Colored Stochastic Multiplicative Processes with Additive Noise Unveil a Third-Order PDE, Defying Conventional FPE and Fick-Law Paradigms.

Marco Bianucci,[1,a)] Mauro Bologna,[2] and Riccardo Mannella[3]

[1)] *Istituto di Scienze Marine, Consiglio Nazionale delle Ricerche (ISMAR - CNR), 19032 Lerici (SP), Italy*

[2)] *Departamento de Ingeniería Eléctrica - Electrónica, Universidad de Tarapacá, Arica, 1000000, Chile*

[3)] *Dipartimento di Fisica, Università di Pisa, 56100 Pisa, Italy*

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Research on stochastic differential equations (SDEs) involving both additive and multiplicative noise has been extensive. In cases where the primary process is driven by a multiplicative stochastic process, additive white noise typically represents an intrinsic and unavoidable fast component. This applies to phenomena such as thermal fluctuations, inherent uncertainties in measurement processes, or rapid wind forcing in ocean dynamics. This study focuses on an important class of such systems, particularly those characterized by linear drift and multiplicative noise, which have been extensively explored in the literature. In many existing studies, multiplicative stochastic processes are often treated as white noise. However, when considering colored multiplicative noise, the emphasis has usually been on characterizing the far tails of the probability density function (PDF), irrespective of the noise's spectral properties. In the absence of additive noise and with a general colored multiplicative SDE, standard perturbation approaches lead to a second-order partial differential equation (PDE) known as the Fokker-Planck Equation (FPE), consistent with Fick's law. This investigation reveals a significant deviation from this standard behavior when additive white noise is introduced. At the leading order of the stochastic process strength, perturbation approaches yield a third-order PDE, regardless of the white noise intensity. The breakdown of the FPE further indicates the breakdown of Fick's law. Additionally, we derive the explicit solution for the equilibrium PDF corresponding to this third-order PDE Master Equation. Through numerical simulations, we demonstrate significant deviations from results obtained using the FPE derived from Fick's law.

[a] marco.bianucci@cnr.it

I. INTRODUCTION

Linear equations driven by both additive and multiplicative noise are common across nearly all scientific disciplines. In the general N-dimensional case (N-D), these equations are expressed as:

$$\dot{\boldsymbol{x}} = -\mathbb{E}\cdot\boldsymbol{x} + \boldsymbol{f}[t] - \Xi[t]\cdot\boldsymbol{x} \tag{1}$$

where $\boldsymbol{x} := (x_1, \ldots, x_N)$, $\mathbb{E}$ and $\Xi[t]$ are $N \times N$ matrices with constant and stochastic components, respectively, and $f[t]$ is a multidimensional white noise with a correlation (or diffusion) matrix D. We adopt the convention of using square brackets for the time argument of a stochastic process, while standard parentheses are used for specific realizations of the stochastic process, which are functions of time. For example, a specific function $f(u)$ with $0 \le u \le t$ is a realization of $f[t]$.

As shown in[1] extending equation (1) to an infinite (or continuous) vector space leads to a general model that describes a wide range of important physical phenomena in fluid dynamics and quantum mechanics. This model, also known as a random multiplicative process (RMP), is a well-established mechanism for generating power-law behaviors.

Widely used to model various systems with both discrete and continuous time, the RMP has been applied to phenomena such as on-off intermittency[2–6] general intermittency with power-law statistics[7,8] (see Fig. 1), lasers[9,10], economic activity[11,12], fluctuations in biological populations within changing environments[13], and the advection of passive scalar fields by fluids[1,14]. It is also a key model in theories of large fluctuations (e.g.,[15] and references therein). Therefore, the significance of the model (1) cannot be overstated.

For simplicity, this work focuses on the one dimensional (1-D) version of the model (1):

$$\dot{x} = -\gamma x + f[t] - \epsilon\, x\xi[t], \tag{2}$$

which is the primary focus of most of the literature cited above. While the extension to the N-D case (1) case is straightforward, it is somewhat intricate, so to avoid burdening the reader with tedious algebra, this generalization is relegated to Appendix A.

In equation (2) $f[t]$ is a white noise with diffusion coefficient D_f and $\xi[t]$ is a Gaussian stochastic process with zero mean, finite correlation time $\bar{\tau}$[16] and normalized autocorrelation function $\varphi(t) = \langle\xi(t)\xi(0)\rangle_\xi/\langle\xi^2\rangle_\xi$. Here, $\langle\ldots\rangle_\xi$ denotes the average over realizations of the random process $\xi[t]$, which is assumed to be at equilibrium. We also define:

$$\tau := \int_0^\infty \varphi(u)\,\mathrm{d}u. \tag{3}$$

The value of τ can be much smaller than the actual decorrelation time $\bar{\tau}$ in cases where the function $\varphi(u)$ decays with oscillations over time.

Without loss of generality, we assume that $\langle \xi^2 \rangle_\xi = 1$, so the intensity of the fluctuations in the stochastic perturbation is governed by ϵ. However, the relevant dimensionless expansion parameter is $\bar{\delta} = \epsilon\bar{\tau}$. Due to the practical difficulty of quantifying $\bar{\tau}$ precisely, we use the parameter $\delta := \epsilon\tau$ instead, where τ is evaluated as defined in equation (3).

The drift field $-\gamma x$ in the SDE (2) can also be interpreted as originating from the same multiplicative stochastic process, in cases where the mean of its fluctuations is non-zero. Indeed, if the unperturbed dynamics of x does not include the friction term and $\langle \xi \rangle_\xi = \gamma/\epsilon$, we recover the SDE (2) by replacing $\xi[t]$ with $\xi[t] + \gamma/\epsilon$.

If x is interpreted as the velocity of a Brownian particle, the SDE (2) has the notable feature that it can be considered a continuous process realization of Lévy random walks with superdiffusive and superballistic regimes[17] for certain parameter ranges.

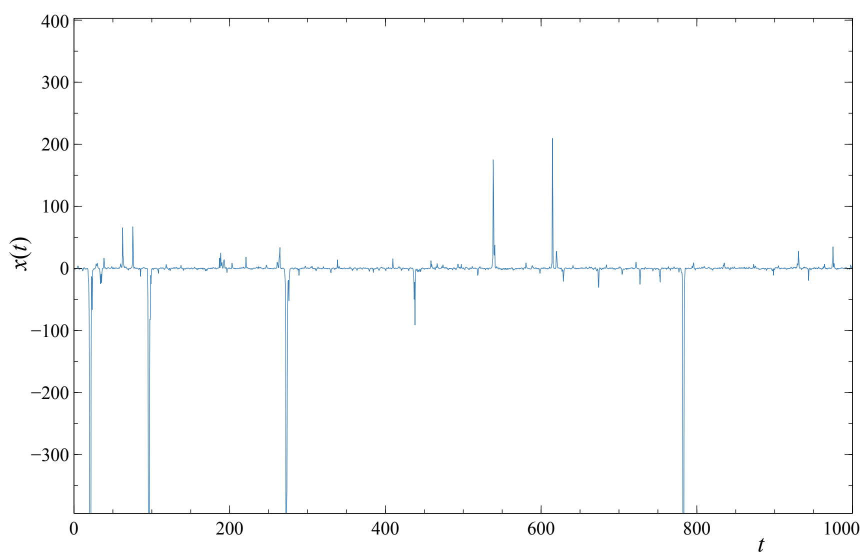

FIG. 1. A representative example, illustrating the intermittent behavior, is depicted in the time evolution of the amplitude x for the SDE (2) with parameters $\tau = 0.5$, $\epsilon = 5.0$, $\gamma = 2.0$, $D_f = 0.5$.

We refer to $f[t]$ in equation (2) as the internal or intrinsic noise. This terminology is appropriate since it typically arises from intrinsic and unavoidable factors such as thermal fluctuations, inherent uncertainties in measurement processes, or rapid wind forcing in the context of ocean dynamics, among other possibilities.

As mentioned, the SDE (2) has been extensively studied in the scientific literature. However, in nearly all these studies, aside from the additive noise $f[t]$, the stochastic process $\xi[t]$ has been treated as white noise. In cases where a colored stochastic process $\xi[t]$ has been considered[1], the focus has been on characterizing the far tail of the probability density function (PDF) of x, which, as we will see later, does not depend on the spectral properties of the multiplicative noise.

To the best of our knowledge, no previous works have derived a simple closed-form expression (i.e., not a formal result involving an infinite series of operators) for the PDF across its entire support range, along with the corresponding equilibrium solution. We will address this gap by focusing on the case where the δ parameter is small and will find surprisingly simple results that do not conform to the structure of the Fokker-Planck equation (FPE) or Fick's law.

To begin, let us assume that the intrinsic noise is absent (i.e., $f[t] = 0$). In this case, it is well-known and straightforward to show (see Appendix B) that, regardless of the values of τ and ϵ, the Master Equation (ME) for the PDF of x in equation (2) coincides with the following FPE (we use the shorthand $\partial_y := \partial/\partial y$):

$$\partial_t P(x;t) = \left\{ \gamma \partial_x x + \frac{\delta^2}{\tau} \partial_x x \partial_x \, x \right\} P(x;t). \tag{4}$$

This result indicates that the process (2), with $D_f = 0$, does not depend on the spectral (or color) features of the stochastic process $\xi[t]$. Consistently, in the white noise limit, i.e., for $\tau \to 0$ and $\delta^2/\tau = \epsilon^2 \tau$ held constant, the FPE (4) remains unchanged and corresponds to the standard FPE for SDEs with multiplicative white noise, under the Stratonovich interpretation of Wiener process differentials.

