ISOPERIMETRIC CLUSTERS IN HOMOGENEOUS SPACES VIA CONCENTRATION COMPACTNESS

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ABSTRACT. We show the existence of generalized clusters of a finite or even infinite number of sets, with minimal total perimeter and given total masses, in metric measure spaces homogeneous with respect to a group acting by measure preserving homeomorphisms, for a quite wide range of perimeter functionals. Such generalized clusters are a natural "relaxed" version of a cluster and can be thought of as "albums" with possibly infinite pages, having a minimal cluster drawn on each page, the total perimeter and the vector of masses being calculated by summation over all pages, the total perimeter being minimal among all generalized clusters with the same masses. The examples include any anisotropic perimeter in a Euclidean space, as well as a hyperbolic plane with the Riemannian perimeter and Heisenberg groups with a canonical left invariant perimeter or its equivalent versions.

1. INTRODUCTION

In a finite-dimensional Euclidean space $\mathbb{R}^n$ we call $\mathbf{E} = (E_1, \ldots, E_N)$ with $N \in \mathbb{N} \cup \{\infty\}$ an N-*cluster* of sets if each $E_j \subset \mathbb{R}^n$ is Borel (possibly empty), and $\mathcal{L}^n(E_i \cap E_j) = 0$ for all $i \neq j$, $\mathcal{L}^n$ standing for the Lebesgue measure. If $P(M)$ stands for the classical (Euclidean) perimeter of the set $M \subset \mathbb{R}^n$, we set

$$\mathbf{m}(\mathbf{E}) := (\mathcal{L}^n(E_1), \ldots, \mathcal{L}^n(E_N))$$

$$P(\mathbf{E}) := \frac{1}{2} \sum_{j \geq 1} P(E_j) + \frac{1}{2} P\left(\bigcup_{j \geq 1} E_j\right)$$

An N-cluster $\mathbf{E}$ is usually called *minimal*, or *isoperimetric*, if

$$P(\mathbf{E}) = \min\{P(\mathbf{F}) \colon \mathbf{m}(\mathbf{F}) = \mathbf{m}(\mathbf{E})\}.$$

Existence of *finite* (i.e. with $N < +\infty$) isoperimetric clusters for the classical (Euclidean) perimeter in a Euclidean space has been proven in [12] for each given vector of masses $\mathbf{m}$. The technique of the proof has been further extended to several other situations, for various different notions of perimeter. More has been done for the problem of existence of *isoperimetric sets*, i.e. in our terminology, minimal 1-clusters. It has been accomplished for different versions of perimeter in finite-dimensional spaces such as Riemannian manifolds (see the list of relevant results

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in the Introduction to [14] as well as the recent papers [5, 9, 3, 7]) and even in more general spaces with Ricci curvature bounds [2, 4]. However these techniques are based on regularity of isoperimetric sets, and are quite difficult even in the Euclidean case. Moreover, such results are only valid for *finite* clusters.

The difficulty to prove the existence by purely variational techniques, even in the Euclidean space and just for isoperimetric sets (i.e. just 1-clusters) is only due to noncompactness of the ambient space (in compact spaces it is trivial), and comes from the fact that minimizing sequences of sets may "escape to infinity" and lose part of their volumes. It is however well-known that the limits of the pieces going to infinity are still isoperimetric sets for their own volumes (which may be lower than the original requirement), see e.g. [8, 16] where the problem of existence of isoperimetric sets has been studied for quite a general family of weighted perimeters and volume measures. The same is known to happen also with *finite* isoperimetric clusters, see [18, 7] where a formula for the perimeter of a minimal finite cluster is proven (again, for possibly weighted perimeters and volumes) suggesting that a minimal cluster is given by the union of the limit cluster (with possibly lower volumes) for a minimizing sequence and a "cluster at infinity" which has precisely the missing volumes. Therefore to prove the existence of an isoperimetric set or cluster one needs to appropriately adjust the sets by adding necessary volume which requires a great deal of heavy regularity techniques. However it is important to emphasize that the problem is not merely technical. In fact, even for $N = 1$ (isoperimetric sets) it might happen that in some Riemannian manifolds there are no isoperimetric sets of some or even every volume, see [17] as well as [15, 14].

Here we consider a more general situation of a metric measure space (X, d, μ) (with distance d and measure μ) instead of an Euclidean space with Lebesgue measure, which is *homogeneous* with respect to some group acting by measure preserving homeomorphisms. We show that in such a situation one can easily prove by means of concentration compactness-like technique (i.e. similar to [11]), the existence of "generalized" minimal clusters (finite or possibly infinite) for each given vector of masses. Such generalized clusters are in a sense a natural "relaxed" version of a cluster and can be thought of as "albums" with possibly infinite pages, having a minimal N-cluster drawn on each page, the total perimeter and the vector of masses being calculated by summation over all pages, the total perimeter being minimal among all generalized clusters with the same vector of masses. Such a generalized cluster in fact keeps track of all the parts of minimal sequences "escaping at infinity" by placing them on possibly different pages.

The toy examples we provide include finite or infinite minimal clusters for any anisotropic perimeter in $\mathbb{R}^n$, as well as in a hyperbolic plane with the Riemannian perimeter and in Heisenberg groups with a canonical left invariant perimeter or its equivalent versions. In all these examples there is some natural discrete group acting on the respective space properly discontinuously and cocompactly by measure preserving isometries (of course, the existence of isoperimetric sets is also known in such situations). Note that the technique developed here is purely variational and does not involve any regularity-type arguments, thus allowing to obtain existence of generalzied minimal clusters for the whole range of perimeter functionals. The cost paid for such a simplicity is of course that one can say nothing a priori about the regularity of generalized clusters; and it has to be noted that even very weak regularity properties of a minimal generalized cluster would allow to conclude the

existence of a solution to the original problem (i.e. the existence of minimal clusters in the original space).

The paper is structured as follows. After introducing the general axiomatic notion of a perimieter functional by a set of quite weak requirements in Section 2, we provide in Section 3 the general semicontinuity and compactness result for this functional for sequences of sets and in Section 4 for sequences of clusters. This allows to prove the existence of generalized optimal clusters in Section 5. Finally, in Section 6 we give some basic applications of such existence results for various types of perimeter functional in different environments, in partcicular, in a finite-dimensional normed space (i.e. with either classical or anisotropic perimeter), in a hyperbolic space and in te Heisenberg group.

2. NOTATION AND PRELIMIARIES

In the sequel we suppose (X, d, μ) to be a metric measure space with distance d and nonnegative σ-finite Radon measure μ with $\mu(X) \neq 0$, and G be a discrete countable topological group acting properly discontinuously on X (i.e. $\{g \in G \colon gK \cap K \neq \emptyset\}$ is finite for all compact $K \subset X$) by homeomorphisms preserving the measure μ. We denote by $\mathcal{B}(X)$ the Borel σ-algebra of X, and by $\mathcal{A}(X)$ the class of open subsets of X. For a set $S \subset X$ and $x \in X$ we denote $\operatorname{dist}(x, S) := \inf\{d(x, y) \colon y \in S\}$ to be a distance from x to S. The metric space (X, d) is said to have Heine-Borel property, if every closed ball is compact.

