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Weakly binary expansions of dense meet-trees

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We compute the domination monoid in the theory DMT of dense meet-trees. In order to show that this monoid is well-defined, we prove *weak binarity* of DMT and, more generally, of certain expansions of it by binary relations on sets of open cones, a special case being the theory DTR from [7]. We then describe the domination monoids of such expansions in terms of those of the expanding relations.

If asked what a *tree* is, a mathematician has a number of options to choose from. The graph theorist's answer will probably contain the words "acyclic" and "connected", while the set theorist may have in mind certain sets of sequences of natural numbers. In this paper we are instead concerned with *lower semilinear orders*: posets where the set of predecessors of each element is linearly ordered.

More specifically, a *meet-tree* is a lower semilinear order $<$ in which each pair of elements a, b has a greatest common lower bound, their *meet* $a \sqcap b$. When viewed as $\{<, \sqcap\}$-structures, finite meet-trees form an amalgamation class, hence have a Fraïssé limit, the universal homogeneous countable meet-tree. Its complete first-order theory DMT is that of *dense meet-trees*: dense lower semilinear orders with meets and everywhere infinite ramification.

Such structures have received a certain amount of model-theoretic attention in the recent (and not so recent) past. They appear in the classification of countable 2-homogeneous trees from [5], and have since been important in the theory of permutation groups; cf., e.g., [1, 3, 4]. More recently, they were shown to be dp-minimal in [17], and the automorphism group of the unique countable one was studied in [10], while the interest in similar structures goes back at the very least to [14, 20], where they were used as a base to produce examples in the context of Ehrenfeucht theories. Here we study DMT, and some of the expansions defined in [7], from the viewpoint of domination, in the sense of [13].

One motivation for such a study comes from valuation theory. The nonzero points of a valued field K can notoriously be identified with the *branches* of a meet-tree, that is, its maximal linearly ordered subsets. This identification is used, for instance, to endow K with a C-relation; cf. [9, 11]. Viewing the residue field k of K as a set of open valuation balls yields a correspondence between k and, for an arbitrary but fixed point g of the underlying tree, the set of *open cones* above g: the equivalence classes of the relation $E(x, y) := x \sqcap y > g$ defined on the set of points above g. If k is pseudofinite, then it interprets a structure elementarily equivalent to the Random Graph (cf. [2, 6][1]); it is therefore interesting to study the theory of a dense meet-tree with a Random Graph structure on each set of open cones above a point. This theory was used in [7], where it was dubbed DTR, to show that restrictions to nonforking bases need not preserve NIP.

Another motivation is rooted in the study of *invariant types*: types over a saturated model $\mathfrak{U}$ of a first order theory which are fixed, under the natural action of $\operatorname{Aut}(\mathfrak{U})$ on the space $S(\mathfrak{U})$ of types, by the stabiliser of some small set. The space of invariant types is a semigroup when equipped with the tensor product, and can be endowed with the preorder of *domination*, where a type $p(x)$ dominates a type $q(y)$ iff $q(y)$ is implied by the union of $p(x)$ with a small type $r(x, y)$ consistent with $p(x) \cup q(y)$. The induced equivalence relation, called *domination-equivalence*, may or may not be a congruence with respect to the tensor product, and some conditions ensuring

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[1] Beyarslan [2] proves a more general version for generic hypergraphs, and attributes the graph version to Duret [6].

this to be the case were isolated in [13]. One of the main results in the present work is a proof that one of them, *weak binarity* (Definition 2.1), is satisfied by DMT, and by certain expansions of the latter by binary structures on sets of open cones, a special case of which is DTR. Therefore the tensor product induces a well-defined operation on the quotient, yielding the *domination monoid* $\widetilde{\mathrm{Inv}}(\mathfrak{U})$, which we then calculate.

The paper is structured as follows. After briefly reviewing standard definitions and facts about dense meet-trees and invariant types in § 1, we recall in § 2 the definition of *weak binarity* and prove that, despite not being binary, DMT and all of its *binary cone-expansions* (Definition 2.5) are weakly binary. This is in particular the case for DTR.

Theorem A (Theorem 2.8) *The theory of dense meet-trees is weakly binary, and so is each of its binary cone-expansions.*

Hence the monoid $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined in such theories, and we proceed to compute it. The case of pure dense meet-trees is handled in § 3.

Theorem B (Theorem 3.14) *If $\mathfrak{U}$ is a monster model of the theory of dense meet-trees, then there is a set X such that*

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup) \times \bigoplus_{g\in\mathfrak{U}} (\mathbb{N}, +),$$

where $(\mathscr{P}_{\mathrm{fin}}(X), \cup)$ is the upper semilattice of finite subsets of X.

In the same section, we take the opportunity to record an instance of a theory where domination differs from $\mathrm{F}^{\mathrm{s}}_{\kappa}$-isolation in the sense of Shelah, Example 3.4. Theorem 3.14 is generalised in § 4 to *purely binary cone-expansions* (Definition 2.5), such as DTR.

Theorem C (Theorem 4.13) *Let $\mathfrak{U}$ be a monster model of a purely binary cone-expansion of* DMT *and, for $g \in \mathfrak{U}$, denote by O_g the structure on the set of open cones above g. Then there is a set X such that*

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup) \times \bigoplus_{g\in\mathfrak{U}} \widetilde{\mathrm{Inv}}(O_g).$$

1 Preliminaries

In what follows, lowercase Latin letters may denote finite tuples of variables or elements of a model. The length of a tuple is denoted by $|\cdot|$, and its coordinates will be denoted by subscripts, starting with 0; we may write, e.g., $a = (a_0, \dots, a_{|a|-1}) \in M^{|a|}$ or, with abuse of notation, simply $a \in M$. Concatenation is denoted by juxtaposition, and elements of a sequence of tuples by superscripts. For instance, if we write $a = a^0 a^1$ then $a_{|a^0|}$ equals a^1_0, the first element of a^1. Tuples may be treated as sets, in which case juxtaposition denotes union, as in $Ab = A \cup \{b_i \mid i < |b|\}$. *Type* means "complete type in finitely many variables".

Proofs regarding trees have a tendency to split in cases and subcases. As they become much easier to follow if the objects in them are drawn as soon as they appear in the proof, the reader is encouraged to reach for writing devices, preferably capable of producing different colours.

1.1 Invariant types

Fix a complete first-order theory T with infinite models, a sufficiently large cardinal κ, and a κ-saturated and κ-strongly homogeneous $\mathfrak{U} \vDash T$. *Small* means "of cardinality strictly less than κ"; if A is a small subset of $\mathfrak{U}$, we denote this by $A \subset^+ \mathfrak{U}$, or $A \prec^+ \mathfrak{U}$ if additionally $A \prec \mathfrak{U}$. *Global type* means "type over $\mathfrak{U}$".

Definition 1.1 1. Let $A \subseteq B$. A type $p(x) \in S(B)$ is *A-invariant* iff for all $\varphi(x; y) \in L$ and $a \equiv_A b$ in $B^{|y|}$ we have $p(x) \vdash \varphi(x; a) \leftrightarrow \varphi(x; b)$. A global type $p(x) \in S(\mathfrak{U})$ is *invariant* iff it is A-invariant for some $A \subset^+ \mathfrak{U}$. Such an A is called a *base* for p.

2. If $p(x) \in S(\mathfrak{U})$ is A-invariant and $\varphi(x; y) \in L(A)$, write
$$(d_p\varphi(x; y))(y) := \{\mathrm{tp}_y(b/A) \mid \varphi(x; b) \in p, b \in \mathfrak{U}\}.$$
The map $\varphi \mapsto d_p\varphi$ is called the *defining scheme* of p over A.
3. We denote by $S^{\mathrm{inv}}_x(\mathfrak{U}, A)$ the space of global A-invariant types in variables x, with A small, and by $S^{\mathrm{inv}}_x(\mathfrak{U})$ the union of all $S^{\mathrm{inv}}_x(\mathfrak{U}, A)$ as A ranges among small subsets of $\mathfrak{U}$. Denote by $S(B)$ the union of all spaces of types over B in finite tuples of variables; similarly for, say, $S^{\mathrm{inv}}(\mathfrak{U})$.

If we say that a type p is invariant, and its domain is not specified and not clear from context, it is usually a safe bet to assume that $p \in S(\mathfrak{U})$. Similarly if we say that a tuple has invariant type without specifying over which set.

1.2 Dense meet-trees

A poset $(M, <)$ is a *lower semilinear order* iff every pair of elements from each set of the form $\{x \in M \mid x < a\}$ is comparable. Let $L_{\mathrm{mt}} = \{<, \sqcap\}$, where $<$ is a binary relation symbol and $\sqcap$ is a binary function symbol. A *meet-tree* is an L_{mt}-structure M such that $(M, <)$ is a lower semilinear order where every pair of elements a, b has a greatest common lower bound, their *meet* $a \sqcap b$. If M is a meet-tree and $g \in M$, classes of the equivalence relation defined on $\{x \in M \mid x > g\}$ by $E(x, y) := x \sqcap y > g$ are called *open cones above* g.

Finite meet-trees are well-known to form a Fraïssé class, hence have a Fraïssé limit, whose theory is complete and eliminates quantifiers.[2] A *dense meet-tree* is a model of the theory DMT of the Fraïssé limit of finite meet-trees. The following fact is well-known, but I could not find a reference including a proof. It can be proven by an easy back-and-forth argument.