The FPE (4) can also be rewritten as a continuity equation:

$$\partial_t P(x;t) = -\partial_x J(x) \tag{5}$$

where

$$J(x) := -\left\{ (\gamma\tau + \delta^2) x/\tau + D_\xi(x) \partial_x \right\} P(x;t), \tag{6}$$

and we have introduced the inhomogeneous diffusion coefficient:

$$D_\xi(x) := \delta \epsilon x^2 = \delta^2 x^2/\tau, \tag{7}$$

characteristic of multiplicative noise. The terms in Eq. (6) have a straightforward interpretation: in the absence of internal noise ($D_f = 0$ in (2)), the multiplicative stochastic process generates an additional friction/drift term proportional to the intensity of the stochastic perturbation and an inhomogeneous diffusion process proportional to the gradient of the PDF (thus, following Fick-s law).

The equilibrium PDF of (5), obtained by setting $J(x) = 0$ in (6), exhibits a singular behavior:

$$P_{\rm eq}(x) = \begin{cases} \propto |x|^{-\left(1+\frac{\gamma\tau}{\delta^2}\right)} & \text{for } x \neq 0 \quad (a) \\ \delta(x) & \text{for } x = 0 \quad (b), \end{cases} \tag{8}$$

where solution (a) is not normalizable, meaning that when normalized, it yields zero everywhere in its domain[18].

Introducing an internal diffusion source effectively addresses this issue and is physically plausible for many realistic models. In fact, by applying Fick's law and including the standard, constant diffusion coefficient D_f in the current (6), we have:

$$J(x) := -\left\{(\gamma\tau + \delta^2)x/\tau + (D_\xi(x) + D_f)\partial_x\right\} P(x;t). \tag{9}$$

By setting $J(x) = 0$ we now obtain $P(x)_{eq} \propto (D_f + D_\xi(x))^{-\frac{1}{2}\left(1+\frac{\gamma\tau}{\delta^2}\right)}$, which no longer displays the singular behavior around $x = 0$.

Note that for $D_\xi(x) \gg D_f$, i.e., for $x \gg \sqrt{D_f\tau/\delta^2}$, the behavior is similar to the previous noiseless case. Thus, the condition for the existence of the moments of x remains unchanged by the introduction of this diffusive term.

We can say that the white noise $f[t]$, corresponding to a diffusion process with diffusion coefficient D_f, introduces a repulsion from the origin, preventing any path from getting trapped at $x = 0$ once reached. While the introduction of such intrinsic noise in multiplicative processes has been acknowledged by many researchers (see the works cited earlier), it has not been emphasized that even though the two fluctuating processes are assumed independent of each other, their contributions to the current $J(x)$ of (5) *don't simply add up*, unless the multiplicative process is also white noise. More precisely, in this work, we will show that instead of (9) (derived assuming Fick's law holds), we have:

$$J(x) = -\left\{(\gamma\tau + \delta^2)x/\tau + (D_\xi(x) + D_f)\partial_x + D_f D_\xi(x)\vartheta\partial_x^2\right\} P(x;t) \tag{10}$$

with θ, given in (22), having the dimension of time and coinciding with 2τ for $\gamma\tau \ll 1$ and with γ^{-1} for $\gamma\tau \gg 1$. It is clear that Eq. (10) implies a breakdown of Fick's law and the corresponding FPE structure.

It should be noted that the new term on the right-hand side of (10) arises due to two key factors: the finite time scale of the external stochastic perturbation (colored noise) and the non-commutativity of its Liouvillian with the Liouvillian associated with the internal noise (the standard diffusion operator).

More specifically, when using any time-dependent perturbation approach (such as cumulants, Zwanzig, Born-Oppenheimer-like methods) to derive a Master Equation (ME) for the reduced probability density function (PDF) of x, it leads to a power series expansion over the dimensionless parameter δ (see Appendix B). In the second order (which is the leading order for weak perturbations), we obtain a correction to the Fokker-Planck Equation (FPE) derived by simply applying Fick's law. This correction is proportional to both δ and D_f and results in a third-order partial differential operator on x.

This result is confirmed by the numerical simulations reported in Section III. A detailed derivation of this phenomenon will be undertaken in the next session.

Before concluding this introduction, we also emphasize that this departure from the standard FPE/Fick's law is a general result, applicable beyond the linear drift case of (1), and always occurs when both additive noise and multiplicative colored stochastic processes are present. For simplicity, we focus here on the linear 1-D case of (2), while a more in-depth exploration of these findings will be presented in future works.

II. A THIRD ORDER PDE FOR THE PDF

We indicate with $\xi(\cdot)$ a realization $\xi(u)$ of the stochastic process $\xi[t]$ with, $0 \leq u \leq t$. Given the infinitely short time correlation of the additive noise $f[t]$, to any realization $\xi(\cdot)$ of $\xi[t]$, the continuity equation for the SDE (2) corresponds to the following Liouville equation for the PDF of x:

$$\begin{aligned}&\partial_t P_{\xi(\cdot)}(x,t)\\&= \{\mathcal{L}_a + \epsilon\,\xi(t)\mathcal{L}_I\} P_{\xi(\cdot)}(x,t),\end{aligned} \tag{11}$$

in which $\mathcal{L}_a$ is the unperturbed Liouville operator given by

$$\mathcal{L}_a := \gamma\partial_x x + D_f\partial_x^2; \tag{12}$$

and $\epsilon\,\xi(t)\mathcal{L}_I$ is the Liouville perturbation operator with:

$$\mathcal{L}_I := \partial_x x. \tag{13}$$

If the perturbing process $\epsilon\xi(t)$ is weak (characterized by small values of the δ parameter), applying a perturbation projection[19–21] or a cumulant[22] approach to Eq. (11), at the leading order of δ, we formally obtain the following standard result for the reduced PDF of x (see Appendix B for a brief overview of the cumulant approach adapted to the present case, note that $P(x;t) := \langle P_{\xi(\cdot)}(x,t)\rangle_\xi$, where $\langle ...\rangle_\xi$ is the average over the realizations of the stochastic process $\xi[t]$):

$$\partial_t P(x;t) = \mathcal{L}_a P(x;t) + \frac{\delta^2}{\tau^2}\mathcal{L}_I \int_0^t \mathrm{d}u\,\varphi(u)\,\tilde{\mathcal{L}}_I(-u)\;P(x;t), \tag{14}$$

where $\varphi(t)$ is the normalized autocorrelation function of $\xi[t]$, as defined in the Introduction, and

$$\tilde{\mathcal{L}}_I(t) := e^{-\mathcal{L}_a t}\mathcal{L}_I e^{\mathcal{L}_a t} \tag{15}$$

is the interaction representation of the perturbing Liouvillian $\mathcal{L}_I$. By exploiting the Hadamard's lemma for exponentials of operators we can also write

$$\tilde{\mathcal{L}}_I(t) = e^{-\mathcal{L}_a^\times t}[\mathcal{L}_I] \tag{16}$$

in which, for any couple of operators $\mathcal{A}$ and $\mathcal{B}$, we have defined $\mathcal{A}^\times[\mathcal{B}] := [\mathcal{A},\mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$. In literature (e.g.[23]), $e^{\mathcal{A}^\times t}[\mathcal{B}]$ is called the Lie evolution of the operator $\mathcal{B}$ along $\mathcal{A}$, for a time t.

Because the perturbing Liouvillian $\mathcal{L}_I$ of (13) is a first order differential operator, the order of the differential operator corresponding to the second addend in the r.h.s. of (14) is one plus the order of differential operator of $\tilde{\mathcal{L}}_I(-u)$. From Eq. (16), we see that this latter is the result of the Lie evolution of $\mathcal{L}_I$ along the unperturbed Liouvillian $\mathcal{L}_a$.

