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Navier–Stokes equations: a new estimate of a possible gap related to the energy equality of a suitable weak solution

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Abstract The paper is concerned with the IBVP of the Navier-Stokes equations. The result of the paper is in the wake of analogous results obtained by the authors in previous articles Crispo et al. (Ricerche Mat 70:235–249, 2021). The goal is to estimate the possible gap between the energy equality and the energy inequality deduced for a weak solution.

Keywords Navier–Stokes equations · Weak solutions · Energy equality

Mathematics subject classification 35Q30 · 35B65 · 76D05

1 Introduction

This note concerns the 3D-Navier–Stokes initial boundary value problem:

$$v_t + v \cdot \nabla v + \nabla \pi_v = \Delta v + f,\ \nabla \cdot v = 0,\ \text{in } (0,T) \times \Omega,$$
$$v = 0 \text{ on } (0,T) \times \partial\Omega, v(0,x) = v_0(x) \text{ on } \{0\} \times \Omega. \tag{1}$$

In system (1) $\Omega \subseteq \mathbb{R}^3$ is assumed bounded or exterior, and its boundary is assumed smooth.

In the two recent papers [5, 6] the authors look for an energy equality for suitable weak solutions. Here, the term suitable is meant in the sense that a new solution is exhibited and not that an improvement is obtained to the one given in [3]. Actually, the crucial result of papers [5, 6] is the strong convergence in $L^p(0,T;W^{1,2}(\Omega))$, for all $T > 0$ and $p \in [1,2)$, of a sequence $\{v^m\}$ of smooth solutions to the "Leray's approximating Navier–Stokes Cauchy problem" (see (4) below), [11].

Since the strong convergence is not in $L^2(0,T;W^{1,2}(\Omega))$, the authors attempt to obtain the energy equality employing the (differential and integral) energy equality of the approximating solutions and some auxiliary functions. Actually, the approaches used so far allow to prove an energy equality which involves other quantities. Here it is proved that a suitable weak solution exists and satisfies the following relation

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2 d\tau + M(s,t) = \|v(s)\|_2^2 + \int_s^t (f,v)d\tau \text{ for all } 0 < s < t \in \mathcal{T}, \tag{2}$$

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where, thanks to the result of strong convergence in $L^p(0,T;W^{1,2}(\Omega))$, $p \in [1,2)$ (see Lemma 1),

$$\mathcal{T} := \left\{ t \in (0,T) \,:\, \|v^m(t)\|_{1,2} \to \|v(t)\|_{1,2} \right\}$$

is of full measure in (0, T) for all $T > 0$, and

$$M(s,t) := 2 \lim_{\alpha\to 1^-} \overline{\lim_m} \int\limits_{J^m(\alpha)} \|\nabla v^m(\tau)\|_2^2 \, d\tau$$
$$= \lim_{\alpha\to 1^-} \overline{\lim_m} \sum_{h\in\mathbb{N}(\alpha,m)} \left[\|v^m(s_h)\|_2^2 - \|v^m(t_h)\|_2^2 \right]$$

where $J^m(\alpha)$ is the union of, at most, a countable sequence ($\mathbb{N}(\alpha,m)$) of disjoint intervals $(s_h, t_h) \subset (s,t)$ and the following holds:

$$\lim_{\alpha\to 1^-} \frac{|J^m(\alpha)|}{1-\alpha} \le \frac{1}{\pi}\|v_0\|_2^2 + \frac{2}{\pi}\int\limits_0^t (f,v)d\tau\,,$$
$$\text{uniformly in } m \in \mathbb{N}.$$

Instead in the case of $s = 0$, one obtains

$$\|v(t)\|_2^2 + 2\int\limits_0^t \|\nabla v(\tau)\|_2^2 d\tau + M(0,t)$$
$$= \|v_0\|_2^2 + \int\limits_0^t (f,v)d\tau \text{ for all } t \in \mathcal{T}, \tag{3}$$

where

$$M(0,t) := \lim_{s_k\to 0} M(s_k,t)\,, \text{ for any } \{s_k\} \subset \mathcal{T}.$$

Roughly speaking the above intervals seem to contain the possible singular points S of the weak solution that, as is known, has $\mathcal{H}^{\frac{1}{2}}(S) = 0$ ($\mathcal{H}^a$ Hausdorff's measure), [16]. Of course, independently of the meaning of the conjecture for the intervals, from a physical view point the energy relation (2) would add a dissipative quantity which is not justifiable. If this is a necessary consequence of an initial datum only in L^2, then from a physical point of view it is a right reason to reject the L^2-class as a class of existence.

Also in [15] the author considers the possibility to add a further dissipative term to the right hand side of the classical energy inequality, but, as already stressed in [5], our result is different, since we obtain the equality (2) with $M(s, t)$ expressed only in terms of energy quantities ("kinetic or dissipated"). We think that this difference is of a special interest.