By $L^1(\mu)$ (resp. $L^1_{\operatorname{loc}}(\mu)$) we denote the usual Lebesgue space of μ-integrable (resp. μ-integrable over compact sets) functions over X, and if $X = \mathbb{R}^n$, we denote by $C_0^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions with compact support. As usual, for an $E \subset X$ we let $\bar{E}$ stand for the closure of E and $\mathbf{1}_E$ for the characteristic function of E, while $B_r(x)$ denotes the open ball of radius $r > 0$ centered at $x \in X$.

We write $g_k \to \infty$ for a sequence of $g_k \in G$, if $\lim_k g_k = \infty$ in the one-point compactifictaion $G \cup \{\infty\}$ of G, that is, for every finite $F \subset G$ the set of $\{k \in \mathbb{N} \colon g_k \in F\}$ is finite.

In what follows we let $P \colon \mathcal{B}(X) \times \mathcal{A}(X) \to \mathbb{R}^+ \cup \{\infty\}$ be a "relative perimeter" functional for which we customarily abbreviate $P(E) := P(E, X)$, satisfying

(semicontinuity) $P(D, U) \leq \liminf_k P(D_k, U)$ whenever $D_k \xrightarrow{L^1_{\operatorname{loc}}(\mu)} D$,

(Beppo Levi) If $U_k \nearrow X$, then $P(D, U_k) \nearrow P(D)$ as $k \to \infty$,

(monotonicity) $P(B, U) \leq P(B, V)$ as $U \subset V$,

(superadditivity) $P(B, U) \geq \sum_{k=1}^M P(B, U_k)$ whenever $U = \bigsqcup_{i=1}^M U_k$,

(G-invariance) $P(gB, gU) = P(B, U)$ for every $g \in G$,

(compactness) if $E_k \subset X$ satisfy $\sup_k P(E_k, U) < +\infty$ for some precompact set U, then

there exists an $E \subset X$ such that, up to a subsequence, $E_k \cap U \xrightarrow{L^1(\mu)} E$.

3. COMPACTNESS AND SEMICONTINUITY FOR SEQUENCES OF SETS

3.1. **Semicontinuity.** We start with the following semicontinuity statement which will be further applied both to perimeters and measures.

Theorem 3.1 (semicontinuity). *Assume that (X, d) has Heine-Borel property. Let $E_k \subset X$ be a sequence of Borel sets, and $E^i \subset X$, $i \in I$, I at most countable,*

$g_k^i \in G$ *such that:*

(1) $$\lim_k (g_k^{i'})^{-1} g_k^i = +\infty \qquad \text{for all } i \neq i'$$

(2) $$(g_k^i)^{-1} E_k \xrightarrow{L^1_{\text{loc}}(\mu)} E^i \qquad \text{as } k \to +\infty.$$

Let also $F \colon \mathcal{B}(X) \times \mathcal{A}(X) \to \mathbb{R}$ *be a functional satisfying*

(semicontinuity) $F(B, U) \leq \liminf_k F(B_k, U)$ *whenever* $B_k \xrightarrow{L^1_{\text{loc}}(\mu)} B$,
  (Beppo Levi) *If* $U_k \nearrow X$, *then* $F(D, U_k) \nearrow F(D, X)$ *as* $k \to \infty$,
  (monotonicity) $F(D, U) \leq F(D, V)$ *as* $U \subset V$,
(superadditivity) $F(B, U) \geq \sum_{k=1}^M F(B, U_k)$ *whenever* $U = \bigsqcup_{i=1}^M U_k$,
  (G-invariance) $F(gB, gU) = F(B, U)$ *for every* $g \in G$.

If G *acts on* X *properly discontinuously, then*

$$\sum_i F(E^i, X) \leq \liminf_k F(E_k, X).$$

Proof. Up to a subsequence (not relabeled) we may suppose that

$$\liminf_k F(E_k, X) = \lim_k F(E_k, X).$$

Fix arbitrary $M \in \mathbb{N}$, $x_0 \in X$ and $R > 0$ and denote for brevity $U := B_R(x_0)$. Since U is precompact by Heine-Borel property, and the action of G on X is properly discontinuous, then $gU \cap U \neq \emptyset$ only for a finite number of $g \in G$. Thus from (1) we get that the sets $g_k^1 U, \ldots, g_k^M U$ are pairwise disjoint for all sufficiently large k. Hence we get

$$\lim_k F(E_k, X) \geq \limsup_k F\left(E_k, \bigsqcup_{i=1}^M g_k^i U\right) \quad \text{by monotonicity of } F$$

(3) $$= \limsup_k \sum_{i=1}^M F(E_k, g_k^i U) \quad \text{by superadditivity of } F$$

$$\geq \sum_{i=1}^M \liminf_k F(E_k, g_k^i U).$$

But $F(E_k, g_k^i U)) = F((g_k^i)^{-1} E_k, U)$ and from (2) using semicontinuity of F we obtain

(4) $$\liminf_k F(E_k, g_k^i U) = \liminf_k F((g_k^i)^{-1} E_k, U) \geq F(E^i, U).$$

The inequalities (3) and (4) together give

$$\lim_k F(E_k, X) \geq \sum_{i=1}^M F(E^i, U) = \sum_{i=1}^M F(E^i, B_R(x_0)).$$

Letting now $R \to +\infty$, we obtain

$$\lim_k F(E_k, X) \geq \sum_{i=1}^M F(E^i, X)$$

and finally letting $M \to +\infty$ we get

$$\lim_k F(E_k, X) \geq \sum_i F(E^i, X).$$

as desired. □

Corollary 3.2. *Let* (X, d, μ), E_k, E^i, I, g_k^i *be as in Theorem 3.1. If* G *acts on* X *properly discontinuously, then*

$$\sum_i P(E^i) \leq \liminf_k P(E_k),$$

$$\sum_i \mu(E^i) \leq \liminf_k \mu(E_k).$$

Proof. Apply Theorem 3.1 with $F(B, U) := P(E, U)$, and then with $F(B, U) := \mu(E \cap U)$. □

3.2. **Compactness.** The following theorem is our main technical tool to prove the existence of generalized isoperimetric clusters.

Theorem 3.3 (concentration compactness). *Assume that* (X, d) *has Heine-Borel property. Let*

$$E_k \in \mathcal{B}(X), \quad \mu(E_k) = m, \quad \sup_k P(E_k) = P < +\infty.$$

Suppose that

(i) *G act on X properly discontinuously,*
(ii) *there is a precompact $B \in \mathcal{B}(X)$ with $\mu(\partial B) = 0$, such that $GB = X$ and $\mu(gB \cap B) = 0$ for all $g \in G$ except $g = 1$.*

Assume further that

(iii) *there is an $\varepsilon > 0$ and a precompact open $V \subset X$ with $B \subset V$ such that the local isoperimetric inequality*

$$(5) \qquad\qquad P(E, V) \geq f(\mu(E \cap V))$$

holds for all Borel $E \subset X$ with $\mu(E \cap V) \leq \varepsilon$ with some nondecreasing function $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 0$ and $f'(0) = +\infty$.

