

## Global regularity for a logarithmically supercritical hyperdissipative dyadic equation

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ABSTRACT. We prove global existence of smooth solutions for a slightly supercritical dyadic model. We consider a generalized version of the dyadic model introduced by Katz-Pavlovic [10] and add a viscosity term with critical exponent and a supercritical correction. This model catches for the dyadic a conjecture that for Navier-Stokes equations was formulated by Tao [13].

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### 1. Introduction

The *a priori* estimate of relevant quantities is a crucial part of the analysis of PDEs. For our purposes, the most interesting example is the system of Navier–Stokes equations in dimension three. In that case the kinetic energy and the energy dissipation are super–critical, hence in a way negligible, quantities with respect to the scaling invariance of the problem. Indeed, proofs of regularity are available only in the so–called hyper–dissipative case, where the Laplace operator is replaced by  $(-\Delta)^\alpha$  for  $\alpha \geq 5/4$  and this additional dissipation makes the energy relevant again (see for instance [11, 9]).

In a recent paper Tao [13] has shown that hyper–dissipativity can be slightly relaxed by a logarithmic factor. The idea originates from the same author [12] and

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has been applied in other problems, mainly from dispersive equations. In [13] Tao adds a small correction to the hyper-dissipative term, replacing  $(-\Delta)^{5/4}$  with

$$\frac{(-\Delta)^{5/4}}{g((-\Delta)^{1/2})^2},$$

and provides a simple and neat proof of global existence if  $\int 1/(sg(s)^4) = \infty$ . He then suggests that the same result should hold, based on some heuristics on the flow of energy, under the weaker condition  $\int 1/(sg(s)^2) = \infty$ .

The aim of this paper is to prove Tao's conjecture for the dyadic model, a simplified version of the Navier–Stokes equations, that nevertheless has shown to be an effective tool in the understanding of the full Navier–Stokes problem [14]. In particular, we believe that the main result of our paper (Theorem 5) gives a complete answer to some questions raised in Remark 5.2 of [14]. As a bonus result, in Section 3.3.1 we prove that the conjecture in [13] is true for the vector-valued dyadic model introduced in [14]. A proof of the conjecture for the full Navier–Stokes equations is a work in progress.

**1.1. The model.** Given  $\beta > 0$  and two real sequences  $\phi = (\phi_n)_{n \geq 1}$  and  $g = (g_n)_{n \geq 1}$ , with  $\phi$  bounded and  $g$  positive, set  $k_n = 2^{\beta n}$  for  $n \geq 0$ . Consider the critical hyper-dissipative generalized dyadic model,

$$(1) \quad \begin{cases} X'_n = \phi_{n-1} k_{n-1} X_{n-1}^2 - \phi_n k_n X_n X_{n+1} - \frac{1}{g_n} k_n X_n, & t > 0, n \geq 1, \\ X_n(0) = x_n, \end{cases}$$

where  $X = (X_n)_{n \geq 0}$  is a family of real functions,  $X_0 \equiv 0$  and  $x = (x_n)_{n \geq 1}$  is the given initial condition.

The classical critical regime here corresponds to  $g \equiv 1$ . Tao's statement for Navier–Stokes equation, transposed on our model, works whenever  $\sum_n g_n^{-2} = \infty$  ( $g_n = \sqrt{n}$  for instance), while the conjecture, on our model, states that global regularity should hold for  $\sum_n g_n^{-1} = \infty$  (e. g.  $g_n = n$ ).

The role of the coefficients  $\phi$  is to break the structure of the non-linearity. Otherwise, as shown in [4], if  $\phi \equiv 1$ , the energy flow is very steady, in the sense that the transfer of energy from  $X_n$  to  $X_{n+1}$  starts before  $X_{n-1}$  is discharged enough and this allows to prove regularity in a full supercritical regime. Further generalizations are possible, see Section 3.3.

The dyadic model has been introduced in [10] and analyzed in several other works [7, 8, 1, 2]. The model with viscosity has been initially introduced in [6] and further analyzed in [5, 4].