Fact 1.2 *The theory* DMT *of dense meet-trees is axiomatised by saying that*

1. $(M, <, \sqcap)$ *is a meet-tree;*
2. *for every $a \in M$, the structure $(\{x \in M \mid x < a\}, <)$ is a dense linear order with no endpoints; and*
3. *for every $g \in M$, there are infinitely many open cones above g.*

The following remark will be used throughout, sometimes tacitly.

Remark 1.3 The operation $\sqcap$ is associative, idempotent, and commutative. Using this and quantifier elimination, and observing, e.g., that for every a, b the set defined by $x \sqcap a = b$ is either empty or infinite, it is easy to see that in DMT the definable closure $\mathrm{dcl}(A)$ of a set A coincides with its closure under meets. In particular, if A is finite, then so is $\mathrm{dcl}(A)$: by the properties of $\sqcap$ we just pointed out, its size cannot exceed that of the powerset of A.[3]

When working in expansions of DMT, we will denote the closure of a set A under meets by $\mathrm{dcl}^{L_{\mathrm{mt}}}(A)$. This is justified by the previous remark.

Definition 1.4 Define the *cut* C_p of a type $p(x) \in S_1(M)$ to be $\{c \in M \mid p \vdash x \geq c\}$ and the cut in M of an element b of some elementary extension of M to be $C^M_b := C_{\mathrm{tp}(b/M)}$. We say that C_p is *bounded* iff it is bounded from above in M. A *cut* is the cut of some type.

Equivalently, a cut is a linearly ordered subset which is downward closed. This usage of the word "cut" is a bit more general than the one traditionally used for linear orders: our cuts have no upper part, only a lower one.

It can be shown by standard techniques that DMT is NIP, in fact dp-minimal (cf. [17, Proposition 4.7]). This makes it amenable to an analysis of invariant types using indiscernible sequences, and it turns out that invariant 1-types are necessarily of one of the six kinds below, as shown by using *eventual types* (cf. [19, § 2.2.3]). We refer the reader to [17] and [19, § 2.3.1]. Alternatively, it is possible to prove this directly via quantifier elimination by considering, for a fixed $p(x) \in S^{\mathrm{inv}}_1(\mathfrak{U})$, what are the possible values of each $d_p\varphi$, as $\varphi(x; y)$ ranges among L-formulas.

[2] For basic Fraïssé theory, cf., e.g., [8, Ch. 7].
[3] While sufficient for our purposes, this upper bound is very far from optimal: one can show that $|\mathrm{dcl}(A)| \leq 2|A|$. Cf. [7, Remark 4.6].

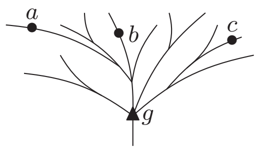

Figure 1 The point a is in the same open cone above g as the point b, while c is in a different open cone above g.

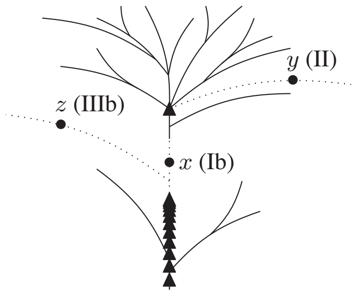

Figure 2 Some nonrealised B-invariant types, where points of B are denoted by triangles. In this picture, the set of triangles below x has no maximum, solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 \mathrel{^{+}\!\!\succ} \mathfrak{U}$. The type of x is of kind (Ib), that of y of kind (II), and that of z of kind (IIIb).

Definition 1.5 Let $\mathfrak{U} \vDash \mathsf{DMT}$ and $p(x) \in S_1(\mathfrak{U})$. We say that p is of kind

(0) iff p is realised by some $a \in \mathfrak{U}$;

(Ia) iff there is a small (nonempty) linearly ordered set $A \subset^+ \mathfrak{U}$ such that $p(x) \vdash \{x < a \mid a \in A\} \cup \{x > b \mid b \in \mathfrak{U}, b < A\}$;

(Ib) iff there is a small linearly ordered set $A \subset^+ \mathfrak{U}$ with no maximum such that $p(x) \vdash \{x > a \mid a \in A\} \cup \{x < b \mid b \in \mathfrak{U}, b > A\}$, or there are a and c in $\mathfrak{U}$ such that $p(x) \vdash \{a < x < c\} \cup \{x < b \mid b \in \mathfrak{U}, a < b < c\}$;

(II) iff there is $g \in \mathfrak{U}$ such that $p(x) \vdash \{x > g\} \cup \{x \sqcap b = g \mid b \in \mathfrak{U}, b > g\}$;

(IIIa) iff $p(x) \vdash \{x \not\leq b \mid b \in \mathfrak{U}\}$ and there is $d \in \mathfrak{U}$ such that, for $e \vDash p$, the type $\operatorname{tp}(e \sqcap d/\mathfrak{U})$ is of kind (Ia);

(IIIb) iff $p(x) \vdash \{x \not\leq b \mid b \in \mathfrak{U}\}$ and there is $d \in \mathfrak{U}$ such that, for $e \vDash p$, the type $\operatorname{tp}(e \sqcap d/\mathfrak{U})$ is of kind (Ib).

Fact 1.6 *If $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$, then p is of one of the six kinds above. Let A' witness that p is of one of these kinds.*[4] *Then p is A'-invariant, and is uniquely determined by its kind and the data in the corresponding part of Definition 1.5.*

So types of kind (0), (Ia), or (Ib) correspond to cuts in a linearly ordered subset of the tree, where in kind (Ib), if the cut of p has a maximum a, we are specifying an existing open cone above a. Kinds (II), (IIIa), and (IIIb) are the corresponding "branching" versions. Types of kind (II) are the types of elements in a new open cone above an existing point. Cf. Figure 2.

We conclude this section by recording some easy observations for later use.

Lemma 1.7 *1. Let $b_0, b_1 \in N \succ M$. If $C^M_{b_0} \subseteq C^M_{b_1}$ then $C^M_{b_0 \sqcap b_1} = C^M_{b_0}$. If none of $C^M_{b_0}$ and $C^M_{b_1}$ is included in the other, then $b_0 \sqcap b_1 \in M$.*

2. For all $b_0, \dots, b_n \in N \succ M$, and every $e \in \operatorname{dcl}(Mb_0, \dots, b_n)$, either $e \in M$ or there is i such that $e \leq b_i$ and $C^M_e = C^M_{b_i}$.

3. If $p \in S_1^{\mathrm{inv}}(\mathfrak{U})$ then C_p is bounded.

P r o o f . The first part is clear from the definitions of cut and meet, the second one follows by induction and Remark 1.3, and the last one follows from Fact 1.6. □

[4] In the notation of Definition 1.5, we can take as A' either a, A, Ac, ac, g, Ad, or Acd, where, for kinds (IIIa)/(IIIb), A, Ac, or ac is a witness that, for $e \vDash p$, $\operatorname{tp}(e \sqcap d/\mathfrak{U})$ is of kind (Ia)/(Ib).

2 Weak binarity

The main result of this section, Theorem 2.8, states that certain expansions of DMT are *weakly binary*. It applies for instance to the theory DTR from [7], obtained by equipping every set of open cones above a point with a structure elementarily equivalent to the Random Graph.

Recall that a theory T is *binary* iff every formula is equivalent modulo T to a Boolean combination of formulas with at most two free variables. Equivalently, for every set of parameters B and tuples a, b,

$$\mathrm{tp}(a/B) \cup \mathrm{tp}(b/B) \cup \mathrm{tp}(ab/\emptyset) \vdash \mathrm{tp}(ab/B).$$

Natural examples of such theories are those which eliminate quantifiers in a binary relational language. On the other hand, binary *function* symbols are usually an obstruction to binarity, as they can be used to write atomic formulas with an arbitrary number of free variables.

This is for instance the case for DMT, whose language contains the binary function symbol $\sqcap$: it is easy to see that DMT is not binary, nor is any of its expansions by constants. Even though DMT is known to be *ternary* (cf. [17, Corollary 4.6]), this is not sufficient for our purposes: the theory from [13, Proposition 2.3] where $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is not well-defined is ternary as well.

Definition 2.1 A theory is *weakly binary* iff, for all monster models $\mathfrak{U}$ and all a, b such that $\mathrm{tp}(a/\mathfrak{U})$ and $\mathrm{tp}(b/\mathfrak{U})$ are invariant, there is $A \subset^{+} \mathfrak{U}$ such that

$$\mathrm{tp}(a/\mathfrak{U}) \cup \mathrm{tp}(b/\mathfrak{U}) \cup \mathrm{tp}(a, b/A) \vdash \mathrm{tp}(a, b/\mathfrak{U}). \qquad (\dagger)$$

Weak binarity was introduced in [13] as a sufficient condition for well-definedness of the domination monoid.[5] The class of weakly binary theories clearly contains any theory which happens to have an expansion by constants which is binary. Examples with no binary expansion by constants include the theory of a generic equivalence relation where every equivalence class carries a circular order (cf. [12, Example 2.2.13]) and, as we will shortly see, DMT.

It follows immediately from the definitions that, in every theory, if $p \in S(\mathfrak{U})$ is invariant then each of its 1-subtypes, that is, each of the restrictions of p to one of its variables, is invariant as well. It is easily seen, say by using [13, Lemma 1.27] and induction, that if T is weakly binary then the converse holds as well. We record this here for later reference.

Remark 2.2 Let T be weakly binary and $p \in S(\mathfrak{U})$. Then p is invariant if and only if every 1-subtype of p is invariant.