Then there are: a subsequence E_k, some Borel sets $E^i \subset X$, and $g_k^i \in G$, $i \in I$, I at most countable, such that

$$(6) \qquad\qquad \lim_k (g_k^{i'})^{-1} g_k^i = +\infty \qquad \text{for all } i \neq i'$$

$$(7) \qquad\qquad (g_k^i)^{-1} E_k \xrightarrow{L^1_{\text{loc}}(\mu)} E^i, \qquad k \to +\infty$$

$$(8) \qquad\qquad \sum_i \mu(E^i) = m.$$

Remark 3.4. In many examples condition (iii) of the above Theorem 3.3 is verified with $f(t) := Ct^\alpha$ for some $\alpha \in (0, 1)$ and $C > 0$, namely, (5) reads as the local isoperimetric inequality

$$(9) \qquad\qquad P(E, V) \geq C\mu(E \cap V)^\alpha.$$

Proof. For each k let h_k^j be an enumeration of G such that the sequence $j \mapsto \mu(E_k \cap h_k^j V)$ is not-increasing. Let U stand for the interior of B. One observes that $U \neq \emptyset$, because $\mu(B) > 0$ (otherwise from $GB = X$ one would have $\mu(X) = 0$), and $\mu(\partial B) = 0$ hence $\mu(U) = \mu(B \setminus \partial B) > 0$.

Note that $Q_k^j := h_k^j U$ are open sets since G acts by homeomorphisms. If there are infinitely many sets $h_k^j V$ in which E_k has positive measure we avoid to enumerate those with zero measure.

By the compactness property of the perimeter functional, for each j we may suppose that up to a subsequence $((h_k^j)^{-1} E_k) \cap U$ converges in $L^1(\mu)$ as $k \to +\infty$ and hence there exists $F^j \subset U$ such that

$$(10) \qquad (h_k^j)^{-1}(E_k \cap Q_k^j) = ((h_k^j)^{-1} E_k) \cap U \xrightarrow{L^1(\mu)} F^j, \qquad \text{as } k \to \infty.$$

By a diagonal argument we may choose a single subsequence of k (not relabeled) such that (10) holds for all $j \in \mathbb{N}$. The rest of the proof is divided in several steps.

STEP 1. We claim that $\sum_j \mu(F^j) = m$.

To show the claim, we first note that since the action of G on X is properly discontinuous, and V is precompact, we have

$$\nu := \#\{g \in G : gV \cap V \neq \emptyset\} < \infty.$$

Applying Lemma A.5 one has a partition of G in at most ν disjoint subsets G_λ such that the sets hV are mutually disjoint for all $h \in G_\lambda$.

Since $j \mapsto \mu(E_k \cap h_k^j(V))$ is non-increasing, we have

$$(11) \quad \begin{aligned} \mu(E_k \cap h_k^n V) &\leq \frac{1}{n} \sum_{j=1}^{n} \mu(E_k \cap h_k^j V) \leq \frac{1}{n} \sum_{j=1}^{+\infty} \mu(E_k \cap h_k^j V) \\ &\leq \frac{1}{n} \sum_{\lambda=1}^{\nu} \sum_{g \in G_\lambda} \mu(E_k \cap gV) \leq \frac{\nu}{n} \mu(E_k) = \frac{\nu m}{n}. \end{aligned}$$

In particular, by (11) for every $\delta > 0$ one can choose an $N = N(\varepsilon, \delta) \in \mathbb{N}$ such that for all $j \geq N$ we have

$$(12) \qquad\qquad\qquad \mu(E_k \cap h_k^j V) \leq \varepsilon, \qquad \text{hence also}$$
$$(13) \qquad\qquad\qquad \mu(E_k \cap Q_k^j) \leq \varepsilon,$$

and, moreover,

$$(14) \qquad\qquad\qquad \frac{\mu(E_k \cap Q_k^j)}{f(\mu(E_k \cap Q_k^j))} \leq \delta.$$

Thus for such $j \in \mathbb{N}$, using (13), (iii) and the invariance of perimeter, we get

$$(15) \quad \begin{aligned} f(\mu(E_k \cap Q_k^j)) &\leq f(\mu((h_k^j)^{-1} E_k \cap V)) \\ &\leq P((h_k^j)^{-1} E_k, V) = P(E_k, h_k^j V). \end{aligned}$$

Therefore, for $n \geq N$ we have

$$
\begin{aligned}
\sum_{j=n}^{+\infty} \mu(E_k \cap Q_k^j) &\leq \sum_{j=n}^{+\infty} \frac{\mu(E_k \cap Q_k^j)}{f(\mu(E_k \cap Q_k^j))} f(\mu(E_k \cap Q_k^j)) \\
&\leq \sum_{j=n}^{+\infty} \frac{\mu(E_k \cap Q_k^j)}{f(\mu(E_k \cap Q_k^j))} P(E_k, h_k^j V) \quad \text{[by (15)]} \\
&\leq \delta \sum_{j=n}^{+\infty} P(E_k, h_k^j V) \quad \text{[by (14)]} \\
&\leq \delta \sum_{\lambda=1}^{\nu} \sum_{g \in G_\lambda} P(E_k, gV) \\
&\leq \delta \sum_{\lambda=1}^{\nu} P\left(E_k, \bigsqcup_{g \in G_\lambda} gV\right) \quad \text{[by superadditivity of perimeter]} \\
&\leq \delta \sum_{\lambda=1}^{\nu} P(E_k) \quad \text{[by monotonicity of perimeter]} \\
&= \nu \delta P(E_k) \leq \nu \delta P.
\end{aligned}
$$

Since $\delta > 0$ is arbitrary, we get

$$
\lim_{n \to +\infty} \sup_k \sum_{j=n}^{+\infty} \mu(E_k \cap Q_k^j) = 0.
$$

But we also have

$$
\mu(F^j) = \lim_k \mu(E_k \cap Q_k^j).
$$

Apply Lemma B.1 with $m_j := \mu(F^j)$, $m_{k,j} := \mu(E_k \cap Q_k^j)$, recalling that

$$
\begin{aligned}
\sum_i m_{k,j} &= \mu\big(E_k \cap \bigcup_j h_k^j U\big) \quad \text{[since } \mu(hU \cap U) = 0 \text{ for all } h \in G\text{]} \\
&= \mu\big(E_k \cap \bigcup_j h_k^j B\big) \quad \text{[since } \mu(h(B \setminus U)) = \mu(B \setminus U) = 0 \text{ for all } h \in G\text{]} \\
&= \mu(E_k) = m \quad \text{[since } GB = X\text{]},
\end{aligned}
$$

to obtain $\sum_j \mu(F^j) = m$ as claimed.

STEP 2: CONSTRUCTION OF g_k^i AND E^i. Define on $\mathbb{N}$ an equivalence relation by letting $j \sim j'$ whenever the set $\{(h_k^{j'})^{-1} h_k^j : k \in \mathbb{N}\} \subset G$ is finite and let $I := \mathbb{N}/\sim$ be the quotient set. For each $i \in I$ let $\underline{i} := \min i$. Passing to a subsequence in k we might and shall suppose that for all $i \in I$ and all $j \in i$ the sequence $\{(h_k^{\underline{i}})^{-1} h_k^j\}_k$ (which by assumption assumes a finite number of values) is actually constant, which will be denoted $h^j \in G$.

Clearly for $j \in i$ all h^j are distinct because so are $h_k^j \in G$. Define $g_k^i := h_k^{\underline{i}}$ and suppose that, up to a subsequence, the sets $(g_k^i)^{-1} E_k$ converge in $L^1_{\mathrm{loc}}(\mu)$ to some Borel set E^i as $k \to +\infty$. Hence (6) and (7) hold by construction.