**1.2. The dyadic version of [13].** It is easy to be convinced that Tao's condition  $\int 1/(sg(s)^4) = \infty$  reads in our case as  $\sum 1/g_n^2 = \infty$ . To this end, we reproduce in this section a non-rigorous sketch of the idea of [13] adapted to the dyadic framework. Assume also, as we do, that  $(g_n)_{n \geq 1}$  and  $(k_n/g_n)_{n \geq 1}$  are non-decreasing. Assume moreover, for simplicity, that  $g_n = g(n)$ , where  $g$  is non-decreasing, continuous, non-zero on  $[0, \infty)$  and  $\int g(x)^{-2} = \infty$ .

Given a solution  $X$ , set for  $s \geq 1$ ,

$$a(t) = \sum_{n=1}^{\infty} \frac{k_n}{g_n} X_n^2, \quad A(t) = \sum_{n=1}^{\infty} k_n^{2s} X_n^2, \quad B(t) = \sum_{n=1}^{\infty} \frac{k_n^{2s+1}}{g_n} X_n^2.$$

We know by the energy estimate that  $a \in L^1([0, \infty))$ . By differentiating and using the Cauchy–Schwartz and Young inequalities,

$$\frac{d}{dt}A + 2B = 2(2^{2\beta s} - 1) \sum_{n=1}^{\infty} \phi_n k_n^{2s+1} X_n^2 X_{n+1} \leq B + c \sum_{n=1}^{\infty} g_n k_n^{2s+1} X_n^2 X_{n+1}^2.$$

Split the sum on the right-hand side in a sum [L] up to  $N$  and in a sum [H] from  $N$  on, where  $N$  will be chosen at the end. On the one hand,

$$[\text{L}] = \sum_{n=1}^N g_n^2 \left( \frac{k_n}{g_n} X_n^2 \right) (k_n^{2s} X_{n+1}^2) \leq c g_N^2 a A,$$

on the other hand

$$[\text{H}] = \sum_{n \geq N} \frac{g_n}{k_n} (k_n^{s+1} X_n^2) (k_n^{s+1} X_{n+1}^2) \leq \frac{g_N}{k_N} A^2.$$

If we choose  $N$  so that  $k_N \approx A$ , that is  $N \approx \log A$ , we have

$$\dot{A} \leq c(1+a)g(\log A)^2 A,$$

whose solutions stay bounded on bounded sets.

**1.3. The dyadic version of Tao’s conjecture.** We present here a heuristic argument that shows, as in Remark 1.2 of [13], that the weaker assumption  $\sum_n g_n^{-1} = \infty$  is sufficient for global regularity.

Indeed, let  $X$  be a weak solution on  $[0, T)$  and consider a blow-up scenario in  $T$ : at some time  $t$  the energy of solution is concentrated in  $n, n+1, \dots, n+m$  and  $n \rightarrow \infty$  when  $t \rightarrow T$ . The balance of energy on  $n, \dots, n+m$  yields:

$$\frac{d}{dt} \left( \frac{1}{2} \sum_{i=n}^{n+m} X_i^2 \right) = \phi_{n-1} k_{n-1} X_{n-1}^2 X_n - \phi_m k_m X_{n+m}^2 X_{n+m+1} - \sum_{i=n}^{n+m} \frac{k_i}{g_i} X_i^2,$$

where we could imagine  $\phi_{n-1} k_{n-1} X_{n-1}^2 X_n$  as the energy moving from  $n-1$  to  $n$ ,  $\phi_m k_m X_{n+m}^2 X_{n+m+1}$  the energy moving from  $n+m$  to  $n+m+1$ , and  $\frac{k_i}{g_i} X_i^2$  the energy dissipated in  $i$ . So, roughly speaking,  $k_n X_n^2$  is the speed at which the energy moves from  $n$  to  $n+1$ , whereas  $\frac{k_n}{g_n} X_n^2$  is the speed at which the energy is dissipated in  $n$ .

