma^p_\mu(V)$ the space of all $L^p_\mu(\mathbb{R}^d)$-sections of V, namely,

$$\Gamma^p_\mu(V) \coloneqq \big\{ v \in L^p_\mu(\mathbb{R}^d; \mathbb{R}^d) \,:\, v(x) \in V(x) \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d \big\}.$$

Observe that $\Gamma^p_\mu(V)$ is a closed vector subspace of $L^p_\mu(\mathbb{R}^d; \mathbb{R}^d)$ which is closed under multiplication by $L^\infty_\mu(\mu)$-functions. As proven in [16, Proposition 2.22], it holds that

$$V \preceq W \quad \Longleftrightarrow \quad \Gamma^2_\mu(V) \subseteq \Gamma^2_\mu(W), \tag{4.1}$$

whenever V and W are bundles in $\mathbb{R}^d$.

Lemma 4.1. *There exists a $\preceq$-minimal bundle T_μ in $\mathbb{R}^d$, uniquely determined up to μ-a.e. equality, such that*

$$\text{given any } v \in \mathrm{D}_\infty(\mathrm{div}_\mu), \text{ it holds that } v(x) \in T_\mu(x) \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{4.2}$$

Proof. We subdivide the proof into three steps:
STEP 1. First of all, let us define

$$\mathcal{V} \coloneqq \big\{ v \in \mathrm{D}_\infty(\mathrm{div}_\mu) : |v| \in L^2_\mu(\mathbb{R}^d) \big\} \quad \text{and} \quad \mathcal{M} \coloneqq \mathrm{cl}_{L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)}(\mathcal{V}). \tag{4.3}$$

We claim that

$$v \in \mathrm{D}_\infty(\mathrm{div}_\mu) \text{ and } f \in C^\infty_c(\mathbb{R}^d) \quad \Longrightarrow \quad fv \in \mathcal{V} \text{ and } \mathrm{div}_\mu(fv) = f\,\mathrm{div}_\mu(v) + \nabla f \cdot v. \tag{4.4}$$

To prove it, notice that for every $g \in C^\infty_c(\mathbb{R}^d)$ we have that $fg \in C^\infty_c(\mathbb{R}^d)$ and $\nabla(fg) = f\nabla g + g\nabla f$, thus accordingly

$$\begin{aligned} \int_{\mathbb{R}^d} \nabla g \cdot (fv)\,\mathrm{d}\mu &= \int_{\mathbb{R}^d} (f\nabla g) \cdot v\,\mathrm{d}\mu = \int_{\mathbb{R}^d} \nabla(fg) \cdot v\,\mathrm{d}\mu - \int_{\mathbb{R}^d} (g\nabla f) \cdot v\,\mathrm{d}\mu \\ &= -\int_{\mathbb{R}^d} g\big(f\,\mathrm{div}_\mu(v) + \nabla f \cdot v\big)\,\mathrm{d}\mu. \end{aligned}$$

Since $fv \in L^\infty_\mu(\mathbb{R}^d; \mathbb{R}^d) \cap L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$ and $f\,\mathrm{div}_\mu(v) + \nabla f \cdot v \in L^\infty_\mu(\mathbb{R}^d)$, we have obtained (4.4).
STEP 2. Next we claim that

$$v \in \mathcal{M} \text{ and } f \in L^\infty_\mu(\mathbb{R}^d) \quad \Longrightarrow \quad fv \in \mathcal{M}. \tag{4.5}$$

To prove it, fix $(v_n)_n \subseteq \mathcal{V}$ such that $v_n \to v$ in $L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$. Moreover, we can find $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $|f_n| \le \|f\|_{L^\infty_\mu(\mathbb{R}^d)}$ for every $n \in \mathbb{N}$ and $f_n \to f$ in the μ-a.e. sense. Indeed, chosen a finite Borel measure $\tilde\mu$ on $\mathbb{R}^d$ having the same null sets as μ, there exists a sequence $(g_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ satisfying $g_n \to f$ in $L^2_{\tilde\mu}(\mathbb{R}^d)$. Up to replacing g_n with $(g_n \wedge \|f\|_{L^\infty_\mu(\mathbb{R}^d)}) \vee (-\|f\|_{L^\infty_\mu(\mathbb{R}^d)})$, we can assume that $|g_n| \le \|f\|_{L^\infty_\mu(\mathbb{R}^d)}$ holds μ-a.e.. Up to passing to a not relabeled subsequence, we can further assume that $g_n \to f$ in the $\tilde\mu$-a.e. sense (thus, in the μ-a.e. sense). Now choose $(\varepsilon_n)_n \subseteq (0,1)$ such that each function $f_n \coloneqq \rho_{\varepsilon_n} * g_n \in C^\infty_c(\mathbb{R}^d)$ satisfies $\|f_n - g_n\|_{C_b(\mathbb{R}^d)} \le 1/n$. In particular, it holds that $|f_n| \le \|f\|_{L^\infty_\mu(\mathbb{R}^d)}$ for every $n \in \mathbb{N}$ and $f_n \to f$ in the μ-a.e. sense, as desired. Therefore, we can estimate

$$\begin{aligned} \|f_n v_n - fv\|^2_{L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)} &\le 2\int_{\mathbb{R}^d} |f_n - f|^2 |v|^2\,\mathrm{d}\mu + 2\int_{\mathbb{R}^d} |f_n|^2 |v_n - v|^2\,\mathrm{d}\mu \\ &\le 2\int_{\mathbb{R}^d} |f_n - f|^2 |v|^2\,\mathrm{d}\mu + 2\,\|f\|^2_{L^\infty_\mu(\mathbb{R}^d)} \|v_n - v\|^2_{L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)}. \end{aligned}$$

Since $|f_n - f|^2|v|^2 \le 2\,\|f\|^2_{L^\infty_\mu(\mathbb{R}^d)}|v|^2 \in L^1_\mu(\mathbb{R}^d)$ for every $n \in \mathbb{N}$, by dominated convergence theorem we deduce that $f_n v_n \to fv$ in $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$. As $(f_n v_n)_n \subseteq \mathcal{V}$ by (4.4), we conclude that $fv \in \mathcal{M}$.

STEP 3. We are now in a position to apply [16, Proposition 2.22]: (4.5) grants that $\mathcal{M}$ is a $L^2_\mu(\mathbb{R}^d)$-normed $L^\infty_\mu(\mathbb{R}^d)$-submodule of $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$ in the sense of [11, Definition 1.2.10], thus there exists a unique bundle T_μ in $\mathbb{R}^d$ such that $\Gamma^2_\mu(T_\mu) = \mathcal{M}$. To show that T_μ satisfies (4.2), let us fix $v \in \mathrm{D}_\infty(\mathrm{div}_\mu)$ and a sequence $(\eta_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $0 \le \eta_n \le 1$ and $\eta_n = 1$ on $B_n(0)$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and notice that $\eta_n v \in \mathcal{V} \subseteq \Gamma^2_\mu(T_\mu)$ by (4.4). Hence, $v(x) = (\eta_n v)(x) \in T_\mu(x)$ for μ-a.e. $x \in B_n(0)$. Thanks to the arbitrariness of $n \in \mathbb{N}$, we obtain that T_μ satisfies (4.2).

Finally, we are left to prove the minimality of T_μ, which also forces uniqueness. Fix an arbitrary bundle S in $\mathbb{R}^d$ satisfying the property in (4.2) with $S(x)$ in place of $T_\mu(x)$. We aim to show that $T_\mu(x) \subseteq S(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Taking into account (4.1), we can equivalently show that $\Gamma^2_\mu(T_\mu) \subseteq \Gamma^2_\mu(S)$. Pick any $v \in \Gamma^2_\mu(T_\mu)$ and a sequence $(v_n)_n \subseteq \mathcal{V}$ such that $v_n \to v$ in $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$. Up to a not relabeled subsequence, we have that $v_n(x) \to v(x)$ for μ-a.e. $x \in \mathbb{R}^d$. Given that for μ-a.e. $x \in \mathbb{R}^d$ it holds that $v_n(x) \in S(x)$ for every $n \in \mathbb{N}$, we conclude that $v(x) \in S(x)$ for μ-a.e. $x \in \mathbb{R}^d$. □

Remark 4.2. Let us point out that there exists a countable family $\mathcal{C} \subseteq \mathrm{D}_\infty(\mathrm{div}_\mu)$, such that

$$(w(x))_{w\in\mathcal{C}} \text{ is dense in } T_\mu(x) \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

To verify this, observe that $\mathcal{V} \subseteq \mathrm{D}_\infty(\mathrm{div}_\mu)$ defined in (4.3) is a linear subspace of $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, it is closed under the multiplication by $C^\infty_c(\mathbb{R}^d)$-functions, due to (4.4). Take now any countable $L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)$-dense subset $\mathcal{C} \subseteq \mathcal{V} \subseteq \mathrm{D}_\infty(\mathrm{div}_\mu)$ and define

$$V(x) \coloneqq \mathrm{cl}\big(\{w(x) : w \in \mathcal{C}\}\big) \text{ for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

By applying [16, Lemma 2.24] (see also [7, Lemma A.1]), we have that $\Gamma^2_\mu(V) = \mathrm{cl}_{L^2_\mu(\mathbb{R}^d;\mathbb{R}^d)}(\mathcal{V}) = \mathcal{M}$. On the other hand, we have from the construction of the bundle T_μ that $\mathcal{M} = \Gamma^2_\mu(T_\mu)$. Thus, $\Gamma^2_\mu(T_\mu) = \Gamma^2_\mu(V)$ which further implies (recalling [16, Proposition 2.22]) that $T_\mu = V$. Therefore, $\{w(x) : w \in \mathcal{C}\}$ is dense in $T_\mu(x)$ for μ-a.e. $x \in \mathbb{R}^d$, as claimed. ■

Once we have the tangent fibers at our disposal, we are in a position to define the *tangential gradient*: namely, we define $\nabla_\mu \colon C^\infty_c(\mathbb{R}^d) \to \Gamma^1_\mu(T_\mu)$ as

$$\nabla_\mu f(x) \coloneqq \mathrm{pr}_{T_\mu(x)}\big(\nabla f(x)\big), \quad \text{for every } f \in C^\infty_c(\mathbb{R}^d) \text{ and } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{4.6}$$

where $\mathrm{pr}_V \colon \mathbb{R}^d \to V$ stands for the orthogonal projection onto the vector subspace V of $\mathbb{R}^d$.

In [2] the space of $W^{1,1}$ functions has been defined as follows:

Definition 4.3 ($W^{1,1}$ space via vector fields). *The Sobolev space $W^{1,1}(\mathbb{R}^d,\mu)$ is defined as the completion of $C^\infty_c(\mathbb{R}^d)$ under the norm*

$$\|f\|_{W^{1,1}_\mu} \coloneqq \|f\|_{L^1_\mu(\mathbb{R}^d)} + \|\nabla_\mu f\|_{\Gamma^1_\mu(T_\mu)}.$$

Observe that, a priori, $\big(W^{1,1}(\mathbb{R}^d,\mu), \|\cdot\|_{W^{1,1}_\mu}\big)$ is an abstract Banach space. Let us now show that $W^{1,1}(\mathbb{R}^d,\mu)$ is actually a space of functions, *i.e.*, that it can be identified with a linear subspace of $L^1_\mu(\mathbb{R}^d)$. The inclusion map $\big(C^\infty_c(\mathbb{R}^d), \|\cdot\|_{W^{1,1}_\mu}\big) \hookrightarrow \big(L^1_\mu(\mathbb{R}^d), \|\cdot\|_{L^1_\mu(\mathbb{R}^d)}\big)$ is a linear contraction. Denoting by $\iota \colon C^\infty_c(\mathbb{R}^d) \hookrightarrow W^{1,1}(\mathbb{R}^d,\mu)$ the canonical isometric embedding of $\big(C^\infty_c(\mathbb{R}^d), \|\cdot\|_{W^{1,1}_\mu}\big)$ into its completion $W^{1,1}(\mathbb{R}^d,\mu)$, there exists a unique linear contraction $\phi \colon W^{1,1}(\mathbb{R}^d,\mu) \to L^1_\mu(\mathbb{R}^d)$

such that the following diagram is commutative:

$$\begin{array}{ccc} C_c^\infty(\mathbb{R}^d) & \hookrightarrow & L^1_\mu(\mathbb{R}^d) \\ \iota \big\downarrow & \nearrow_{\phi} & \\ W^{1,1}(\mathbb{R}^d,\mu) & & \end{array} \tag{4.7}$$

Our aim is to prove that ϕ is injective. To achieve this goal, we need the following key result.