STEP 3. It remains to prove (8). To this aim observe that from (10) with $j \in i$ we get

$$\mu(F^j) \leftarrow \mu(E_k \cap Q_k^j) = \mu\left((g_k^i)^{-1}(E_k \cap Q_k^j)\right)$$

$$= \mu\left((g_k^i)^{-1}E_k \cap (g_k^i)^{-1}h_k^j U\right)$$

$$= \mu\left((g_k^i)^{-1}E_k \cap h^j U\right)$$

$$\rightarrow \mu\left(E^i \cap h^j U\right)$$

as $k \to \infty$, so that $\mu(F^j) = \mu(E^i \cap h^j U)$ whenever $j \in i$. Therefore

$$\lim_k \mu(E_k) = m = \sum_j \mu(F^j) \quad \text{[by Step 1]}$$

$$= \sum_i \sum_{j \in i} \mu(E^i \cap h^j U) = \sum_i \mu\left(E^i \cap \bigcup_{j \in i} h^j U\right) \leq \sum_i \mu\left(E^i\right),$$

the reverse inequality coming from Theorem 3.1. $\qquad\square$

4. COMPACTNESS AND SEMICONTINUITY FOR SEQUENCES OF CLUSTERS

The following definition of an isoperimetric cluster is an obvious extension of the classical one provided in the Introduction from a Euclidean to a general metric measure space.

Definition 4.1 (isoperimetric clusters). We say that $\mathbf{E} = (E_j)_{j=1}^N$ with $N \in \mathbb{N} \cup \{\infty\}$, is an N-cluster in X if each $E_j \subset X$ is a Borel set and $\mu(E_i \cap E_j) = 0$ for all $i \neq j$. If N is finite we have $\mathbf{E} = (E_1, \ldots, E_N)$, if $N = \infty$ we have $\mathbf{E} = (E_1, \ldots, E_n, \ldots)$. We set

$$\mathbf{m}(\mathbf{E}) := (\mu(E_j))_{j=1}^N$$

$$P(\mathbf{E}) := \frac{1}{2}\sum_{j \geq 1} P(E_j) + \frac{1}{2}P\left(\bigcup_{j \geq 1} E_j\right)$$

An N-cluster $\mathbf{E}$ will be called *minimal*, or *isoperimetric*, if

$$P(\mathbf{E}) = \inf\left\{P(\mathbf{F}) : \mathbf{m}(\mathbf{F}) = \mathbf{m}(\mathbf{E})\right\}.$$

Theorem 4.2 (concentration compactness for clusters). *Assume that (X, d) has Heine-Borel property. Let $\mathbf{E}_k$ be a sequence of N-clusters in X, $N \in \mathbb{N} \cup \{\infty\}$, with*

$$\mathbf{m}(\mathbf{E}_k) = \mathbf{m} \in \mathbb{R}^N \quad \text{and} \quad P(\mathbf{E}_k) \leq P < +\infty.$$

Then under conditions (i), (ii) and (iii) of Theorem 3.3 there exist N-clusters $\mathbf{E}^i$ with $i \in I$, I at most countable, such that

$$\sum_i \mathbf{m}(\mathbf{E}^i) = \mathbf{m}$$

$$\sum_i P(\mathbf{E}^i) \leq \liminf_k P(\mathbf{E}_k)$$

and there exist $g_k^i \in G$ such that up to a subsequence

$$(g_k^i)^{-1}(E_k)_j \to (E^i)_j \qquad \text{as } k \to +\infty \text{ in } L_{\text{loc}}^1(\mu)$$

for all $j = 1, \ldots, N$ and $i \in I$.

Proof. Up to a subsequence suppose that $\liminf_k P(\mathbf{E}_k) = \lim_k P(\mathbf{E}_k)$. Let

$$F_k := \bigsqcup_{j=1}^{N} (E_k)_j.$$

Clearly $P(F_k) \leq 2P(\mathbf{E}_k) \leq 2P$, and hence we can apply Theorem 3.3 to find $F^i \subset X$, $g_k^i \in G$, $\lim_k (g_k^{i'})^{-1} g_k^i = \infty$ for $i \neq i'$ such that, up to a subsequence, $(g_k^i)^{-1} F_k \to F^i$ in $L_{\text{loc}}^1(\mu)$ as $k \to \infty$ with $\sum_i \mu(F^i) = \mu(F_k) = \sum_{j=1}^{N} m_j$. Since $P((E_k)_j) \leq P(\mathbf{E}_k) \leq P$, by the compactness assumption on the perimeter functional, up to subsequences we can define the sets

$$E_j^i := L_{\text{loc}}^1(\mu)\text{-}\lim_k (g_k^i)^{-1} (E_k)_j.$$

By Lemma B.2 then

$$F^i = \bigsqcup_{j=1}^{N} E_j^i.$$

By Theorem 3.1 we have $\sum_i \mu(E_j^i) \leq \mu((E_k)_j)$. Therefore, since

$$\sum_j \sum_i \mu(E_j^i) = \sum_i \sum_j \mu(E_j^i) = \sum_i \mu(F^i) = \mu(F_k) = \sum_j \mu((E_k)_j),$$

one gets $\sum_i \mu(E_j^i) = \mu((E_k)_j)$. Hence $\mathbf{E}^i = (E_1^i, \ldots, E_N^i)$ is an M-cluster, $M \leq N$, with

$$\sum_i \mathbf{m}(\mathbf{E}^i) = \mathbf{m},$$

$$P(\mathbf{E}^i) = \frac{1}{2} \sum_{j=1}^{N} P(E_j^i) + \frac{1}{2} P(E^i).$$

Again by Theorem 3.1 we have

$$\sum_i P(F^i) \leq \liminf_k P(F_k),$$

$$\sum_i P(E_j^i) \leq \liminf_k \sum_j P((E_k)_j), \qquad j = 1, \ldots, N$$

Summing up these relationships we get

$$\sum_i P(\mathbf{E}^i) \leq \lim_k P(\mathbf{E}_k)$$

as claimed. $\qquad\square$

5. Existence of generalized isoperimeric clusters

We introduce now the notion of a *generalized isoperimeric cluster*.

Definition 5.1 (generalized isoperimetric clusters)**.** Let $X^\infty := \mathbb{Z} \times X$ and let $\phi_i \colon X \to X^\infty$ be the inclusion $\phi_i(x) = (i, x)$. We call $\mathbf{E} = (E_j)_{j=1}^N$, $E_j \subset X^\infty$ a generalized N-cluster in X with $N \in \mathbb{N} \cup \{\infty\}$, if for each $i \in \mathbb{Z}$ the N-uple $\mathbf{E}^i$ with components $E_j^i := \phi_i^{-1}(E_j) \subset X$, $j = 1, \ldots, N$ is a N-cluster in X. We will also write $\phi_i^{-1}(\mathbf{E})$ instead of $\mathbf{E}^i$. We set

$$\mathbf{m}^\infty(\mathbf{E}) := \sum_i \mathbf{m}(\mathbf{E}^i) = \left(\sum_i \mu(E_1^i), \ldots, \sum_j \mu(E_N^i) \right)$$

$$P^\infty(\mathbf{E}) := \sum_i P(\mathbf{E}^i) = \frac{1}{2} \sum_i \sum_j P(E_j^i).$$

$\mathbf{E}$ will be called *minimal*, or *isoperimetric*, if

$$P^\infty(\mathbf{E}) = \min \left\{ P^\infty(\mathbf{F}) \colon \mathbf{m}(\mathbf{F}) = \mathbf{m}(\mathbf{E}) \right\}.$$

It is convenient to think of a generalized cluster as a sequence of N-clusters $\mathbf{E}^i$ drawn each on a page $X_i := \{i\} \times X$ of the "album" X^∞ and ϕ_i^{-1} as an extraction of the i-th page from this album.