Now in the blow-up scenario, to go to high “ $n$ ”s, the energy has to go through all the states. The ratio between the energy dissipated and the energy that goes through  $n$  is  $\frac{1}{g_n X_n} \geq \frac{c}{g_n}$ . So, to have a non-trivial amount of energy reaching the infinite state, we have to require  $\sum g_n^{-1} < \infty$ .

Our proof is a rigorous version of the above argument. We find a recursive formula (9) for the tail energy and dissipation. Then we prove that any sequence satisfying the recursion decays super-exponentially fast.

## 2. Preliminaries

### 2.1. Basic definitions.

**DEFINITION 1.** *A weak solution is a sequence of  $X = (X_n)_{n \geq 1}$  of differentiable functions on all  $[0, \infty)$ , satisfying (1).*

Whenever  $X$  denotes a weak solution,  $E_n(t)$  and  $F_n(t)$  will denote the energy of the tails: for all  $n \geq 1$  and  $t \geq 0$ ,

$$E_n(t) := \sum_{i \leq n} X_i^2(t) < \infty, \quad \text{and} \quad F_n(t) := \sum_{i \geq n} X_i^2(t) \leq \infty.$$

We will also denote by  $E$  the total energy of the solution  $X$ : for all  $t \geq 0$ ,

$$E(t) := \sum_{n \geq 1} X_n^2(t) = \lim_{n \rightarrow \infty} E_n(t) = \|X(t)\|_H^2.$$

Clearly  $E(t) = E_n(t) + F_{n+1}(t)$  for all  $n \geq 1$ . From (1) we get

$$\frac{d}{dt}(X_n^2) = 2\phi_{n-1}k_{n-1}X_{n-1}^2X_n - 2\phi_nk_nX_n^2X_{n+1} - \frac{2}{g_n}k_nX_n^2,$$

so that if  $X$  is a weak solution, for all  $n \geq 1$ ,

$$(2) \quad E'_n = -2\phi_nk_nX_n^2X_{n+1} - \sum_{i \leq n} \frac{2}{g_i}k_iX_i^2.$$

To compute the variation of  $F_n$  we need an extra condition on solutions.

DEFINITION 2. *A weak solution  $X$  satisfies the energy inequality on  $[0, T]$  if*

$$(3) \quad E(t) + \int_0^t \sum_{n \geq 1} \frac{2}{g_n}k_nX_n^2(s)ds \leq E(0), \quad t \in [0, T].$$

*A weak solution satisfies the energy equality if there is equality in the above formula.*

We remark that, as is expected in this class of problems, regularity readily implies uniqueness and that the energy inequality holds (there is no anomalous dissipation). The converse is not true in general (see for instance [1, 3]).

By (2) and (3) it follows that, if  $X$  satisfies the energy inequality, then

$$(4) \quad F_n(t) \leq F_n(0) + \int_0^t 2\phi_{n-1}k_{n-1}X_{n-1}^2X_n ds - \int_0^t \sum_{i \geq n} \frac{2}{g_i}k_iX_i^2 ds.$$

The following proposition gives a sufficient condition for the energy equality. We first introduce some functional spaces. For all  $s \in \mathbb{R}$  and  $p \geq 1$ , let  $W^{s,p}$  denote the Banach space

$$W^{s,p} = \{x = (x_n)_{n \geq 1} \in \mathbb{R}^{\mathbb{N}} : \|x\|_{W^{s,p}}^p := \sum_{n \geq 1} 2^{psn}|x_n|^p < \infty\}.$$

In particular, we set  $H^s = W^{s,2}$  and  $H := H^0 = \ell^2(\mathbb{R})$ .