The proof of our result is based on a new existence theorem, where our weak solution is the limit of the sequence $\{v^m\}$ of solutions to problem (4). In addition to the usual weak convergences of $\{v^m\}$, there is the peculiarity that our weak solution is strong limit in $L^p(0,T;W^{1,2}(\Omega))$, for all $T > 0$ and $p \in [1,2)$. This result, proved for the first time in [5] (as far as we know it is also the unique known proof), is obtained under the minimal assumption of $v_0 \in L^2(\Omega)$ and divergence free. As already said, it is important in order to obtain that $\lim_m \|\nabla v^m(t) - \nabla v(t)\|_2 = 0$ almost every where in $t > 0$. This is a main difference with other results of existence of weak solutions, classical or more recent, as the ones furnished in [8] and in [9], obtained with stronger assumptions on the initial datum v_0.

By making the minimal requirement on v_0, from one hand we match the result[1] obtained in [13], and from another hand we better match the questions of counterexamples, as we remark below.

The validity of an energy equality, without requiring extra conditions, is interesting to better delimit the case of validity of possible counterexamples.

Actually, in the papers [2] and [1] two examples of non-uniqueness are furnished.

The former works for very-weak solutions, which are continuous in L^2-norm, but do not verify an energy inequality of the kind given by Leray-Hopf, in other words neglecting the term $M(s, t)$ with ≥ 0. Further, in the case of Leray-Hopf weak solutions their counterexample does not work.

[1] In this connection in paper [13], the so called Prodi-Serrin condition for the energy equality for a weak solution is not required on the whole interval of existence, but just on (ε, T), that is $L^4(\varepsilon,T;L^4(\Omega))$, for all $\varepsilon > 0$. This means that no extra assumption on the initial datum in L^2 is needed for the validity of the energy equality.

In [8], from a different point of view, the extra condition $L^4(\varepsilon,T;L^4(\Omega))$ is deduced for a special weak solution. Consequently, a local energy equality holds too.

Following the approach given in [10], under the same weaker extra assumption, the energy equality holds in the set of very-weak solutions.

The latter works with a homogeneous initial datum. Actually, the non-uniqueness is exhibited for solutions corresponding to a suitable data force, that, among other things, allows an energy equality.

The plan of the paper is the following. In Sect. 2 some preliminary lemmas are recalled and some new results of strong convergence are furnished. In Sect. 3 the statement and the proof of the chief result are performed.

2 Preliminary results

We set $J^{1,2}(\Omega)$:=completion of $\mathscr{C}_0(\Omega)$ in $W^{1,2}$-norm, where $\mathscr{C}_0(\Omega)$ is the set of the test functions of the hydrodynamics.

Definition 1 For weak solution to the IBVP (1) we mean a field $v : (0,\infty)\times\Omega \to \mathbb{R}^3$ such that for all $T > 0$

1. $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;J^{1,2}(\Omega))$,
2. the field v solves the integral equation

$$\int_s^t \Big[(v,\varphi_\tau) - (\nabla v,\nabla\varphi) + (v\cdot\nabla\varphi, v) + (\pi_v, \nabla\cdot\varphi)\Big] d\tau + (v(s),\varphi(s)) = (v(t),\varphi(t)),$$

for all $\varphi \in C_0^1([0,T)\times\Omega)$,

3. $\lim\limits_{t\to 0}\|v(t)-v_0\|_2 = 0\,.$

For our goals we consider a mollified Navier–Stokes system. Hence problem (1) becomes

$$\begin{aligned}&v_t^m + J_m[v^m]\cdot\nabla v^m + \nabla\pi_{v^m} = \Delta v^m + f,\ \nabla\cdot v^m = 0,\\ &\quad\text{in } (0,T)\times\Omega,\\ &v^m = 0 \text{ on } (0,T)\times\partial\Omega, v^m(0,x) = v_0^m(x) \text{ on } \{0\}\times\Omega,\end{aligned}\tag{4}$$

where $f \in L^2(0,T,L^2(\Omega))$, $\{v_0^m\} \subset J^{1,2}(\Omega)$ converges to v_0 in $J^2(\Omega)$ and $J_m[\cdot] \equiv \widetilde{J}_{\frac{1}{m}}[\cdot]$ where $\widetilde{J}_{\frac{1}{m}}[\cdot]$ is Friedrichs' (spatial) mollifier and we suppose that v^m is extended to zero in $\mathbb{R}^3 - \Omega$.

Lemma 1 *For all $m \in \mathbb{N}$ there exists a unique solution to problem (4) such that for all $T > 0$*

$$\begin{aligned}&\|v^m(t)\|_2^2 + 2\int_0^t \|\nabla v^m(\tau)\|_2^2\\ &\quad= \|v_0^m\|_2^2 + 2\int_0^t (f(\tau), v^m(\tau))d\tau\,,\ \text{for all } t > 0\,,\end{aligned}\tag{5}$$

$$v^m \in C([0,T);J^{1,2}(\Omega)) \cap L^2(0,T;W^{2,2}(\Omega))\,,$$
$$v_t^m, \nabla\pi^m \in L^2(0,T;L^2(\Omega))\,.$$

Moreover, the sequence $\{v^m\}$ is strong convergent to a limit v in $L^p(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;L^2(\Omega))$, for all $p \in [1,2)$, and the limit v is a weak solution to problem (1) with $(v(t),\varphi) \in C([0,T))$, for all $\varphi \in J^2(\Omega)$.