If the decay time of $\varphi(u)$ is significantly shorter than $1/\gamma$, we can safely assume the approximation $\tilde{\mathcal{L}}_I(-u) \approx \mathcal{L}_I$ inside the integral on the r.h.s. of (14). Consequently, the ME (14) effectively reduces to a FPE. However, when this is not the case, we must address the challenge of evaluating the full Lie evolution of $\mathcal{L}_I$ along $\mathcal{L}_a$. In-depth exploration of this topic, from a formal and general perspective, can be found in[23]. Specifically, Proposition 1 in[23] is of particular relevance. However, for the simple 1-D case with linear drift, corresponding to the present SDE (2), we can easily derive the Lie evolution of $\mathcal{L}_I$ along the unperturbed Liouvillian $\mathfrak{L}_a$ in (16) as follows. From (16) we have

$$\frac{d}{dt}\tilde{\mathfrak{L}}_I(t) = -\mathfrak{L}_a^{\times}\left[e^{-\mathfrak{L}_a^{\times}t}[\mathfrak{L}_I]\right] = -e^{-\mathfrak{L}_a^{\times}t}\left[[\mathfrak{L}_a, \mathfrak{L}_I]\right]. \tag{17}$$

By using (12) and (13), we get

$$[\mathfrak{L}_a, \mathfrak{L}_I] = [\gamma\partial_x x + D_f\partial_x^2\,,\, \partial_x x] = 2D_f\partial_x^2 \tag{18}$$

thus, Eq. (17) can be written as

$$\begin{aligned}\frac{d}{dt}\tilde{\mathfrak{L}}_I(t) =& -2e^{-\mathfrak{L}_a^{\times}t}\left[D_f\partial_x^2\right]\\ =& -2e^{-\mathfrak{L}_a^{\times}t}\left[\gamma\partial_x x + D_f\partial_x^2 - \gamma\partial_x x\right] = -2(\gamma\partial_x x + D_f\partial_x^2) + 2\gamma\tilde{\mathfrak{L}}_I(t)\end{aligned} \tag{19}$$

of which the solution is

$$\tilde{\mathfrak{L}}_I(t) = \partial_x\, x + D_f\frac{1-e^{2\gamma t}}{\gamma}\partial_x^2. \tag{20}$$

By using Eq. (20) into the ME (14), and exploiting again (12) and (13), we finally obtain

$$\begin{aligned}\partial_t P(x;t) =&\gamma\partial_x\, x\, P(x;t)\\ &+\frac{\delta^2}{\tau^2}\partial_x x\left[\int_0^t \mathrm{d}u\,\varphi(u)\,\partial_x\, x + D_f\frac{1}{\gamma}\int_0^t \mathrm{d}u\,\varphi(u)\left(1-e^{-2\gamma u}\right)\partial_x^2\right]P(x;t).\end{aligned} \tag{21}$$

For $t \gg \tau$ and $t \gg 1/\gamma$, Eq. (21) can be safely approximated as

$$\begin{aligned}\partial_t P(x;t) \approx& \left\{\gamma\partial_x\, x + D_f\partial_x^2 + \frac{\delta^2}{\tau}\partial_x x\partial_x x + D_f\delta^2\frac{\vartheta}{\tau}\partial_x\, x\partial_x^2\right\}P(x;t)\\ =& -\partial_x J(x)\end{aligned} \tag{22}$$

with $J(x)$ is given in (10) and the time ϑ defined as

$$\vartheta := \frac{1}{\gamma\tau}\left(\tau - \hat{\varphi}(2\gamma)\right). \tag{23}$$

The hat over a function indicates its Laplace transform: $\hat{\varphi}(s) := \int_0^\infty \mathrm{d}u\,\varphi(u)e^{-su}$. Note that, if $\tau > 0$, then, from (23) it follows that also $\vartheta > 0$.

The third-order PDE (22) with (23) is the main result of this work. At the leading order in powers of the δ parameter, Eq. (22) is exact, regardless of the value of the diffusion coefficient D_f.

Thus, upon introducing the internal noise, alongside the standard diffusion process, *an additional mutual contribution is activated*. As we can see from (22), the mutual contribution of the white internal noise and the external multiplicative stochastic process exhibits an odd nature in terms of partial derivatives. As previously emphasized in the Introduction, we reiterate that the time parameter ϑ of (23) is order of τ for $\gamma\tau \ll 1$ and of γ^{-1} for $\gamma\tau \gg 1$. Consequently, the adimensional parameter $r := \epsilon\vartheta$ is similar the δ parameter, but is rescaled based on the time scale relationship between the stochastic process and the unperturbed dynamics.

Imposing the equilibrium condition to the ME (22), i.e., setting $J(x) = 0$, we obtain two different analytical solutions, both involving the Kummer confluent hypergeometric function of first kind[24]:

$$P_1(x) = {}_1F_1\left(\frac{1}{2}\left(\frac{\gamma\tau}{\delta^2}+1\right); \frac{1}{2}\left(\frac{1}{\delta r}+1\right); -\frac{x^2}{2D_f\vartheta}\right)$$
$$P_2(x) = |x|^{1-\frac{1}{\delta r}}\, {}_1F_1\left(\frac{r\,(2\delta\epsilon+\gamma)-1}{2\delta r}; \frac{3}{2}-\frac{1}{2\delta r}; -\frac{x^2}{2D_f\vartheta}\right). \tag{24}$$

This fact is due to the third order nature of the PDE (22). From a mathematical point of view any linear combination of these two functions is also a possible solution. However, it is easy to show that the second one is not physically acceptable. In fact, let us consider the behaviour of these two functions around $x = 0$. We have

$$P_1(x) \approx 1 - \frac{x^2\left[\gamma\tau+\delta^2\right]}{(2D_f\tau)\,(\delta r+1)} + O\left(x^3\right) \tag{25}$$

$$P_2(x) \approx |x|^{1-\frac{1}{\delta r}} + O\left(x^3\right). \tag{26}$$

We see that if $R := \delta r < 1$, a condition which is typically met in our perturbation approach, the solution $P_2(x)$ is not integrable, therefore it must be discarded. The expression of the function $P_1(x)$ in (25) implies that the presence of $r > 0$ smears the equilibrium PDF around $x = 0$.

Thus, the final result is given by

$$P_{eq}(x) \propto {}_1F_1\left(\frac{1}{2}\left(\frac{\gamma\tau}{\delta^2}+1\right); \frac{1}{2}\left(\frac{1}{\delta r}+1\right); -\frac{x^2}{2D_f\vartheta}\right). \tag{27}$$

We note that the Kummer confluent hypergeometric function ${}_1F_1\left(a; b; -z^2\right)$ is positive (and also well-defined) when its second argument (b) is greater than the first one (a). By the definitions of $\delta := \epsilon\tau$, $r := \epsilon\vartheta$ and of ϑ given in (23), it is straightforward to verify that in (27) this condition leads to $\hat{\varphi}(2\gamma)/\tau > 0$ that, if $\tau > 0$, is always satisfied.

In the case where the support of the PDF is not restricted (e.g., when there are no reflecting boundary conditions at finite values of x), we can derive a simple analytical expression for the far tails ($x \gg 2D_f\vartheta$) of the equilibrium PDF (27). The result is $P_{eq}(x) \sim |x|^{-\left(\frac{\gamma\tau}{\delta^2}+1\right)}$. From this expression, we observe that even when considering the contribution from the third partial derivative, the far-tail behavior of the equilibrium PDF of x remains unaffected by the presence of the additive white noise $f[t]$. This implies that, for an unbounded PDF domain, the condition for the existence of the n-th moment of x depends only on the ratio $\gamma\tau/\delta^2$, and remains independent of the spectral properties of $\xi[t]$. Furthermore, in the case of an unbounded PDF domain, from the PDE (22) it is possible to derive the following ODE for the n-th moment of x:

$$\partial_t\langle x^n\rangle = -n\gamma\langle x^n\rangle\left(1 - n\delta^2/(\gamma\tau)\right) + n(n-1)D_f\left(1 - n\delta r\right)\langle x^{n-2}\rangle. \tag{28}$$

For any fixed n, Eq. (28) represents a closed linear relationship between the first n moments of x. The eigenvalues of the corresponding matrix are $-n\gamma\left(1 - n\delta^2/(\gamma\tau)\right)$, meaning they do not depend on D_f and r. Therefore, the relaxation behavior of the moments is also independent of D_f and r, and the moments exist only if $(1 - n\delta^2/(\gamma\tau)) > 0$. On the other hand, it is clear from the same Eq. (28) that the equilibrium values of the moments (if they exist) do depend on D_f, and also on the value of $R := \delta r$.

When it exists, the equilibrium solution of Eq. (28) is given by

$$\langle x^n\rangle_{eq} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \left(\frac{D_f}{\gamma}\right)^{n/2}(n-1)!!\prod_{j=1}^{n/2}\frac{(1-2j\,\delta r)}{(1-2j\,\delta^2/(\gamma\tau))} & \text{for } n \text{ even}\end{cases} \tag{29}$$

From (23) we always have $\vartheta < 1/\gamma$, and considering that $r = \epsilon\vartheta$ and $\delta = \epsilon\tau$, we obtain $(1 - 2j\delta^2/(\gamma\tau)) < (1 - 2j\,\delta r)$. Because $(1 - n\delta^2/(\gamma\tau)) > 0$ (for the n-th moment to exist), then we have $(1 - 2j\,\delta r) > (1 - j\delta^2/(\gamma\tau)) \geq (1 - n\delta^2/(\gamma\tau)) > 0$. I.e., $\langle x^n\rangle_{eq} > 0$ (for even n), as it must be.

From Eq. (29), we also obtain that the third-order partial differential term in the PDE (22), which is proportional to the parameter $R := \delta r = \epsilon\delta\vartheta$, decreases the value of the moments of the PDF. This observation seems to contradict the above findings that suggest the equilibrium PDF broadens around $x = 0$ as R increases and that the far tails of this PDF do not depend on R. The resolution to this apparent contradiction lies in the fact that, as we move away from the origin, where the expansion (25) holds, but before reaching the asymptotic tails, the equilibrium PDF decays more rapidly as a function of x due to the presence of $R > 0$. This behavior can be easily confirmed by comparing plots of the equilibrium PDF for $R = 0$ and $R \neq 0$ (see next section).