Lemma 4.4 (Closability of the tangential gradient)**.** *Let $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ and $v \in \Gamma^1_\mu(T_\mu)$ be such that*

$$f_n \rightharpoonup 0 \ \text{ in } L^1_\mu(\mathbb{R}^d) \quad \text{ and } \quad \nabla_\mu f_n \rightharpoonup v \ \text{ in } \Gamma^1_\mu(T_\mu).$$

Then $v(x) = 0$ for μ-a.e. $x \in \mathbb{R}^d$.

Proof. Fix $g \in C_c^\infty(\mathbb{R}^d)$ and $w \in \mathcal{C}$, where $\mathcal{C} \subseteq D_\infty(\mathrm{div}_\mu)$ is a countable family as in Remark 4.2. Then, taking into account the property (4.4), we have that

$$\int_{\mathbb{R}^d} g\, w \cdot v \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} g\, w \cdot \nabla_\mu f_n \,\mathrm{d}\mu = -\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \,\mathrm{div}_\mu(g\, w)\,\mathrm{d}\mu = 0.$$

Now, let $f \in \mathrm{LIP}_c(\mathbb{R}^d)$ and let $(g_n)_n \subseteq C_c^\infty(\mathbb{R}^d)$ be such that $g_n \to f$ pointwise μ-a.e. and $|g_n| \leq \|f\|_{L^\infty_\mu(\mathbb{R}^d)}$ for every $n \in \mathbb{N}$ (the existence of such a sequence follows from a standard mollification argument, cf. Lemma 2.1). Then, by applying the dominated convergence theorem and using the above equality, we get that

$$\int_{\mathbb{R}^d} f\, w \cdot v \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} g_n\, w \cdot v \,\mathrm{d}\mu = 0.$$

Since the function $f \in \mathrm{LIP}_c(\mathbb{R}^d)$ was arbitrary, we deduce that $w \cdot v = 0$ holds μ-a.e. in $\mathbb{R}^d$. By Remark 4.2, we know that the elements of the family $\mathcal{C}$ are fiberwise dense in μ-a.e. fiber of T_μ. Thus, by the arbitrariness of $w \in \mathcal{C}$, we finally conclude that $v(x) = 0$ for μ-a.e. $x \in \mathbb{R}^d$. □

Corollary 4.5. *The map ϕ as in (4.7) is injective. In particular, $W^{1,1}(\mathbb{R}^d,\mu)$ can be identified with a linear subspace of $L^1_\mu(\mathbb{R}^d)$.*

Proof. To prove the claim amounts to showing that if $\mathsf{f} \in W^{1,1}(\mathbb{R}^d,\mu)$ and $\phi(\mathsf{f}) = 0$, then $\mathsf{f} = 0$; we are using the different font f to underline that, a priori, the elements of $W^{1,1}(\mathbb{R}^d,\mu)$ are not functions. Choose a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that $\|\iota(f_n) - \mathsf{f}\|_{W^{1,1}_\mu} \to 0$. In particular, the sequences $(f_n)_{n\in\mathbb{N}} \subseteq L^1_\mu(\mathbb{R}^d)$ and $(\nabla_\mu f_n)_{n\in\mathbb{N}} \subseteq \Gamma^1_\mu(T_\mu)$ are Cauchy, thus there exist elements $f \in L^1_\mu(\mathbb{R}^d)$ and $v \in \Gamma^1_\mu(T_\mu)$ such that $f_n \to f$ and $\nabla_\mu f_n \to v$. Being ϕ continuous, we have that $f_n = \phi(\iota(f_n)) \to \phi(\mathsf{f}) = 0$ in $L^1_\mu(\mathbb{R}^d)$, whence it follows that $f = 0$. Hence, an application of Lemma 4.4 yields the identity $v = 0$. All in all, we proved that $\|f_n\|_{W^{1,1}_\mu} \to 0$, so that $\mathsf{f} = 0$. □

In light of Corollary 4.5, hereafter we will tacitly regard $W^{1,1}(\mathbb{R}^d,\mu)$ as a subspace of $L^1_\mu(\mathbb{R}^d)$. Next we show that the tangential gradient ∇_μ can be extended to the whole of $W^{1,1}(\mathbb{R}^d,\mu)$:

Proposition 4.6. *There exists a unique linear extension $\nabla_\mu \colon W^{1,1}(\mathbb{R}^d,\mu) \to \Gamma^1_\mu(T_\mu)$ of the tangential gradient $\nabla_\mu \colon C_c^\infty(\mathbb{R}^d) \to \Gamma^1_\mu(T_\mu)$ having the following property: if $(f_n)_{n\in\mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d,\mu)$ satisfies $f_n \rightharpoonup f$ in $L^1_\mu(\mathbb{R}^d)$ and $\nabla_\mu f_n \rightharpoonup v$ in $\Gamma^1_\mu(T_\mu)$ for some $f \in L^1_\mu(\mathbb{R}^d)$ and $v \in \Gamma^1_\mu(T_\mu)$, then $f \in W^{1,1}(\mathbb{R}^d,\mu)$ and $\nabla_\mu f = v$. Moreover, it holds that*

$$\|f\|_{W^{1,1}_\mu} = \|f\|_{L^1_\mu(\mathbb{R}^d)} + \|\nabla_\mu f\|_{\Gamma^1_\mu(\mathbb{R}^d)}, \qquad \textit{for every } f \in W^{1,1}(\mathbb{R}^d,\mu). \tag{4.8}$$

Proof. Let $f \in W^{1,1}(\mathbb{R}^d, \mu)$ be given. Pick any sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ strongly in $W^{1,1}(\mathbb{R}^d, \mu)$. In particular, $(\nabla_\mu f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\Gamma^1_\mu(T_\mu)$. Then we define $\tilde\nabla_\mu f \in \Gamma^1_\mu(T_\mu)$ as the limit of $\nabla_\mu f_n$ as $n \to \infty$. Notice that this definition is well-posed, *i.e.*, $\tilde\nabla_\mu f$ does not depend on the specific choice of $(f_n)_n$: indeed, given another sequence $(g_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that $g_n \to f$ in $W^{1,1}(\mathbb{R}^d, \mu)$, we have that $f_n - g_n \to 0$ in $L^1_\mu(\mathbb{R}^d)$, thus Lemma 4.4 ensures that $\nabla_\mu f_n - \nabla_\mu g_n = \nabla_\mu(f_n - g_n)$ converges to 0 in $\Gamma^1_\mu(T_\mu)$. Moreover, if $f \in C_c^\infty(\mathbb{R}^d)$, then by taking the constant sequence $f_n \equiv f$ we see that $\tilde\nabla_\mu f = \nabla_\mu f$. Then we can omit the tilde from our notation and obtain an extension $\nabla_\mu \colon W^{1,1}(\mathbb{R}^d, \mu) \to \Gamma^1_\mu(T_\mu)$ of the tangential gradient. Linearity readily follows from the fact that ∇_μ is linear when restricted to $C_c^\infty(\mathbb{R}^d)$. Uniqueness is granted by its very construction. Moreover, observe that if $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ satisfy $f_n \to f$ in $W^{1,1}(\mathbb{R}^d, \mu)$, then we have both $\|f_n\|_{W^{1,1}_\mu} \to \|f\|_{W^{1,1}_\mu}$ and

$$\|f_n\|_{W^{1,1}_\mu} = \|f_n\|_{L^1_\mu(\mathbb{R}^d)} + \|\nabla_\mu f_n\|_{\Gamma^1_\mu(T_\mu)} \to \|f\|_{L^1_\mu(\mathbb{R}^d)} + \|\nabla_\mu f\|_{\Gamma^1_\mu(T_\mu)},$$

whence (4.8) follows. Finally, it remains to show that ∇_μ is a closed operator, namely, that if $(f_n)_{n\in\mathbb{N}} \subseteq W^{1,1}(\mathbb{R}^d, \mu)$ satisfies $f_n \rightharpoonup f \in L^1_\mu(\mathbb{R}^d)$ and $\nabla_\mu f_n \rightharpoonup v \in \Gamma^1_\mu(T_\mu)$, then $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $\nabla_\mu f = v$. Given any $n \in \mathbb{N}$, we can find $g_n \in C_c^\infty(\mathbb{R}^d)$ such that $\|g_n - f_n\|_{L^1_\mu(\mathbb{R}^d)} \le 1/n$ and $\|\nabla_\mu g_n - \nabla_\mu f_n\|_{\Gamma^1_\mu(T_\mu)} \le 1/n$. In particular, $g_n \rightharpoonup f$ in $L^1_\mu(\mathbb{R}^d)$ and $\nabla_\mu g_n \rightharpoonup v$ in $\Gamma^1_\mu(T_\mu)$. Thanks to Mazur lemma (item ii) of Proposition 2.2) and the linearity of ∇_μ, we may assume (possibly replacing the g_n's by their convex combinations) that $g_n \to f$ in $L^1_\mu(\mathbb{R}^d)$ and $\nabla_\mu g_n \to v$ in $\Gamma^1_\mu(T_\mu)$. This implies that $(g_n)_{n\in\mathbb{N}}$ is Cauchy in $W^{1,1}(\mathbb{R}^d, \mu)$ and that its $L^1_\mu(\mathbb{R}^d)$-limit coincides with f, so that $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $\nabla_\mu f = v$. Therefore, the statement is achieved. □

We conclude the current section by proving that the (extended) tangential gradient introduced in Proposition 4.6 satisfies the following Leibniz rule:

Lemma 4.7 (Leibniz rule for the tangential gradient)**.** *Let $f, g \in W^{1,1}(\mathbb{R}^d, \mu) \cap L^\infty_\mu(\mathbb{R}^d)$. Then*

$$fg \in W^{1,1}(\mathbb{R}^d, \mu) \quad \text{and} \quad \nabla_\mu(fg) = g\nabla_\mu f + f\,\nabla_\mu g. \tag{4.9}$$

Proof. We divide the proof into several steps:
STEP 1. We first prove that (4.9) holds for any $f, g \in W^{1,1}(\mathbb{R}^d, \mu) \cap L^\infty_\mu(\mathbb{R}^d)$ having bounded support. To verify this claim, let us fix sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ and $g_n \to g$ in $W^{1,1}(\mathbb{R}^d, \mu)$. Without loss of generality, we may assume that there exists $C > 0$ and a compact set $K \subseteq \mathbb{R}^d$ such that $|f_n|, |g_n| \le C$ and $\mathrm{supp}(f_n), \mathrm{supp}(g_n) \subseteq K$, for all $n \in \mathbb{N}$. Proposition 4.6 ensures that $\nabla_\mu f_n \to \nabla_\mu f$ and $\nabla_\mu g_n \to \nabla_\mu g$ in $\Gamma^1_\mu(T_\mu)$. Since also $f_n \to f$ and $g_n \to g$ in $L^1_\mu(\mathbb{R}^d)$, we know from Proposition 2.2 i) that (up to a not relabelled subsequence) the convergence $(f_n, \nabla_\mu f_n, g_n, \nabla_\mu g_n) \to (f, \nabla_\mu f, g, \nabla_\mu g)$ is both dominated and in the pointwise μ-a.e. sense. Now define $h_n \coloneqq f_n g_n \in C_c^\infty(\mathbb{R}^d)$ and $v_n \coloneqq g_n \nabla_\mu f_n + f_n \nabla_\mu g_n \in \Gamma^1_\mu(T_\mu)$ for every $n \in \mathbb{N}$. Observe that $\nabla_\mu h_n = v_n$, as one can easily verify:

$$\begin{aligned}\nabla_\mu h_n(x) &= \mathrm{pr}_{T_\mu(x)}\big(\nabla(f_n g_n)(x)\big) = \mathrm{pr}_{T_\mu(x)}\big(g_n(x)\nabla f_n(x) + f_n(x)\nabla g_n(x)\big)\\ &= g_n(x)\mathrm{pr}_{T_\mu(x)}\big(\nabla f_n(x)\big) + f_n(x)\mathrm{pr}_{T_\mu(x)}\big(\nabla g_n(x)\big) = v_n(x),\end{aligned}$$

for μ-a.e. $x \in \mathbb{R}^d$. Fixed a non-negative function $H \in L^1_\mu(\mathbb{R}^d)$ such that $|\nabla_\mu f_n|, |\nabla_\mu g_n| \le H$ is satisfied μ-a.e. for every $n \in \mathbb{N}$, we can estimate $|h_n| \le C^2\chi_K \in L^1_\mu(\mathbb{R}^d)$ and $|v_n| \le 2C\chi_K H \in L^1_\mu(\mathbb{R}^d)$ in the μ-a.e. sense for every $n \in \mathbb{N}$. By using the dominated convergence theorem, we can finally conclude that

$$h_n \to fg \ \text{ in } L^1_\mu(\mathbb{R}^d) \quad \text{and} \quad \nabla_\mu h_n = v_n \to g\nabla_\mu f + f\nabla_\mu g \ \text{ in } \Gamma^1_\mu(T_\mu).$$

Therefore, Proposition 4.6 implies that $fg \in W^{1,1}(\mathbb{R}^d, \mu)$ and $\nabla_\mu(fg) = g\nabla_\mu f + f\nabla_\mu g$.
STEP 2. We next show that (4.9) holds for $f \in W^{1,1}(\mathbb{R}^d, \mu) \cap L^\infty_\mu(\mathbb{R}^d)$ and $\eta \in C_c^\infty(\mathbb{R}^d)$. Choosing

a sequence of smooth functions $(f_n)_n \subseteq C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $L^1_\mu(\mathbb{R}^d)$ and $\nabla_\mu f_n \to \nabla_\mu f$ in $\Gamma^1_\mu(T_\mu)$, we have that $\nabla_\mu(f_n\eta) = \eta\nabla_\mu f_n + f_n\nabla_\mu\eta$ holds for every $n \in \mathbb{N}$, and consequently

$$\int_{\mathbb{R}^d} \big|\nabla_\mu(f_n\eta) - (\eta\nabla_\mu f + f\nabla_\mu\eta)\big| \,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |\eta||(\nabla_\mu f_n - \nabla_\mu f)| \,\mathrm{d}\mu + \int_{\mathbb{R}^d} |(f_n - f)||\nabla_\mu\eta| \,\mathrm{d}\mu.$$

Therefore, by passing to the limit as $n \to \infty$, we get (4.9).