Proposition 5.2. *Every page of a generalized minimal cluster is a minimal cluster, namely, for every $i \in \mathbb{Z}$ one has that $\mathbf{E}^i = \phi_i^{-1}(\mathbf{E})$ is a minimal cluster in X.*

Proof. If not, there is an N-cluster $\mathbf{G}$ in X such that

$$P(\mathbf{G}) < P(\mathbf{E}^i) \quad \text{and } \mathbf{m}(\mathbf{G}) = \mathbf{m}(\mathbf{E}^i).$$

Then for the generalized cluster $\mathbf{F}$ defined by

$$\mathbf{F}^h = \begin{cases} \mathbf{E}^h, & \text{if } h \neq i, \\ \mathbf{G}, & \text{if } h = i, \end{cases}$$

one has $P^\infty(\mathbf{F}) < P^\infty(\mathbf{E})$ contradicting the minimality of $\mathbf{E}$. $\qquad \square$

Notice that a generalized N-cluster $\mathbf{E}$ in X is nothing else than an N-cluster in X^∞ by considering $X^\infty = \mathbb{Z} \times X$ as a metric space with the euclidean distance:

$$d^\infty((i, x), (h, y)) = \sqrt{d(x, y)^2 + (i - h)^2}.$$

The measure μ^∞ and perimeter P^∞ are naturally defined by:

$$\mu^\infty(E) := \sum_i \mu(\phi_i^{-1}(E)),$$

$$P^\infty(E, U) := \sum_i P(\phi_i^{-1}(E), \phi_i^{-1}(U)).$$

This allows us to prove the existence of minimal generalized clusters.

Theorem 5.3 (existence of generalized minimal clusters)**.** *Assume that (X, d) has Heine-Borel property. If conditions (i), (ii) and (iii) of Theorem 3.3 hold, condition (iii) being satisfied with $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ subadditive (this is true in particular when $f(t) := Ct^\alpha$, with $C > 0$, $\alpha \in (0, 1)$), then for an arbitrary $\mathbf{m} \in \mathbb{R}_+^N$ there exists a minimal generalized N-cluster $\mathbf{E}$ with $\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}$.*

Proof. Define $G^\infty = \mathbb{Z} \times G$ the product group acting over X^∞ by

$$(n,g)(m,x) = (n+m, g(x)) \qquad g \in G, x \in X, n, m \in \mathbb{Z}.$$

Notice that d^∞ has the Heine-Borel property, P^∞ and μ^∞ satisfy monotonicity, superadditivity, G^∞-invariance and compactness properties of section 2. The semicontinuity property is simply an application of Fatou Lemma and Beppo Levi property is an application of Beppo Levi theorem.

We claim that (i) of Theorem 3.3 holds. In fact if K^∞ is a compact set in X^∞ then the set $I := \{i \in \mathbb{Z} : (i, x) \in K^\infty \text{ for some } x \in X\}$ is finite. Let $K := \{x \in X : (i, x) \in K^\infty \text{ for some } i \in I\}$. Since I is finite K is compact in X. Then the set

$$\{(n, g) \in G^\infty : (n, g) K^\infty \cap K^\infty \neq \emptyset\} \subset (I - I) \times \{g \in G : gK \cap K \neq \emptyset\}$$

is finite because if $(i, x) \in (n, g) K^\infty \cap K^\infty$ then $x \in K$, $g(x) \in K$, $i \in I$ and $n + i \in I$.

Also the condition (ii) of Theorem 3.3 is satisfied for the metric measure space X^∞ with this group action. In fact the set $U^\infty := \{0\} \times U$ satisfies $G^\infty \overline{U^\infty} = X^\infty$ and $\mu^\infty(g\overline{U^\infty} \cap \overline{U^\infty}) = 0$ for all $g \in G^\infty$ except when g is the neutral element in G^∞.

For condition (iii) of Theorem 3.3 just notice that if $\mu^\infty(E) < \varepsilon$ implies that $\mu(E^i) < \varepsilon$ (where $E^i = \phi_i^{-1}(E)$) hence $P(E^i, V) \geq f(\mu(E^i \cap V))$ and summing up, using the subadditivity of f,

$$P^\infty(E, \{0\} \times V) = P(E^0, V) + \sum_{i \neq 0} P(E^i, \emptyset) \geq P(E^0, V)$$

$$= P(E^0, V) + \sum_{i \neq 0} f(\mu(\emptyset)) \geq f(\mu(E^0 \cap V))$$

$$= f(\mu^\infty(E \cap (\{0\} \times V))).$$

Let $\mathbf{E}_k$ be a sequence of generalized N-clusters in X satisfying $\mathbf{m}^\infty(\mathbf{E}_k) = \mathbf{m} \in \mathbb{R}^N$. By Theorem 4.2 applied to $\mathbf{E}_k$, X^∞, μ^∞, P^∞ in place of X, μ, P respectively, we have the existence of N-clusters $\mathbf{E}^h$ in X^∞ for $h \in H$ with H at most countable, such that

(16) $$\sum_h \mathbf{m}^\infty(\mathbf{E}^h) = \mathbf{m}, \qquad \sum_h P^\infty(\mathbf{E}^h) \leq \liminf_k P^\infty(\mathbf{E}_k).$$

Given an injective map $f \colon H \times \mathbb{Z} \to \mathbb{Z}$ we can define the generalized N-cluster $\mathbf{E} = (E_1, \ldots, E_N)$ in X such that

$$\phi_{f(h,i)}^{-1}(E_j) = \phi_i^{-1}((\mathbf{E}^h)_j), \qquad \text{for every } j = 1, \ldots N, i \in \mathbb{Z}, h \in H.$$

Hence (16) reads now

$$\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}, \qquad P^\infty(\mathbf{E}) \leq \liminf_k P^\infty(\mathbf{E}_k)$$

which means that $\mathbf{E}$ is a generalized minimal N-cluster in X. $\qquad \square$

Remark 5.4. Note that a minimal *generalized* cluster can be thought in many particular cases as a kind of natural relaxation of the notion of a minimal cluster. In fact, if it is possible, say, to cut away from each page everything outside of a sufficiently large ball, without changing too much the perimeter and the volumes, so that what remains on each a page is a bouunded cluster, then put by the group action all the bounded clusters from each page to just one page, and finally adjust the volumes, say, by adding small isoperimetric sets or at least sets with sufficiently

small perimeters, we get that for every $\varepsilon > 0$ and for each generalized c N-cluster $\mathbf{E}$ there is an N-cluster $\mathbf{E}'$ with the same volumes, i.e. $\mathbf{m}^\infty(\mathbf{E}') = \mathbf{m}(\mathbf{E})$ and

$$P(\mathbf{E}') \le P^\infty(\mathbf{E}) + \varepsilon.$$

This can be done for instance in a Euclidean space (the cutting of a large bounded set from each page without changing too much the volumes and the perimeter can be done in view of the coarea formula; in more general spaces even a coarea type inequality would suffice).