Proof This lemma for data force $f = 0$ is Theorem 6.1.1 proved in [5]. It is not difficult to image that the proof can be modified without difficulty assuming $f \neq 0$. So that we consider as achieved the proof of the lemma. □

Lemma 2 *Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in W^{2,2}(\Omega) \cap J^{1,2}(\Omega)$. Then there exists a constant c independent of u such that*

$$\|u\|_r \le c\|P\Delta u\|_2^a\|u\|_q^{1-a}, \qquad a\Big(\frac{1}{2}-\frac{2}{n}\Big) + (1-a)\frac{1}{q} = \tfrac{1}{r},\tag{6}$$

provided that $a \in [0,1)$.

Proof See [12, 14] . □

The following lemma furnishes an integrability property of derivatives with respect to t of the sequence $\{\|\nabla v^m\|_2\}$. This is made following the approach given in paper [5]. However, there are similar results directly concerning weak solutions. For the sake of completeness, we give the following references [4, 7, 17]. In any case, our proof is different from those given in the quoted papers.

Lemma 3 *For any $T > 0$, there exists a constant $M > 0$, not depending on m, such that*

$$\int_0^T \frac{\Big|\frac{d}{dt}\|\nabla v^m(t)\|_2^2\Big|}{\big(1+\|\nabla v^m\|_2^2\big)^2}\,dt \le M$$

where v^m is the solution of problem (4) stated in Lemma 1.

Proof By virtue of the regularity of (v^m, π^m) stated in (5), we multiply Eq. $(4)_1$ by $P\Delta v^m - v_t^m$. Integrating by parts on Ω, and applying the Hölder inequality, we get

$$\|P\Delta v^m - v_t^m\|_2^2 \le 2\|J_m[v^m]\cdot\nabla v^m\|_2^2 + 2\|f\|_2^2\,, \text{ a.e. in } t>0\,. \tag{7}$$

Applying inequality (6) with $r=\infty$ and $q=6$, by virtue of the Sobolev inequality, we obtain

$$\begin{aligned}\|J_m[v^m]\cdot\nabla v^m\|_2 &\le \|v^m\|_\infty\|\nabla v^m\|_2\\ &\le c\|P\Delta v^m\|_2^{\frac12}\|\nabla v^m\|_2^{\frac32}.\end{aligned} \tag{8}$$

By inequalities (7) and (8), we get

$$\begin{aligned}&\|P\Delta v^m - v_t^m\|_2^2 \le c\|P\Delta v^m\|_2\|\nabla v^m\|_2^3 + 2\|f\|_2^2\\ &\le \frac12\|P\Delta v^m\|_2^2 + c\|\nabla v^m\|_2^6 + 2\|f\|_2^2,\end{aligned} \tag{9}$$

for all $m\in\mathbb{N}$ and a.e. in $t>0$. Substituting in inequality (9) the identity

$$\frac{d}{dt}\|\nabla v^m\|_2^2 + \|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2 = \|P\Delta v^m - v_t^m\|_2^2 \tag{10}$$

and dividing by $(1+\|\nabla v^m(t)\|_2^2)^2$, we get the following estimate

$$\frac{\dot\rho_m}{(1+\rho_m)^2} + \frac{\frac12\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1+\rho_m)^2} \le c\rho_m + \frac{2\|f\|_2^2}{\left(1+\rho_m\right)^2}\,,$$

where we set $\rho_m(t) := \|\nabla v^m(t)\|_2^2$. Integrating on $(0,T)$ we have

$$\begin{aligned}&\frac{1}{1+\|\nabla v_0^m\|_2^2} - \frac{1}{1+\|\nabla v^m(T)\|_2^2}\\ &\qquad + \int_0^T \frac{\frac12\|P\Delta v^m\|_2^2 + \|v_t^m\|_2^2}{(1+\rho_m)^2}\,dt\\ &\le c\int_0^T \rho_m\,dt + 2\int_0^T \frac{2\|f\|_2^2}{\left(1+\rho_m\right)^2}\,dt \le C.\end{aligned}$$

It follows that

$$\int_0^T \frac{\|P\Delta v^m\|_2^2}{(1+\rho_m)^2}\,dt \le 2C+2, \quad \int_0^T \frac{\|v_t^m\|_2^2}{(1+\rho_m)^2}\,dt \le C+1.$$

Using the identity (10) we get

$$\begin{aligned}&\int_0^T \frac{\|P\Delta v^m - v_t^m\|_2^2}{(1+\rho_m)^2}\,dt = \int_0^T \frac{\frac{d}{dt}\rho_m}{(1+\rho_m)^2}\,dt + \int_0^T \frac{\|P\Delta v^m\|_2^2}{(1+\rho_m)^2}\,dt\\ &+ \int_0^T \frac{\|v_t^m\|_2^2}{(1+\rho_m)^2}\,dt\\ &\le \frac{1}{1+\|\nabla v_0^m\|_2^2} - \frac{1}{1+\|\nabla v^m(T)\|_2^2} + 3C + 3 \le 3C+4.\end{aligned}$$