Before returning to trees, note that this converse is in general false. For example, in the theory of Divisible Ordered Abelian Groups, let $p(x_0, x_1)$ be a 2-type prescribing x_0, x_1 to be larger than $\mathfrak{U}$, and such that both the cofinality of $\{d \in \mathfrak{U} \mid p \vdash x_0 - x_1 > d\}$ and the coinitiality of its complement are not small. Then p is not invariant, even if both of its 1-subtypes are.

Notation 2.3 *We write $x \parallel y$ to mean that $x \nleq y$ and $y \nleq x$.*

Lemma 2.4 *Let $M \vDash$ DMT and let b be a finite tuple. There is a finite tuple d such that Mbd is closed under meets. Moreover, d can be chosen such that additionally, if we let $c := M \cap d$, then $bd = \mathrm{dcl}(bc)$, and for every $e \in bd \setminus M$ such that C_e^M is bounded, the following happens.*

1. *There is $a_e \in c$ such that $a_e > C_e^M$.*
2. *If C_e^M has a maximum g and e is in an existing open cone above g, then this is the open cone of a_e.*

P r o o f . Define a tuple a as follows. If $C_{b_i}^M$ is not bounded, choose a_i to be an arbitrary point of M (or, if the reader prefers, leave a_i undefined). If $C_{b_i}^M$ has a maximum g and b_i is in an open cone above g which intersects M, let $a_i \in M$ be such that $a_i \sqcap b_i > g$ (cf. first half of Figure 3); otherwise (second half of the same figure), choose any $a_i > C_{b_i}^M$. Note that the closure $\mathrm{dcl}(ba)$ of ba under meets is finite by Remark 1.3, and let d be a tuple enumerating $\mathrm{dcl}(ba) \setminus b$. Recall that we defined $c := M \cap d$, and note that, by construction, $bd = \mathrm{dcl}(bc)$.

[5] The quantification on $\mathfrak{U}$ is not explicit in [13].

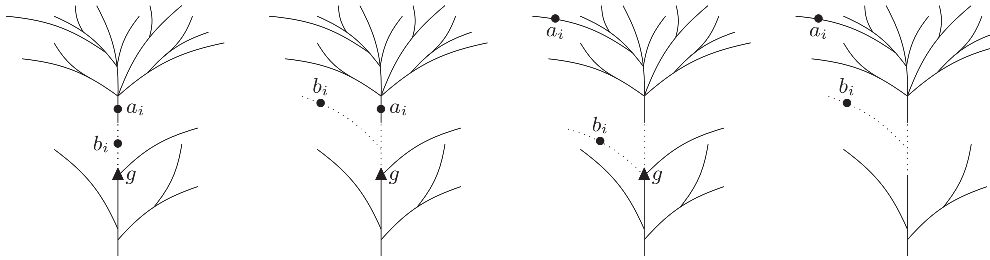

Figure 3 How to choose a_i in the proof of Lemma 2.4. In the first three pictures, $C^M_{b_i}$ has a maximum, g, denoted by a triangle. In the last picture it does not have one. Solid lines lie in M, and dotted lines lie in a bigger $M_1 \succ M$.

We now prove the "moreover" part, and then show how closure under meets of Mbd follows. Let $e \in bd \setminus M$ have bounded cut. By Lemma 1.7, construction, and the fact that $e \notin M$, there is $i < |b|$ such that $e \le b_i$ and $C^M_e = C^M_{b_i}$.

(1) Let i be as above. Since $C^M_e = C^M_{b_i}$, we have $a_e := a_i > C^M_e$.

(2) Let i and a_e be as above. By choice of $a_i = a_e$, we have $a_i \sqcap b_i > g$. By construction and the fact that $e \notin M$, we have $g < e \le b_i$, so $e \sqcap b_i = e > g$ and e and b_i are in the same open cone above g, which is that of a_i. This completes the proof of (2), hence of the "moreover" part.

We are left to prove that Mbd is closed under meets. As both M and bd are, and $\sqcap$ is commutative, all we need to show is that if $e \in bd \setminus M$ and $f \in M$ then $f \sqcap e \in Mbd$. If e and f and comparable there is nothing to prove, so assume they are not, i.e., that $e \parallel f$. If C^M_e is unbounded, neither of C^M_e and C^M_f is included in the other, because $f \in M$ and $e \parallel f$. Hence, by the first point of Lemma 1.7, we have $e \sqcap f \in M$. Assume now that C^M_e is bounded.

Claim *To conclude, it is enough to show that $f \sqcap e \le f \sqcap a_e$.*

P r o o f o f C l a i m . By assumption, commutativity, and idempotency of $\sqcap$ we have $f \sqcap e = (f \sqcap e) \sqcap (f \sqcap a_e) = (f \sqcap a_e) \sqcap (a_e \sqcap e)$. Since $f \sqcap a_e$ and $a_e \sqcap e$ are both predecessors of a_e they are comparable, so their meet is one of them. But $a_e \sqcap e \in bd$ and $f \sqcap a_e \in M$, so $f \sqcap e \in Mbd$. □

We prove that $f \sqcap e \le f \sqcap a_e$ by cases. Note that, since $f \sqcap a_e$ and $f \sqcap e$ are both predecessors of f, they are comparable.

1. If $f > C^M_e$ then $C^M_{f \sqcap e} = C^M_e$. Suppose additionally that $f \sqcap a_e > C^M_e = C^M_{f \sqcap e}$. Since $f \sqcap a_e \in M$, having $f \sqcap a_e \le f \sqcap e$ would contradict $f \sqcap a_e > C^M_{f \sqcap e}$, and therefore $f \sqcap e < f \sqcap a_e$.
2. If $f > C^M_e$ and we are not in the previous case, then C_e has a maximum g and $f \sqcap a_e = g$, i.e., f and a_e are in different open cones above g. Now, e can be either in the same open cone as a_e, or in a new one, but in both cases $f \sqcap e = g = f \sqcap a_e$.
3. If $f \not> C^M_e$ then there is $h \in C^M_e$ such that $f \not> h$, and then $f \sqcap h = f \sqcap (h \sqcap e) = f \sqcap e$. As $a_e > C^M_e$ in particular $a_e > h$, hence by definition of meet we must have $f \sqcap a_e = f \sqcap h = f \sqcap e$. □

We introduce the following notion, motivated by [7, § 4.3].

Definition 2.5 A *binary cone-expansion* of DMT is a theory T in a language $L = L_{\mathrm{mt}} \cup \{R_j, P_{j'} \mid j \in J, j' \in J'\}$ satisfying the following properties.

1. Every $P_{j'}$ is a unary relation symbol; every R_j is a binary relation symbol.
2. T is a completion of DMT and eliminates quantifiers in L.
3. Every R_j is *on open cones*, in the sense that
   (a) $R_j(x, y) \to x \parallel y$, and
   (b) if $x \parallel y$ and x', y' are such that $x \sqcap x' > x \sqcap y$ and $y \sqcap y' > x \sqcap y$ then $R_j(x, y) \leftrightarrow R_j(x', y')$.

If additionally $J' = \emptyset$ we say that T is a *purely binary cone-expansion* of DMT.

Example 2.6 One example of a purely binary cone-expansion of DMT is DTR, axiomatised by taking $J = \{R\}$, $J' = \emptyset$, and saying that, for all g, the structure induced by R on the (imaginary sort of) open cones above g is elementarily equivalent to the Random Graph. Cf. [7] for DTR, and for a more general analysis of theories of trees with relations on sets of open cones.

Example 2.7 Another theory examined in [7], called DTE_2, is defined similarly to DTR, but instead of the Random Graph it uses the Fraïssé limit of all finite structures with two equivalence relations. More generally, one can define DTE_n in an analogous fashion. The results of this paper apply to these theories as well even if, strictly speaking, they do not satisfy Definition 2.5, since the latter requires the R_j to be irreflexive. This can easily be circumvented by observing that, if E is an equivalence relation and Δ is the diagonal, then E and $E \setminus \Delta$ are interdefinable.

Theorem 2.8 *Let T be a binary cone-expansion of* DMT *and $M \vDash T$. For all tuples b^0, b^1 there is a finite tuple c from M such that*

$$\mathrm{tp}(b^0/M) \cup \mathrm{tp}(b^1/M) \cup \mathrm{tp}(b^0b^1/c) \vdash \mathrm{tp}(b^0b^1/M).$$

In particular, every binary cone-expansion of DMT *is weakly binary.*

In the following proof, as well as later in the paper, we abuse the notation as follows. If, for instance, $e \in \mathrm{dcl}(Mb)$, as witnessed by an M-definable function h such that $e = h(b)$, then we may write, e.g., that $\mathrm{tp}(b/M)$ entails $\mathrm{tp}(e/M)$ to mean that, for every $L(M)$-formula φ satisfied by e, we have $\mathrm{tp}(b/M) \vdash \varphi(h(x))$. Similarly if we say, e.g., that a partial type implies $C_e^M = C_{b_i}^M$.

P r o o f . By quantifier elimination it is enough to find a finite tuple $c \in M$ such that $\mathrm{tp}_x(b^0/M) \cup \mathrm{tp}_y(b^1/M) \cup \mathrm{tp}_{xy}(b^0b^1/c)$ decides all the atomic relations in L between points of b^0, b^1, M, and their meets. Apply Lemma 2.4 to M and $b := b^0b^1$, let d be the resulting tuple and set $c := M \cap d$. We want to show that

$$\pi := \mathrm{tp}(b^0/M) \cup \mathrm{tp}(b^1/M) \cup \mathrm{tp}(b/c) \vdash \mathrm{tp}(b/M).$$

If e and f are both in bd then $e, f \in \mathrm{dcl}^{L_{\mathrm{mt}}}(bc)$, hence $\mathrm{tp}(b/c)$ entails $\mathrm{tp}(ef/\emptyset)$, and in particular decides all formulas of the forms $R_j(e, f)$ and $P_{j'}(e)$.