If there is a significant separation of time scales, i.e. $\gamma\tau \ll 1$, then from (23) and the assumption given in Eq. (B13) (see also note[16]), we have

$$R = \delta r \approx \epsilon^2 \int_0^\infty \mathrm{d}u \, \varphi(u) u \sim (\epsilon\tau)^2 = \delta^2, \tag{30}$$

which does not depend on γ. It should be noted that in the white noise limit, i.e., as $\tau \to 0$ while keeping $\epsilon^2\tau = \delta\epsilon$ fixed, R in (30) tends to zero. Consequently, the non-Fick contribution to J becomes negligible. Conversely, if δ, the relevant parameter for the perturbation/cumulant series, is kept fixed (small enough to truncate the series at the second order), while changing the time scale of the noise, R in (30) remains constant.

In essence, even with a weak stochastic process $\xi[t]$ (which diverges in the white noise limit), the breakdown of Fick's law and the associated Fokker-Planck equation (FPE) for model (2) persists, resulting in a PDE that includes a third-derivative term.

III. THE CASE OF ORNSTEIN UHLENBECK EXTERNAL STOCHASTIC PROCESS: ANALYTICAL AND NUMERICAL RESULTS

To explore the full range of τ and γ, we consider the specific classical case of exponentially decaying correlation function: $\varphi(u) = \exp(-u/\tau)$, from which, by also exploiting (23), we have:

$$\vartheta = \frac{2\tau}{(2\gamma\tau + 1)} \tag{31}$$

i.e.,

$$R := \delta r = \frac{2\delta^2}{(2\gamma\tau + 1)}. \tag{32}$$

We observe that R depends solely on δ (the small parameter in the cumulant expansion) and $\gamma\tau$ (quantifying the time scale separation between the unperturbed relaxation process and the relaxation of the correlation function of $\xi[t]$). It is evident from (32) that R decreases when the time scale separation decreases ($\gamma\tau$ increases) and increases quadratically with δ. The equilibrium PDF (27) in this case reads:

$$P_{eq}(x) = N \, {}_1F_1\left(\frac{1}{2}\left(\frac{\gamma\tau}{\delta^2} + 1\right); \frac{1}{4}\left(\frac{2\gamma\tau + 1}{\delta^2} + 2\right); -\frac{(2\gamma\tau + 1)x^2}{4D_f\tau}\right). \tag{33}$$

where N is a normalization factor. We note that, except for the quantity $D_f\tau$, which acts as a scale factor for x, also the equilibrium PDF (33) depends only on δ and $\gamma\tau$.

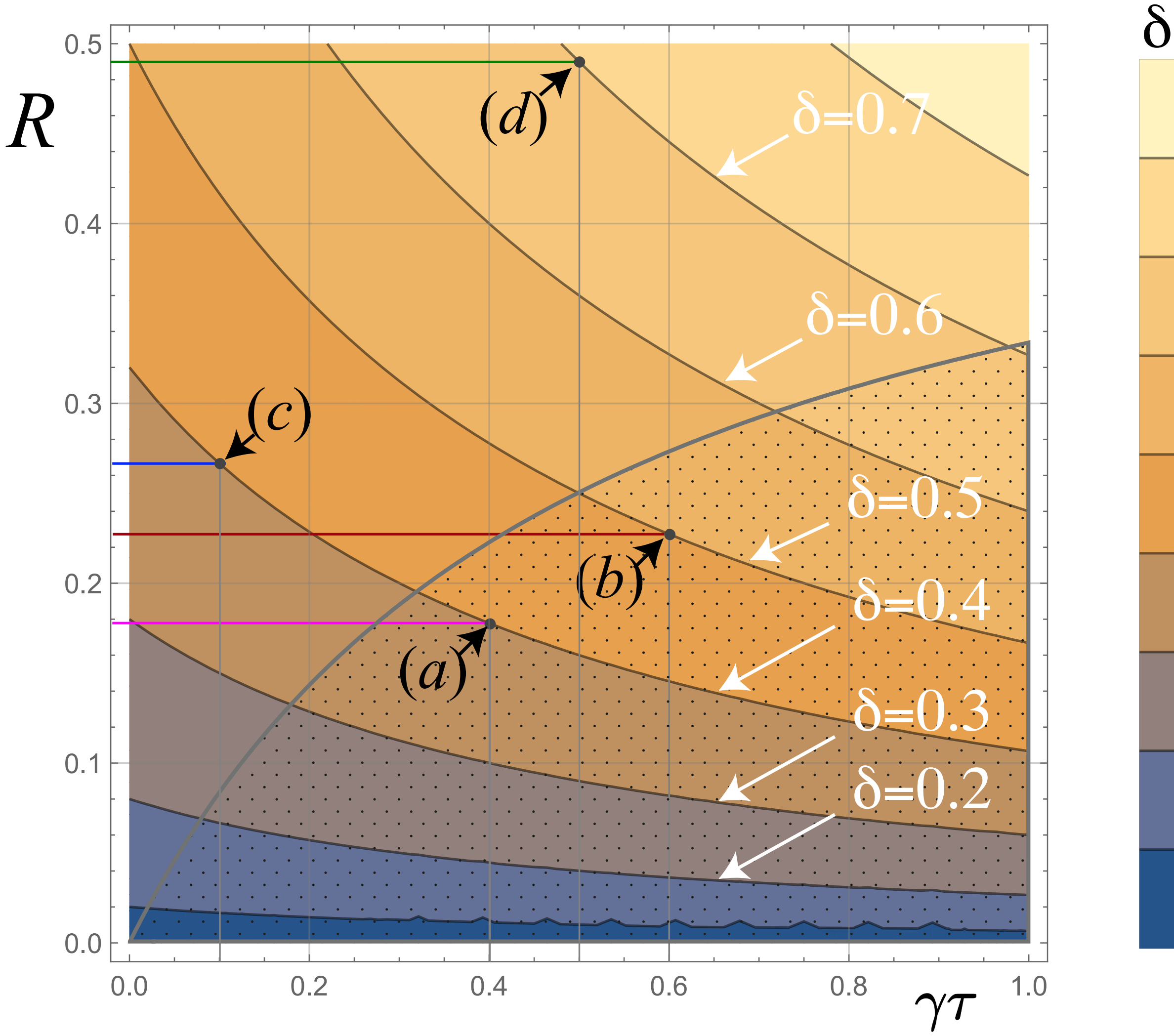

FIG. 2. Plot of R of (32) vs $\gamma\tau$, for various values of δ (distinct curves). We see that at fixed δ, as $\gamma\tau$ decreases, R increases. The same happens increasing δ, at $\gamma\tau$ fixed. The area with dotted background correspond to δ and $\gamma\tau$ values for which the variance of x is finite ($\gamma\tau - 2\delta^2 > 0$). The points in the graph labeled with the letters (a) and (b) ((c) and (d)) corresponds to the $\gamma\tau$ and δ values, used for the four plots of the PDF of figure 3 (figure 4).

In figures 3-4, solid lines depict the plots of the PDF (33) for a fixed $D_f\tau = 0.5$ and different values of δ and $\gamma\tau$ corresponding to the points $(a)-(d)$ in the diagram (2). We have also included the corresponding results of the numerical simulation of the SDE (2) (circles), where $\xi[t]$ is the Ornstein-Uhlenbeck process. Additionally, to assess the relevance of the non Fick contribution to the current, we have also plotted, with dashed lines, the normalized function $P_{eq,FPE}(x) \propto (D_f + D_\xi(x))^{-\frac{1}{2}\left(1+\frac{\gamma\tau}{\delta^2}\right)}$ which is the solution for the vanishing "Fick" current of (9) (or the equilibrium PDF of the corresponding FPE). The excellent agreement of the analytical result (33) with numerical simulations is evident, while when relying on the $P_{eq,FPE}(x)$, the comparison with numerical simulations is not at all so good.