PROPOSITION 3. *Let  $T > 0$  and  $X$  be a weak solution with initial condition  $x \in H$ . If  $X \in L^3([0, T]; W^{\beta/3,3})$ , then  $X$  satisfies the energy equality on  $[0, T]$ .*

PROOF. Let  $t \in [0, T]$  and  $n \geq 1$ . By equation (2),

$$(\star) \quad 0 \leq E_n(t) + \int_0^t \sum_{i \leq n} \frac{2}{g_i}k_iX_i^2(u)du = E_n(0) - \int_0^t 2\phi_nk_nX_n^2(u)X_{n+1}(u)du.$$

To prove the energy equality, it is sufficient to take the limit for  $n \rightarrow \infty$  and show that the last term of  $(\star)$  converges to zero. By Young's inequality,

$$\int_0^t k_nX_n^2(u)|X_{n+1}(u)|du \leq \frac{2}{3} \int_0^t k_n|X_n(u)|^3du + \frac{1}{3} \int_0^t k_{n+1}|X_{n+1}(u)|^3du,$$

and the terms on the right-hand side converge to zero, since  $X \in L^3([0, T]; W^{\beta/3, 3})$ .  $\square$

## 2.2. Local existence and uniqueness.

PROPOSITION 4. *Let  $s > 0$  and suppose  $x \in H^s$ ,  $g \in H^{-s}$ . Then there exist  $\eta > 0$ , depending only on  $\|x\|_{H^s}$ , and a unique solution in the class  $\mathcal{H} := L^\infty([0, \eta]; H^s)$ .*

PROOF. In view of applying Banach's fixed point theorem, we introduce the operator  $\mathcal{F}$  on  $\mathcal{H}$  defined as follows. For all  $n \geq 1$  and  $t \in [0, \eta]$ , let

$$(\mathcal{F}V)_n(t) := x_n e^{-\frac{k_n}{g_n} t} + \int_0^t e^{-\frac{k_n}{g_n}(t-u)} \left[ \phi_{n-1} k_{n-1} V_{n-1}^2(u) - \phi_n k_n V_n(u) V_{n+1}(u) \right] du,$$

so that  $X$  is a solution if and only if it is a fixed point of  $\mathcal{F}$ . To apply Banach's fixed point theorem we must show that  $\mathcal{F}$  maps some ball  $B_{\mathcal{H}}(M) := \{v \in \mathcal{H} : \|v\|_{\mathcal{H}} \leq M\}$  into itself and that  $\mathcal{F}$  is a contraction on the ball. To this end, we will often use that if  $v \in \mathcal{H}$ , then  $|v_n(t)| \leq k_n^{-s} \|v\|_{\mathcal{H}}$  for all  $t \geq 0$ ,  $n \geq 1$ .

We deal with the first requirement, so suppose  $V \in B_{\mathcal{H}}(M)$ . For all  $n \geq 1$  and  $t \in [0, \eta]$ ,

$$\begin{aligned} |(\mathcal{F}V)_n(t)| &\leq |x_n| e^{-\frac{k_n}{g_n} t} + \|\phi\|_{\ell^\infty} \int_0^t e^{-\frac{k_n}{g_n}(t-u)} \left[ k_{n-1} V_{n-1}^2 + k_n |V_n V_{n+1}| \right] du \\ &\leq |x_n| e^{-\frac{k_n}{g_n} t} + \|\phi\|_{\ell^\infty} (k_{n-1}^{1-2s} + k_n^{1-s} k_{n+1}^{-s}) \|V\|^2 \int_0^t e^{-\frac{k_n}{g_n}(t-u)} du \\ &\leq |x_n| + 2\|\phi\|_{\ell^\infty} k_{n-1}^{-2s} g_n (1 - e^{-\frac{k_n}{g_n} t}) \|V\|^2, \end{aligned}$$

so  $\|\mathcal{F}V\|_{\mathcal{H}} \leq \|x\|_{H^s} + 2\|\phi\|_{\ell^\infty} \|V\|^2 L(\eta)$ , where we defined

$$L(t) := \left[ \sum_{n \geq 1} k_n^{2s} k_{n-1}^{-4s} g_n^2 (1 - e^{-\frac{k_n}{g_n} t})^2 \right]^{1/2},$$

and  $\sup_{0 \leq t \leq \eta} L(t) = L(\eta)$  by monotonicity. We claim that  $\lim_{\eta \rightarrow 0} L(\eta) = 0$ . Consider

$$L^2(\eta) = 2^{4\beta s} \sum_{n \geq 1} k_n^{-2s} g_n^2 (1 - e^{-\frac{k_n}{g_n} \eta})^2 \leq 2^{4\beta s} \sum_{n=1}^N k_n^{-2s} g_n^2 \left( \frac{k_n}{g_n} \eta \right)^2 + 2^{4\beta s} \sum_{n > N} k_n^{-2s} g_n^2.$$

Since  $g \in H^{-s}$ , we can choose  $N$  such that the second term is arbitrarily small, and then choose  $\eta$  in such a way that the first term is small too, hence  $L(\eta) \rightarrow 0$  as  $\eta \rightarrow 0$ .