Claim *We have $\pi \vdash \mathrm{tp}^{L_{\mathrm{mt}}}(b/M)$.*

P r o o f o f C l a i m . Since Mbd is closed under meets we only need to show that the position of all the $e \in d \setminus Mb$ with respect to M is determined. By Lemma 1.7 and the fact that $e \in \mathrm{dcl}^{L_{\mathrm{mt}}}(bc) \setminus Mb$ there is $i < |b|$ such that $e < b_i$ and $C_e^M = C_{b_i}^M$; note that this information is deduced by π, because e is a meet of points in bc. If C_e^M is unbounded, we are done. Otherwise, if $a_e \in c$ is as in Lemma 2.4, all we need to decide is whether e is below or incomparable to $\{h \in M \mid h > a_e \sqcap e\}$ (because every point of M below [resp. incomparable to] $a_e \sqcap e$ is automatically below [resp. incomparable to] e). This is decided by whether $a_e > e$ or not, and this information is in $\mathrm{tp}(b/c)$. □

We then need to take care of formulas of the form $R_j(e, f)$ for $e \in d \setminus Mb$ and $f \in M$; the argument for formulas of the form $R_j(f, e)$ is identical *mutatis mutandis*. If $e \leq f$ or $f \leq e$, by hypothesis we must have $\neg R_j(e, f)$, so we may assume that $e \parallel f$. We distinguish three cases; the fact that, by the Claim, π implies the position of e with respect to M will be used tacitly.

1. Assume first $e \sqcap f > C_e^M$. Some subcases of this case are depicted in Figure 4. By assumption C_e^M is bounded and, if $a_e \in c$ is as in Lemma 2.4, we have $a_e \sqcap f > C_e^M = C_{e \sqcap f}^M$. Since $a_e \sqcap f$ and $e \sqcap f$ must be comparable, and $a_e \sqcap f \in M$, this implies $a_e \sqcap f > e \sqcap f$, so a_e and f are in the same open cone above $e \sqcap f$. Since R_j is on open cones, $R_j(e, f) \leftrightarrow R_j(e, a_e)$, but $a_e \in c$ and $e \in \mathrm{dcl}^{L_{\mathrm{mt}}}(bc)$, so since $\pi \vdash \mathrm{tp}(b/c)$ we are done.
2. Assume now that $e \sqcap f \not> C_e^M$ and there is $h \in M$ such that $e \sqcap h > e \sqcap f$. Then e is in the same open cone above $e \sqcap f$ as h, hence $R_j(e, f) \leftrightarrow R_j(h, f)$. Since $f, h \in M$ we are done.
3. If $e \sqcap f \not> C_e^M$ but there is no h as in the previous point, then C_e^M must have a maximum g, which needs to equal $e \sqcap f$, and since $e \parallel f$ we need to have $f > g$. If e is in an existing open cone above g, since the R_j are on open cones, we are done, so assume it is in a new one. Since $e \in \mathrm{dcl}^{L_{\mathrm{mt}}}(bc)$, by Lemma 1.7 this can only happen if there is $i < |b|$ such that $e \leq b_i$, hence e shares the same open cone above g as b_i. Again, since the R_j are on open cones, we are done. □

Figure 4 Two subcases of case (1) in the proof of Theorem 2.8, where $e \sqcap f > C_e^M$. In the first picture, C_e^M does not have a maximum. In the second picture it has one, denoted by a triangle. Solid lines lie in M, and dotted lines lie in a bigger $M_1 \succ M$. Other subcases are similar, and correspond to different arrangements of a_e and f, e.g., $a_e > f$.

3 The domination monoid: pure trees

We now compute the domination monoid in DMT. We first recall briefly its definition and some of its basic properties for the reader's convenience, and otherwise refer to [13]. Cf. also [12] for a more extensive treatment.

Below, when considering $p(x)$, $q(y)$, say, we assume x and y to be disjoint.

It is well-known that, if $A \subset^+ \mathfrak{U} \subseteq B$ and $p \in S_x^{\mathrm{inv}}(\mathfrak{U}, A)$, then there is a unique $p \mid B$ extending p to an A-invariant type over B, given by requiring, for each $\varphi(x; y) \in L(A)$ and $b \in B$,

$$\varphi(x; b) \in p \mid B \overset{\mathrm{def}}{\iff} \mathrm{tp}(b/A) \in (d_p\varphi(x; y))(y).$$

This allows to define the *tensor product* of $p \in S_x^{\mathrm{inv}}(\mathfrak{U}, A)$ with any $q \in S_y(\mathfrak{U})$ as follows. Fix $b \vDash q$; for each $\varphi(x, y) \in L(\mathfrak{U})$, define

$$\varphi(x, y) \in p(x) \otimes q(y) \overset{\mathrm{def}}{\iff} \varphi(x, b) \in p \mid \mathfrak{U}b.$$

Some authors denote by $q(y) \otimes p(x)$ what we denote by $p(x) \otimes q(y)$.

It is an easy exercise to show that the product $\otimes$ does not depend on $b \vDash q$, nor on the choice of a base of invariance for p, that it is associative, and that if p, q are both A-invariant then so is $p \otimes q$.

Notation 3.1 *For $p(x)$, $q(y)$ types over $B \supseteq A$, we denote*

$$S_{pq}(A) := \{r \in S_{xy}(A) \mid r \supseteq (p \upharpoonright A) \cup (q \upharpoonright A)\}.$$

Definition 3.2 Let $p \in S_x(\mathfrak{U})$ and $q \in S_y(\mathfrak{U})$. We say that p *dominates* q, and write $p \geq_{\mathrm{D}} q$, iff there are some small A and some $r \in S_{xy}(A)$ such that

1. $r \in S_{pq}(A)$, and
2. $p(x) \cup r(x, y) \vdash q(y)$.

In this case, we say that r is a *witness* to, or *witnesses* $p \geq_{\mathrm{D}} q$. We say that p and q are *domination-equivalent*, and write $p \sim_{\mathrm{D}} q$, iff $p \geq_{\mathrm{D}} q$ and $q \geq_{\mathrm{D}} p$.

Example 3.3 Suppose that $q(y)$ is the *pushforward* of $p(x)$ under the A-definable function f, namely $q(y) := \{\varphi(y) \mid p(x) \vdash \varphi(f(x))\}$. In this case, and in the more general one where $|y| > 1$ and f is a tuple of definable functions, we have $p \geq_{\mathrm{D}} q$, witnessed by any completion of $(p(x) \upharpoonright A) \cup (q(y) \upharpoonright A) \cup \{y = f(x)\}$.

In Definition 3.2 we are not requiring $p \cup r$ to be a complete global type in variables xy; in other words, domination is "small-type semi-isolation", as opposed to "small-type isolation", i.e., $\mathrm{F}_\kappa^{\mathrm{s}}$-isolation in the notation of [16, Ch. IV]. While it is easy to see that $\mathrm{F}_\kappa^{\mathrm{s}}$-isolation is the same as domination in every weakly binary theory, the two relations are in general distinct. This can be seen in the theory below; the reader who dislikes random digraphs may feel free to replace them with generic equivalence relations.

Example 3.4 Work in a 2-sorted language, with sorts O ("objects") and D ("digraphs"). Let $L := \{E^{(O^2)}, P^{(O)}, R^{(O^2\times D)}\}$, a relational language with arities indicated as superscripts. Consider the following universal axioms.

1. E is an equivalence relation.
2. $R(x, y, w) \to E(x, y)$.
3. $R(x, y, w) \to \neg R(y, x, w)$.

The finite structures satisfying these axioms form a Fraïssé class; let T be the theory of its limit. In a model of T, the equivalence relation E partitions the sort O into infinitely many classes. On each class a/E the predicate P is infinite and coinfinite, and each point of D induces a random digraph on each a/E. Different random digraphs, on the same a/E or on different ones, interact generically with P and with each other, but no digraph has an edge across different classes. Let x be a variable of sort O, define $\pi(x) := \{\neg E(x, d) \mid d \in \mathfrak{U}\}$, and let $p(x) := \pi(x) \cup \{P(x)\}$ and $q(y) := \pi(y) \cup \{\neg P(y)\}$. By quantifier elimination and the lack of edges across different classes, p and q are complete global types, in fact $\emptyset$-invariant ones. If $\rho(x, y) := E(x, y) \wedge P(x) \wedge \neg P(y)$, then $p \cup \{\rho\} \vdash q$ and $q \cup \{\rho\} \vdash p$, hence if $r \in S_{pq}(\emptyset)$ contains ρ then it witnesses simultaneously that $p \geq_{\mathrm{D}} q$ and that $q \geq_{\mathrm{D}} p$. Note that the predicate P forbids r from containing $x = y$. Let A be a small set and let $r \in S_{pq}(A)$. If $r \vdash \neg E(x, y)$, then $p(x) \cup r(x, y)$ does not imply $\{\neg E(y, c) \mid c \in O(\mathfrak{U})\}$, therefore r cannot witness domination, let alone $\mathrm{F}^{\mathrm{s}}_{\kappa}$-isolation. If instead $r \vdash E(x, y)$ then, by genericity, $p \cup r$ cannot decide, for all $d \in \mathfrak{U}$, whether $R(x, y, d)$ holds. Since similar arguments hold for $q \cup r$, we have shown that, for all $a \vDash p$ and $b \vDash q$, neither $\operatorname{tp}(a/\mathfrak{U}b)$ nor $\operatorname{tp}(b/\mathfrak{U}a)$ is $\mathrm{F}^{\mathrm{s}}_{\kappa}$-isolated.