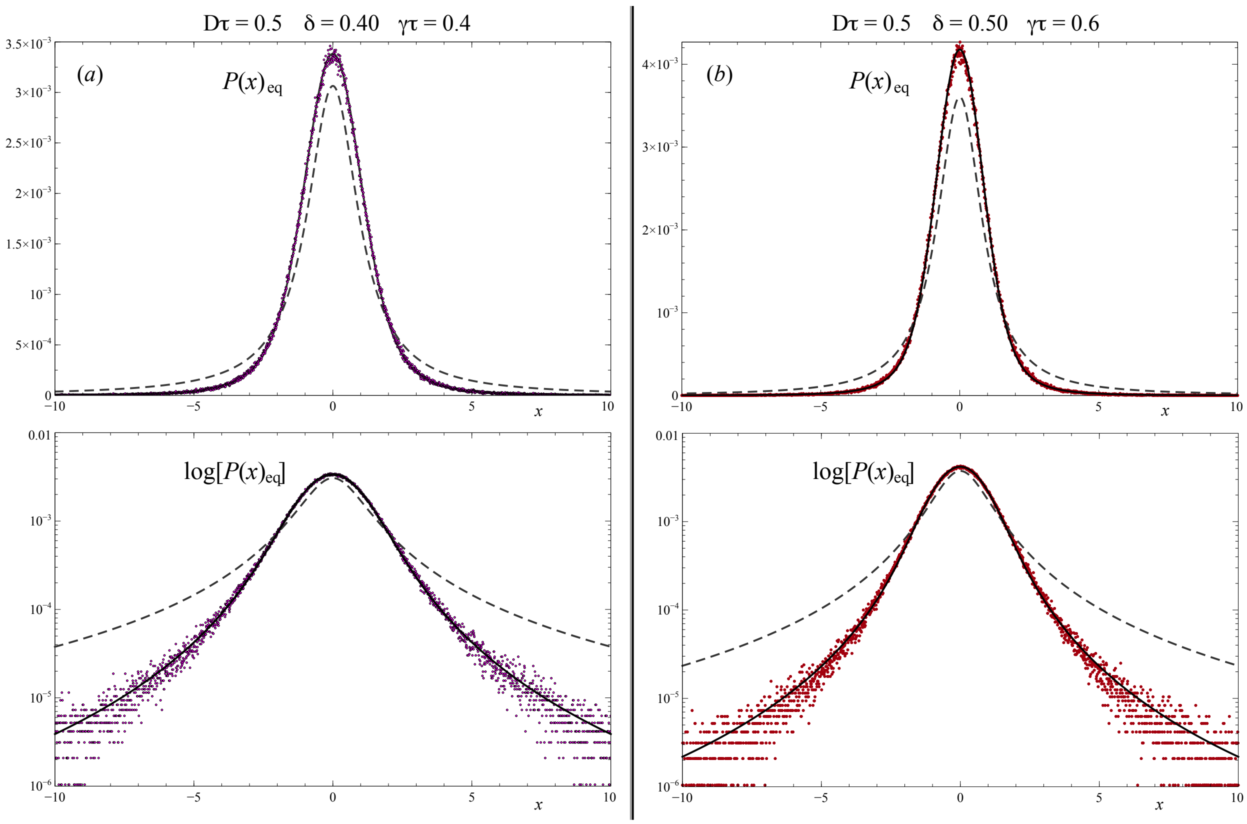

FIG. 3. Two vertical panels (a) and (b), respectively, displaying the Equilibrium PDF of the SDE (2) for the case in which the stochastic process $\xi[t]$ is the Ornstein Uhlenbeck process. Panel (a): $D_f\tau = 0.5$, $\delta = 0.4$ and $\gamma\tau = 0.4$. Panel (b): $D_f\tau = 0.5$, $\delta = 0.5$ and $\gamma\tau = 0.6$. In the bottom part, the same data as the upper part are presented in semi-logarithmic scale. Circles are the results of the numerical simulation. Solid lines represent the theoretical result (33), i.e., the equilibrium solution of the PDE (22), with r given in (31). Dashed lines depict $P_{eq,FPE}(x) \propto (D_f + D_\xi(x))^{-\frac{1}{2}\left(1+\frac{\gamma\tau}{\delta^2}\right)}$, the solution for the vanishing "Fick" current of (9) (or the equilibrium PDF of the corresponding FPE). In these two cases, corresponding to the two points (a) and (b) in the diagram of figure 2, we have $\gamma\tau > 2\delta^2$, thus the variance of x is finite (see text for details).

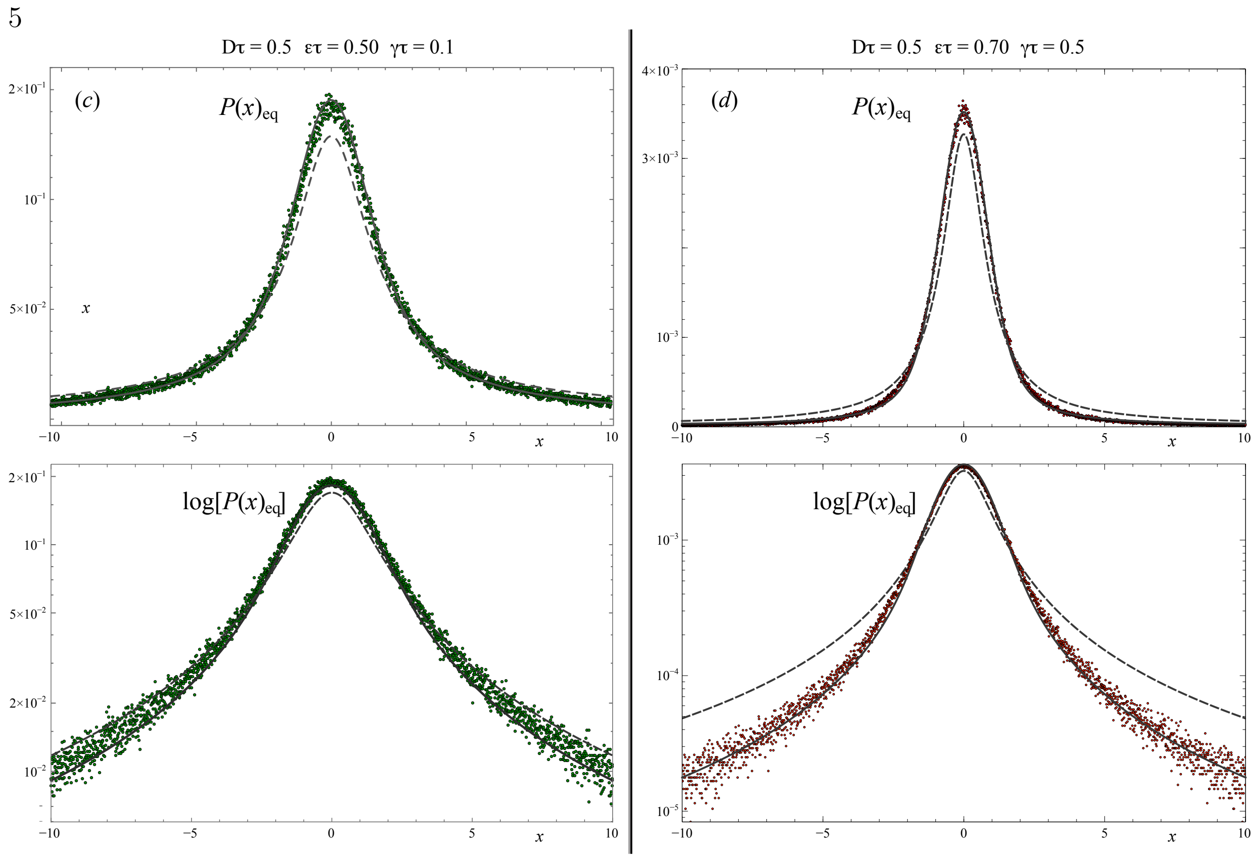

FIG. 4. The same as figure 3, but for different values of $\gamma\tau$ and δ as indicated in the header of the panels. In these two cases, corresponding to the two points (c) and (d) in the diagram of figure 2, we have $\gamma\tau < 2\delta^2$, thus, at equilibrium all the moments of x diverge.

IV. CONCLUSIONS

The Fokker-Planck Equation (FPE) holds a central role in statistical mechanics. Initially derived as a Kramers-Moyal expansion of the Master Equation (ME), limited to Markovian systems, it is now recognized as applicable to non-Markovian processes as well. In fact, the FPE arises by eliminating irrelevant or fast variables -those weakly interacting with the system of interest- through perturbation techniques, such as Zwanzig and Mori's projection methods, or by considering the order of magnitude of generalized cumulants. Thus, it stands as the most important equation for deriving the time evolution of the PDF under these approximations.

Moreover, the FPE has the advantage of being a second-order classical parabolic PDE with well-established properties regarding the existence and positivity of its solutions. Its importance and widespread use are undeniable.

The connection between the FPE and Fick's law is no coincidence. When expressed as a continuity equation, the FPE shows that the current associated with the stochastic process involves a diffusion term proportional to the gradient of the PDF, which constitutes Fick's law. Conversely, assuming Fick's law holds, the continuity equation takes the form of a second-order PDE, resembling the structure of the FPE. Therefore, the validity of Fick's law and the ME with the structure of the FPE are deeply interconnected.

The extensive use of the FPE has led to the development of numerous methods for extracting key statistical information. Standard spectral analysis techniques, similar to those used for the Schrödinger equation in quantum mechanics, can be applied. Furthermore, the diffusion and drift coefficients of the FPE allow for the derivation of an analytical expression for the mean first-passage time, an important quantity representing the average time it takes for a trajectory, starting from an initial position x_0, to reach a target point x_T for the first time.

In our study, we demonstrated that when the system of interest is inherently noisy - featuring sources such as Nyquist noise in electric circuits, various thermal fluctuations, rapid internal dynamics, intrinsic measurement errors, and more- the standard procedures for eliminating fast or weakly interacting variables (often, but not necessarily, modeled as stochastic processes) lead to a third-order PDE instead of a Fokker-Planck Equation (FPE). Specifically, an additive third-order partial differential operator emerges from the interaction between the standard diffusion process caused by internal noise and the diffusion process induced by an external colored stochastic process (or by irrelevant degrees of freedom that are projected out).