6. Basic examples

For merely illustrative purposes we provide here several examples of existence of finite or infinite generalized isoperimetric clusters in a finite-dimensional vector space (for any anisotropic perimeter related to some norm), in the hyperbolic plane and in the Heisenberg groups. Note that even such simple examples in fact provide immediately the existence resluts for the whole range of equivalent perimeters in the mentioned spaces, without requiring the study of regularity properties of such clsuters and/or just isoperimetric sets for each particular perimeter. It is also worth mentioning that in the same way one can formulate similar existence results in many more interestng geometries (in particular, in higher dimensional hyperbolic spaces, or more general Carnot groups).

6.1. **Finite-dimensional space.**
Let X be an n-dimensional finite dimensional space equipped with any norm $\|\cdot\|$, μ be the Lebesgue measure on X, $G := \mathbb{Z}^n$ acting by translations, and P be the relative perimeter functional in X corresponding to the chosen norm, i.e.

$$P(E, U) := \sup \left\{ \int_U \mathbf{1}_E \mathrm{div} v \, d\mu \colon \|v\|_\infty \le 1 \right\},$$

the sup being taken over smooth vector fields v with compact support on H, with $\|v\|_\infty := \sup_{p \in X} \|v(p)\|$. The following corollary is a direct consequence of Theorem 5.3.

Corollary 6.1. *For every* $\mathbf{m} \in \mathbb{R}_+^N$ *there exists a minimal generalized N-cluster* $\mathbf{E}$ *in X with* $\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}$.

It is worth mentioning that there is nothing obligatory in the choice of the group $G = \mathbb{Z}^n$; instead, some other crystallographic group could have been chosen.

6.2. **Hyperbolic space.**
Taking H to be the hyperbolic plane, μ be its canonical volume measure, G be any countable Fuchsian group (i.e. a discrete subgroup of isometries of H) acting properly discontinuously and cocompactly on H (e.g. one may take G to be the classical Fucsian group providing the tiling of H into isometric Schwartz triangles) and P be the classical Riemannian relative perimeter functional in H, i.e.

$$P(E, U) := \sup \left\{ \int_U \mathbf{1}_E \mathrm{div}_H v \, d\mu \colon \|v\|_\infty \le 1 \right\},$$

the sup being taken over smooth vector fields v with compact support on H, with $\|v\|_\infty := \sup_{p \in H} |v_p|_p$, $|\cdot|_p$ standing for the Riemannian norm of a vector in $T_p H$.

Corollary 6.2. *For every* $\mathbf{m} \in \mathbb{R}_+^N$ *there exists a minimal generalized N-cluster* $\mathbf{E}$ *in H with* $\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}$.

Proof. Note that the perimeter functional P clearly satisfies the semicontinuity, monotonicity, superadditivity, compactness, G-invariance and Beppo Levi properties, and conditions (i) and (ii) of Theorem 3.3 are satisfied as well with B given by Lemma A.1. Condition (iii) of Theorem 3.3 is satisfied as in Remark 3.4 with $V \supset B$ being a sufficiently large ball containing B, and $\alpha := 1/2$, $\varepsilon < \mu(V)/2$ due to the relative isoperimetric inequality over a compact Riemannian manifold $\bar{V}$ [10]. The claim is now a direct application of Theorem 5.3. $\square$

Remark 6.3. If, chosen a Fuchsian group G, there is an equivalent Finslerian structure on H (with the norm $\| \cdot \|_p$ in each T_pH equivalent to the Riemannian one $| \cdot |_p$), invariant under the action of G, then the same argument gives the existence for every $\mathbf{m} \in \mathbb{R}_+^N$ of a minimal generalized N-cluster $\mathbf{E}$ in H with $\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}$, for the perimeter P_G relative to this Finslerian structure instead of P, i.e. for

$$P_G(E,U) := \sup \left\{ \int_U \mathbf{1}_E \mathrm{div}_H v \, d\mu \colon \|v\|_{G,\infty} \leq 1 \right\},$$

the sup being taken over smooth vector fields v with compact support on H, with $\|v\|_{G,\infty} := \sup_{p \in H} \|v_p\|_p$. To see this it is enough to note that P_G and P are equivalent.

6.3. Heisenberg groups.
Taking H^n to be the Heisenberg group of topological dimension $2n + 1$, μ be its Haar measure, G be the respective discrete Heisenberg group (i.e. once H^n is canonically associated with a group of matrices with real entries, G is associated with the subgroup of such matrices with integer entries), $\{X_i\}_{i=1}^{2n}$ be left-invariant vector fields satisfying the Hörmander condition, and P be the sub-Riemannian relative perimeter functional corresponding to the choice of X_i, defined as the total variation measure of the vector ($\mathbb{R}^{2n}$-valued) measure

$$D_X \mathbf{1}_1 := (D_{X_1} \mathbf{1}_E, \ldots, D_{X_{2n}} \mathbf{1}_E),$$

where the distribution $D_{X_i} f$ is defined by its action $\langle \varphi, D_{X_i} f \rangle$ on every test function $\varphi_{\in} C_0^\infty(\mathbb{R}^{2n})$ by the formula

$$\langle \varphi, D_{X_i} f \rangle := - \int_{\mathbb{R}^{2n}} f X_i \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^{2n}} f \varphi \mathrm{div} \, X_i \, dx.$$

Note that there are several equivalent definitions of this perimeter given by theorem 3.1 of [1]. We also observe that one of the equivalent definitions is given in [section 5.3][13] and used in particular in [6].

Corollary 6.4. *For every* $\mathbf{m} \in \mathbb{R}_+^N$ *there exists a minimal generalized* N-*cluster* $\mathbf{E}$ *in* H^n *with* $\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}$.

Proof. We equip H^n with either the Carnot-Caratheodory distance or any equivalent left-invariant distance d, so that now G acts by isometries. The claim follows from Theorem 5.3 since he perimeter functional P satisfies the semicontinuity, monotonicity, superadditivity, Beppo Levi and G-invariance properties, as well as conditions (i) and (ii) of Theorem 3.3 (with B given by Lemma A.1), while

- condition (iii) of Theorem 3.3 is satisfied as in Remark 3.4 with $V \supset B$ sufficiently large ball containing U, and $\alpha := (Q - 1)/Q$, $Q := 2n + 2$ standing for the homogeneous dimension of H^n, $\varepsilon < \mu(V)/2$ due to the relative isoperimetric inequality in a ball of a Carnot-Caratheodory space (theorem 1.6 from [6]),

- compactness property is given by theorem 3.7 from [13].

$\square$

Remark 6.5. Similarly to Remark 6.3 if one takes in H^n any perimeter P defined as $P(E, U) := |D\mathbf{1}_E|(U)$ with the metric total variation $u \mapsto |Du|$ defined with respect to any chosen left G-invariant distance in H^n, then word-to-word repetition of the proof of the above Corollary 6.4 gives the existence for every $\mathbf{m} \in \mathbb{R}_+^N$ of a minimal generalized N-cluster $\mathbf{E}$ in H with $\mathbf{m}^\infty(\mathbf{E}) = \mathbf{m}$ for such a perimeter.