It can be shown that $\geq_{\mathrm{D}}$ is a preorder, hence $\sim_{\mathrm{D}}$ is an equivalence relation. Let $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ be the quotient of $S^{\mathrm{inv}}(\mathfrak{U})$ by $\sim_{\mathrm{D}}$. The partial order induced by $\geq_{\mathrm{D}}$ on $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ will, with abuse of notation, still be denoted by $\geq_{\mathrm{D}}$, and we call $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \geq_{\mathrm{D}})$ the *domination poset*. This poset has a minimum, the (unique) class of *realised types*, i.e., global types realised in $\mathfrak{U}$, denoted by $[\![0]\!]$.

If T is such that $(S^{\mathrm{inv}}(\mathfrak{U}), \otimes, \geq_{\mathrm{D}})$ is a preordered semigroup, we say that $\otimes$ *respects* $\geq_{\mathrm{D}}$. In particular, then $\sim_{\mathrm{D}}$ is a congruence with respect to $\otimes$, and induces a well-defined operation on $\widetilde{\operatorname{Inv}}(\mathfrak{U})$, still denoted by $\otimes$, easily seen to have neutral element $[\![0]\!]$. Call the structure $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes, [\![0]\!], \geq_{\mathrm{D}})$ the *domination monoid*. We usually denote it simply by $\widetilde{\operatorname{Inv}}(\mathfrak{U})$, and say that $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ *is well-defined* to mean that $\otimes$ respects $\geq_{\mathrm{D}}$; this should cause no confusion since $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ is *always* well-defined as a poset.

As shown in [13], $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ need not be well-defined in general, but it is in certain classes of theories, such as stable ones. More relevantly to the present endeavour, we recall the following.

Fact 3.5 ([13, Corollary 1.30]) *In every weakly binary theory, the partially ordered monoid* $(\widetilde{\operatorname{Inv}}(\mathfrak{U}), \otimes, [\![0]\!], \geq_{\mathrm{D}})$ *is well-defined.*

Recall that two types $p(x), q(y) \in S(B)$ are *weakly orthogonal*, denoted by $p \perp^{\mathrm{w}} q$, iff $p(x) \cup q(y)$ is a complete type in $S_{xy}(B)$. In particular, if $p, q \in S^{\mathrm{inv}}(\mathfrak{U})$ are weakly orthogonal, then $p(x) \otimes q(y) = q(y) \otimes p(x)$, since both products extend $p(x) \cup q(y)$. We will also need the following two facts.

Fact 3.6 ([18, Corollary 4.7]) *Assume T is* NIP *and let $\{p_i \mid i \in I\}$ be a family of types $p_i(x^i) \in S^{\mathrm{inv}}(\mathfrak{U})$ such that if $i \neq j$ then $p_i \perp^{\mathrm{w}} p_j$. Then $\bigcup_{i\in I} p_i(x^i)$ is complete.*

Fact 3.7 ([13, Proposition 3.13 & Corollary 3.14]) *Let $p_0, p_1 \in S^{\mathrm{inv}}(\mathfrak{U})$, $q \in S(\mathfrak{U})$, and assume that $p_0 \geq_{\mathrm{D}} p_1$. If $p_0 \perp^{\mathrm{w}} q$, then $p_1 \perp^{\mathrm{w}} q$. If $p_0 \geq_{\mathrm{D}} q$ and $p_0 \perp^{\mathrm{w}} q$, then q is realised.*

In particular we may endow $\widetilde{\operatorname{Inv}}(\mathfrak{U})$ with an additional relation, induced by $\perp^{\mathrm{w}}$ and denoted by the same symbol.

Remark 3.8 In what follows, if $r \in S_{pq}(A)$ witnesses $p \geq_{\mathrm{D}} q$, by passing to a suitable extension of r there is no harm in enlarging A, provided it stays small, which we may do tacitly; if p, q are invariant, we will furthermore assume A to be large enough so that $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$. Sometimes, we say that $r(x, y)$ witnesses domination even if it is not complete, but merely consistent with $(p(x) \cup q(y)) \upharpoonright A$. In that case, we mean that any of its completions to a type in $S_{pq}(A)$ does. Similarly, we sometimes just write, e.g., "put in r the formula $\varphi(x, y)$".

For the next proposition, recall the notion of *kind* of an invariant type from Definition 1.5.

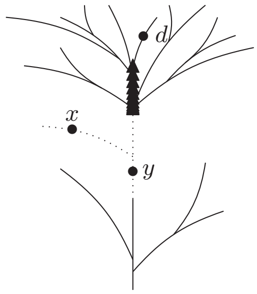

Figure 5 Proof of Proposition 3.9, how to show that $q(y) \geq_{\mathrm{D}} p(x)$. Points of A are denoted by a triangle. Solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 \succ \mathfrak{U}$.

Proposition 3.9 *The following statements hold in* DMT.

1. *Suppose all 1-subtypes of $p \in S(\mathfrak{U})$ have the same cut C_0, all 1-subtypes of $q \in S(\mathfrak{U})$ have the same cut C_1, and $C_0 \neq C_1$. Then $p \perp^{\mathrm{w}} q$.*
2. *Let C be a cut with maximum g. Suppose that all 1-subtypes of p are of kind (Ib) with cut C and all 1-subtypes of q are of kind (II) with cut C, or that all 1-subtypes of p, q are of kind (Ib) with cut C, but no open cone above g contains both a coordinate of p and one of q. Then $p \perp^{\mathrm{w}} q$.*
3. *Every 1-type of kind (IIIa) is domination-equivalent to the unique 1-type of kind (Ia) with the same cut. Every 1-type of kind (IIIb) is domination-equivalent to the unique 1-type of kind (Ib) with the same cut and, if this cut has a maximum g, the same open cone above g.*

In particular, if $p, q \in S_1^{\mathrm{inv}}(\mathfrak{U})$, then either $p \perp^{\mathrm{w}} q$ or $p \sim_{\mathrm{D}} q$.

Proof. (1) By quantifier elimination and the first two points of Lemma 1.7.

(2) This does not follow from the previous point because such types have the same cut, but it is still easy from quantifier elimination and the fact that the open cones in which types of kind (II) concentrate are new, while those of types of kind (Ib) are realised.

(3) We give a proof for kind (IIIa) which may be easily modified to yield one for kind (IIIb). By definition of kind (IIIa) there are $d \in \mathfrak{U}$ and $A \subset^{+} \mathfrak{U}$ such that

$$p(x) \vdash \{x \not\leq b \mid b \in \mathfrak{U}\} \cup \{x \sqcap d < a \mid a \in A\} \cup \{x \sqcap d > b \mid b \in \mathfrak{U}, b < A\}.$$

Let q be the pushforward of p under the definable function $x \mapsto x \sqcap d$. By this very description $p(x) \geq_{\mathrm{D}} q(y)$ (cf. Example 3.3) and, clearly, q is of kind (Ia) and $C_q = C_p$. To prove $q(y) \geq_{\mathrm{D}} p(x)$, let A be as in the definition of kind (Ia), let $d > C_q$, and use some $r \in S_{pq}(Ad)$ containing $x \sqcap d > y$; since r contains $p(x) \upharpoonright A$, which, for all $a \in A$, proves $x \not\leq a$ and $a > x \sqcap d$, we are done. Cf. Figure 5.

The "in particular" statement follows by considering the different possibilities for p, q and applying what we just proved. □

In Proposition 3.9, it is important to work with $\sim_{\mathrm{D}}$, as opposed to the finer relation $\equiv_{\mathrm{D}}$ of *equidominance*, obtained by requiring that domination of q by p and of p by q can be witnessed by the same r.

Definition 3.10 Let $p \in S_x(\mathfrak{U})$ and $q \in S_y(\mathfrak{U})$. We say that p is *equidominant* to q, and write $p \equiv_{\mathrm{D}} q$, iff there are a small A and $r \in S_{xy}(A)$ such that

1. $r \in S_{pq}(A)$,
2. $p(x) \cup r(x, y) \vdash q(y)$, and
3. $q(x) \cup r(x, y) \vdash p(y)$.

In this case, we say that r is a *witness* to, or *witnesses* $p \equiv_{\mathrm{D}} q$.

While using some r containing[6] $x \sqcap d = y$ would still work to show that every type of kind (IIIa) is equidominant to one of type (Ia), this would not work for kind (IIIb), as shown below.

[6] Here, the domain A of r has to be large enough for p, q to be A-invariant. Using $q(y) \cup \{x \sqcap d = y\}$ alone is not enough to show $x \neq d$, and if $\{a \in \mathfrak{U} \mid p \vdash x \sqcap d < a\}$ does not have a minimum then no single formula is enough to show $q \geq_{\mathrm{D}} p$.

Remark 3.11 Let $p(x)$ and $q(y)$ be the types respectively of kind (IIIb) and (Ib) with cut $\emptyset$ (so, q is the type of an infinitely small element). Then $p \not\equiv_{\mathrm{D}} q$.