Given the inevitability of such internal noise (of any intensity), we conclude that the third-order PDE should be considered more fundamental than the FPE in statistical physics. This also implies the breakdown of Fick's law.

While this approach can be extended to more general drift fields, our current focus in this work is on the simpler case of linear drift, which is widely used across various disciplines. The analytical expressions for the moments of the PDF reveal that the unexpected third-derivative term significantly tightens the equilibrium PDF compared to the results we would obtain by omitting this term and retaining only the standard FPE structure.

Figures 3 and 4 support this observation, showing perfect agreement between numerical simulations of the SDE and the third-order PDE. The figures clearly illustrate that the actual PDFs, accurately captured by the third-order PDE, exhibit much narrower equilibrium distributions compared to those derived from the FPE. This aligns with the general conclusion emphasized in section II, that the third-order differential contribution to the Master Equation leads to a reduction in the moments of x. As a result, the tails of the actual equilibrium PDF (and those of the third-order PDE equilibrium PDF) decay more rapidly than those of the FPE. This indicates that crucial statistical quantities, such as the mean first-passage time, computed using standard FPE techniques, would yield inaccurate results.

The fact that, for the statistical behavior of a specific part of a complex system, the third-order PDE should be considered more fundamental than the FPE raises the question of how to extend the general methods and results that allow the extraction of relevant statistical information from the FPE to this higher-order PDE. For instance, in the 1-D case, it would be valuable to derive a closed-form expression for the mean first-passage time. This question will be the subject of future work.

As a final remark, it is worth noting the apparent conflict between our result (the third-order PDE) and the well-known Pawula's "truncation lemma"[25]. This lemma, which pertains to the coefficients of the Kramers-Moyal expansion of the differential Chapman-Kolmogorov equation, concludes that the expansion can terminate after the first or second term. If the expansion continues beyond the second term, it must contain an infinite number of terms to ensure that the solution of the differential equation remains a valid PDF. In this context, as stated in Risken's book[26], "this theorem does not say that expansions truncated at $n \geq 3$ are of no use,...one may very well use Kramers-Moyal expansions trucated at $n \geq 3$ for calculating distribution functions. Though the transition probability must then have negative values at least for sufficiently small times, these negative values may be very small". Although it is crucial to verify that the solutions of any third- (or higher-) order PDE are valid PDFs -i.e., positive and normalizable (a check we have indeed performed for the general equilibrium PDF in Eq. (27))- it is also important to recall that our approach does not originate from a Markovian system. Consequently, it does not rely on the Chapman-Kolmogorov equation: our stochastic process is non-Markovian, and the PDE for the PDF is derived through a forcing noise cumulant expansion, rather than a moment expansion as in the Kramers-Moyal series.

Beyond the formal development presented in Appendix B, a simple argument provided below should convince the reader that the cumulant expansion, to which Pawula's lemma does not apply, is more general and accurate than the moment expansion.

Let us *assume* that it is possible to expand the equation of evolution of the PDF in a series such as

$$\frac{\partial}{\partial t}P(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} D_n(t) \frac{\partial^n}{\partial x^n} P(x,t), \tag{34}$$

where $D_n(t)$ are some time-dependent coefficients. Applying the Fourier transform to this series, we obtain (here, the hat over the PDF means its Fourier transform)

$$\frac{\partial}{\partial t}\hat{P}(k,t) = \sum_{n=1}^{\infty} (\mathrm{i}k)^n \frac{1}{n!} D_n(t) \hat{P}(k,t), \tag{35}$$

which, from a formal point of view, has the solution

$$\hat{P}(k,t) = e^{\sum_{n=1}^{\infty} (\mathrm{i}k)^n \frac{1}{n!} \int_0^t D_n(u)\mathrm{d}u} \hat{P}(k,0). \tag{36}$$

Identifying $P(x,t)$ with the Green function of the PDE (or with the conditional PDF), we can set $\hat{P}(k,0) = 1$. Using this fact, it is apparent that the terms $\int_0^t D_n(u)\mathrm{d}u$ of Eq. (36) coincide, by definition, with the cumulants of x, while the coefficients $D_n(u)$ are trivially related to the cumulants of the fluctuations of the stochastic process forcing the variable x.

V. AKNOWLEDGEMENT

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Appendix A: The multidimensional case

In this Appendix we generalize the result (22) to the multi-dimensional case. For the reader convenience, we copy here the general N-D extension of the SDE (2), already introduced in (1):

$$\dot{\boldsymbol{x}} = -\mathbb{E} \cdot \boldsymbol{x} + \boldsymbol{f}[t] - \Xi[t] \cdot \boldsymbol{x} \tag{A1}$$

where $\boldsymbol{x} := (x_1, \dots, x_N)$, $\mathbb{E}$ and $\Xi[t]$ are $N \times N$ matrices with constant and stochastic components, respectively. Moreover, $\boldsymbol{f}[t]$ is a multidimensional white noise with correlation, or diffusion matrix given by $\mathbb{D}$.

As for the 1-D case, to any realization $\Xi(\cdot)$ of the matrix stochastic process $\Xi(u)$, $0 \leq u \leq t$, from (A1) we can write the following Liouville equation for the PDF of $\boldsymbol{x}$, that we indicate with $P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t)$:

$$\begin{aligned} &\partial_t P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t) \\ &= \{\mathcal{L}_a + \mathcal{L}_{\boldsymbol{\Xi}(t)}\} P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t), \end{aligned} \tag{A2}$$

in which the unperturbed Liouvillian is ($\boldsymbol{\partial}$ is the N-D gradient operator and the superscript "T" means "transpose"):

$$\mathcal{L}_a := \boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} + +\boldsymbol{\partial}^T \cdot \mathbb{D} \cdot \boldsymbol{\partial} \tag{A3}$$

and the Liouville perturbation operator is

$$\mathcal{L}_{\boldsymbol{\Xi}(t)} := \boldsymbol{\partial}^T \cdot \Xi(t) \cdot \boldsymbol{x}. \tag{A4}$$

We rewrite the Liouville equation (A2) in interaction representation:

$$\partial_t \tilde{P}_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t) = \tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t) \tilde{P}_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t), \tag{A5}$$

where

$$\tilde{P}_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t) := e^{-\mathcal{L}_a t} P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t) \tag{A6}$$

and

$$\tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(t) := e^{-\mathcal{L}_a t} \mathcal{L}_{\boldsymbol{\Xi}(t)} e^{\mathcal{L}_a t} = e^{-\mathcal{L}_a^{\times} t} [\mathcal{L}_{\boldsymbol{\Xi}(t)}]. \tag{A7}$$

Integrating (A5) and averaging over the realization of $\Xi[t]$, we get

$$\tilde{P}(\boldsymbol{x}; t) = \langle \overleftarrow{\exp} \left[\int_0^t du \, \tilde{\mathcal{L}}_{\boldsymbol{\Xi}(t)}(u) \right] \rangle_{\boldsymbol{\Xi}} \, P(\boldsymbol{x}; 0) \tag{A8}$$

in which $\overleftarrow{\exp}[...]$ is the standard chronological ordered exponential (from right to left) and $\tilde{P}(\boldsymbol{x}; t) := e^{-\mathcal{L}_a t} P(\boldsymbol{x}; t)$ with $P(\boldsymbol{x}; t) := \langle P_{\boldsymbol{\Xi}(\cdot)}(\boldsymbol{x}, t) \rangle_{\boldsymbol{\Xi}}$. By using the generalized cumulant approach and retaining only the second cumulant we get the following ME for the PDF of $\boldsymbol{x}$:

$$\partial_t P(\boldsymbol{x};t) = \mathfrak{L}_a P(x;t) + \int_0^\infty \mathrm{d}u \, \langle \mathfrak{L}_{\boldsymbol{\Xi}(t)} \tilde{\mathfrak{L}}_{\boldsymbol{\Xi}(-u)}(-u) \rangle_{\boldsymbol{\Xi}} \; P(\boldsymbol{x};t), \tag{A9}$$

corresponding to the N-D version of Eq. (14). To obtain the explicit expression, as PDE, of the ME (A9), we must solve the Lie evolution of $\mathfrak{L}_{\boldsymbol{\Xi}(t)}$ along the Liouvillian $\mathfrak{L}_a$, i.e. we have to explicitly evaluate $\tilde{\mathfrak{L}}_{\boldsymbol{\Xi}(u)}(u)$ of (A7), in which $\mathfrak{L}_a$ and $\mathfrak{L}_{\boldsymbol{\Xi}(t)}$ are given in (A3) and (A4), respectively. For that, let us start considering the operator identity $e^{(\mathfrak{L}_A+\mathfrak{L}_B)t} = e^{\mathfrak{L}_A t} \cdot \overleftarrow{\exp}\left(\int_0^t \mathrm{d}u \, \bar{\mathfrak{L}}_B(u)\right)$ in which $\mathfrak{L}_A$ and $\mathfrak{L}_B$ are operators that in general do not commute with each other, and where $\bar{\mathfrak{L}}_B(u) := e^{-\mathfrak{L}_A u} \mathfrak{L}_B e^{\mathfrak{L}_A u}$. From this identity, by making the associations $\mathfrak{L}_A = \boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x}$ and $\mathfrak{L}_B = \boldsymbol{\partial}^T \cdot \mathbb{D} \cdot \boldsymbol{\partial}$ (thus, $\mathfrak{L}_A + \mathfrak{L}_B = \mathfrak{L}_a$), with a few algebra we easily obtain:

$$e^{\mathfrak{L}_a t} = e^{\boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} \, t} \cdot \overleftarrow{\exp}\left(\int_0^t \mathrm{d}u \, \boldsymbol{\partial}^T \cdot e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial}\right). \tag{A10}$$