Using once again identity (10) we get

$$\begin{aligned}\int_0^T \frac{\left|\frac{d}{dt}\rho_m\right|}{(1+\rho_m)^2}\,dt &\le \int_0^T \frac{\|P\Delta v^m - v_t^m\|_2^2}{(1+\rho_m)^2}\,dt\\ &+ \int_0^T \frac{\|P\Delta v^m\|_2^2}{(1+\rho_m)^2}\,dt + \int_0^T \frac{\|v_t^m\|_2^2}{(1+\rho_m)^2}\,dt\\ &\le 6C + 7 =: M.\end{aligned}$$

□

Lemma 4 *Let $\{h_m(t)\}$ be a sequence of non-negative functions bounded in $L^1(0,T)$. Also, assume that $h_m(t)\to h(t)$ a.e. in $t\in(0,T)$ with $h(t)\in L^1(0,T)$. Let be $g:(0,\alpha_0)\longrightarrow\mathbb{R}$ a continuous and strictly increasing function such that $\lim_{\alpha\to\alpha_0} g(\alpha)=+\infty$ and $p:[0,1)\times[0,\infty)\longrightarrow[0,1]$ a continuous function such that $p(\alpha,\rho)=1$ if $0\le\rho\le g(\alpha)$, $p(\alpha,\cdot)$ is weakly decreasing and $\lim_{\rho\to+\infty} p(\alpha,\rho)=0$ for any $\alpha\in(0,\alpha_0)$.*

Then we get

$$\lim_{\alpha\to\alpha_0}\lim_m \int_0^T h_m(t)p(\alpha,h_m(t))dt = \int_0^T h(t)dt\,, \tag{11}$$

Proof We have

$$\begin{aligned}&\int_0^T h_m(t)p(\alpha,h_m(t))dt\\ &= \int_0^T (h_m(t)-h(t))p(\alpha,h_m(t))dt\\ &\qquad + \int_0^T h(t)p(\alpha,h_m(t))\\ &=: I_1(\alpha,m) + I_2(\alpha,m)\,.\end{aligned}$$

We fix $\alpha \in (0, \alpha_0)$ and we consider the first integral. For any $\varepsilon \in (0, \alpha_0 - \alpha)$ we set

$$J_m^-(\varepsilon) = \{t : h_m(t) \le g(\alpha_0 - \varepsilon)\}, \quad J_m^+(\varepsilon) = \{t : g(\alpha_0 - \varepsilon) < h_m(t)\}. \tag{12}$$

Hence we have

$$\begin{aligned} I_1(\alpha, m) &= \int_0^T \chi_{J_m^-(\varepsilon)}(t)(h_m(t) - h(t))p(\alpha, h_m(t))dt \\ &+ \int_0^T \chi_{J_m^+(\varepsilon)}(t)(h_m(t) - h(t))p(\alpha, h_m(t))dt \\ &=: I_1^-(\alpha, m, \varepsilon) + I_1^+(\alpha, m, \varepsilon). \end{aligned}$$

By (12) we get

$$|\chi_{J_m^-(\varepsilon)}(t)(h_m(t) - h(t))p(\alpha, h_m(t))| \le g(\alpha_0 - \varepsilon) + |h(t)|$$

hence, by the dominated convergence theorem, we have

$$\lim_m I_1^-(\alpha, m, \varepsilon) = 0, \qquad \forall\, \alpha, \varepsilon. \tag{13}$$

Since $p(\alpha, \cdot)$ is decreasing, we get

$$\begin{aligned} &\left|\chi_{J_m^+(\varepsilon)}(t)(h_m(t) - h(t))p(\alpha, h_m(t))\right| \\ &\quad \le p(\alpha, g(\alpha_o - \varepsilon))\big(|h_m(t)| + |h(t)|\big). \end{aligned}$$

Using the boundedness of the sequence $\{h_m\}$ in L^1 we obtain that

$$|I_1^+(\alpha, m, \varepsilon)| \le cp(\alpha, g(\alpha_0 - \varepsilon)), \qquad \forall m \in \mathbb{N}. \tag{14}$$

By (13) and (14) we get

$$0 \le \overline{\lim_m} |I_1(\alpha, m)| \le cp(\alpha, g(\alpha_0 - \varepsilon)), \qquad \forall\, \alpha, \varepsilon.$$