Step 3. We finally prove (4.9) for any $f, g \in W^{1,1}(\mathbb{R}^d, \mu) \cap L^\infty_\mu(\mathbb{R}^d)$. To this aim, fix a sequence $(\eta_n)_n \subseteq C_c^\infty(\mathbb{R}^d)$ of cut-off functions, *i.e.*, $0 \le \eta_n \le 1$, $\eta_n = 1$ on $B_n(0)$ and $|\nabla\eta_n| \le 1$, for every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$, call $f_n \coloneqq \eta_n f$ and $g_n \coloneqq \eta_n g$ and observe that f_n, g_n are compactly supported functions belonging to $W^{1,1}(\mathbb{R}^d, \mu) \cap L^\infty_\mu(\mathbb{R}^d)$ by Step 2. Thus, by Step 1 we have that $f_n g_n \in W^{1,1}(\mathbb{R}^d, \mu)$ and that

$$\begin{aligned}\nabla_\mu(f_n g_n) =& f_n\nabla_\mu g_n + g_n\nabla_\mu f_n = \eta_n f\nabla_\mu(\eta_n g) + \eta_n g\nabla_\mu(\eta_n f)\\ =& \underbrace{\left(\eta_n^2 f\nabla_\mu g + \eta_n f g\nabla_\mu\eta_n\right)}_{\text{A}} + \underbrace{\left(\eta_n^2 g\nabla_\mu f + \eta_n g f\nabla_\mu\eta_n\right)}_{\text{B}}\end{aligned}$$

holds μ-a.e. in $\mathbb{R}^d$. Clearly, the sequence $(f_n g_n)_n$ converges to fg strongly in $L^1_\mu(\mathbb{R}^d)$. In order to prove (4.9), we will show that $\nabla_\mu(f_n g_n)$ converges to $f\nabla_\mu g + g\nabla_\mu f$ strongly in $L^1_\mu(\mathbb{R}^d)$. We prove that the part of $\nabla_\mu(f_n g_n)$ denoted by A above converges to $f\nabla_\mu g$ strongly in $L^1_\mu(\mathbb{R}^d)$. Indeed, it holds that

$$\int_{\mathbb{R}^d} |\eta_n^2 f\nabla_\mu g - f\nabla_\mu g| \,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |f||\nabla_\mu g|(1-\eta_n^2) \,\mathrm{d}\mu \le \int_{B_n^c(0)} |f||\nabla_\mu g| \,\mathrm{d}\mu \overset{n\to\infty}{\longrightarrow} 0$$

and that

$$\int_{\mathbb{R}^d} |\eta_n f g\nabla_\mu\eta_n| \,\mathrm{d}\mu \le \int_{B_n^c(0)} |fg| \,\mathrm{d}\mu \overset{n\to\infty}{\longrightarrow} 0,$$

proving the claim. Similarly, one can show that the part B converges to $g\nabla_\mu f$ strongly in $L^1_\mu(\mathbb{R}^d)$, concluding the proof. □

4.2. **$W^{1,1}$ space via relaxed slope.** The relaxation type approach to the definition of $W^{1,1}$ space proposed in [8] is based on the concept of the 'relaxed slope':

Definition 4.8 (Relaxed slope). *Let $f \in L^1_\mu(\mathbb{R}^d)$. A non-negative function $G \in L^1_\mu(\mathbb{R}^d)$ is said to be a* relaxed slope *of f if there exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ such that $f_n \to f$ in $L^1_\mu(\mathbb{R}^d)$ and $\mathrm{lip}_a(f_n) \rightharpoonup G'$ weakly in $L^1_\mu(\mathbb{R}^d)$, for some $G' \in L^1_\mu(\mathbb{R}^d)$ with $G' \le G$ μ-a.e. in $\mathbb{R}^d$. We denote the set of all relaxed slopes of f by $\mathrm{RS}(f)$.*

Definition 4.9 ($W^{1,1}$ space via relaxation). *We say that a function $f \in L^1_\mu(\mathbb{R}^d)$ belongs to the space $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d, \mu)$ if $\mathrm{RS}(f) \neq \emptyset$. The minimal element (in the μ-a.e. sense) of $\mathrm{RS}(f)$ will be denoted by $|\nabla f|_{rs}$ and called the* minimal relaxed slope *of f.*

Remark 4.10. Given any $f \in W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d, \mu)$, there exist $(f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ and $H \in L^1_\mu(\mathbb{R}^d)$ such that $f_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$, $\mathrm{lip}_a(f_n) \rightharpoonup |\nabla f|_{rs}$ weakly in $L^1_\mu(\mathbb{R}^d)$, and $\mathrm{lip}_a(f_n) \le H$ μ-a.e. for every $n \in \mathbb{N}$. Indeed, by exploiting the minimality of $|\nabla f|_{rs}$ we can find $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ such that $g_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$ and $\mathrm{lip}_a(g_n) \rightharpoonup |\nabla f|_{rs}$ weakly in $L^1_\mu(\mathbb{R}^d)$. By Proposition 2.2 ii), we can find $(\alpha_i^n)_{i=n}^{N_n} \subseteq [0,1]$ with $\sum_{i=n}^{N_n} \alpha_i^n = 1$ and $\sum_{i=n}^{N_n} \alpha_i^n \mathrm{lip}_a(g_i) \to |\nabla f|_{rs}$ strongly in $L^1_\mu(\mathbb{R}^d)$ as $n \to \infty$. By Proposition 2.2 i), we know that there exists $H \in L^1_\mu(\mathbb{R}^d)$ such that (up to a not relabeled subsequence in n) it holds $\sum_{i=n}^{N_n} \alpha_i^n \mathrm{lip}_a(g_i) \le H$ μ-a.e. for every $n \in \mathbb{N}$. Now we define $f_n \coloneqq \sum_{i=n}^{N_n} \alpha_i^n g_i \in \mathrm{LIP}_c(\mathbb{R}^d)$ for every $n \in \mathbb{N}$. Notice that $f_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$ and

$$\mathrm{lip}_a(f_n) = \mathrm{lip}_a\left(\sum_{i=n}^{N_n} \alpha_i^n g_i\right) \le \sum_{i=n}^{N_n} \alpha_i^n \mathrm{lip}_a(g_i) \le H, \qquad \text{in the } \mu\text{-a.e. sense,}$$

for every $n \in \mathbb{N}$. By Proposition 2.2 iii), we can find $G \in L^1_\mu(\mathbb{R}^d)$ such that $G \leq |\nabla f|_{rs}$ μ-a.e. and (up to a further subsequence in n) it holds $\mathrm{lip}_a(f_n) \rightharpoonup G$ weakly in $L^1_\mu(\mathbb{R}^d)$. Finally, the minimality of $|\nabla f|_{rs}$ ensures that $G = |\nabla f|_{rs}$, thus accordingly the claim is proved. ■

4.3. $W^{1,1}$ **space via tangential relaxed slope.** We introduce here an auxiliary notion of $W^{1,1}$ space, which is intermediate between the approaches $W^{1,1}(\mathbb{R}^d, \mu)$ and $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d, \mu)$, as it is based upon the relaxation of the modulus of the tangential gradient ∇_μ. Its equivalence with $W^{1,1}(\mathbb{R}^d, \mu)$ will be proved in Theorem 4.13. As a consequence of this characterization, we will show in Proposition 4.14 that, as one might expect, the compactly-supported Lipschitz functions belong to $W^{1,1}(\mathbb{R}^d, \mu)$.

Definition 4.11 (Tangential relaxed slope)**.** *Given any $f \in L^1_\mu(\mathbb{R}^d)$, we say that $G \in L^1_\mu(\mathbb{R}^d)$ is a* tangential relaxed slope *of f provided there exists a sequence $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ converging to f in $L^1_\mu(\mathbb{R}^d)$ and such that $|\nabla_\mu f_n| \rightharpoonup G'$ weakly in $L^1_\mu(\mathbb{R}^d)$, for some function $G' \in L^1_\mu(\mathbb{R}^d)$ with $G' \leq G$ μ-a.e. in $\mathbb{R}^d$. We denote by $\mathrm{TRS}(f)$ the family of all tangential relaxed slopes of f.*

Lemma 4.12. *Let $f \in L^1_\mu(\mathbb{R}^d)$ be such that $\mathrm{TRS}(f) \neq \emptyset$. Then the set $\mathrm{TRS}(f)$ is a closed convex sublattice of $L^1_\mu(\mathbb{R}^d)$. In particular, it admits a unique μ-a.e. minimal element $G_f \in \mathrm{TRS}(f)$, namely $G_f \leq G$ holds μ-a.e. for every $G \in \mathrm{TRS}(f)$.*

Proof.
CLOSURE. Let $(G_i)_i \subseteq \mathrm{TRS}(f)$ satisfy $G_i \to G$ strongly in $L^1_\mu(\mathbb{R}^d)$ for some $G \in L^1_\mu(\mathbb{R}^d)$. We aim to show that $G \in \mathrm{TRS}(f)$. For any $i \in \mathbb{N}$, we can find $G'_i \leq G_i$ and $(f^i_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $f^i_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$ as $n \to \infty$ and $|\nabla_\mu f^i_n| \rightharpoonup G'_i$ weakly in $L^1_\mu(\mathbb{R}^d)$ as $n \to \infty$. Since $G_i \to G$ in $L^1_\mu(\mathbb{R}^d)$, we know from Proposition 2.2 i) that (up to a not relabeled subsequence) it holds $G'_i \leq G_i \leq H$ μ-a.e. for all $i \in \mathbb{N}$, for some $H \in L^1_\mu(\mathbb{R}^d)$. Hence, Proposition 2.2 iii) ensures that (up to a further subsequence) it holds $G'_i \rightharpoonup G'$ weakly in $L^1_\mu(\mathbb{R}^d)$, for some $G' \in L^1_\mu(\mathbb{R}^d)$. Since $G'_i \leq G_i$ μ-a.e. for all $i \in \mathbb{N}$, we deduce that $G' \leq G$ μ-a.e.. We now perform a diagonalization argument: for any $i \in \mathbb{N}$ we can find $n(i) \in \mathbb{N}$ such that the elements $f_i := f^i_{n(i)}$ satisfy $f_i \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$ and $|\nabla_\mu f_i| \rightharpoonup G'$ weakly in $L^1_\mu(\mathbb{R}^d)$. This shows that $G \in \mathrm{TRS}(f)$, as desired.
CONVEXITY. Fix any $G, H \in \mathrm{TRS}(f)$ and $\lambda \in [0,1]$. Then there exist $G' \leq G$, $H' \leq H$, and $(f_n)_n, (g_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n \to f$, $g_n \to f$, $|\nabla_\mu f_n| \rightharpoonup G'$, and $|\nabla_\mu g_n| \rightharpoonup H'$ in $L^1_\mu(\mathbb{R}^d)$. Proposition 2.2 yields the existence of some coefficients $(\alpha^n_i)_{i=n}^{N_n}, (\beta^n_i)_{i=n}^{M_n} \subseteq [0,1]$ with $\sum_{i=n}^{N_n} \alpha^n_i = \sum_{i=n}^{M_n} \beta^n_i = 1$ such that $\big(\sum_{i=n}^{N_n} \alpha^n_i |\nabla_\mu f_i|\big)_n$ and $\big(\sum_{i=n}^{M_n} \beta^n_i |\nabla_\mu g_i|\big)_n$ are dominated and converge strongly in $L^1_\mu(\mathbb{R}^d)$ to G' and H', respectively. For any $n \in \mathbb{N}$ we define

$$h_n := \lambda \sum_{i=n}^{N_n} \alpha^n_i f_i + (1-\lambda) \sum_{i=n}^{M_n} \beta^n_i g_i \in C^\infty_c(\mathbb{R}^d).$$