Let  $M := 2\|x\|_{H^s}$ . If  $\eta$  is small enough so that  $L(\eta) \leq (4\|\phi\|_{\ell^\infty} M)^{-1}$ , then  $\|\mathcal{F}V\|_{\mathcal{H}} \leq M/2 + 2\|\phi\|_{\ell^\infty} M^2 L(\eta) \leq M$ , so the first requirement is satisfied for all  $\eta$  such that  $L(\eta) \leq (8\|\phi\|_{\ell^\infty} \|x\|_{H^s})^{-1}$ .

To prove that  $\mathcal{F}$  is a contraction, suppose  $X, Y \in B_{\mathcal{H}}(M)$ . For all  $n \geq 1$  and  $t \in [0, \eta]$ ,

$$|(\mathcal{F}X - \mathcal{F}Y)_n(t)| \leq \|\phi\|_{\ell^\infty} \int_0^t e^{-\frac{k_n}{g_n}(t-u)} \left[ k_{n-1} |X_{n-1}^2 - Y_{n-1}^2| + k_n |X_n X_{n+1} - Y_n Y_{n+1}| \right] du.$$

With the obvious decomposition  $ab - cd = \frac{1}{2}(a - c)(b + d) + \frac{1}{2}(b - d)(a + c)$  and recalling that for all  $j$ ,  $|v_j(t)| \leq k_j^{-s} \|v\|_{\mathcal{H}}$ , we get

$$\begin{aligned} |(\mathcal{F}X - \mathcal{F}Y)_n(t)| &\leq 2M \|\phi\|_{\ell^\infty} \|X - Y\|_{\mathcal{H}} \left[ k_{n-1} k_{n-1}^{-2s} + k_n k_n^{-s} k_{n+1}^{-s} \right] \int_0^t e^{-\frac{k_n}{g_n}(t-u)} du \\ &\leq 4M \|\phi\|_{\ell^\infty} \|X - Y\|_{\mathcal{H}} k_{n-1}^{-2s} g_n (1 - e^{-\frac{k_n}{g_n}t}), \end{aligned}$$

hence

$$\|\mathcal{F}X - \mathcal{F}Y\|_{\mathcal{H}} \leq 4M \|\phi\|_{\ell^\infty} \|X - Y\|_{\mathcal{H}} L(\eta).$$

Let  $\theta \in (0, 1)$ . Choose  $\eta$  small enough that  $L(\eta) \leq \theta(8\|\phi\|_{\ell^\infty} \|x\|_{H^s})^{-1}$ . Then the first requirement is satisfied and  $\|\mathcal{F}X - \mathcal{F}Y\|_{\mathcal{H}} \leq \theta \|X - Y\|_{\mathcal{H}}$ , and we conclude by Banach's fixed point theorem.  $\square$

### 3. The main result

In this section we prove our main result. The theorem follows immediately from our Theorem 13, which works in a slightly more general setting.

**THEOREM 5.** *Suppose that  $g_n$  is non-decreasing,  $\frac{k_n}{g_n}$  is eventually non-decreasing and that  $\sum_{n \geq 1} g_n^{-1} = \infty$ . If  $x \in H^s$  for all  $s > 0$ , then there exists a solution  $X$  with initial condition  $x$  such that  $X \in L^\infty([0, \infty); H^s)$  for all  $s > 0$ . This solution is unique in the class  $L_{\text{loc}}^3([0, \infty); W^{\beta/3, 3})$ .*

**3.1. The bounding sequence.** For all initial condition in  $H$ , we introduce a sequence of positive numbers which will be fundamental to bound all weak solutions.