Since $\lim_{\varepsilon \to 0} p(\alpha, g(\alpha_0 - \varepsilon)) = 0$ we have that

$$\lim_m I_1(\alpha, m) = 0, \qquad \forall\, \alpha.$$

Now we consider the integral $I_2(\alpha, m)$. Since $\left|p(\alpha, h_m(t))h(t)\right| \le 1$ and $\lim_m h_m(t) = h(t)$ a.e. in $t \in (0, T)$, by the dominated convergence theorem, we get

$$\lim_m I_2(\alpha, m) = \int_0^T h(t)p(\alpha, h(t))dt.$$

Finally, since $\lim_{\alpha \to \alpha_0} p(\alpha, h(t)) = 1$ we have that

$$\lim_{\alpha \to \alpha_0} \lim_m I_2(\alpha, m) = \int_0^T h(t)dt,$$

and this completes the proof. □

3 The chief result

We recall the definition

$$\mathcal{T} := \left\{t \in (0, T) : \|v^m(t)\|_{1,2} \to \|v(t)\|_{1,2}\right\}, \tag{15}$$

where $\{v^m\}$ is the sequence of solutions to problem (4). By virtue of the strong convergence stated in Lemma 1, the set $\mathcal{T}$ is certainly not empty and, as matter of fact, it is of full measure in $(0, T)$ for all $T > 0$.

Theorem 1 *Let v be the weak solution and $\{v^m\}$ the related approximating sequence stated in Lemma 1. Then, for all $t, s \in \mathcal{T}$, v satisfies the relation*

$$\begin{aligned} &\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2 d\tau + M(s, t) = \|v(s)\|_2^2 \\ &\quad + \int_s^t (f, v)d\tau\,, \end{aligned} \tag{16}$$

with

$$\begin{aligned} M(s, t) &:= 2 \lim_{\alpha \to 1^-} \overline{\lim_m} \int_{J^m(\alpha)} \|\nabla v^m(\tau)\|_2^2\, d\tau \\ &= \lim_{\alpha \to 1^-} \overline{\lim_m} \sum_{h \in \mathbb{N}(\alpha, m)} \left[\|v^m(s_h)\|_2^2 - \|v^m(t_h)\|_2^2\right] \end{aligned}$$

where, for a suitable positive α_0 depending on (s, t), for all $\alpha \in (\alpha_0, 1)$, $J^m(\alpha) \equiv \bigcup_{i \in \mathbb{N}(\alpha, m)} (s_i, t_i)$ with $\mathbb{N}(\alpha, m)$ which is, at most, a sequence of integers, and for all $i \in \mathbb{N}(\alpha, m)$ $(s_i, t_i) \subset (s, t)$ with $(s_i, t_i) \cap (s_j, t_j) = \emptyset$ for any $i \neq j$, and

$$\lim_{\alpha\to 1^-}\frac{|J^m(\alpha)|}{1-\alpha}\le\frac{1}{\pi}\|v_0\|_2^2 + \frac{2}{\pi}\int_0^t (f,v)d\tau\,,\ \text{uniformly with respect to } m\,. \tag{17}$$

Moreover, if $s=0$*, the relation* (16) *holds with* $M(0,t)=\lim_k M(s_k,t)$ *where* $\{s_k\}$ *is any sequence in* $\mathcal{T}$ *converging to* 0.

Proof We consider the sequence $\{v^m\}$ of solutions to problem (4) whose existence is ensured by Lemma 1. For all $m\in\mathbb{N}$ the Reynolds-Orr equation holds:

$$\frac{d}{d\tau}\|v^m(\tau)\|_2^2+2\|\nabla v^m(\tau)\|_2^2=(f,v^m)\,. \tag{18}$$

We set $\rho_m(t)\,:=\|\nabla v^m(t)\|_2^2$, and we consider

$$\alpha\in(0,1)\,,\ p(\alpha,\rho_m)\,:=\begin{cases}1 & \text{if } \rho_m\in[0,\tan\alpha\frac{\pi}{2}]\\ \frac{\frac{\pi}{2}-\arctan\rho_m}{(1-\alpha)\frac{\pi}{2}} & \text{if } \rho_m\in(\tan\alpha\frac{\pi}{2}\,,\infty)\end{cases}. \tag{19}$$

Fix $s,t\in\mathcal{T}$, with $s<t$, $\mathcal{T}$ given in (15) . Let α_1 be such that

$$\max\{\|\nabla v(s)\|_2^2,\|\nabla v(t)\|_2^2\}<\tan\alpha\frac{\pi}{2}\,,\ \text{for all } \alpha\in(\alpha_1,1)\,.$$

Hence, by virtue of the pointwise convergence, we claim the existence of m_0 such that

$$\max\{\|\nabla v^m(s)\|_2^2,\|\nabla v^m(t)\|_2^2\}<\tan\alpha\frac{\pi}{2}\,,\quad \text{for all } m\ge m_0 \text{ and } \alpha\in(\alpha_1,1)\,. \tag{20}$$

We set $A^m\,:=\max\limits_{[s,t]}\rho_m(t)$. We denote by

$$J^m(\alpha)\,:=\{\tau\,:\,\rho_m(\tau)\in(\tan\alpha\frac{\pi}{2},A^m]\}.$$

If $A_m\le\tan\alpha\frac{\pi}{2}$, then $J^m(\alpha)$ is an empty set. If $A_m>\tan\alpha\frac{\pi}{2}$ holds, since $\rho_m(s)<\tan\alpha\frac{\pi}{2}$, there exists the minimum $\bar{s}>s$ such that $\rho_m(\bar{s})=\tan\alpha\frac{\pi}{2}$, as well, being $\rho_m(t)<\tan\alpha\frac{\pi}{2}$, there exists the maximum $\bar{t}<t$ such that $\rho_m(\bar{t})=\tan\alpha\frac{\pi}{2}$. Thus, if $J^m(\alpha)$ is a non-empty set, by the regularity of $\rho_m(t)$, we get that $J^m(\alpha)$ is at most the union of a sequence of open interval (s_h,t_h) such that $\rho_m(s_h)=\rho_m(t_h)=\tan\alpha\frac{\pi}{2}$. We justify the claim.