Observe that $h_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$. Moreover, from the inequality

$$|\nabla_\mu h_n| \leq \lambda \sum_{i=n}^{N_n} \alpha^n_i |\nabla_\mu f_i| + (1-\lambda) \sum_{i=n}^{M_n} \beta^n_i |\nabla_\mu g_i|$$

we deduce that $\big(|\nabla_\mu h_n|\big)_n$ is dominated, thus (by Proposition 2.2 iii) and up to subsequence) it holds $|\nabla_\mu h_n| \rightharpoonup L'$ weakly in $L^1_\mu(\mathbb{R}^d)$, for some $L' \leq \lambda G' + (1-\lambda)H' \leq \lambda G + (1-\lambda)H$. This shows that $\lambda G + (1-\lambda)H \in \mathrm{TRS}(f)$, thus proving the convexity of the set $\mathrm{TRS}(f)$.
LATTICE PROPERTY. We aim to show that, given any $G, H \in \mathrm{TRS}(f)$, it holds $G \vee H \in \mathrm{TRS}(f)$ and $G \wedge H \in \mathrm{TRS}(f)$. The former is trivial, so let us focus on the latter. Define $E := \{G \leq H\}$. By convolution, we can find a sequence $(\eta_j)_j \subseteq C^\infty_c(\mathbb{R}^d)$ with $0 \leq \eta_j \leq 1$ that weakly* converges to χ_E in $L^\infty_\mu(\mathbb{R}^d)$. In particular, $\eta_j G + (1-\eta_j)H \rightharpoonup \chi_E G + \chi_{E^c} H = G \wedge H$ weakly in $L^1_\mu(\mathbb{R}^d)$.

The set $\mathrm{TRS}(f) \subseteq L^1_\mu(\mathbb{R}^d)$ is weakly closed (as it is strongly closed and convex), thus in order to prove that $G \wedge H \in \mathrm{TRS}(f)$ it suffices to show that $\eta_j G + (1-\eta_j)H \in \mathrm{TRS}(f)$ for all $j \in \mathbb{N}$. To this aim, pick $G' \leq G$, $H' \leq H$, and $(f_n)_n, (g_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n \to f$, $g_n \to f$, $|\nabla_\mu f_n| \rightharpoonup G'$, and $|\nabla_\mu g_n| \rightharpoonup H'$ in $L^1_\mu(\mathbb{R}^d)$. Thanks to Proposition 2.2, for any $n \in \mathbb{N}$ we can find coefficients $(\alpha^n_i)_{i=n}^{N_n}, (\beta^n_i)_{i=n}^{M_n} \subseteq [0,1]$ with $\sum_{i=n}^{N_n} \alpha^n_i = \sum_{i=n}^{M_n} \beta^n_i = 1$ such that the sequences $\big(\sum_{i=n}^{N_n} \alpha^n_i |\nabla_\mu f_i|\big)_n$ and $\big(\sum_{i=n}^{M_n} \beta^n_i |\nabla_\mu g_i|\big)_n$ are dominated and converge strongly in $L^1_\mu(\mathbb{R}^d)$ to G' and H', respectively. Now fix $j \in \mathbb{N}$ and define $h^j_n := \eta_j \sum_{i=n}^{N_n} f_i + (1-\eta_j)\sum_{i=n}^{M_n} g_i \in C^\infty_c(\mathbb{R}^d)$ for every $n \in \mathbb{N}$. Observe that $h^j_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$ as $n \to \infty$. For any $n \in \mathbb{N}$ we have that

$$|\nabla_\mu h^j_n| \leq \eta_j \sum_{i=n}^{N_n} \alpha^n_i |\nabla_\mu f_i| + (1-\eta_j) \sum_{i=n}^{M_n} \beta^n_i |\nabla_\mu g_i|,$$

thus in particular $\big(|\nabla_\mu h^j_n|\big)_n$ is dominated. Therefore, Proposition 2.2 iii) ensures that (up to a not relabeled subsequence in n) it holds $|\nabla_\mu h^j_n| \rightharpoonup L'$ weakly in $L^1_\mu(\mathbb{R}^d)$ as $n \to \infty$, for some function $L' \leq \eta_j G' + (1-\eta_j)H' \leq \eta_j G + (1-\eta_j)H$. This yields $\eta_j G + (1-\eta_j)H \in \mathrm{TRS}(f)$. □

Theorem 4.13. *Let $f \in L^1_\mu(\mathbb{R}^d)$ be given. Then $f \in W^{1,1}(\mathbb{R}^d, \mu)$ if and only if $\mathrm{TRS}(f) \neq \emptyset$. In this case, the function $|\nabla_\mu f|$ coincides with the μ-a.e. minimal element of $\mathrm{TRS}(f)$.*

Proof. First, we aim to show that if $f \in W^{1,1}(\mathbb{R}^d, \mu)$, then $\mathrm{TRS}(f) \neq \emptyset$ and $G_f \leq |\nabla_\mu f|$ μ-a.e., where G_f stands for the minimal element of $\mathrm{TRS}(f)$. Pick $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n \to f$ and $\nabla_\mu f_n \to \nabla_\mu f$ strongly in $L^1_\mu(\mathbb{R}^d)$ and $\Gamma^1_\mu(T_\mu)$, respectively. In particular, $|\nabla_\mu f_n| \to |\nabla_\mu f|$ strongly in $L^1_\mu(\mathbb{R}^d)$, whence it follows that $|\nabla_\mu f| \in \mathrm{TRS}(f)$, thus $G_f \leq |\nabla_\mu f|$ in the μ-a.e. sense.

Conversely, let us show that if $\mathrm{TRS}(f) \neq \emptyset$, then $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $|\nabla_\mu f| \leq G$ μ-a.e. for every $G \in \mathrm{TRS}(f)$. There exist $G' \leq G$ and $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that $f_n \to f$ and $|\nabla_\mu f_n| \rightharpoonup G'$ in $L^1_\mu(\mathbb{R}^d)$. Proposition 2.2 yields the existence of $(\alpha^n_i)_{i=n}^{N_n} \subseteq [0,1]$ with $\sum_{i=n}^{N_n} \alpha^n_i = 1$ such that the sequence $\big(\sum_{i=n}^{N_n} \alpha^n_i |\nabla_\mu f_i|\big)_n$ is dominated and converges (both strongly in $L^1_\mu(\mathbb{R}^d)$ and in the pointwise μ-a.e. sense) to G' as $n \to \infty$. Define

$$g_n := \sum_{i=n}^{N_n} \alpha^n_i f_i \in C^\infty_c(\mathbb{R}^d), \qquad \text{for every } n \in \mathbb{N}.$$

Then $g_n \to f$ strongly in $L^1_\mu(\mathbb{R}^d)$. Moreover, from the inequality $|\nabla_\mu g_n| \leq \sum_{i=n}^{N_n} \alpha^n_i |\nabla_\mu f_i|$ we deduce that the sequence $(\nabla_\mu g_n)_n$ is dominated. Hence, by applying Proposition 2.2 iii) we obtain that there exists a vector field $v \in \Gamma^1_\mu(T_\mu)$ such that (up to a subsequence in n) it holds $\nabla_\mu g_n \rightharpoonup v$ weakly in $\Gamma^1_\mu(T_\mu)$. Lemma 4.4 ensures that $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $v = \nabla_\mu f$. Finally, let us show that $|\nabla_\mu f| \leq G$ μ-a.e. in $\mathbb{R}^d$. Given any $v \in \mathbb{R}^d$ with $|v| \leq 1$ and any $0 \leq h \in L^\infty_\mu(\mathbb{R}^d)$ it holds that

$$\int_{\mathbb{R}^d} h\, v \cdot \nabla_\mu f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} h\, v \cdot \nabla_\mu g_n \,\mathrm{d}\mu \leq \lim_{n\to\infty} \int_{\mathbb{R}^d} h |\nabla_\mu g_n| \,\mathrm{d}\mu \leq \int_{\mathbb{R}^n} h G' \,\mathrm{d}\mu \leq \int_{\mathbb{R}^d} h G \,\mathrm{d}\mu.$$

By the arbitrariness of h, we deduce that $v \cdot \nabla_\mu f \leq G$ holds μ-a.e. in $\mathbb{R}^d$. Then, we conclude that $|\nabla_\mu f|(x) = \sup\{v \cdot \nabla_\mu f(x) : v \in \mathbb{R}^d, |v| \leq 1\} \leq G$ for μ-a.e. $x \in \mathbb{R}^d$. Therefore, the statement is achieved. □

Proposition 4.14. *Let $f \in \mathrm{LIP}_c(\mathbb{R}^d)$. Then $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $|\nabla_\mu f| \leq \mathrm{lip}_a(f)$ holds μ-a.e..*

Proof. Denote by K the closed 1-neighbourhood of $\mathrm{supp}(f)$. Fix a sequence $(\varepsilon_n)_n \subseteq (0,1)$ such that $\varepsilon_n \to 0$ and define $f_n := \rho_{\varepsilon_n} * f \in C^\infty_c(\mathbb{R}^d)$ for every $n \in \mathbb{N}$. Notice that each $\mathrm{supp}(f_n)$ is contained in K. Thanks to (2.1b), we see that $|f_n| \leq \|f\|_{L^\infty_\mu(\mathbb{R}^d)} \chi_K$ and $f_n(x) \to f(x)$ for all

$x \in \mathbb{R}^d$, thus by using the dominated convergence theorem we obtain that $f_n \to f$ in $L^1_\mu(\mathbb{R}^d)$. Moreover,

$$|\nabla_\mu f_n| \leq |\nabla f_n| \overset{(2.1c)}{\leq} \operatorname{Lip}\big(f; B_{\varepsilon_n}(\cdot)\big) \leq \operatorname{Lip}(f)\chi_K \in L^1_\mu(\mathbb{R}^d), \quad \text{for every } n \in \mathbb{N},$$

thus in particular $\big(|\nabla_\mu f_n|\big)_n$ is dominated. By Proposition 2.2 iii) we get the existence of a function $G \in L^1_\mu(\mathbb{R}^d)$ such that (up to a subsequence) it holds $|\nabla_\mu f_n| \rightharpoonup G$ weakly in $L^1_\mu(\mathbb{R}^d)$. Hence, we conclude that for every $0 \leq h \in L^\infty_\mu(\mathbb{R}^d)$ it holds that

$$\begin{aligned}\int_{\mathbb{R}^d} hG \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\mathbb{R}^d} h|\nabla_\mu f_n| \,\mathrm{d}\mu \;&\leq\; \lim_{n\to\infty} \int_{\mathbb{R}^d} h|\nabla f_n| \,\mathrm{d}\mu \\ &\overset{(2.1c)}{\leq} \lim_{n\to\infty} \int_{\mathbb{R}^d} h \operatorname{Lip}\big(f; B_{\varepsilon_n}(\cdot)\big) \,\mathrm{d}\mu = \int_{\mathbb{R}^d} h \operatorname{lip}_a(f) \,\mathrm{d}\mu,\end{aligned} \tag{4.10}$$

where, in order to get the last equality above, we have used a simple technical result given in Lemma 4.15 below. It follows from (4.10) that $G \leq \operatorname{lip}_a(f)$ holds μ-a.e. and thus that $\operatorname{lip}_a(f) \in \operatorname{TRS}(f)$. Accordingly, $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and $|\nabla_\mu f| \leq \operatorname{lip}_a(f)$ holds μ-a.e. by Theorem 4.13. The proof is complete. □

In the proof of the lemma above the following easy technical lemma has been used. It will be useful also later on, in the proof of Theorem 5.8.