Proof. Suppose that $r(x, y)$ witnesses equidominance. If $r(x, y) \vdash x \sqcap y < y$, then $p(x) \cup r(x, y) \nvdash q(y)$, since by quantifier elimination and compactness it cannot prove all formulas $y < d$, for $d \in \mathfrak{U}$. If $r(x, y) \vdash x \sqcap y = y$, then $q(y) \cup r(x, y) \nvdash p(x)$, since it cannot prove all formulas $x \not\leq d$. □

Proposition 3.12 *In the theory of dense meet-trees the following hold.*

1. *Types of kind (Ia) and (Ib) are idempotent modulo equidominance.*
2. *If p is of kind (II) and $m < n \in \omega$ then $p^{(m)} \not\geq_{\mathrm{D}} p^{(n)}$.*

Proof. (1) Let A be such that p is A-invariant. It follows easily from quantifier elimination that in order to show $p(x_1) \otimes p(x_0) \equiv_{\mathrm{D}} p(y)$ it is enough to put in $r \in S_{p^{(2)}p}(A)$ the formula $x_0 = y$.

(2) For notational simplicity we show the case $m = 1$, $n = 2$, the general case being analogous. Suppose that p is the type of an element in a new open cone above g, i.e., $p(y) \vdash \{y > g\} \cup \{y \sqcap b = g \mid b \in \mathfrak{U}, b > g\}$. Towards a contradiction, let $r \in S_{p,p^{(2)}}(A)$ be such that $p(y) \cup r(y, x_0, x_1) \vdash p(x_1) \otimes p(x_0)$. We may assume that $g \in A$. Since $p^{(2)} \upharpoonright \{g\}$ proves $x_0 \sqcap x_1 = g$, i.e., that the cones of x_0 and x_1 are distinct, there is $i < 2$ such that $r \vdash y \sqcap x_i = g$. Since r is small there is $d > g$ in $\mathfrak{U}$ such that $p(y) \cup r \nvdash x_i \sqcap d = g$; in other words it is not possible, with a small type, to say that x_i is in a new open cone, unless it is the same cone as y, but y cannot be in the open cones of x_0 and x_1 simultaneously. □

Since by Theorem 2.8 dense meet-trees are weakly binary, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined by Fact 3.5. By Proposition 3.12 and Proposition 3.9, (domination-equivalence classes of) 1-types of kind (II) generate a copy of $\mathbb{N}$, while all other (classes of) 1-types are idempotent. We have also seen in Proposition 3.9 that if p, q are nonrealised 1-types, then either $p \perp^{\mathrm{w}} q$ or $p \sim_{\mathrm{D}} q$. In particular, all pairs of 1-types commute modulo domination-equivalence. To complete our study we need one last ingredient.

Proposition 3.13 *In* DMT, *every invariant type is domination-equivalent to a product of invariant 1-types.*

Proof. By Fact 3.6 and Proposition 3.9 we reduce to showing the conclusion for types $p(x)$ whose 1-subtypes all have the same cut C_p. We may furthermore assume that no 1-subtype of p is realised.

Assume first that C_p does not have a maximum, let $c \vDash p(x)$ and let $d \in \mathfrak{U}$ be such that $d > C_p$, which exists by the last point of Lemma 1.7. Let $H = \{h_0(c), \dots, h_n(c)\}$ be the (finite, by Remark 1.3) set of points in $\mathrm{dcl}(cd)$ such that $d > h_i(c)$, where each $h_i(x)$ is a $\{d\}$-definable function. By semilinearity H is linearly ordered; suppose, by reindexing, that $h_0(c) = \min H$ and $h_n(c) = \max H$. We have two subcases. If C_p has small cofinality, let $q(y)$ be of kind (Ib) with $C_q = C_p$. We show that $p \sim_{\mathrm{D}} q$. Let A be such that $p, q \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ and A contains a set cofinal in C_p. By density, the formula $h_n(x) < y$ is consistent with $(p(x) \cup q(y)) \upharpoonright Ad$. Let $r(x, y) \in S_{pq}(Ad)$ contain the formula $h_n(x) < y$, and note that $q(y) \cup r(x, y)$ implies the type over $\mathfrak{U}$ of each point of $\mathrm{dcl}(cd)$, i.e., of the closure of cd under meets.[7] It follows from quantifier elimination that $q \cup r \vdash p$. To prove $p \cup r \vdash q$, use instead some r containing, for an arbitrary $i \leq n$, the formula $y < h_i(x)$. In the other subcase, $\{e \in \mathfrak{U} \mid C_p < e < d\}$ has small coinitiality. The argument is analogous, except we take q of kind (Ia), use an r containing $h_0(x) > y$ to show $q \cup r \vdash p$, and an r containing $h_i(x) < y$ to show $p \cup r \vdash q$.

Suppose now that C_p has maximum g. Assume without loss of generality that $c_0, \dots, c_{k-1}$ are the points of c such that there is $d_i \in \mathfrak{U}$ with $d_i \sqcap c_i > g$. In other words, these are the points in existing open cones above g, and $c_k, \dots, c_{|c|-1}$ are in new open cones. Again by quantifier elimination, we have $\mathrm{tp}(c_0, \dots, c_{k-1}/\mathfrak{U}) \perp^{\mathrm{w}} \mathrm{tp}(c_k, \dots, c_{|c|-1}/\mathfrak{U})$, so we can deal with the two subtypes separately. Similarly, by Proposition 3.9 we may split $c_{<k}$ further, and we may assume that for $i < \ell$, say, all c_i are in the same open cone, say that of the point $d \in \mathfrak{U}$. It is now enough to proceed as in the previous case, by taking $q(y)$ to be the type of kind (Ib) with the same cut and open cone above g. As for $c_k, \dots, c_{|c|-1}$, let H be the set of minimal elements of $\mathrm{dcl}(c_k, \dots, c_{|c|-1}) \setminus \mathfrak{U}$. Let $q(y)$ be the type of kind (II) above g. To conclude, let r identify elements of H with the coordinates of a realisation of $q^{(|H|)}$. □

7 Here we use the same abuse of notation as in the proof of Theorem 2.8.

The previous results yield the following characterisation of $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ in DMT. Before stating it, recall that $\bigoplus_{i\in I} A_i$ denotes the submonoid of $\prod_{i\in I} A_i$ consisting of I-sequences with finite support. The order is the product order, that is, $(a_i)_{i\in I} \le (b_i)_{i\in I}$ iff $a_i \le b_i$ for all $i \in I$.

Theorem 3.14 *In dense meet-trees there is an isomorphism of partially ordered monoids*

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup) \times \bigoplus_{g\in\mathfrak{U}} (\mathbb{N}, +).$$

Generators of copies of $\mathbb{N}$ *correspond to types of elements in a new open cone above a point* $g \in \mathfrak{U}$*, i.e., to types of kind (II), while each point of* X *corresponds to, either:*

1. *a linearly ordered subset of* $\mathfrak{U}$ *with small coinitiality, modulo mutual coinitiality; this corresponds to types of kind (Ia)/(IIIa);*
2. *a cut with no maximum, but with small cofinality; this corresponds to some types of kind (Ib)/(IIIb);*
3. *an existing open cone above an existing point; this corresponds to the rest of the types of kind (Ib)/(IIIb).*

4 The domination monoid: expansions

In this section we generalise Theorem 3.14 to purely binary cone-expansions of DMT, such as DTR, by replacing the direct summands isomorphic to $\mathbb{N}$ with the domination monoids of the structures induced on sets of open cones. In DMT these are pure sets, which are easily seen to have domination monoid isomorphic to $\mathbb{N}$ and generated by the $\sim_{\mathrm{D}}$-class of the unique nonrealised 1-type.

In this section we will have to take reducts of monster models of binary cone-expansions of DMT to L_{mt}. While saturation is preserved by taking reducts, in general a reduct of a κ-strongly homogeneous $\mathfrak{U}$ need not still be such. Therefore, in this section we work on a monster model $\mathfrak{U}$ such that its reducts are still strongly κ-homogeneous. Models with such properties always exist: for instance, we may take $\mathfrak{U}$ to be κ-special (cf. [8, § 10.4]).

Assumption 4.1 *In this section,* T *is a a binary cone-expansion of* DMT*, and* $\mathfrak{U}$ *is a monster model of* T *such that* $\mathfrak{U} \upharpoonright L_{\mathrm{mt}}$ *is a monster model of* DMT.

Before restricting our attention to purely binary cone-expansions, we observe a phenomenon which can arise in the presence of unary predicates. Suppose for instance that $L = L_{\mathrm{mt}} \cup \{P\}$, where P is a unary predicate symbol interpreted as a branch of $\mathfrak{U}$, i.e., a maximal linearly ordered subset. In this case, there is an $\emptyset$-invariant type p with cut $C_p = P(\mathfrak{U})$, and by the last point of Lemma 1.7 $p \upharpoonright L_{\mathrm{mt}}$ is not invariant. Another binary cone-expansion of DMT where there is an invariant type p such that $p \upharpoonright L_{\mathrm{mt}}$ is not invariant can be obtained by taking as $P(\mathfrak{U})$ a bounded linearly ordered subset with no supremum. However, using unary predicates is the *only* way to obtain such behaviour in a binary cone-expansion of DMT, as we are about to show. We refer the reader interested in preservation of invariance under reducts to [15].

Denote by G_g the *closed cone* above g, namely $\{b \in \mathfrak{U} \mid b \ge g\}$.

Definition 4.2 Let T be an expansion of DMT. We call a formula $\varphi(x)$ with $|x| = 1$ *tame* iff it has the following property: there is a finite set $D \subseteq \mathfrak{U}$ such that, for every $a \in \varphi(\mathfrak{U})$, either there is $d \in D$ such that $a \le d$, or $G_a \subseteq \varphi(\mathfrak{U})$.