The set $J^m(\alpha)$ is an open set, hence it is at most the countable union of maximal intervals (s_h,t_h). We set $E^m\,:=\overline{(s,t)-\bigcup_{h\in\mathbb{N}}(s_h,t_h)}$.

For all $\tau\in E^m$ we have $\rho_m(\tau)\le\tan\alpha\frac{\pi}{2}$, thus, by continuity of ρ_m, we get $\rho_m(s_h)=\tan\alpha\frac{\pi}{2}=\rho_m(t_h)$ for all $h\in\mathbb{N}$. For the measure of $J^m(\alpha)$ we get

$$|J^m(\alpha)|\tan\alpha\frac{\pi}{2}\le\int_{J^m(\alpha)}\rho_m(\tau)d\tau<\frac{1}{2}\|v(s)\|_2^2+\int_s^t(f,v)d\tau\,, \tag{21}$$

where we took the energy relation (18) into account and the strong convergence of the right-hand side too. Estimate (21) leads to (17). Recalling the definition of $p(\alpha,\rho_m(t))$, we have

$$\frac{d}{d\tau}p(\alpha,\rho_m(\tau))=\begin{cases}0 & \text{a.e. in } \tau\in E^m\,,\\ \dfrac{-2}{(1-\alpha)\pi}\dfrac{\dot\rho_m(\tau)}{1+(\rho_m(\tau))^2} & \text{for all } \tau\in J^m(\alpha)\,,\end{cases} \tag{22}$$

where we took into account that, for all $\alpha\in(0,1)$, function p is a Lipschitz's function in ρ_m, and $\rho_m(t)$ is a regular function in t. Hence, we get $p(\alpha,\rho_m(t))$ is a Lipschitz's function with respect to t. We multiply Eq. (18) for $p(\alpha,\rho_m(\tau))$, with $\alpha>\alpha_1$, and we integrate by parts on (s,t):

$$\|v^m(t)\|_2^2+2\int_s^t p(\alpha,\rho_m(\tau))\|\nabla v^m(\tau)\|_2^2d\tau+\frac{2A(t,s,m,\alpha)}{(1-\alpha)\pi}=\|v^m(s)\|_2^2+\int_s^t(f,v^m)p(\alpha,\rho_m(\tau))d\tau\,,$$

where we set

$$A(t,s,m,\alpha)\,:=\int_{J^m(\alpha)}\frac{\|v^m(\tau)\|_2^2}{1+\rho_m^2(\tau)}\dot\rho_m(\tau)d\tau$$

where we took (20) and definition of p into account. Letting $m\to\infty$ and $\alpha\to1$, by virtue of the pointwise convergence in s and in t, and Lemma 4, we arrive at

$$\|v(t)\|_2^2+M(s,t)+2\int_s^t\|\nabla v(\tau)\|_2^2d\tau=\|v(s)\|_2^2+\int_s^t(f,v)d\tau\,, \tag{23}$$

where we set

$$M(s,t) := \lim_{\alpha\to 1^-} \overline{\lim_m} \frac{2}{(1-\alpha)\pi} \int_{J^m(\alpha)} \frac{\|v^m(\tau)\|_2^2}{1+(\rho_m(\tau))^2} \dot{\rho}_m(\tau) d\tau \,.$$

Recalling the properties of $J^m(\alpha)$, for all α and m, integrating by parts, we get

$$\begin{aligned}\int_{J^m(\alpha)} \frac{\|v^m\|_2^2}{1+\rho_m^2}\dot{\rho}_m d\tau &= \sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2}{1+\rho_m^2}\dot{\rho}_m d\tau \\ &= \frac{\tan\alpha\frac{\pi}{2}}{1+\tan^2\alpha\frac{\pi}{2}} \sum_{h\in\mathbb{N}(\alpha,m)} \left[\|v^m(t_h)\|_2^2 - \|v^m(s_h)\|_2^2\right] \\ &+ \sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{2\rho_m^2}{1+\rho_m^2} d\tau - 2\sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\rho_m(f,v_m)}{1+\rho_m^2} d\tau \\ &+ 2\sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2\rho_m^2}{(1+\rho_m^2)^2}\dot{\rho}_m d\tau \,.\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}&\sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2}{1+\rho_m^2}\dot{\rho}_m d\tau \\ &- 2\sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\|v^m\|_2^2\rho_m^2}{(1+\rho_m^2)^2}\dot{\rho}_m d\tau + 2\sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{\rho_m(f,v_m)}{1+\rho_m^2} d\tau \\ &= \frac{\tan\alpha\frac{\pi}{2}}{1+\tan^2\alpha\frac{\pi}{2}} \sum_{h\in\mathbb{N}(\alpha,m)} \left[\|v^m(t_h)\|_2^2 - \|v^m(s_h)\|_2^2\right] \\ &+ \sum_{h\in\mathbb{N}(\alpha,m)} \int_{s_h}^{t_h} \frac{2\rho_m^2}{1+\rho_m^2} d\tau \,.\end{aligned} \tag{24}$$