Lemma 4.15. *Let $f \in \mathrm{LIP}_c(\mathbb{R}^d)$ be given. Then it holds that*

$$\operatorname{Lip}\big(f; B_r(\cdot)\big) \to \operatorname{lip}_a(f), \qquad \textit{strongly in } L^1_\mu(\mathbb{R}^d) \textit{ as } r \searrow 0. \tag{4.11}$$

Proof. Call K the closed 1-neighbourhood of $\operatorname{supp}(f)$. Observe that for any $r \in (0,1)$ we have that $\operatorname{Lip}\big(f; B_r(\cdot)\big) \leq \operatorname{Lip}(f)\chi_K \in L^1_\mu(\mathbb{R}^d)$ on $\mathbb{R}^d$. Recalling that $\lim_{r\searrow 0} \operatorname{Lip}\big(f; B_r(x)\big) = \operatorname{lip}_a(f)(x)$ for every $x \in \mathbb{R}^d$ by the very definition of $\operatorname{lip}_a(f)$, by applying the dominated convergence theorem we conclude that (4.11) is verified, as desired. □

5. Relation between BV and $W^{1,1}$ spaces

5.1. **The total variation measure.** Given that the relaxation-type approach to the BV space presented in Subsection 3.2 comes from the general metric measure space setting, one has to use Lipschitz functions in the relaxation process. When we stick to the specific case of the weighted Euclidean space, one would expect that Lipschitz functions can be replaced by smooth ones. Indeed, in the next results we confirm it. In this subsection we will use the notation $\mathrm{LIP}_{bs}(\mathrm{X})$ to denote the space of boundedly-supported Lipschitz functions on a given metric space (X, d). We start with two preparatory lemmata:

Lemma 5.1. *Let $f \in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d, \mu)$ be given. Let $\Omega \subseteq \mathbb{R}^d$ be an open set with $|D_\mu f|_{\mathrm{Lip}}(\partial\Omega) = 0$. Then $f|_{\bar\Omega} \in \mathrm{BV}_{\mathrm{Lip}}(\bar\Omega, \mu|_{\bar\Omega})$ and $\big|D_{\mu|_{\bar\Omega}}(f|_{\bar\Omega})\big|_{\mathrm{Lip}}(\bar\Omega) = |D_\mu f|_{\mathrm{Lip}}(\Omega)$.*

Proof. For brevity, call $C := \bar\Omega$ and $g := f|_C$. Given that

$$|D_\mu f|_{\mathrm{Lip}}(\Omega) = |D_\mu f|(C) = \inf_U |D_\mu f|_{\mathrm{Lip}}(U),$$

where the infimum is among all open sets $U \subseteq \mathbb{R}^d$ containing C, for any $\varepsilon > 0$ we can find an open set $U \subseteq \mathbb{R}^d$ with $C \subseteq U$ and a sequence $(f_n)_n \subseteq \mathrm{LIP}_{loc}(U) \cap L^1_\mu(U)$ such that $f_n \to f|_U$ in $L^1_\mu(U)$ and $\underline{\lim}_n \int_U \operatorname{lip}_a(f_n) \,\mathrm{d}\mu \leq |D_\mu f|_{\mathrm{Lip}}(\Omega) + \varepsilon$. Then $g_n := f_n|_C \in \mathrm{LIP}_{loc}(C) \cap L^1_\mu(C)$ satisfies $g_n \to g$ and $\underline{\lim}_n \int_C \operatorname{lip}_a(g_n) \,\mathrm{d}\mu \leq \underline{\lim}_n \int_U \operatorname{lip}_a(f_n) \,\mathrm{d}\mu \leq |D_\mu f|_{\mathrm{Lip}}(\Omega) + \varepsilon$. Hence, we have $g \in \mathrm{BV}_{\mathrm{Lip}}(C, \mu|_C)$ and $|D_{\mu|_C} g|_{\mathrm{Lip}}(C) \leq |D_\mu f|_{\mathrm{Lip}}(\Omega)$. Conversely, we can find a sequence

$(g'_n)_n \subseteq \mathrm{LIP}_{bs}(C)$ such that $g'_n \to g$ in $L^1_\mu(C)$ and $\int_C \mathrm{lip}_a(g'_n)\,\mathrm{d}\mu \to |D_{\mu|_C} g|_{\mathrm{Lip}}(C)$, thus the functions $f'_n := g'_n|_\Omega \in \mathrm{LIP}_{loc}(\Omega) \cap L^1_\mu(\Omega)$ satisfy $f'_n \to f|_\Omega$ in $L^1_\mu(\Omega)$ and

$$\varliminf_{n\to\infty} \int_\Omega \mathrm{lip}_a(f'_n)\,\mathrm{d}\mu \le \lim_{n\to\infty} \int_C \mathrm{lip}_a(g'_n)\,\mathrm{d}\mu = |D_{\mu|_C} g|_{\mathrm{Lip}}(C),$$

whence it follows that $|D_\mu f|_{\mathrm{Lip}}(\Omega) \le |D_{\mu|_C} g|_{\mathrm{Lip}}(C)$. Therefore, the statement is achieved. □

Lemma 5.2. *Let $f \in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ be given. Define*

$$\mathcal{O}_f := \big\{\Omega \subseteq \mathbb{R}^d \text{ open} \,\big|\, \mu(\Omega) < +\infty,\ \mu(\partial\Omega) = |D_\mu f|_{\mathrm{Lip}}(\partial\Omega) = 0\big\}. \tag{5.1}$$

Then for any σ-finite Borel measure ν on $\mathbb{R}^d$ and for any compact set $K \subseteq \mathbb{R}^d$ it holds that

$$\nu(K) = \inf\big\{\nu(\Omega) \,\big|\, \Omega \in \mathcal{O}_f,\ K \subseteq \Omega\big\}. \tag{5.2}$$

Proof. For any $r > 0$, denote by Ω_r the open r-neighbourhood of K. Since the sets $\{\Omega_r\}_{r>0}$ have pairwise disjoint boundaries, we deduce that $\Omega_r \in \mathcal{O}_f$ for a.e. $r > 0$. In particular, we can find a sequence $r_i \searrow 0$ such that $(\Omega_{r_i})_i \subseteq \mathcal{O}_f$. Given that $K = \bigcap_i \Omega_{r_i}$, we conclude that $\nu(K) = \lim_i \nu(\Omega_{r_i})$, whence the claim (5.2) follows. □

Now, given a function $f \in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$ and an open set $\Omega \subseteq \mathbb{R}^d$, we define

$$|D_\mu f|_{C^\infty}(\Omega) := \inf\left\{\varliminf_{n\to\infty} \int_\Omega |\nabla f_n|\,\mathrm{d}\mu \,\middle|\, (f_n)_n \subseteq C^\infty(\Omega) \cap L^1_\mu(\Omega),\ f_n \to f \text{ in } L^1_\mu(\Omega)\right\}. \tag{5.3}$$

We can extend it via Carathéodory construction to a set-function on all Borel sets, as follows:

$$|D_\mu f|_{C^\infty}(B) := \inf\big\{|D_\mu f|_{C^\infty}(\Omega) \,\big|\, \Omega \subseteq \mathbb{R}^d \text{ open},\ B \subseteq \Omega\big\}.$$

By suitably adapting the arguments in [8, Lemma 4.4.2 and Lemma 4.4.3], one can show that $|D_\mu f|_{C^\infty}$ is a finite Borel measure on $\mathbb{R}^d$.

Theorem 5.3. *Let $f \in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$. Then the measures $|D_\mu f|_{C^\infty}$ and $|D_\mu f|_{\mathrm{Lip}}$ coincide.*

Proof. It suffices to show that $|D_\mu f|_{C^\infty}(K) = |D_\mu f|_{\mathrm{Lip}}(K)$ for every $K \subseteq \mathbb{R}^d$ compact. Thanks to Lemma 5.2, this is verified as soon as $|D_\mu f|_{C^\infty}(\Omega) = |D_\mu f|_{\mathrm{Lip}}(\Omega)$ for every $\Omega \in \mathcal{O}_f$, where $\mathcal{O}_f$ is defined as in (5.1). Then let $\Omega \in \mathcal{O}_f$ be fixed. Since $C^\infty(\Omega) \subseteq \mathrm{LIP}_{loc}(\Omega)$ and $|\nabla g| = \mathrm{lip}_a(g)$ for all $g \in C^\infty(\Omega)$, we have that $|Df|_{\mathrm{Lip}}(\Omega) \le |D_\mu f|_{C^\infty}(\Omega)$. To prove the converse inequality, we apply Lemma 5.1: given that $f|_{\bar\Omega} \in \mathrm{BV}_{\mathrm{Lip}}(\bar\Omega, \mu|_{\bar\Omega})$ and $\big|D_{\mu|_{\bar\Omega}}(f|_{\bar\Omega})\big|_{\mathrm{Lip}}(\bar\Omega) = |D_\mu f|_{\mathrm{Lip}}(\Omega)$, there exists a sequence $(f_n)_n \subseteq \mathrm{LIP}_{bs}(\bar\Omega)$ such that $f_n \to f$ in $L^1_\mu(\bar\Omega)$ and

$$\int_\Omega \mathrm{lip}_a(f_n)\,\mathrm{d}\mu = \int_{\bar\Omega} \mathrm{lip}_a(f_n)\,\mathrm{d}\mu \to \big|D_{\mu|_{\bar\Omega}}(f|_{\bar\Omega})\big|_{\mathrm{Lip}}(\bar\Omega) = |D_\mu f|_{\mathrm{Lip}}(\Omega), \tag{5.4}$$

where the first identity is granted by the fact that $\partial\Omega$ is μ-negligible. Now let $n \in \mathbb{N}$ be fixed. Extend f_n to some $\mathrm{Lip}(f_n)$-Lipschitz function $\bar f_n \colon \mathbb{R}^d \to \mathbb{R}$. Define $f^m_n := \bar f_n * \rho_{1/m} \in C^\infty(\mathbb{R}^d)$ and $g^m_n := f^m_n|_\Omega \in C^\infty(\Omega)$ for every $m \in \mathbb{N}$. Since $|g^m_n - f_n| \le \mathrm{Lip}(f_n)/m$ by (2.1b), we may estimate

$$\int_\Omega |g^m_n|\,\mathrm{d}\mu \le \int_\Omega |f_n|\,\mathrm{d}\mu + \frac{\mathrm{Lip}(f_n)\mu(\Omega)}{m} < +\infty,$$

so that $g^m_n \in L^1_\mu(\Omega)$. By dominated convergence theorem, we also obtain that $g^m_n \to f_n$ in $L^1_\mu(\Omega)$ as $m \to \infty$. Moreover, we have $|\nabla g^m_n|(x) \le \mathrm{Lip}\big(\bar f_n; B_{2/m}(x)\big)$ for all $m \in \mathbb{N}$ and $x \in \Omega$ by (2.1c), so that $|\nabla g^m_n| \le \mathrm{Lip}(f_n)\chi_\Omega \in L^1_\mu(\Omega)$ on Ω. An application of the reverse Fatou lemma yields

$$\begin{aligned}\varlimsup_{m\to\infty} \int_\Omega |\nabla g^m_n|\,\mathrm{d}\mu &\le \varlimsup_{m\to\infty} \int_\Omega \mathrm{Lip}\big(\bar f_n; B_{2/m}(x)\big)\,\mathrm{d}\mu(x) \le \int_\Omega \lim_{m\to\infty} \mathrm{Lip}\big(\bar f_n; B_{2/m}(x)\big)\,\mathrm{d}\mu(x)\\ &= \int_\Omega \mathrm{lip}_a(f_n)\,\mathrm{d}\mu.\end{aligned}$$

Hence, we can choose $m_n \in \mathbb{N}$ such that the function $g_n \coloneqq g_n^{m_n}$ satisfies $\int_\Omega |g_n - f_n| \, \mathrm{d}\mu \leq 1/n$ and $\int_\Omega |\nabla g_n| \, \mathrm{d}\mu \leq \int_\Omega \mathrm{lip}_a(f_n) \, \mathrm{d}\mu + 1/n$. Then $C^\infty(\Omega) \cap L^1_\mu(\Omega) \ni g_n \to f$ in $L^1_\mu(\Omega)$, so that accordingly

$$|D_\mu f|_{C^\infty}(\Omega) \leq \varliminf_{n\to\infty} \int_\Omega |\nabla g_n| \, \mathrm{d}\mu \leq \varliminf_{n\to\infty} \int_\Omega \mathrm{lip}_a(f_n) \, \mathrm{d}\mu \overset{(5.4)}{=} |D_\mu f|_{\mathrm{Lip}}(\Omega).$$

All in all, we have proved that $|D_\mu f|_{C^\infty}(\Omega) = |D_\mu f|_{\mathrm{Lip}}(\Omega)$ for every $\Omega \in \mathcal{O}_f$, as desired. □