APPENDIX A. USEFUL FACTS ABOUT GROUP ACTIONS

We collect here the following lemma, useful in applications of our results to verify condition (ii) of Theorem 3.3, and some easy remarks related to it.

Lemma A.1. *Let (X, d, μ) be a complete locally compact metric space with non-negative σ-finite Radon measure. Suppose G is a topological group acting on X by homeomorphisms preserving μ-nullsets, and*

- (i) *the action of G is proper discontinuous,*
- (ii) *the set S of $x \in X$ having nontrivial stabilizer subgroup*

$$G_x := \{g \in G \colon gx = x\},$$

   i.e. $S := \{x \in X \colon G_x \neq \{1\}\}$, is μ-negligible, that is, $\mu(S) = 0$,
- (iii) *and X is compactly generated, i.e. there exists a compact $K \subset X$ such that $GK = X$.*

Then there is a precompact Borel set $B \subset X$ (a fundamental domain for the action of G) such that $GB = X$, $\mu(\partial B) = 0$, and $\mu(gB \cap B) = 0$ for all $g \in G$, $g \neq 1$. Moreover, if the action of G on X is free, then one can choose B so as to have $gB \cap B = \emptyset$ for all $g \in G$, $g \neq 1$.

Remark A.2. Condition (iii) of Lemma A.1 for locally compact metric space X is equivalent to *cocompactness* of the action of G, that is, to compactness of the quotient space X/G. In fact, since the natural projection map $\pi \colon X \to X/G$ is continuous, then condition (iii) implies that $X/G = \pi(K)$ is compact. Vice versa, note that π is also an open map (as a projection map under group action, namely, because for every open $U \subset X$ one has $\pi^{-1}(\pi(U))) = \cup_{g \in G} gU$ is open since so is each gU). Therefore, taking a cover $\{U_\lambda\}$ of X by precompact open sets, we have that $\{\pi(U_\lambda)\}$ is an open cover of X/G, and if X/G is compact, extracting a finite subcover $\{\pi(U_{\lambda_j})\}_{j=1}^m$, $m \in \mathbb{N}$ of $\{\pi(U_\lambda)\}$, we get that

$$K := \bigcup_{j=1}^m \bar{U}_{\lambda_j}$$

is a compact set satisfying $GK = X$.

Remark A.3. Another immediate observation worth mentioning is that for the condition (iii) of Lemma A.1 to hold, it is necessary that G be infinite, unless, of course, X is compact.

Proof. We divide the proof in two steps.

STEP 1. For all $x \in X \setminus S$ we show that there exists a $\rho > 0$ (which can be taken arbitrarily small) such that $\bar{B}_\rho(x)$ is compact, $B_\rho(x) \cap S = \emptyset$, $\mu(\partial B_\rho(x)) = 0$ and $gB_\rho(x) \cap B_\rho(x) = \emptyset$ for all $g \in G$, $g \neq 1$. In particular this proves that S is closed.

By proper dicontinuity of the action of G for a precompact neighborhood W of x the set $\{g \in G \colon g\bar{W} \cap \bar{W} \neq \emptyset, g \neq 1\}$ is finite; let $g_1, \dots, g_n$ be its elements. For each $k = 1, \dots, n$ since $x \notin S$ we have $x \neq g_k x$ and hence we can find U_k neighbourhood of x and V_k neighbourhood of $g_k x$ which are disjoint, i.e. $U_k \cap v_k = \emptyset$. Since $W, U_1, \dots, U_n$ and $g_1^{-1} V_1, \dots, g_n^{-1} V_n$ are all neighbourhoods of x (because g_k are homeomorphisms), their interesection U is also a neighbourhood of x. Since $U \subset W$ we have $gU \cap U = \emptyset$ unless $g = g_k$ for some $k = 1, \dots, n$. But in the latter case, i.e. when $g = g_k$, since $U \subset U_k$ and $gU = g_k U \subset V_k$ we find that $gU \cap U \subset U_k \cap V_k = \emptyset$. Clearly $U \cap S = \emptyset$ since otherwise there would exist an $x \in U \cap S$ and $g \neq 1$ such that $gx = x \in U$.

Now since U is a neighbourhood of x, there exists an $\varepsilon > 0$ such that for all $\rho \leq \varepsilon$ the ball $B_\rho(x)$ is precompact and $\bar{B}_\rho(x) \subset U$. To complete the proof of the claim of this step notice that all but a countable number of $\rho < \varepsilon$ have the property $\mu(\partial B_\rho(x)) = 0$, because otherwise $\mu(B_\varepsilon(x))$ would be infinite, contrary to the fact that is should be finite since which should be finite since $B_\varepsilon(x)$ is precompact and μ is a Radon measure.

STEP 2. Let $\tilde{K} \supset K$ be a cover of K by a finite number of open balls so small to be precompact, so that in particular, $\tilde{K}$ is open and precompact. Let S_k be a sequence of open sets satisfying

$$S \cap K \subset S_k \subset \bar{S}_k \subset \tilde{K}$$

and such that $\mu(\bar{S}_k) \to 0$ as $k \to \infty$ (one can take for instance $S_k := \{x \in X \colon \operatorname{dist}(x, S \cap K) < t_k\}$ for a sequence of numbers $t_k \searrow 0$). By compactness we can take a finite cover $U_1, \dots, U_{n_1}$ of $K \setminus S_1$ with open balls U_i satisfying the properties stated in Step 1 and such that $\bar{U}_i \subset \tilde{K}$. Inductively we can take a finite cover $U_{n_k+1}, \dots, U_{n_{k+1}}$ of the compact set $(K \setminus S_{k+1}) \setminus \bigcup_{i=1}^{n_k} U_i$ by open balls U_i satisfying the properties stated in Step 1 which are all contained in S_k so that we have

$$K \setminus S \subset \bigcup_{i=1}^{+\infty} U_i, \qquad \bigcup_{i=n_k}^{+\infty} U_i \subset S_k.$$

Define inductively the disjoint sets

$$V_1 := U_1,$$

$$V_{i+1} := U_{i+1} \setminus \bigcup_{j=1}^{i} GU_j.$$

Note that since U_j are all precompact and G acts properly discontinuously, then each GU_j in the definition of V_{i+1} can be substituted by a finite union of gU_j over a finite subset of $g \in G$ (depending of course on i and j): in fact,

$$gU_j \cap U_{i+1} \subset g\overline{(U_j \cup U_{i+1})} \cap \overline{(U_j \cup U_{i+1})} \neq \emptyset$$

for an at most finite set of $g \in G$. Therefore, recalling also that $\partial(gU_j) = g\partial U_j$ (since the action of g is a homeomorphism), and hence $\mu(\partial(gU_j)) = \mu(g\partial U_j) = 0$ (since the action preserves μ-nullsets), we get $\mu(\partial V_i) = 0$ for all i.

Let

$$\tilde{B} := \bigcup_{i=1}^{+\infty} V_i, \qquad B := (S \cap \tilde{K}) \cup \tilde{B},$$

so that $B \subset \tilde{K}$ is a precompact Borel set.