By using (A10) and (A3) in (A7) we get

$$\begin{aligned}
\tilde{\mathfrak{L}}_{\boldsymbol{\Xi}(t)}(t) :=& e^{-\mathfrak{L}_a t} \mathfrak{L}_{\boldsymbol{\Xi}(t)} e^{\mathfrak{L}_a t} \\
=& e^{-\boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} \, t} \cdot \overleftarrow{\exp}\left(-\int_{-t}^0 \mathrm{d}u \, \boldsymbol{\partial}^T \cdot e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial}\right) \\
& \cdot \mathfrak{L}_{\boldsymbol{\Xi}(t)} \cdot e^{\boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} \, t} \cdot \overleftarrow{\exp}\left(\int_0^t \mathrm{d}u \, \boldsymbol{\partial}^T \cdot e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial}\right) \\
=& \overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u \, \boldsymbol{\partial}^T \cdot e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial}\right)^\times \left[e^{-\boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} \, t^\times} \left[\mathfrak{L}_{\boldsymbol{\Xi}(t)}\right]\right].
\end{aligned} \tag{A11}$$

In the last side of the following equation we have exploited the following identity, easily demonstrated:

$$\begin{aligned}
& e^{-\mathfrak{L}_A \theta} \cdot \overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u \, \bar{\mathfrak{L}}_B(u)\right) = e^{-\mathfrak{L}_A \theta^\times} \left[\overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u \, \bar{\mathfrak{L}}_B(u)\right)\right] e^{-\mathfrak{L}_A \theta} \\
& = \overleftarrow{\exp}\left(-\int_\theta^{t+\theta} \mathrm{d}u \, \bar{\mathfrak{L}}_B(u)\right) e^{-\mathfrak{L}_A \theta}.
\end{aligned} \tag{A12}$$

By using (A4) and the results of[23], in particular those in Section VIA, we have

$$e^{-\boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} \, t^\times} \left[\mathfrak{L}_{\boldsymbol{\Xi}(t)}\right] = e^{-\boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} \, t^\times} \left[\boldsymbol{\partial}^T \cdot \boldsymbol{\Xi}(t) \cdot \boldsymbol{x}\right] = \boldsymbol{\partial}^T \cdot e^{\mathbb{E} t^\times} \left[\boldsymbol{\Xi}(t)\right] \cdot \boldsymbol{x}. \tag{A13}$$

Inserting this result in (A11) we obtain

$$\begin{aligned}
& \tilde{\mathfrak{L}}_{\boldsymbol{\Xi}(t)}(t) = \\
& \overleftarrow{\exp}\left(-\int_0^t \mathrm{d}u \, \boldsymbol{\partial}^T \cdot e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial}\right)^\times \left[\boldsymbol{\partial}^T \cdot e^{\mathbb{E} t^\times} \left[\boldsymbol{\Xi}(t)\right] \cdot \boldsymbol{x}\right].
\end{aligned} \tag{A14}$$

By expanding the above series of nested commutators we see that all the terms are zero, apart the zeroth and the first ones. Therefore, we get

$$\tilde{\mathcal{L}}_{\mathbf{\Xi}(t)}(t) = \boldsymbol{\partial}^T \cdot e^{\mathbb{E}t^\times}\left[\mathbf{\Xi}(t)\right] \cdot \boldsymbol{x} \\ - \boldsymbol{\partial}^T \cdot e^{\mathbb{E}t^\times}\left[\mathbf{\Xi}(t)\right] \cdot \int_0^t \mathrm{d}u \left\{ \left(e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \right)^T + e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \right\} \cdot \boldsymbol{\partial}, \tag{A15}$$

that, given the symmetry property of the diffusion coefficient matrix, yields the final explicit differential form for the interaction representation of the Liouvillian $\mathcal{L}_{\mathbf{\Xi}(t)}$:

$$\tilde{\mathcal{L}}_{\mathbf{\Xi}(t)}(t) = \boldsymbol{\partial}^T \cdot e^{\mathbb{E}t^\times}\left[\mathbf{\Xi}(t)\right] \cdot \left\{ \boldsymbol{x} - 2\int_0^t \mathrm{d}u\, e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial} \right\}. \tag{A16}$$

Thus, by using this expression in the ME (A9), together with Eqs. (A3) and (A4), we arrive to the final general PDE of third order for the PDF of $\boldsymbol{x}$ for the multi-dimensional case:

$$\partial_t P(\boldsymbol{x};t) = \left\{ \boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} + + \boldsymbol{\partial}^T \cdot \mathbb{D} \cdot \boldsymbol{\partial} \right\} P(\boldsymbol{x};t) \\ + \int_0^\infty \mathrm{d}u \langle \boldsymbol{\partial}^T \cdot \mathbf{\Xi}(t) \cdot \boldsymbol{x} \left(\boldsymbol{\partial}^T \cdot e^{-\mathbb{E}u^\times}\left[\mathbf{\Xi}(-u)\right] \cdot \left\{ \boldsymbol{x} + 2\int_0^u \mathrm{d}u\, e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial} \right\} \right) \rangle_{\mathbf{\Xi}}\, P(\boldsymbol{x};t). \tag{A17}$$

In the simplified case in which $\mathbf{\Xi}[t] = \epsilon \mathbb{G} \xi[t]$, where $\mathbb{G}$ is a $N \times N$ matrix with constant components, and $\langle \xi(t)\xi(0) \rangle_{\mathbf{\Xi}} = \varphi(t)$, then we have

$$\partial_t P(\boldsymbol{x};t) = \left\{ \boldsymbol{\partial}^T \cdot \mathbb{E} \cdot \boldsymbol{x} + + \boldsymbol{\partial}^T \cdot \mathbb{D} \cdot \boldsymbol{\partial} \right\} P(x;t) \\ + \frac{\delta^2}{\tau^2} \int_0^\infty \mathrm{d}u\, \varphi(u) \boldsymbol{\partial}^T \cdot \mathbb{G} \cdot \boldsymbol{x} \left(\boldsymbol{\partial}^T \cdot e^{-\mathbb{E}u^\times}\left[\mathbb{G}\right] \cdot \left\{ \boldsymbol{x} + 2\int_0^u \mathrm{d}u\, e^{\mathbb{E}u} \cdot \mathbb{D} \cdot e^{\mathbb{E}^T u} \cdot \boldsymbol{\partial} \right\} \right) P(\boldsymbol{x};t), \tag{A18}$$

where we have also used the definition of the adimensional parameter $\delta := \epsilon\tau$, that is the relevant small quantity in the cumulant expansion.

Appendix B: The cumulant approach as a systematic way to obtain a ME for the reduced PDF of x

In this Appendix we outline a few minima key steps to obtain the FPE (4) and the ME (14), starting from the generalized cumulant (or M-cumulant) approach formally presented in[22]. We begin with the generic Liouville equation (11) (the stochastic process is one-dimensional, but the extension to multi-dimensional cases is straightforward), expressed in interaction representation:

$$\partial_t \tilde{P}_{\xi(\cdot)}(x,t) = \epsilon \xi(t), \tilde{\mathcal{L}}_I(t) \tilde{P}_{\xi(\cdot)}(x,t). \tag{B1}$$

Here,

$$\tilde{P}_{\xi(\cdot)}(x,t) := e^{-\mathcal{L}_a t} P_{\xi(\cdot)}(x,t) \tag{B2}$$

and

$$\tilde{\mathcal{L}}_I(t) := e^{-\mathcal{L}_a t} \tilde{\mathcal{L}}_I e^{\mathcal{L}_a t} = e^{-\mathcal{L}_a^\times t}[\mathcal{L}_I]. \tag{B3}$$

In[23], $\tilde{\mathcal{L}}_I(t)$ of (B3) is also referred to as the Lie evolution of the operator $\mathcal{L}_I$ along the Liouvillian $\mathcal{L}_a$, for a time $-t$.

Integrating in time (B1) and averaging over the realization of $\xi[t]$, we get

$$\tilde{P}(x;t) = \langle \overleftarrow{\exp} \left[\epsilon \int_0^t du\, \xi(u) \tilde{\mathcal{L}}_I(u) \right] \rangle_\xi P(x;0) \tag{B4}$$

in which $\overleftarrow{\exp}[...]$ is the standard chronological ordered exponential (from right to left) and $\tilde{P}(x;t) := e^{-\mathcal{L}_a t} P(x;t)$ with $P(x;t) := \langle P_{\xi(\cdot)}(x,t) \rangle_\xi$. Moreover, we have exploited the assumption that at the initial time $t = 0$ the total PDF factorizes as $P_{\xi(\cdot)}(x,0) = P(x;0)p(\xi)$. This is equivalent to stating that at the initial time the PDF of x does not depend on the possible values of the process ξ, or alternatively, we wait long enough so that the initial conditions became irrelevant. Apart that, Eq. (B4) is exact; no approximations have been introduced at this level.