**DEFINITION 6.** *A sequence  $y = (y_n)_{n \geq 1}$  is the bounding sequence for  $x \in H$  if it is defined by*

$$(5) \quad y_1 := y_2 := 2\|x\|_H^2,$$

$$(6) \quad y_{n+2} := C_{n+2}(y_{n+1}^{1/2})y_n + \sum_{i \geq n+2} x_i^2, \quad n \geq 1,$$

where for  $n \geq 3$ ,  $C_n : \mathbb{R}_+ \mapsto (0, 1)$  is the following increasing function,

$$C_n(v) := \left( 1 + \frac{1}{\frac{1}{2}g_n \|\phi\|_{\ell^\infty} v} \right)^{-1}, \quad v > 0.$$

**LEMMA 7.** *Suppose  $g$  is non-decreasing. Let  $T > 0$ ,  $x \in H$  and  $y$  be the bounding sequence for  $x$ . Suppose  $X$  is a weak solution with initial condition  $x$  that satisfies the energy inequality on  $[0, t]$  for all  $t < T$ . Then  $X_n^2(t) \leq y_n$  for all  $t \in [0, T)$  and all  $n \geq 1$ .*

**PROOF.** Define

$$(7) \quad d_n^2(t) = F_n(t) + \sum_{i \geq n+1} \int_0^t \frac{2}{g_i} k_i X_i^2(s) ds.$$

Roughly speaking,  $d_n^2$  represents the combination of the amount of energy stored in all modes larger than  $n$  at time  $t$  and the energy dissipated by these modes up to time  $t$ .

Notice that  $X_n^2(t) \leq F_n(t) \leq d_n^2(t) \leq \|x\|_H^2$  by the definition of  $d_n$  and (3). By (4) we deduce that

$$(8) \quad d_n^2(t) \leq \int_0^t 2\phi_{n-1}k_{n-1}X_{n-1}^2(s)X_n(s)ds - \int_0^t \frac{2}{g_n}k_nX_n^2(s)ds + F_n(0).$$

Define

$$\bar{d}_n := \sup_{0 \leq t < T} d_n(t) < \infty.$$

We claim that for all  $n \geq 1$

$$(9) \quad \bar{d}_{n+2}^2 \leq C_{n+2}(\bar{d}_{n+1})\bar{d}_n^2 + F_{n+2}(0).$$

Then, since  $y_1 := y_2 := 2\|x\|_H^2$  and since  $C_n$  is monotone increasing, an easy induction argument yields  $\bar{d}_n^2 \leq y_n$  for all  $n \geq 1$  and hence that

$$X_n^2(t) \leq d_n^2(t) \leq \bar{d}_n^2 \leq y_n,$$

for all  $n$  and all  $t$ .

We turn to the proof of the claim (9). By the Cauchy-Schwarz inequality applied to (8)

$$\begin{aligned} d_n^2(t) &\leq \int_0^t \|\phi\|_{\ell^\infty} k_{n-1} |X_{n-1}(s)| (X_{n-1}^2(s) + X_n^2(s)) ds + F_n(0) \\ &\leq \|\phi\|_{\ell^\infty} \bar{d}_{n-1} \int_0^t k_{n-1} (X_{n-1}^2(s) + X_n^2(s)) ds + F_n(0) \\ &\leq g_n \|\phi\|_{\ell^\infty} \bar{d}_{n-1} \int_0^t \left( \frac{k_{n-1}}{g_{n-1}} X_{n-1}^2(s) + \frac{k_n}{g_n} X_n^2(s) \right) ds + F_n(0), \end{aligned}$$

where we used the fact that  $g_n$  and  $k_n$  are non-decreasing with  $n$ . We get another bound from (7),