We estimate the last integral. Let be

$$\widetilde{J}(\alpha) := \overline{\lim_m} J^m(\alpha) = \bigcap_{j=0}^{\infty} \bigcup_{m=j}^{\infty} J^m(\alpha). \tag{25}$$

It results that

$$\begin{aligned}&\tau \in \widetilde{J}(\alpha) \iff \exists\, m_k \to \infty \text{ s.t. } \tau \in J^{m_k}(\alpha) \\ &\quad \forall k \in \mathbb{N} \iff \overline{\lim_m} \chi_{J^m(\alpha)}(\tau) = 1.\end{aligned}$$

Hence, if $\tau \in \widetilde{J}(\alpha) \cap \mathcal{T}$ we get that

$$\rho(\tau) = \lim_{k\to\infty} \rho_{m_k}(\tau) \geq \tan\frac{\alpha\pi}{2}. \tag{26}$$

On the complement of the set $\mathcal{T}$ we can set $\rho = 0$, since the value on a null measure set does not change the estimates. Since $0 \leq \chi_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} \leq 1$, by Fatou's lemma, it follows that

$$\begin{aligned}&\frac{1}{1-\alpha}\overline{\lim_m} \int_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} d\tau = \frac{1}{1-\alpha}\overline{\lim_m} \int_s^t \chi_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} d\tau \\ &\leq \frac{1}{1-\alpha} \int_s^t \overline{\lim_m} \chi_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} d\tau \\ &= \frac{1}{1-\alpha} \int_s^t \chi_{\widetilde{J}(\alpha)} \frac{\rho^2}{1+\rho^2} d\tau = \frac{1}{1-\alpha} \int_{\widetilde{J}(\alpha)} \frac{\rho^2}{1+\rho^2} d\tau \\ &\leq \frac{1}{1-\alpha} \frac{1}{\tan\frac{\alpha\pi}{2}} \int_{\widetilde{J}(\alpha)} \rho(\tau)\, d\tau\end{aligned}$$

Since $\rho \in L^1$ and, by (26),

$$\left|\widetilde{J}(\alpha)\right| \leq \frac{\|\rho\|_1}{\tan\frac{\alpha\pi}{2}} \tag{27}$$

the last integral vanishes as α tends to 1^-. Moreover

$$\lim_{\alpha\to 1^-} \frac{1}{(1-\alpha)\tan\alpha\frac{\pi}{2}} = \frac{\pi}{2} \tag{28}$$

hence

$$\lim_{\alpha\to 1^-} \overline{\lim_m} \frac{1}{1-\alpha} \int_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} d\tau = 0. \tag{29}$$

Concerning the force term we have

$$\begin{aligned}&\left|\int_{J^m(\alpha)} \frac{\rho_m(f,v^m)}{1+\rho_m^2} d\tau\right| \leq \left(\int_{J^m(\alpha)} \frac{\rho_m^2}{(1+\rho_m^2)^2} d\tau\right)^{\frac{1}{2}} \\ &\left(\int_{J^m(\alpha)} \|f(\tau)\|_2^2 \|v^m(\tau)\|_2^2\, d\tau\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{1+(\tan\frac{\alpha\pi}{2})^2}\right)^{\frac{1}{2}} \left(\int_{J^m(\alpha)} \frac{\rho_m^2}{1+\rho_m^2} d\tau\right)^{\frac{1}{2}} \\ &\sup_{m,t} \|v^m(t)\|_2 \left(\int_{J^m(\alpha)} \|f(\tau)\|_2^2\, d\tau\right)^{\frac{1}{2}} \\ &\leq \frac{C}{\tan\frac{\alpha\pi}{2}} \left[\frac{(\tan\frac{\alpha\pi}{2})^2}{1+(\tan\frac{\alpha\pi}{2})^2} |J^m(\alpha)|\right]^{\frac{1}{2}} \leq \frac{C}{\tan\frac{\alpha\pi}{2}} \left[\frac{C}{\tan\frac{\alpha\pi}{2}}\right]^{\frac{1}{2}}.\end{aligned}$$

It follows that

$$\lim_{\alpha\to 1^-}\frac{1}{1-\alpha}\overline{\lim_m}\int_{J^m(\alpha)}\frac{\rho_m(f,v^m)}{1+\rho_m^2}\,d\tau=0. \tag{30}$$