We can look at the r.h.s. of (B4) as a sort of characteristic function (i.e., Fourier transform of the PDF), or moment generating function, with wave number $k := \mathrm{i}\epsilon$, for the stochastic operator

$$\Omega(u) := \xi(u) \tilde{\mathcal{L}}_I(u). \tag{B5}$$

Formally, we can then introduce a generalized cumulant generating function[22]:

$$\langle \overleftarrow{\exp} \left[\epsilon \int_0^t du\, \xi(u) \tilde{\mathcal{L}}_I(u) \right] \rangle_\xi := \overleftarrow{\exp} \left[\mathcal{K}(\epsilon, t) \right] \tag{B6}$$

with

$$\mathcal{K}(\epsilon, t) = \sum_{i=1}^{\infty} \epsilon^i \mathcal{K}_i(t). \tag{B7}$$

As for standard stochastic processes, we define the n-times joint M-cumulant of $\Omega(u)$, that we indicate as $\langle\langle \Omega(u_1)\Omega(u_2)...\Omega(u_n)\rangle\rangle$, by setting

$$\mathcal{K}_i(t) := \int_0^t \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}u_2 ... \int_0^{u_{n-1}} \mathrm{d}u_n \langle\langle \Omega(u_1)\Omega(u_2)...\Omega(u_n)\rangle\rangle. \tag{B8}$$

Using (B8) in the r.h.s. of (B6) and expanding both exponential functions, we get the standard relationship among cumulants and moments. For example, the joint two and four times M-cumulants are given in terms of moments as (to improve readability, until the end of this paragraph we will avoid putting the subscript "ξ" to the angle brackets):

$$\langle\langle \Omega(u_1)\,\Omega(u_2)\rangle\rangle = \langle \Omega(u_1)\,\Omega(u_2)\rangle = \tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_2)\langle \xi(u_1)\,\xi(u_2)\rangle \tag{B9}$$

and

$$\begin{aligned}
&\langle\langle \Omega(u_1)\,\Omega(u_2)\Omega(u_3)\,\Omega(u_4)\rangle\rangle = \\
&\langle \Omega(u_1)\,\Omega(u_2)\Omega(u_3)\,\Omega(u_4)\rangle - \langle \Omega(u_1)\,\Omega(u_2)\rangle\langle \Omega(u_3)\,\Omega(u_4)\rangle \\
&- \langle \Omega(u_1)\,\Omega(u_3)\rangle\langle \Omega(u_2)\,\Omega(u_4)\rangle - \langle \Omega(u_1)\,\Omega(u_4)\rangle\langle \Omega(u_2)\,\Omega(u_3)\rangle = \\
&\tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_2)\tilde{\mathcal{L}}_I(u_3)\tilde{\mathcal{L}}_I(u_4)\left[\langle \xi(u_1)\,\xi(u_2)\,\xi(u_3)\,\xi(u_4)\rangle - \langle \xi(u_1)\xi(u_2)\langle \xi(u_3)\xi(u_4)\rangle\right] \\
&- \tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_3)\tilde{\mathcal{L}}_I(u_2)\tilde{\mathcal{L}}_I(u_4)\langle \xi(u_1)\xi(u_3)\rangle\langle \xi(u_2)\xi(u_4)\rangle \\
&- \tilde{\mathcal{L}}_I(u_1)\tilde{\mathcal{L}}_I(u_4)\tilde{\mathcal{L}}_I(u_2)\tilde{\mathcal{L}}_I(u_3)\langle \xi(u_1)\xi(u_4)\rangle\langle \xi(u_2)\xi(u_3)\rangle,
\end{aligned} \tag{B10}$$

respectively. From (B10) it is clear that the Gaussian nature of $\xi[t]$ does not implies the same for $\Omega[t]$ of Eq. (B5), as the time-dependent Liouvillian $\tilde{\mathcal{L}}_I(u)$ generally does not commute with itself evaluated at different times. However, when the unperturbed Liouvillian $\mathcal{L}_a$ and perturbation Liouvillian $\mathcal{L}_I$ commute with each other, as in the case of Eq. (12) with $D_f = 0$ and $\mathcal{L}_I$ of Eq. (13), we have $\tilde{\mathcal{L}}_I(u) = \mathcal{L}_I$, that does not depend on time. Hence, in this case the Gaussian nature of $\xi[t]$ is transferred to the stochastic operator $\Omega[t]$. Therefore, in this specific scenario, the M-cumulant series appearing in the exponential function of (B6) reduces to only the second term containing the second M-cumulant, simplifying to (without loss of generality, we consider the average value of $\xi[t]$ to be zero):

$$\tilde{P}(x;t) = \exp\left[\epsilon^2 \mathcal{L}_I \mathcal{L}_I \int_0^t \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}u_2 \langle \xi(u_1)\,\xi(u_2)\rangle\right] P(x;0). \tag{B11}$$

Time-deriving this result we obtain

$$\begin{aligned}
\partial_t \tilde{P}(x;t) =& \epsilon^2 \mathcal{L}_I \mathcal{L}_I \int_0^t \mathrm{d}u \langle \xi(t)\,\xi(u)\rangle\ \tilde{P}(x;t) \\
=& \epsilon^2 \mathcal{L}_I \mathcal{L}_I \tau \tilde{P}(x;t).
\end{aligned} \tag{B12}$$

Getting rid of the interaction representation and by using (12) with $D_f = 0$ and (13), Eq. (B12) becomes exactly the FPE (4).

In the more general case, the Liouvillians $\mathfrak{L}_a$ and $\mathfrak{L}_I$ do not commute with each other, so $\tilde{\mathfrak{L}}_I(u)$ of (B3) depends on time. The advantage of utilizing the M-cumulants lies in the fact that, similar to standard cumulants, they are exactly zero when referring to independent random variables[22]. Thus, if the time lag between two events increases until they become independent of each other, any joint M-cumulant containing these two events must tend to zero. To model this situation more realistically, we assume that independence does not occur abruptly at a fixed time lag $\bar{\tau}$ but instead follows a smoother pattern, characterized by an exponential trend. Formally, for a series of events $\xi(t_1), \xi(t_2), ..., \xi(t_n)$ with $t_1 \geq t_2 \geq ... \geq t_n$, we assume that the corresponding joint n-cumulant decays at least exponentially with the time lag $u_1 - u_n$:

$$|\langle\langle\Omega(u_1)\Omega(u_2)...\Omega(u_n)\rangle\rangle| \lesssim |\xi^n| \exp(-(u_1 - u_n)/\bar{\tau}). \tag{B13}$$

In this scenario, along with the definitions (B7) and (B8), it is evident that the argument of the exponential function in the right-hand side of (B6) now yields a power series of $\bar{\delta} := \epsilon\bar{\tau}$. For a sufficiently small $\bar{\delta}$, we can truncate this series to the first non-zero term, which is the second one. Thus, Eq. (B6), combined with Eq. (B6) and (B9), gives

$$\tilde{P}(x;t) = \overleftarrow{\exp}\left[\epsilon^2 \int_0^t \mathrm{d}u_1 \int_0^{u_1} \mathrm{d}u_2 \tilde{\mathfrak{L}}_I(u_1)\tilde{\mathfrak{L}}_I(u_2)\langle\xi(u_1)\,\xi(u_2)\rangle + O(\bar{\delta}^4)\right] P(x;0). \tag{B14}$$

Time-deriving this result we obtain

$$\partial_t \tilde{P}(x;t) = \epsilon^2 \int_0^t \mathrm{d}u \tilde{\mathfrak{L}}_I(t)\tilde{\mathfrak{L}}_I(u)\langle\xi(t)\,\xi(u)\rangle\ \tilde{P}(x;t) + O(\bar{\delta}^4 t/\bar{\tau}) \tag{B15}$$

Getting rid of the interaction representation and by using again (12) (but now letting $D_f \neq 0$) and (13), Eq. (B15) becomes the ME (14).

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$p_k(\xi, t_1'; \xi_2, t_2'; ...; \xi_k, t_k')\, p_h(\xi_{k+1}, t_1; ...; \xi_n, t_h) + O(\bar{\tau})$ with $k, h, n \in \mathbb{N}$, $k + h = n$ and $t_i' > t_j + \bar{\tau}$. For example, $p_2(\xi, t'; \xi_2, t) = p_1(\xi, t')\, p_1(\xi_2, t) + O(\bar{\tau})$..

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[18] The cause of this issue lies in the fact that $x = 0$ is both a stable attraction point for the unpertubed motion ($\epsilon\tau^2 = 0$) and a point where the perturbation vanishes. Hence, given that any perturbed trajectory will eventually arrive at $x = 0$, this is an accumulation point of any initial 'ensemble".

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