$$\begin{aligned} d_n^2(t) - d_{n-2}^2(t) &= F_n(t) - F_{n-2}(t) - \int_0^t \frac{2k_{n-1}}{g_{n-1}} X_{n-1}^2(s) ds - \int_0^t \frac{2k_n}{g_n} X_n^2(s) ds \\ &\leq -2 \int_0^t \left( \frac{k_{n-1}}{g_{n-1}} X_{n-1}^2(s) + \frac{k_n}{g_n} X_n^2(s) \right) ds, \end{aligned}$$

hence putting the former into the latter,

$$d_n^2(t) \leq d_{n-2}^2(t) - 2 \int_0^t \left( \frac{k_{n-1}}{g_{n-1}} X_{n-1}^2(s) + \frac{k_n}{g_n} X_n^2(s) \right) ds \leq d_{n-2}^2(t) - \frac{d_n^2(t) - F_n(0)}{\frac{1}{2}g_n \|\phi\|_{\ell^\infty} \bar{d}_{n-1}},$$

yielding

$$d_n^2(t) \leq \left( 1 + \frac{1}{\frac{1}{2}g_n \|\phi\|_{\ell^\infty} \bar{d}_{n-1}} \right)^{-1} d_{n-2}^2(t) + F_n(0) = C_n(\bar{d}_{n-1}) d_{n-2}^2(t) + F_n(0).$$

Taking the sup for  $s \in [0, T]$  yields the claimed inequality (9).  $\square$

Lemma 7 states that the variables  $X_n(t)$  can be bounded by the the bounding sequence  $y$ , so we will spend the rest of the section to show exponential decay for the bounding sequence  $y_n$ . As a first step we see that bounding sequences converge to 0.

LEMMA 8. *Suppose  $g$  is non-decreasing and  $\sum_{n \geq 1} g_n^{-1} = \infty$ . Let  $x \in H^s$  for some  $s > 0$  and let  $y$  be the bounding sequence for  $x$ . For all  $n \geq 1$ , let  $h_n := \sum_{j \geq n} \sum_{i \geq j} x_i^2$ . Then*

$$(10) \quad y_{n+2m} \leq y_n \prod_{i=1}^m C_{n+2i}(y_{n+2i-1}^{1/2}) + h_n, \quad \text{for all } n \geq 1, m \geq 0.$$

Moreover  $y_n \rightarrow 0$  as  $n \rightarrow \infty$ .

PROOF. Since  $C_j \leq 1$  for all  $j$ , inequality (10) is easily proved by induction on  $m$  using (6).

From this we deduce that  $y$  is bounded. Since  $v \mapsto C_j(v)$  is monotone increasing, we may replace the bound for  $y$  inside  $C_j$  yielding that

$$C_j(y_{j-1}^{1/2}) \leq (1 + cg_j^{-1})^{-1}, \quad j \geq 1,$$

for some constant  $c > 0$ . Since  $\sum_{j \geq 1} g_j^{-1} = \infty$ , then  $\prod_{j \geq 1} (1 + cg_j^{-1})^{-1} = 0$ . Since  $g$  is monotone, then  $\prod_{i \geq 1} (1 + cg_{n+2i}^{-1})^{-1} = 0$  too, hence by considering (10) for  $n$  and  $n + 1$ , we get,

$$\limsup_{j \geq n} y_j \leq h_n + h_{n+1}.$$

Since  $x \in H^s$ ,  $\lim_{n \rightarrow \infty} h_n = 0$ , therefore  $y_n \rightarrow 0$ .  $\square$

The next step is to introduce in Definition 9 below a special sub-sequence of the indices of  $g_n$ , this step is necessary because the hypothesis  $\sum_n g_n = \infty$  does not provide enough information on the rate of divergence of the series.

DEFINITION 9. *Given a sequence  $g$  with  $\sum_{n \geq 1} g_n^{-1} = \infty$ , a positive integer  $n_0$  and real numbers  $\theta > 0, s > 0$ , define by induction on  $k \geq 0$ ,*

$$(11) \quad n_{k+1} := \inf \left\{ n \geq n_k + 2 : \sum_{j=n_k+2}^n g_j^{-1} \geq 2^{-sk\theta} \right\} < \infty.$$

Notice that the definition above gives a finite number, because  $\sum_{j \geq 1} g_j^{-1} = \infty$ . The importance of this definition will be clear with the next two lemmas.