Proposition 4.3 *If* T *is a purely binary cone-expansion of* DMT*, then every formula in one free variable is tame.*

P r o o f . It is routine to verify that every atomic and negated atomic $\varphi(x) \in L_{\mathrm{mt}}(\mathfrak{U})$ is tame. Fix a point c, and consider $\varphi(x) := R_j(x, c)$; if $a \in \varphi(\mathfrak{U})$, since R_j is on open cones we also have $\varphi(b)$ for every $b > a$, hence $G_a \subseteq \varphi(\mathfrak{U})$. Consider now $\varphi(x) := \neg R_j(x, c)$, and let $D = \{c\}$. Suppose that $a \not\le c$. If $a \parallel c$ and $\varphi(a)$ holds, we can argue as above, so assume that $a > c$. For any $b \ge a$ we have in particular $b > c$, hence $\varphi(b)$ holds by assumption and $G_a \subseteq \varphi(\mathfrak{U})$; therefore $\neg R_j(x, c)$ is tame. The formulas $R_j(x, x \sqcap c)$ and $R_j(x \sqcap c_0, x \sqcap c_1)$ are always false, hence they are tame, together with their negations. Since the same arguments apply to $R_j(c, x)$, $R_j(x \sqcap c, x)$, and their negations, we conclude that all atomic and negated atomic formulas are tame.

Tame formulas in the variable x are easily seen to be closed under conjunctions and disjunctions: if D_φ and D_ψ witness tameness of $\varphi(x)$ and $\psi(x)$ respectively, then $D_\varphi \cup D_\psi$ witnesses tameness of both $\varphi(x) \wedge \psi(x)$ and $\varphi(x) \vee \psi(x)$. By quantifier elimination, we have the conclusion. □

Corollary 4.4 *If T is a purely binary cone-expansion of* DMT *and $p \in S^{\mathrm{inv}}(\mathfrak{U})$, then $(p \upharpoonright L_{\mathrm{mt}}) \in S^{\mathrm{inv}}(\mathfrak{U} \upharpoonright L_{\mathrm{mt}})$.*

Proof. By Remark 2.2 & Theorem 2.8, it is enough to show that if $p(x) \in S^{\mathrm{inv}}(\mathfrak{U}, A)$ and $q(y)$ is a 1-subtype of p, then $q \upharpoonright L_{\mathrm{mt}}$ is invariant. We may assume that no 1-subtype of p is realised.

Similarly to the final step in the previous proof, we see by taking unions of witnesses that, if $\Phi(y)$ is a small disjunction of types over a fixed small set A, then it satisfies the analogue of tameness where we allow D to have size $|\Phi| + |A| + |T|$. By saturation, if $\Phi(\mathfrak{U})$ is linearly ordered, it must be bounded.

By invariance, the linearly ordered set C_q is the set of realisations of a disjunction of 1-types over A. By the previous paragraph, C_q is bounded. By observing that, if a small intersection of open cones is nonempty, then it cannot be linearly ordered, we see that C_q can be defined by a disjunction of small L_{mt}-types.

Assume now that there is $c \in \mathfrak{U}$ such that $q(y) \vdash y < c$, and let $\varphi(y; w) := y < w$. Again by invariance, $d_q\varphi$ is a disjunction of small types over A, which by assumption is nonempty. By quantifier elimination, the fact that the R_j are on open cones, and the fact that if y is below two distinct open cones above g then necessarily $y \le g$, we see that $d_q\varphi$ is equivalent to a disjunction of small L_{mt}-types over A, and it follows that $q \upharpoonright L_{\mathrm{mt}}$ is an invariant type of kind (Ia) or (Ib). If instead, for every $c \in \mathfrak{U}$, we have $q(y) \vdash y \not< c$, let $c \in \mathfrak{U}$ be greater than C_q; in particular, $q(y) \vdash y \sqcap c < c$. Assume, up to enlarging A, that $c \in A$. By an analogous argument with the formula $\varphi(y; w) := y < (w \sqcap c)$, we see that $q \upharpoonright L_{\mathrm{mt}}$ is an A-invariant type of kind (II), (IIIa), or (IIIb). □

Remark 4.5 The conclusion of Corollary 4.4 fails if we only assume that T is a binary cone-expansion of DMT where all formulas are tame. In fact, if P is a predicate for a bounded linearly ordered subset with no supremum, then the formula $P(x)$ is easily seen to be tame.

Assumption 4.6 *From now on, unless we say that T is arbitrary, we work in a purely binary cone-expansion T of* DMT*, in a language $L = L_{\mathrm{mt}} \cup \{R_j \mid j \in J\}$.*

We saw in Theorem 3.14 that, in DMT, domination-equivalence classes of invariant 1-types correspond to either new open cones above existing points, or to certain cuts in linearly ordered subsets of $\mathfrak{U}$. In what follows, restrictions of invariant 1-types to L_{mt}, which are still invariant by Corollary 4.4, will play a special role; we therefore introduce some terminology for these cones and cuts.

Definition 4.7 Let $p, q \in S_1^{\mathrm{inv}}(\mathfrak{U})$ be nonrealised, and suppose that $(p \upharpoonright L_{\mathrm{mt}}) \sim_{\mathrm{D}} (q \upharpoonright L_{\mathrm{mt}})$ in DMT. If these restrictions are of kind (II), in a new open cone above the same $g \in \mathfrak{U}$, we say that p, q have the same *sprout*, and that each of them *sprouts from* g. If the restrictions are of another kind, we say that p, q have the same *graft*.

So, in Theorem 3.14, X corresponds to the set of grafts, and there is a copy of $\mathbb{N}$ for each sprout. The reason behind the choice of terminology should be clear from Figure 2.

Lemma 4.8 *Let $p(x), q(y) \in S^{\mathrm{inv}}(\mathfrak{U})$. Denote by $q \upharpoonright i$ the restriction of q to the variable y_i, and similarly for p. If, for all $i < |y|$ and $i' < |x|$, the types $q \upharpoonright i$ and $p \upharpoonright i'$ have the same graft, then $p \sim_{\mathrm{D}} q$.*

Proof. As the roles of p and q are symmetric, it is enough to prove $p \ge_{\mathrm{D}} q$. By assumption and Theorem 3.14, $(p \upharpoonright L_{\mathrm{mt}}) \ge_{\mathrm{D}} (q \upharpoonright L_{\mathrm{mt}})$, witnessed by some r' over a small set A, and all 1-subtypes of p and q have the same cut C. Recall that C must be bounded by Corollary 4.4 and the last point of Lemma 1.7. Up to enlarging A, b and q, we may assume that (cf. Lemma 2.4), if $b \vDash q$,

1. there is $a \in A$ such that $a > C$ and, if C has a maximum g, such that a is in the same open cone above g of each coordinate of p and q; and
2. $\mathfrak{U}b$ is closed under meets and $\mathrm{dcl}^{L_{\mathrm{mt}}}(\mathfrak{U}b) \setminus \mathfrak{U} = b$.

By the second point of Lemma 1.7, this enlargement does not break the hypothesis that all $q \upharpoonright i$ have the same graft. Fix any $r \in S_{pq}(A)$ extending r', and recall that $p \cup r \vdash (q \upharpoonright L_{\mathrm{mt}}) \cup (q \upharpoonright A)$. By quantifier elimination and our assumptions on y, we are only left to deal with the formulas $R_j(y_i, f)$ and $R_j(f, y_i)$, where $f \in \mathfrak{U}$ and $i < |y|$. We have three possibilities for $y_i \sqcap f$. If $q(y) \vdash y_i \sqcap f = y_i$, then $q(y) \vdash y_i \le f$. If instead there is $h \in \mathfrak{U}$ such that $q(y) \vdash y_i \sqcap f = h$, then there must be a point of $\mathfrak{U}$ in the same open cone as b_i above $b_i \sqcap f$, because otherwise

$(q \upharpoonright i) \upharpoonright L_{\mathrm{mt}}$ would be of kind (II). In the only other possible case, which can only arise if $(q \upharpoonright i) \upharpoonright L_{\mathrm{mt}}$ is of kind (IIIa) or (IIIb), it is easy to see that f must be in the same open cone above $b_i \sqcap f$ as a. In each case, since the R_j are on open cones, $a \in A$, and $r \in S_{pq}(A)$, the partial type $p \cup r$ decides whether $R_j(y_i, f)$ and $R_j(f, y_i)$ hold, and we are done. □

Remark 4.9 The set of grafts of types in $S_1^{\mathrm{inv}}(\mathfrak{U})$ can be identified with that of grafts of types in $S_1^{\mathrm{inv}}(\mathfrak{U} \upharpoonright L_{\mathrm{mt}})$.

P r o o f . The natural map from the former to the latter, well-defined by Corollary 4.4, is injective by definition of graft, and is surjective because, since T is a purely binary cone-expansion of DMT, if $p \in S_1^{\mathrm{inv}}(\mathfrak{U} \upharpoonright L_{\mathrm{mt}})$ is of kind (Ia) or (Ib), then p implies a unique type in $S_1(\mathfrak{U})$, easily seen to be invariant. □

Lemma 4.10 *Let* $p_0, \dots, p_n \in S^{\mathrm{inv}}(\mathfrak{U})$ *be such that* $\Phi := \bigcup_{i \le n}(p_i(x^i) \upharpoonright L_{\mathrm{mt}})$ *is a complete type in* DMT. *Then* $\bigcup_{i \le n} p_i(x^i)$ *is a complete type in* T.