Using algebraic manipulation we obtain the following relation:

$$\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\|v^m\|_2^2}{1+\rho_m^2}\dot\rho_m d\tau-2\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\|v^m\|_2^2\rho_m^2}{(1+\rho_m^2)^2}\dot\rho_m d\tau$$
$$=-\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\|v^m\|_2^2}{1+\rho_m^2}\dot\rho_m d\tau+2\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\|v^m\|_2^2}{(1+\rho_m^2)^2}\dot\rho_m d\tau\,.$$

Substituting the above relation in Eq. (24) we get

$$-\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\|v^m\|_2^2}{1+\rho_m^2}\dot\rho_m d\tau+2\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\|v^m\|_2^2}{(1+\rho_m^2)^2}\dot\rho_m d\tau$$
$$+\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{\rho_m(f,v_m)}{1+\rho_m^2}\,d\tau$$
$$=\frac{\tan\alpha\frac{\pi}{2}}{1+\tan^2\alpha\frac{\pi}{2}}\sum_{h\in\mathbb{N}(\alpha,m)}\Big[\|v^m(t_h)\|_2^2-\|v^m(s_h)\|_2^2\Big]$$
$$+\sum_{h\in\mathbb{N}(\alpha,m)}\int_{s_h}^{t_h}\frac{2\rho_m^2}{1+\rho_m^2}d\tau \tag{31}$$

At last we estimate the integral

$$\left|\int_{J^m(\alpha)}\frac{\|v^m\|_2^2}{(1+\rho_m^2)^2}\dot\rho_m d\tau\right|\le\int_{J^m(\alpha)}\frac{\|v^m\|_2^2}{(1+\rho_m^2)^2}|\dot\rho_m|\,d\tau$$
$$\le\sup_{t,m}\|v^m(t)\|_2^2\int_{J^m(\alpha)}\frac{|\dot\rho_m|}{(1+\rho_m^2)^2}\,d\tau$$
$$\le c\,\frac{1}{1+(\tan\frac{\alpha\pi}{2})^2}\int_{J^m(\alpha)}\frac{|\dot\rho_m|}{1+\rho_m^2}\,d\tau$$
$$\le\frac{2c}{1+(\tan\frac{\alpha\pi}{2})^2}\int_{J^m(\alpha)}\frac{|\dot\rho_m|}{(1+\rho_m)^2}\,d\tau\le\frac{2cM}{1+(\tan\frac{\alpha\pi}{2})^2} \tag{32}$$

where the last inequality follows by Lemma 3. Hence, by (28), we get

$$\lim_{\alpha\to 1^-}\overline{\lim_m}\left|\frac{2}{1-\alpha}\int_{J^m(\alpha)}\frac{\|v^m\|_2^2}{(1+\rho_m^2)^2}\dot\rho_m d\tau\right|$$
$$\le\lim_{\alpha\to 1^-}\frac{2}{1-\alpha}\frac{2cM}{1+(\tan\frac{\alpha\pi}{2})^2}=0.$$

Multiplying Eq. (31) by $\frac{2}{(1-\alpha)\pi}$ and passing to the limit using (32), (30) and (29), we get

$$M(s,t)=\lim_{\alpha\to 1^-}\overline{\lim_m}\sum_{h\in\mathbb{N}(\alpha,m)}\Big[\|v^m(s_h)\|_2^2-\|v^m(t_h)\|_2^2\Big].$$

By Eq. (18) we get

$$\sum_{h\in\mathbb{N}(\alpha,m)}\Big[\|v^m(t_h)\|_2^2-\|v^m(s_h)\|_2^2\Big]=$$
$$-2\int_{J^m(\alpha)}\|\nabla v^m(\tau)\|_2^2\,d\tau+\int_{J^m(\alpha)}(f,v^m)\,d\tau.$$

Let us consider the last integral. Since $|(f(\tau),v^m(\tau))|\le\|f(\tau)\|_2\|v^m(\tau)\|_2\le c\|f(\tau)\|_2$ we can apply the Fatou's lemma to get

$$\overline{\lim_m}\left|\int_{J^m(\alpha)}(f,v^m)\,d\tau\right|=\overline{\lim_m}\left|\int_s^t\chi_{J^m(\alpha)}(f,v^m)\,d\tau\right|$$
$$\le\int_s^t\overline{\lim_m}\left|\chi_{J^m(\alpha)}(f,v^m)\right|d\tau$$
$$\le c\int_s^t\|f\|_2\overline{\lim_m}\,\chi_{J^m(\alpha)}\,d\tau\le c\int_s^t\|f\|_2\chi_{\widetilde J(\alpha)}\,d\tau$$

with $\widetilde J(\alpha)$ defined in (25). Since $\|f(\tau)\|_2$ is summable, considering (27), we get

$$\lim_{\alpha\to 1^-}\overline{\lim_m}\left|\int_{J^m(\alpha)}(f,v^m)\,d\tau\right|=0$$

and this completes the proof in the case of $s,t\in\mathcal{T}$. In order to complete the proof of the theorem, we limit ourselves to remark that, letting $s\to 0$, the left-hand side tends to values in 0, in particular on any sequence $\{s_k\}\subset\mathcal{T}$ letting to 0, and as a consequence the limit on $\{s_k\}$ of the right hand side is well posed.

□

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