We now verify the claimed properties of B. Let us prove first that $GB \supset K$. For $x \in K$ we have either $x \in S$ or $x \in U_i$ for some $i \in \mathbb{N}$. If $x \in S$ we have $x \in B$ because $x \in S \cap K \subset S \cap \tilde{K}$. If $x \in U_i$ then either $x \in V_i$ and hence $x \in B$ or there exists a $g \in G$ and $j < i$ such that $x \in gU_j$. If j is the minimum possibile index with this property, then $V_j = U_j$ and we obtain that $g^{-1}x \in V_j \subset B$. Hence $GB \supset K$ and since $GK = X$, we conclude that $GB = X$.

To prove $\mu(\partial B) = 0$, note that

$$\partial \tilde{B} \subset \bigcup_{i=1}^{n_k} \partial V_i \cup \partial \left(\bigcup_{i=n_k+1}^{\infty} V_i \right) \subset \bigcup_{i=1}^{n_k} \partial V_i \cup \bar{S}_k,$$

the latter inclusion being due to the fact that $V_i \subset S_k$ for $i \geq n_k$. Thus from $\mu(\partial V_i) = 0$ we get

$$\mu(\partial \tilde{B}) \leq \mu(\bar{S}_k) \to 0$$

as $k \to \infty$, which gives $\mu(\tilde{B}) = 0$. Recalling that $\mu(S) = 0$ we get the claim.

It remains to prove that $\mu(gB \cap B) = 0$ for all $g \in G$, $g \neq 1$. To this aim first notice that $g\tilde{B} \cap \tilde{B} = \emptyset$ because $V_i \cap gV_j = \emptyset$ for all i, j and $g \neq 1$. Now

$$gB \cap B \subset gS \cup S \cup (g\tilde{B} \cap \tilde{B}) = gS \cup S$$

hence $\mu(gB \cap B) \leq \mu(gS) + \mu(S) = 0$ for all $g \neq 1$. Finally note that when $S = \emptyset$, then $B \subset \tilde{B}$, hence $gB \cap B \subset g\tilde{B} \cap \tilde{B} = \emptyset$ concluding the proof. $\qquad\square$

Remark A.4. It is worth remarking that in many applications already the Dirichlet-Voronoi fundamental domain

$$D_x := \{y \in X \colon d(y, x) < d(gy, x) \quad \text{for all } g \in G, \, g \neq 1\}$$

for some $x \in X$ satisfies the statement of Lemma A.1. Note in fact that

$$gD_x \cap D_x = \emptyset \quad \text{for all } g \in G, g \neq 1,$$

because if $y \in gD_x$, that is, $gy \in D_x$, then $y := g^{-1}gy \notin D_x$. Thus one can take $B := D_x$ when $\mu(\partial B) = 0$. For instance, when $X = \mathbb{R}^n$, and $G = \mathbb{Z}^n$ acting by translations (say, by integer valued vectors), then one can take for B the Dirichlet-Voronoi fundamental domain for this action which is the cube containing part of its boundary, once μ is absolutely continuous with respect to the Lebesgue measure.

The following easy technical lemma is also used in the paper.

Lemma A.5. *Let G be a group acting on a set X and $V \subset X$. If*

$$\nu := \#\{g \in G \colon gV \cap V \neq \emptyset\} < \infty,$$

then there is a partition of G into ν pairwise disjoint subsets $G_1, \ldots, G_\nu$, such that $hV \cap gV = \emptyset$ for every $h, g \in G_j$, $h \neq j$, $j = 1, \ldots, \nu$.

Proof. We consider the family $\mathcal{F}$ of all subsets $F \subset G$ with the property that $hV \cap gV = \emptyset$ for all $h, g \in F$, $h \neq g$. Thanks to Zorn's lemma we can find G_1 in $\mathcal{F}$ which is maximal with respect to inclusion. Then define inductively G_{n+1} by taking any maximal subset of $G \setminus \bigcup_{j=1}^{n} G_j$ in the family $\mathcal{F}$. We claim that $G_n = \emptyset$ for all $n > \nu$ because if $g \in G_n$ then for all $j < n$ by the maximality of G_j there exists $h_j \in G_j$ such that $h_j V \cap gV \neq \emptyset$. This means $g^{-1}h_j V \cap V \neq \emptyset$ for $j = 1, \ldots, \nu$ and, of course, also $V \cap V \neq \emptyset$ (if V is empty the lemma is trivial). So we have found $\nu + 1$ distinct elements of G in the set $\{g \in G \colon gV \cap V \neq \emptyset\}$ against the hypothesis. $\qquad\square$

Appendix B. Auxiliary lemmata

We collect here, mainly for the sake of completeness and readers' convenience, some auxiliary lemmata of more or less folkloric nature.

Lemma B.1 (equisummability). *Suppose that $m_{k,i} \geq 0$, and:*

$$\lim_{k \to +\infty} \sum_i m_{k,i} = m,$$

$$\lim_k m_{k,i} = m_i$$

$$\lim_n \left(\sup_k \sum_{i=n}^{+\infty} m_{k,i} \right) = 0.$$

Then

$$\sum_i m_i = m.$$

Proof. For every $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ and $n \geq n_\varepsilon$ one has

$$\sum_{i=n}^{+\infty} m_{k,i} < \varepsilon, \quad \text{and} \quad \sum_{i<n} m_{k,i} \leq \sum_i m_{k,i} \leq \sum_{i<n} m_{k,i} + \varepsilon.$$

Hence either $m = +\infty$ and $m_i = +\infty$ for some $i < n_\varepsilon$, or $m \in \mathbb{R}$. In the first case the thesis follows. In the second letting $k \to +\infty$ we obtain for $n \geq n_\varepsilon$ the estimate

$$m - \varepsilon \leq \sum_{i=1}^n m_i \leq m.$$

Letting now $n \to +\infty$, one gets

$$m - \varepsilon \leq \sum_i m_i \leq m$$

for all $\varepsilon > 0$ implying the thesis. $\qquad\square$

Lemma B.2. *Suppose that for all $k \in \mathbb{N}$, $j = 1, \ldots, N$, where $N \in \mathbb{N} \cup \{\infty\}$ the sets $A_k \subset X$, $A_k^j \subset X$ be μ-measurable and*

$$\mu \left(A_k \triangle \bigcup_{j=1}^N A_k^j \right) = 0, \qquad \mu \left(A_k^j \cap A_k^i \right) = 0 \quad \text{whenever } i \neq j.$$

If $A_k \to A$ and $A_k^j \to A^j$ in $L_{\mathrm{loc}}^1(\mu)$ as $k \to \infty$, then

$$\mu \left(A \triangle \bigcup_{j=1}^N A^j \right) = 0, \qquad \mu \left(A^j \cap A^i \right) = 0 \quad \text{whenever } i \neq j.$$

Proof. It suffices to write for every compact $K \subset X$ the relationships

$$\mu \left(\left(A_k \triangle \bigcup_{j=1}^N A_k^j \right) \cap K \right) = \int_K \left| \mathbf{1}_{A_k}(x) - \left(\sum_{j=1}^N \mathbf{1}_{A_k^j}(x) \right) \right| d\mu(x),$$

$$\mu \left(\left(A_k^j \cap A_k^i \right) \cap K \right) = \int_K \mathbf{1}_{A_k^j}(x) \mathbf{1}_{A_k^i}(x) \, d\mu(x),$$

and pass to the limit as $k \to \infty$. $\qquad\square$

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