Note that it also follows from Theorem 5.3 that

$$\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d, \mu) = \mathrm{BV}_{C^\infty}(\mathbb{R}^d, \mu) \coloneqq \big\{ f \in L^1_\mu(\mathbb{R}^d) : |D_\mu f|_{C^\infty}(\mathbb{R}^d) < +\infty \big\}. \tag{5.5}$$

Moreover, the total variation measure of the entire space can be recovered by using only compactly-supported smooth functions:

Lemma 5.4. *Let $f \in \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d, \mu)$. Then*

$$|Df|_{\mathrm{Lip}}(\mathbb{R}^d) = \inf \Big\{ \varliminf_{n\to\infty} \int_{\mathbb{R}^d} |\nabla f_n| \, \mathrm{d}\mu : \ (f_n)_n \subseteq C^\infty_c(\mathbb{R}^d), \ f_n \to f \text{ in } L^1_\mu(\mathbb{R}^d) \Big\}. \tag{5.6}$$

Proof. Denote by $\mathrm{R}(f)$ the right-hand side of (5.6). Pick a sequence $(f_n)_n \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$ converging to f in $L^1_\mu(\mathbb{R}^d)$ (whose existence is guaranteed by the characterization of BV functions in item 2) of Definition 3.2). Fix $n \in \mathbb{N}$ and denote by $(f_n^m)_m \subseteq C^\infty_c(\mathbb{R}^d)$ the sequence satisfying

$$\big| f_n^m(x) - f_n(x) \big| \leq \frac{1}{m} \quad \text{ and } \quad |\nabla f_n^m(x)| \leq \mathrm{Lip}\big(f_n; B_{1/m}(x)\big) \ \text{ for all } x \in \mathbb{R}^d, \tag{5.7}$$

whose existence is provided by Lemma 2.1. Calling K_n the closed 1-neighbourhood of $\mathrm{supp}(f_n)$, observe that

$$|\nabla f_n^m| \leq \mathrm{Lip}(f_n)\chi_{K_n} \quad \text{ holds for every } m \in \mathbb{N}$$

and (by passing to the limsup in the second inequality in (5.7)) that

$$\varlimsup_m |\nabla f_n^m| \leq \mathrm{lip}_a(f_n).$$

Thus, we may apply the reverse Fatou lemma and get that

$$\varlimsup_{m\to\infty} \int_{\mathbb{R}^d} |\nabla f_n^m| \, \mathrm{d}\mu \leq \int_{\mathbb{R}^d} \varlimsup_{m\to\infty} |\nabla f_n^m| \, \mathrm{d}\mu \leq \int_{\mathbb{R}^d} \mathrm{lip}_a(f_n) \, \mathrm{d}\mu.$$

Now pick $m_n \in \mathbb{N}$ so that

$$\|f_n^{m_n} - f_n\|_{L^1_\mu(\mathbb{R}^d)} \leq \frac{1}{n} \quad \text{ and } \quad \int_{\mathbb{R}^d} |\nabla f_n^{m_n}| \, \mathrm{d}\mu \leq \int_{\mathbb{R}^d} \mathrm{lip}_a(f_n) \, \mathrm{d}\mu + \frac{1}{n}.$$

By setting $g_n \coloneqq f_n^{m_n} \in C^\infty_c(\mathbb{R}^d)$, we get (via a diagonalization argument) that

$$g_n \to f \ \text{ in } L^1_\mu(\mathbb{R}^d) \quad \text{ and } \quad \mathrm{R}(f) \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^d} |\nabla g_n| \, \mathrm{d}\mu \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^d} \mathrm{lip}_a(f_n) \, \mathrm{d}\mu.$$

This gives that $\mathrm{R}(f) \leq |Df|(\mathbb{R}^d)$. Given that also the opposite inequality holds, by the fact that $C^\infty_c(\mathbb{R}^d) \subseteq \mathrm{LIP}_c(\mathbb{R}^d)$, the proof of (5.6) is done. □

5.2. **Relation between vector fields and derivations.** In this subsection we show that the space of bounded derivations with bounded divergence is isometrically isomorphic to the space of bounded vector fields with bounded divergence. The main tool we are going to use is the following result that we refer to as the *superposition principle for derivations*:

Theorem 5.5. *Let $\mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^n, \mu)$ be such that $|\mathbf{b}|, \mathrm{div}(\mathbf{b}) \in L^1_\mu(\mathbb{R}^d)$. Then there exists a finite, non-negative Borel measure π on $C([0,1], \mathbb{R}^d)$ concentrated on non-constant absolutely continuous curves having constant speed and such that*

$$\int_{\mathbb{R}^d} g\,\mathbf{b}(f)\,\mathrm{d}\mu = \iint_0^1 g(\gamma_t)\,(f\circ\gamma)'_t\,\mathrm{d}t\,\mathrm{d}\pi(\gamma) \quad \textit{for every } (g,f) \in \mathrm{LIP}(\mathbb{R}^d)\times \mathrm{LIP}_c(\mathbb{R}^d), \tag{5.8a}$$

$$\int_{\mathbb{R}^d} g\,|\mathbf{b}|\,\mathrm{d}\mu = \iint_0^1 g(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma) \quad \textit{for every } g \in \mathrm{LIP}_c(\mathbb{R}^d). \tag{5.8b}$$

The above result is a consequence of a metric version (provided by Paolini and Stepanov in [19, 18]) of the superposition principle for normal 1-currents proven by Smirnov in [20] and of the fact that any element of $\mathrm{Der}_b(\mathbb{R}^d, \mu)$ induces a normal 1-current. Namely, for any $\mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu)$, the map given by

$$T_{\mathbf{b}}(g,f) := \int g\,\mathbf{b}(f)\,\mathrm{d}\mu \quad \text{for every } (g,f) \in \mathrm{LIP}(\mathbb{R}^d)\times \mathrm{LIP}_c(\mathbb{R}^d),$$

defines a normal 1-current on $\mathbb{R}^d$. The formulation given in Theorem 5.5 is due to [10].

Theorem 5.6. *The operator $\Phi\colon \mathrm{D}_\infty(\mathrm{div}_\mu) \to \mathrm{Der}_b(\mathbb{R}^d, \mu)$, given by*

$$\Phi(v)(f) := v\cdot\nabla_\mu f \in L^\infty_\mu(\mathbb{R}^d), \quad \textit{for every } v \in D_\infty(\mathrm{div}_\mu) \textit{ and every } f \in \mathrm{LIP}_c(\mathbb{R}^d),$$

is a bijection, $\mathrm{LIP}_c(\mathbb{R}^d)$-linear and satisfies

$$|\Phi(v)| = |v| \quad \textit{and} \quad \mathrm{div}\big(\Phi(v)\big) = \mathrm{div}_\mu(v) \quad \mu\textit{-a.e., for every } v \in \mathrm{D}_\infty(\mathrm{div}_\mu). \tag{5.9}$$

Proof.
Step 1. First of all, given $v \in \mathrm{D}_\infty(\mathrm{div}_\mu)$, we verify that $\Phi(v) \in \mathrm{Der}_b(\mathbb{R}^d, \mu)$. The linearity of $\Phi(v)$ clearly holds true, while the properties 1) and 2) in the Definition 3.4 follow from the fact that the gradient operator ∇_μ satisfies the Leibniz rule (see Lemma 4.7) and from the μ-a.e. inequality (granted by Proposition 4.14)

$$\big|\Phi(v)(f)\big| \le |v||\nabla_\mu f| \le |v|\,\mathrm{lip}_a(f) \quad \text{for every } f \in \mathrm{LIP}_c(\mathbb{R}^d),$$

respectively. We now prove that $|\Phi(v)| = |v| \in L^\infty_\mu(\mathbb{R}^d)$. Recalling formula (3.5), we have that

$$\begin{aligned} |\Phi(v)| &= \text{ess sup}\left\{|v\cdot\nabla_\mu f| :\ f \in \mathrm{LIP}_c(\mathbb{R}^d),\ \mathrm{Lip}(f)\le 1\right\} \\ &\le \text{ess sup}\left\{|v|\,|\nabla_\mu f| :\ f \in \mathrm{LIP}_c(\mathbb{R}^d),\ \mathrm{Lip}(f)\le 1\right\} \le |v|, \end{aligned}$$

holds μ-a.e.. To prove the opposite inequality, take a dense sequence $(w_i)_i \subseteq \mathbb{S}^{d-1} := \{w \in \mathbb{R}^d : |w| = 1\}$. Then for μ-a.e. $x \in \mathbb{R}^d$ we have that $|v|(x) = \sup_{i\in\mathbb{N}} v(x)\cdot w_i$. Now, for every $i,k \in \mathbb{N}$ choose $f_{i,k} \in C^\infty_c(\mathbb{R}^d)$ such that $\nabla f_{i,k} = w_i$ on $B_k(0)$. Then, given $k \in \mathbb{N}$, we have that

$$\begin{aligned} |v|(x) = \sup_{i\in\mathbb{N}} v(x)\cdot w_i &= \sup_{i\in\mathbb{N}} v(x)\cdot\nabla f_{i,k}(x) = \sup_{i\in\mathbb{N}} v(x)\cdot\nabla_\mu f_{i,k}(x) \\ &= \sup_{i\in\mathbb{N}} \Phi(v)(f_{i,k})(x) \le |\Phi(v)|(x)\,\mathrm{lip}_a(f_{i,k})(x) \le |\Phi(v)|(x), \end{aligned}$$

for μ-a.e. $x \in B_k(0)$. By the arbitrariness of k, we conclude that $|v| \le |\Phi(v)|$ holds μ-a.e. in $\mathbb{R}^d$.

To see that $\Phi(v)$ admits bounded divergence, let us first observe that for every $f \in C^\infty_c(\mathbb{R}^d)$ it holds that

$$\int_{\mathbb{R}^d} \Phi(v)(f)\,\mathrm{d}\mu = \int_{\mathbb{R}^d} v\cdot\nabla_\mu f\,\mathrm{d}\mu \overset{(4.6)}{=} \int_{\mathbb{R}^d} v\cdot\nabla f\,\mathrm{d}\mu = -\int_{\mathbb{R}^d} f\,\mathrm{div}_\mu(v)\,\mathrm{d}\mu. \tag{5.10}$$

Now, given any $f \in \mathrm{LIP}_c(\mathbb{R}^d)$, we know that $f \in W^{1,1}(\mathbb{R}^d, \mu)$ and thus we can find a sequence $(f_n)_n \subseteq C^\infty_c(\mathbb{R}^d)$ such that

$$f_n \to f \ \text{ in } L^1_\mu(\mathbb{R}^d) \quad \text{and} \quad \nabla_\mu f_n \to \nabla_\mu f \ \text{ in } L^1_\mu(\mathbb{R}^d;\mathbb{R}^d).$$

Thus, we can pass to the limit in (5.10) and get that

$$\int_{\mathbb{R}^d} \Phi(v)(f)\,\mathrm{d}\mu = \lim_{n\to\infty}\int_{\mathbb{R}^d} v\cdot\nabla_\mu f_n\,\mathrm{d}\mu = -\lim_{n\to\infty}\int_{\mathbb{R}^d} f_n\,\mathrm{div}_\mu(v)\,\mathrm{d}\mu = -\int_{\mathbb{R}^d} f\,\mathrm{div}_\mu(v)\,\mathrm{d}\mu.$$

By the arbitrariness of $f\in \mathrm{LIP}_c(\mathbb{R}^d)$, this proves that $\Phi(v)$ admits divergence and that $\mathrm{div}\big(\Phi(v)\big) = \mathrm{div}_\mu(v)\in L^\infty_\mu(\mathbb{R}^d)$.