P r o o f . Let $b^i \vDash p_i$. In order for Φ to be complete in DMT, given $i < i' \le n$, no 1-subtype of p_i can have the same graft as a 1-subtype of $p_{i'}$: if this was the case for the types of b^i_0 and $b^{i'}_0$, say, then there would be $a \in \mathfrak{U}$ such that Φ does not decide whether $x^i_0 \sqcap a = x^{i'}_0 \sqcap a$ holds. Similarly, no 1-subtype of p_i, say that of b^i_0 again, can have the same sprout as a 1-subtype of $p_{i'}$, say that of $b^{i'}_0$, otherwise Φ does not decide whether $x^i_0 = x^{i'}_0$ holds. It follows from this observation and Lemma 1.7 that $\mathrm{dcl}^{L_{\mathrm{mt}}}(\mathfrak{U}b^0, \dots, b^n) = \bigcup_{i \le n} \mathrm{dcl}^{L_{\mathrm{mt}}}(\mathfrak{U}b^i)$. Therefore, we only need to show that, for each $i < i' \le n$ and each pair of $\{\sqcap\}$-terms (with parameters) h_i, h'_i with $h_i(b^i) \in \mathrm{dcl}^{L_{\mathrm{mt}}}(\mathfrak{U}b^i) \setminus \mathfrak{U}$ and $h_{i'}(b^{i'}) \in \mathrm{dcl}^{L_{\mathrm{mt}}}(\mathfrak{U}b^{i'}) \setminus \mathfrak{U}$, we have that every formula of the form $R_j(h_i(x^i), h_{i'}(x^{i'}))$ is decided by $p_i(x^i) \cup p_{i'}(x^{i'})$. Since the R_j are on open cones, it is enough to show that at least one between $h_i(b^i)$ and $h_{i'}(b^{i'})$ must be in the same open cone above $h_i(b^i) \sqcap h_{i'}(b^{i'})$ as a point of $\mathfrak{U}$. Again because Φ is complete, the types of $h_i(b^i)$ and $h_{i'}(b^{i'})$ cannot have the same graft, nor the same sprout.

We have two cases. Suppose first that $\Phi \vdash h_i(x^i) \sqcap h_{i'}(x^{i'}) = g$ for some $g \in \mathfrak{U}$. This happens, e.g., if $\mathrm{tp}(h_i(b^i)/\mathfrak{U})$ sprouts from g and the graft of $\mathrm{tp}(h_{i'}(b^{i'})/\mathfrak{U})$ is in an existing open cone above g, or if none of $C^{\mathfrak{U}}_{h_i(b^i)}$ and $C^{\mathfrak{U}}_{h_{i'}(b^{i'})}$ is included in the other. Then, at least one between $h_i(b^i)$ and $h_{i'}(b^{i'})$ must be in an open cone above g represented in $\mathfrak{U}$, because otherwise both would be sprouting from g, contradicting completeness of Φ.

If instead for all $d \in \mathfrak{U}$ we have $\Phi \vdash h_i(x^i) \sqcap h_{i'}(x^{i'}) \neq d$ then, up to swapping i and i', we must have $\Phi \vdash$ "$C^{\mathfrak{U}}_{h_{i'}(x^{i'})} \subsetneq C^{\mathfrak{U}}_{h_i(x^i)}$", because otherwise the types of $h_i(b^i)$ and $h_{i'}(b^{i'})$ have the same graft. Let $a \in C^{\mathfrak{U}}_{h_i(b^i)} \setminus C^{\mathfrak{U}}_{h_{i'}(b^{i'})}$. Then $\Phi \vdash h_i(x^i) > a > h_i(x^i) \sqcap h_{i'}(x^{i'})$, and in particular $h_i(b^i)$ is in the same open cone above $h_i(b^i) \sqcap h_{i'}(b^{i'})$ as a. □

Remark 4.11 Suppose that, for each given i, all the 1-subtypes of p_i have the same graft or the same sprout and, for $i \neq i'$, no 1-subtype of p_i has the same sprout, nor the same graft, as a 1-subtype of $p_{i'}$. Then, by quantifier elimination in DMT and easy observations such as those in Lemma 1.7, the assumptions of Lemma 4.10 are satisfied.

Recall that a sort Y of a multi-sorted $\mathfrak{U}$ is said to be *stably embedded* iff, whenever $D \subseteq \mathfrak{U}^m$ is definable, then $D \cap Y^m$ is definable with parameters from Y, in the sense that it is definable with parameters when we view Y as a structure on its own, the atomic relations being the traces on Y of $\emptyset$-definable relations of $\mathfrak{U}$. It is easy to obtain a proof of the following fact; the reader may find one in [12, Proposition 2.3.31].

Fact 4.12 (T arbitrary) *Let* Y *be a stably embedded sort of* $\mathfrak{U}$. *There is an embedding of posets* $\widetilde{\mathrm{Inv}}(Y) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U})$. *This embedding is a* $\perp^{\mathrm{w}}$*-homomorphism, a* $\not\perp^{\mathrm{w}}$*-homomorphism, and, if* $\otimes$ *respects* $\ge_{\mathrm{D}}$*, an embedding of monoids.*

For $g \in \mathfrak{U}$, denote by O_g the set of open cones above g equipped with the $\{R_j \mid j \in J\}$-structure induced by $\mathfrak{U}$. This may be regarded as an imaginary sort of the expansion of $\mathfrak{U}$ obtained by naming the point g. While it is not known whether well-definedness of the domination monoid is preserved by adding imaginaries, in our case this can be circumvented by using the following observation. Each type $p \in S_n(O_g)$ may be seen as the pushforward under the projection map of a suitable $q \in S_n(\mathfrak{U})$ with all non-realised 1-subtypes sprouting from g. Since T eliminates quantifiers, q is axiomatised by its quantifier-free part. It follows easily that the same is true of p, and therefore $\mathrm{Th}(O_g)$ eliminates quantifiers in a binary language, hence is (weakly) binary. By Fact 3.5, $I_g := \widetilde{\mathrm{Inv}}(O_g)$ is well-defined. Note that we are not claiming well-definedness of the domination monoid for $\mathfrak{U}$ with the sort O_g added, but only for the sort O_g in isolation.

Theorem 4.13 *Let T be a purely binary cone-expansion of* DMT*, and let X be the set of grafts of types in $S_1^{\mathrm{inv}}(\mathfrak{U})$. Then there is an isomorphism of partially ordered monoids*

$$\widetilde{\mathrm{Inv}}(\mathfrak{U}) \cong (\mathscr{P}_{\mathrm{fin}}(X), \cup) \times \bigoplus_{g\in\mathfrak{U}} I_g.$$

P r o o f . Recall that by Theorem 2.8 $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is well-defined, and that by Corollary 4.4 taking restrictions to L_{mt} preserves invariance. By Remark 4.11 & Lemma 4.10, $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ is generated by the $\sim_{\mathrm{D}}$-classes of those types p where all 1-subtypes of p have all the same graft, or have all the same sprout. If all 1-subtypes of p have the same graft, by Lemma 4.8 p is domination-equivalent to any 1-type with such a graft, and by using Lemma 4.10 a second time we see that $(\mathscr{P}_{\mathrm{fin}}(X), \cup)$ embeds in $\widetilde{\mathrm{Inv}}(\mathfrak{U})$. A third use of Lemma 4.10 yields $\widetilde{\mathrm{Inv}}(\mathfrak{U}) = (\mathscr{P}_{\mathrm{fin}}(X), \cup) \times \bigoplus_{g\in\mathfrak{U}} \tilde{I}_g$, where $\tilde{I}_g$ is the monoid of $\sim_{\mathrm{D}}$-classes of types whose every 1-subtype sprouts from g.

We are only left to show that $\tilde{I}_g \cong I_g$. Fix $g \in \mathfrak{U}$. Since $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ does not change after naming a small number of constants, we may add to L a constant symbol to be interpreted as g. For the time being, we also adjoin to the language a sort for O_g and its natural projection map π_g. Call the resulting structure $\mathfrak{U}_g$. Clearly $\mathfrak{U}$ is stably embedded in $\mathfrak{U}_g$, so by Fact 4.12 we have an embedding of posets[8] $\widetilde{\mathrm{Inv}}(\mathfrak{U}) \hookrightarrow \widetilde{\mathrm{Inv}}(\mathfrak{U}_g)$. Similarly, O_g is stably embedded, hence $\widetilde{\mathrm{Inv}}(O_g) = I_g$ embeds as a poset in $\widetilde{\mathrm{Inv}}(\mathfrak{U}_g)$. Let p be a type with all 1-subtypes sprouting from g (different coordinates might be in different open cones), and let q be the pushforward of p along π_g. Clearly $p \geq_{\mathrm{D}} q$. Moreover, that $q \geq_{\mathrm{D}} p$ is easily seen to be witnessed by any r containing all the formulas $y_i = \pi_g(x_i)$ for $i < |x|$: the only information lost when taking the projection concerns points in the same new open cone, but this information is in r. For instance, if $x_0 \sqcap x_1 > g$, we need to recover whether $R_j(x_0, x_1)$ holds, and whether any inequality holds between x_0 and x_1. More generally, the information we need to recover is implied by $p \upharpoonright \emptyset$, which is included in r by Definition 3.2. Therefore, we obtain an injective embedding of $\widetilde{\mathrm{Inv}}(O_g)$ into $\widetilde{\mathrm{Inv}}(\mathfrak{U})$ with image $\tilde{I}_g$. By Fact 4.12, this is an embedding of monoids, and we are done. □

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