Step 2. What remains to show is that Φ is bijective. The injectivity of Φ is granted by the μ-a.e. equality $|\Phi(v)| = |v|$ proved in Step 1. Let us now fix $\mathbf{b}\in \mathrm{Der}_b(\mathbb{R}^d,\mu)$ such that $|\mathbf{b}|,\mathrm{div}(\mathbf{b})\in L^1_\mu(\mathbb{R}^d)$. Let π be the measure on the space of curves $C\big([0,1],\mathbb{R}^d\big)$ given by the superposition principle in Theorem 5.5. Define the map $\mathsf{D}\colon C\big([0,1],\mathbb{R}^d\big)\times[0,1]\to\mathbb{R}^d\times\mathbb{R}^d$ as

$$\mathsf{D}(\gamma,t)\coloneqq(\gamma_t,\dot\gamma_t),\qquad \text{for every } (\gamma,t)\in C\big([0,1],\mathbb{R}^d\big)\times[0,1].$$

We further set $\nu\coloneqq \mathsf{D}_*\big(\pi\otimes\mathcal{L}^1|_{[0,1]}\big)$. Calling $p\colon\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ the canonical projection map, i.e. $p(x,v)=x$ for every $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$, we disintegrate the measure ν with respect to the map p, getting a measurable family $\{\nu_x\}_{x\in\mathbb{R}^d}$ of probability measures ν_x on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} g(x,v)\,\mathrm{d}\nu(x,v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x,\cdot)\,\mathrm{d}\nu_x\,\mathrm{d}p_*\nu(x),\qquad \text{for every } g\in L^1_\nu(\mathbb{R}^d\times\mathbb{R}^d).$$

We claim that $p_*\nu\ll\mu$. Indeed, by using (5.8b) we have for every $g\in\mathrm{LIP}_c(\mathbb{R}^d)$ that

$$\begin{aligned}\int_{\mathbb{R}^d} g|\mathbf{b}|\,\mathrm{d}\mu &= \iint_0^1 g(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\\ &= \int_{\mathbb{R}^d} g(x)\bigg(\int_{\mathbb{R}^d}|w|\,\mathrm{d}\nu_x(w)\bigg)\mathrm{d}p_*\nu(x).\end{aligned}$$

By the arbitrariness of $g\in\mathrm{LIP}_c(\mathbb{R}^d)$, we have that

$$|\mathbf{b}|\mu = \int_{\mathbb{R}^d}|w|\,\mathrm{d}\nu_{(\cdot)}(w)\,p_*\nu. \tag{5.11}$$

Since the measure π is concentrated on non-constant curves having constant speed, we have that $\dot\gamma_t\neq 0$ for $(\pi\otimes\mathcal{L}^1|_{[0,1]})$-a.e. (γ,t). This implies that $\int_{\mathbb{R}^d}|w|\,\mathrm{d}\nu_x(w)>0$ holds for $p_*\nu$-a.e. $x\in\mathbb{R}^d$. Therefore, we conclude that $p_*\nu\ll\mu$.

Step 3. Now, we define

$$v(x)\coloneqq\frac{\mathrm{d}p_*\nu}{\mathrm{d}\mu}(x)\int_{\mathbb{R}^d} w\,\mathrm{d}\nu_x(w),\qquad \text{for } \mu\text{-a.e. } x\in\mathbb{R}^d.$$

Our aim is to show that $v\in \mathrm{D}_\infty(\mathrm{div}_\mu)$ and that $\mathbf{b}(f)=\nabla_\mu f\cdot v=\Phi(v)(f)$ for every $f\in\mathrm{LIP}_c(\mathbb{R}^d)$. First of all, observe that the formula (5.11) ensures that $|v|\le|\mathbf{b}|$ holds μ-a.e., thus $v\in L^\infty_\mu(\mathbb{R}^d)$. By using formula (5.8a) and by unwrapping the above definitions we have the following: given any $g\in\mathrm{LIP}_c(\mathbb{R}^d)$ and $f\in C^\infty_c(\mathbb{R}^d)$ it holds that

$$\begin{aligned}\int_{\mathbb{R}^d} g\,\mathbf{b}(f)\,\mathrm{d}\mu &= \iint_0^1 g(\gamma_t)\,\nabla f(\gamma_t)\cdot\dot\gamma_t\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)\\ &= \int_{\mathbb{R}^d} g(x)\bigg(\int\nabla f(x)\cdot w\,\mathrm{d}\nu_x(w)\bigg)\mathrm{d}p_*\nu(x)\\ &= \int_{\mathbb{R}^d} g(x)\bigg(\frac{\mathrm{d}p_*\nu}{\mathrm{d}\mu}(x)\int_{\mathbb{R}^d}\nabla f(x)\cdot w\,\mathrm{d}\nu_x(w)\bigg)\mathrm{d}\mu(x)\\ &= \int_{\mathbb{R}^d} g(x)\,\nabla f(x)\cdot v(x)\,\mathrm{d}\mu(x).\end{aligned}$$

Thus, since $g \in \mathrm{LIP}_c(\mathbb{R}^d)$ was arbitrary, we deduce that $\mathbf{b}(f)(x) = \nabla f(x) \cdot v(x)$ for μ-a.e. $x \in \mathbb{R}^d$. This also ensures that

$$\int_{\mathbb{R}^d} \nabla f \cdot v \, \mathrm{d}\mu = \int_{\mathbb{R}^d} \mathbf{b}(f) \, \mathrm{d}\mu = -\int_{\mathbb{R}^d} \mathrm{div}(\mathbf{b}) \, f \, \mathrm{d}\mu$$

holds for every $f \in C_c^\infty(\mathbb{R}^d)$. Hence, v is a vector field with divergence and $\mathrm{div}_\mu(v) = \mathrm{div}(\mathbf{b})$. All in all, we have proved that $v \in \mathrm{D}_\infty(\mathrm{div}_\mu)$. Consequently, we have that

$$\mathbf{b}(f) = \nabla f \cdot v = \nabla_\mu f \cdot v \quad \text{holds for every } \ f \in C_c^\infty(\mathbb{R}^d). \tag{5.12}$$

By approximation, we can obtain (5.12) for every $f \in \mathrm{LIP}_c(\mathbb{R}^d)$, proving that $\Phi\big(\mathrm{D}_\infty(\mathrm{div}_\mu)\big) \subseteq \{\mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu) : \, |\mathbf{b}|, \mathrm{div}(\mathbf{b}) \in L^1_\mu(\mathbb{R}^d)\} =: \mathcal{D}$.

Step 4. It remains to show that Φ is surjective. Fix $\mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu)$ and fix a sequence $(\eta_n)_n \subseteq C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ on $B_n(0)$ and $\mathrm{Lip}(\eta_n) = 1$ for each $n \in \mathbb{N}$. We set $\mathbf{b}_n := \eta_n \mathbf{b}$ and note that $\mathbf{b}_n \in \mathcal{D}$. Moreover, $|\mathbf{b}_n| \leq |\mathbf{b}|$ and $|\mathrm{div}(\mathbf{b}_n)| = |\mathbf{b}(\eta_n) + \eta_n \mathrm{div}(\mathbf{b})| \leq |\mathbf{b}| + |\mathrm{div}(\mathbf{b})|$. By Step 3 for every $n \in \mathbb{N}$ we have the existence of an element $v_n \in \mathrm{D}_\infty(\mathrm{div}_\mu)$ such that $\Phi(v_n) = \mathbf{b}_n$. Also, $|v_n| = |\mathbf{b}_n| \leq |\mathbf{b}|$ and $|\mathrm{div}_\mu(v_n)| = |\mathrm{div}(\mathbf{b}_n)| \leq |\mathbf{b}| + |\mathrm{div}(\mathbf{b})|$, thus (up to a subsequence) we have that $v_n \rightharpoonup v$ weakly* in $L^\infty_\mu(\mathbb{R}^d; \mathbb{R}^d)$ for some $v \in L^\infty_\mu(\mathbb{R}^d; \mathbb{R}^d)$ and $\mathrm{div}_\mu(v_n) \rightharpoonup h$ weakly* in $L^\infty_\mu(\mathbb{R}^d)$ for some $h \in L^\infty_\mu(\mathbb{R}^d)$. Moreover, due to the closure of the operator div_μ we have that $v \in \mathrm{D}_\infty(\mathrm{div}_\mu)$ and that $h = \mathrm{div}_\mu(v)$. We only need to check that $\Phi(v) = \mathbf{b}$. Let us fix $f \in \mathrm{LIP}_c(\mathbb{R}^d)$. Then, for every $n \in \mathbb{N}$ we have that $\mathbf{b}_n(f) = \Phi(v_n)(f) = v_n \cdot \nabla_\mu f$. Since $v_n \cdot \nabla_\mu f$ and $\mathbf{b}_n(f) = \eta_n \mathbf{b}(f)$ converge weakly* in $L^\infty_\mu(\mathbb{R}^d)$ to $v \cdot \nabla_\mu f$ and $\mathbf{b}(f)$, respectively, we get that $\mathbf{b}(f) = v \cdot \nabla_\mu f = \Phi(v)(f)$. By the arbitrariness of $f \in \mathrm{LIP}_c(\mathbb{R}^d)$, we get the surjectivity of Φ and conclude the proof. □

5.3. **Equivalence of BV spaces.** As our main result, we have the following equivalent characterizations of the BV space:

Theorem 5.7 (Equivalent characterizations of BV function)**.** *It holds that*

$$\mathrm{BV}(\mathbb{R}^d, \mu) = \mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu) = \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d, \mu) = \mathrm{BV}_{C^\infty}(\mathbb{R}^d, \mu).$$

Moreover, it holds that $|D_\mu f| = |D_\mu f|_{\mathrm{Der}} = |D_\mu f|_{\mathrm{Lip}} = |D_\mu f|_{C^\infty}$ *for every* $f \in \mathrm{BV}(\mathbb{R}^d, \mu)$.

Proof.
Step 1. First of all, the fact that $\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu) = \mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d, \mu) = \mathrm{BV}_{C^\infty}(\mathbb{R}^d, \mu)$ and $|D_\mu f|_{\mathrm{Der}} = |D_\mu f|_{\mathrm{Lip}} = |D_\mu f|_{C^\infty}$ for every $f \in \mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu)$ follows from [8, Theorem 4.5.3] and Theorem 5.3. Moreover, it follows from Theorem 5.6 that $|D_\mu f|(\mathbb{R}^d) = |D_\mu f|_{\mathrm{Der}}(\mathbb{R}^d)$ for every $f \in \mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu)$, which implies $\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu) \subseteq \mathrm{BV}(\mathbb{R}^d, \mu)$.

For the purposes of the next step, we recall that it was proved in [2, Theorem 5.1] that a given function $f \in L^1_\mu(\mathbb{R}^d)$ belongs to $\mathrm{BV}(\mathbb{R}^d, \mu)$ if and only if

$$\exists (f_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d) : \quad f_n \to f \ \text{ in } L^1_\mu(\mathbb{R}^d), \quad \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^d} |\nabla_\mu f_n| \, \mathrm{d}\mu < +\infty. \tag{5.13}$$

Step 2. Next we claim that

$$\mathrm{BV}(\mathbb{R}^d, \mu) \subseteq \mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d, \mu). \tag{5.14}$$

In order to prove it, fix any $f \in \mathrm{BV}(\mathbb{R}^d, \mu)$. Define $\mathcal{L}_f \colon \mathrm{Der}_b(\mathbb{R}^d, \mu) \to \big(\mathrm{LIP}_c(\mathbb{R}^d), \| \cdot \|_{C_b(\mathbb{R}^d)}\big)^*$ as

$$\mathcal{L}_f(\mathbf{b})(h) := -\int_{\mathbb{R}^d} f \, \mathrm{div}(h\mathbf{b}) \, \mathrm{d}\mu, \qquad \text{for every } \mathbf{b} \in \mathrm{Der}_b(\mathbb{R}^d, \mu) \text{ and } h \in \mathrm{LIP}_c(\mathbb{R}^d). \tag{5.15}$$

To see that $\mathcal{L}_f(\mathbf{b})$ is indeed an element of $\big(\mathrm{LIP}_c(\mathbb{R}^d),\|\cdot\|_{C_b(\mathbb{R}^d)}\big)^*$, note that for any $\mathbf{b}\in\mathrm{Der}_b(\mathbb{R}^d,\mu)$ and $h\in\mathrm{LIP}_c(\mathbb{R}^d)$ with $h\neq 0$ we may compute

$$\mathcal{L}_f(\mathbf{b})(h)=-\int_{\mathbb{R}^d} f\,\mathrm{div}\left(\frac{h\|h\|_{C_b(\mathbb{R}^d)}}{\|h\|_{C_b(\mathbb{R}^d)}}\mathbf{b}\right)\mathrm{d}\mu=-\|h\|_{C_b(\mathbb{R}^d)}\int_{\mathbb{R}^d} f\,\mathrm{div}\left(\frac{h}{\|h\|_{C_b(\mathbb{R}^d)}}\mathbf{b}\right)\mathrm{d}\mu.$$

We now apply Theorem 5.6: since $v\coloneqq\frac{h}{\|h\|_{C_b(\mathbb{R}^d)}}\Phi^{-1}(\mathbf{b})$ is a competitor in the definition of $\|D_\mu f\|$ and $\mathrm{div}_\mu(v)=\mathrm{div}\big(\frac{h}{\|h\|_{C_b(\mathbb{R}^d)}}\boldsymbol{b}\big)$, we get $|\mathcal{L}_f(\mathbf{b})|\leq\|h\|_{C_b(\mathbb{R}^d)}\|D_\mu f\|$, thus the continuity of $\mathcal{L}_f(\mathbf{b})$. The linearity is clear from the very definition of $\mathcal{L}_f(\mathbf{b})$, thus we conclude that $\mathcal{L}_f(\mathbf{b})$ is an element of $\big(\mathrm{LIP}_c(\mathbb{R}^d),\|\cdot\|_{C_b(\mathbb{R}^d)}\big)^*$. Being $\mathrm{LIP}_c(\mathbb{R}^d)$ dense in $C_0(\mathbb{R}^d)$ with respect to the $C_b(\mathbb{R}^d)$-norm, we can uniquely extend $\mathcal{L}_f(\mathbf{b})$ to an element of $\big(C_0(\mathbb{R}^d),\|\cdot\|_{C_b(\mathbb{R}^d)}\big)^*$, which we still call $\mathcal{L}_f(\mathbf{b})$. Given that $\big(C_0(\mathbb{R}^d),\|\cdot\|_{C_b(\mathbb{R}^d)}\big)^*$ can be identified with $\mathscr{M}(\mathbb{R}^d)$ (recall Remark 2.3), there exists a unique $L_f(\mathbf{b})\in\mathscr{M}(\mathbb{R}^d)$ such that

$$\mathcal{L}_f(\mathbf{b})(h)=\int_{\mathbb{R}^d} h\,\mathrm{d}L_f(\mathbf{b}),\qquad\text{for every } h\in C_0(\mathbb{R}^d).\tag{5.16}$$

To verify that $L_f(v)(\mathbb{R}^d)=-\int_{\mathbb{R}^d} f\,\mathrm{div}_\mu(v)\,\mathrm{d}\mu$, pick a sequence $(h_n)_{n\in\mathbb{N}}\subseteq\mathrm{LIP}_c(\mathbb{R}^d)$ of 1-Lipschitz functions $h_n\colon\mathbb{R}^d\to[0,1]$ such that $h_n=1$ on $B_n(0)$ for every $n\in\mathbb{N}$. In particular, $h_n(x)\to 1$ and $\mathrm{lip}_a(h_n)(x)\to 0$ for every $x\in\mathbb{R}^d$. Since for any $n\in\mathbb{N}$ we have the μ-a.e. inequality

$$\big|f\big(h_n\mathrm{div}(\mathbf{b})+\mathbf{b}(h_n)\big)\big|\leq|f|\big(|\mathrm{div}(\mathbf{b})|+|\mathbf{b}|\,\mathrm{lip}_a(h_n)\big)\leq|f|\big(|\mathrm{div}(\mathbf{b})|+|\mathbf{b}|\big)\in L^1_\mu(\mathbb{R}^d),$$

by using twice the dominated convergence theorem we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d}\mathrm{d}L_f(\mathbf{b}) &= \lim_{n\to\infty}\int_{\mathbb{R}^d} h_n\,\mathrm{d}L_f(\mathbf{b})\overset{(5.16)}{=}\lim_{n\to\infty}\mathcal{L}_f(\mathbf{b})(h_n)\overset{(5.15)}{=}-\lim_{n\to\infty}\int_{\mathbb{R}^d} f\,\mathrm{div}(h_n\mathbf{b})\,\mathrm{d}\mu\\ &\overset{(3.6)}{=}-\lim_{n\to\infty}\int_{\mathbb{R}^d} f\big(h_n\mathrm{div}(\mathbf{b})+\mathbf{b}(h_n)\big)\,\mathrm{d}\mu=-\int_{\mathbb{R}^d} f\,\mathrm{div}(\mathbf{b})\,\mathrm{d}\mu.\end{aligned}$$

The linearity of the map $\mathrm{Der}_b(\mathbb{R}^d,\mu)\ni\mathbf{b}\mapsto L_f(\mathbf{b})\in\mathscr{M}(\mathbb{R}^d)$ is clear from the very definition. Moreover, given any $g,h\in\mathrm{LIP}_c(\mathbb{R}^d)$, we can compute

$$\int_{\mathbb{R}^d} h\,\mathrm{d}L_f(g\mathbf{b})=\mathcal{L}_f(g\mathbf{b})(h)=-\int_{\mathbb{R}^d} f\,\mathrm{div}(hg\mathbf{b})\,\mathrm{d}\mu=\mathcal{L}_f(\mathbf{b})(hg)=\int_{\mathbb{R}^d} hg\,\mathrm{d}L_f(\mathbf{b}),$$

which, thanks to the arbitrariness of $h\in\mathrm{LIP}_c(\mathbb{R}^d)$, implies $L_f(g\mathbf{b})=gL_f(\mathbf{b})$ for all $g\in\mathrm{LIP}_c(\mathbb{R}^d)$. Hence, we have proved that the operator L_f is $\mathrm{LIP}_c(\mathbb{R}^d)$-linear. We are just left to prove the continuity of L_f with respect to the $\|\cdot\|_b$-norm on $\mathrm{Der}_b(\mathbb{R}^d,\mu)$. By applying (5.13), we can find a sequence $(f_n)_{n\in\mathbb{N}}\subseteq C_c^\infty(\mathbb{R}^d)$ and $C\geq 0$ such that $f_n\to f$ in $L^1_\mu(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d}|\nabla_\mu f_n|\,\mathrm{d}\mu\to C$. For any $n\in\mathbb{N}$ and $h\in\mathrm{LIP}_c(\mathbb{R}^d)$, we can estimate

$$\begin{aligned}\left|\int f_n\,\mathrm{div}(h\mathbf{b})\,\mathrm{d}\mu\right| &\overset{(5.9)}{=}\left|\int f_n\,\mathrm{div}_\mu\big(h\,\Phi^{-1}(\mathbf{b})\big)\,\mathrm{d}\mu\right|=\left|\int_{\mathbb{R}^d} h\,\nabla_\mu f_n\cdot\Phi^{-1}(\mathbf{b})\,\mathrm{d}\mu\right|\\ &\leq\ \|h\|_{C_b(\mathbb{R}^d)}\|\Phi^{-1}(\mathbf{b})\|_{L^\infty_\mu(\mathbb{R}^d;\mathbb{R}^d)}\int_{\mathbb{R}^d}|\nabla_\mu f_n|\,\mathrm{d}\mu\\ &\overset{(5.9)}{=}\ \|h\|_{C_b(\mathbb{R}^d)}\|\mathbf{b}\|_b\int_{\mathbb{R}^d}|\nabla_\mu f_n|\,\mathrm{d}\mu,\end{aligned}$$

whence by letting $n\to\infty$ it follows $|\mathcal{L}_f(\mathbf{b})(h)|=\lim_n\big|\int f_n\,\mathrm{div}(h\mathbf{b})\,\mathrm{d}\mu\big|\leq\|h\|_{C_b(\mathbb{R}^d)}\|\mathbf{b}\|_b C$, thus

$$\|L_f(\mathbf{b})\|_{\mathsf{TV}}=\sup_{\substack{h\in\mathrm{LIP}_c(\mathbb{R}^d):\\ \|h\|_{C_b(\mathbb{R}^d)}\leq 1}}|\mathcal{L}_f(\mathbf{b})(h)|\leq C\|\mathbf{b}\|_b.$$

This yields continuity of the map $\mathrm{Der}_b(\mathbb{R}^d,\mu)\ni\mathbf{b}\mapsto L_f(\mathbf{b})\in\mathscr{M}(\mathbb{R}^d)$. All in all, we have shown that $f\in\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)$ with $Df=L_f$, thus accordingly the claim (5.14) is proved.

STEP 3. So far, we have shown that $\mathrm{BV}(\mathbb{R}^d,\mu)=\mathrm{BV}_{\mathrm{Der}}(\mathbb{R}^d,\mu)=\mathrm{BV}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)=\mathrm{BV}_{C^\infty}(\mathbb{R}^d,\mu)$. To conclude, fix a function $f\in\mathrm{BV}(\mathbb{R}^d,\mu)$ and an open set $\Omega\subseteq\mathbb{R}^d$. The fact that $|D_\mu f|_{C^\infty}(\Omega)=|D_\mu f|_{\mathrm{Lip}}(\Omega)=|D_\mu f|_{\mathrm{Der}}(\Omega)$ is granted by Theorem 5.3 and [8, Theorem 4.5.3]. Moreover, it readily follows from Theorem 5.6 that $|D_\mu f|(\Omega)=|D_\mu f|_{\mathrm{Der}}(\Omega)$ as well. This is sufficient to conclude that $|D_\mu f|=|D_\mu f|_{\mathrm{Der}}=|D_\mu f|_{\mathrm{Lip}}=|D_\mu f|_{C^\infty}$ as measures, thus completing the proof of the statement. □

5.4. **Relation between $W^{1,1}$ spaces.** Aim of this brief section is to investigate the relation between the Sobolev spaces $W^{1,1}(\mathbb{R}^d,\mu)$ and $W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$.

Theorem 5.8 (Relation between $W^{1,1}$ spaces)**.** *It holds that*

$$W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)\subseteq W^{1,1}(\mathbb{R}^d,\mu).$$

Moreover, it holds that $|\nabla_\mu f|\le|\nabla f|_{rs}$ μ-a.e. for every $f\in W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)$.

Proof. To prove the statement amounts to showing that

$$f\in W^{1,1}_{\mathrm{Lip}}(\mathbb{R}^d,\mu)\quad\Longrightarrow\quad f\in W^{1,1}(\mathbb{R}^d,\mu)\text{ and }|\nabla_\mu f|\le|\nabla f|_{rs}\text{ in the }\mu\text{-a.e. sense.}\tag{5.17}$$

Taking Remark 4.10 into account, we can find a sequence $(f_n)_{n\in\mathbb{N}}\subseteq\mathrm{LIP}_c(\mathbb{R}^d)$ and a non-negative function $H\in L^1_\mu(\mathbb{R}^d)$ such that $f_n\to f$ strongly in $L^1_\mu(\mathbb{R}^d)$, $\mathrm{lip}_a(f_n)\rightharpoonup|\nabla f|_{rs}$ weakly in $L^1_\mu(\mathbb{R}^d)$, and $\mathrm{lip}_a(f_n)\le H$ μ-a.e. for every $n\in\mathbb{N}$. Given any $n\in\mathbb{N}$, define $f_n^k\coloneqq\rho_{1/k}*f_n\in C^\infty_c(\mathbb{R}^d)$ for every $k\in\mathbb{N}$. Lemma 2.1 says that $f_n^k\to f_n$ strongly in $L^1_\mu(\mathbb{R}^d)$ as $k\to\infty$. Lemma 4.15 gives

$$|\nabla_\mu f_n^k|\le|\nabla f_n^k|\overset{(2.1c)}{\le}\mathrm{Lip}\big(f_n;B_{2/k}(\cdot)\big)\to\mathrm{lip}_a(f_n),\qquad\text{strongly in }L^1_\mu(\mathbb{R}^d)\text{ as }k\to\infty.$$

Then Proposition 2.2 yields the existence of a function $G_n\in L^1_\mu(\mathbb{R}^d)$ such that (up to a subsequence in k) it holds $G_n\le\mathrm{lip}_a(f_n)\le H$ μ-a.e. and $|\nabla_\mu f_n^k|\rightharpoonup G_n$ weakly in $L^1_\mu(\mathbb{R}^d)$ as $k\to\infty$. By applying Proposition 2.2 again, we can also find a function $G\in L^1_\mu(\mathbb{R}^d)$ such that $G\le|\nabla f|_{rs}$ μ-a.e. and (up to a subsequence in n) $G_n\rightharpoonup G$ weakly in $L^1_\mu(\mathbb{R}^d)$. Thanks to a diagonalization argument, we can construct a sequence $(k(n))_{n\in\mathbb{N}}\subseteq\mathbb{N}$ such that the functions $g_n\coloneqq f_n^{k(n)}\in C^\infty_c(\mathbb{R}^d)$ satisfy $g_n\to f$ strongly in $L^1_\mu(\mathbb{R}^d)$ and $|\nabla_\mu g_n|\rightharpoonup G$ weakly in $L^1_\mu(\mathbb{R}^d)$. This implies that $|\nabla f|_{rs}\in\mathrm{TRS}(f)$, whence (by Theorem 4.13) it follows that $f\in W^{1,1}(\mathbb{R}^d,\mu)$ and $|\nabla_\mu f|\le|\nabla f|_{rs}$ μ-a.e., getting (5.17). Therefore, the statement is achieved. □

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