LEMMA 10. *Suppose  $g$  is non-decreasing and  $\sum_{n \geq 1} g_n^{-1} = \infty$ . Let  $x \in H^s$  for some  $s > 0$  and let  $y$  be the bounding sequence for  $x$ . Then there exist  $n_0 \geq 1$  and  $\theta > 0$  such that the sequence  $(n_k)_{k \geq 0}$  given in Definition 9 satisfies the following inequality:*

$$(12) \quad \sup_{j \geq n_k} y_j \leq 2^{-2sk}, \quad k \geq 0.$$

PROOF. In view of applying Lemma 8, we need to bound  $y_n$  and  $h_n$  for  $n$  large. Since  $x \in H^s$ , then for any  $\epsilon > 0$ ,  $x_n \leq \epsilon 2^{-sn}$  eventually and in particular for  $n$  large,

$$h_n = \sum_{j \geq n} \sum_{i \geq j} x_i^2 = \sum_{j \geq 1} j x_{j+n-1}^2 \leq 2^{-2sn} \epsilon \sum_{j \geq 1} j 2^{-2s(j-1)}.$$

so for any  $\eta > 0$ ,  $h_n \leq \eta 2^{-2sn}$  eventually. We also know from Lemma 8 that  $y_n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus fix some  $\eta > 0$  and let  $n_0$  be large enough that for all  $n \geq n_0$ ,

$$(13) \quad y_n \leq 1 \quad \text{and} \quad h_n \leq \eta 2^{-2sn}.$$

We now proceed to prove (12) by induction on  $k \geq 0$ . The initial step is simply given by the definition of  $n_0$ .



We turn to the induction step. Suppose  $\sup_{j \geq n_k} y_j \leq 2^{-2sk}$ . Then for  $j \geq n_k + 1$ ,

$$C_j(y_{j-1}^{1/2})^{-1} := 1 + \frac{1}{\frac{1}{2}g_j \|\phi\|_{\ell^\infty} y_{j-1}^{1/2}} \geq 1 + \frac{1}{\frac{1}{2}g_j \|\phi\|_{\ell^\infty} 2^{-sk}} = 1 + c2^{sk} g_j^{-1},$$

where  $c = 2/\|\phi\|_{\ell^\infty}$ . By (10) we have then, for  $n \geq n_k - 1$ ,

$$y_{n+2m} \leq y_n \prod_{i=1}^m (1 + c2^{sk} g_{n+2i}^{-1})^{-1} + h_n,$$

By the monotonicity of  $g$ ,

$$\prod_{i=1}^m (1 + c2^{sk} g_{n+2i}^{-1}) \geq 1 + c2^{sk} \sum_{i=1}^m g_{n+2i}^{-1} \geq 1 + c2^{sk} \frac{1}{2} \sum_{j=n+2}^{n+2m} g_j^{-1}.$$

By the definition of  $n_{k+1}$  in (11), if  $n \leq n_k$  and  $n + 2m \geq n_{k+1}$  we have

$$\sum_{j=n+2}^{n+2m} g_j^{-1} \geq 2^{-sk} \theta.$$

Collecting all conditions, we have proved that if  $n \in \{n_k - 1, n_k\}$  and  $n + 2m \geq n_{k+1}$ , then

$$y_{n+2m} \leq y_n (1 + \frac{1}{2}c\theta)^{-1} + h_n.$$

Since  $n \geq n_k - 1 \geq n_{k-1}$ , then by inductive hypothesis  $y_n \leq 2^{-2s(k-1)}$ ; moreover since  $n \geq n_k - 1 \geq k$ , then by the second one of (13),  $h_n \leq \eta 2^{-2sk}$ , so the bound above becomes

$$y_{n+2m} \leq 2^{-2s(k-1)} (1 + \frac{1}{2}c\theta)^{-1} + \eta 2^{-2sk}.$$

Now we choose  $\theta$  large enough and  $\eta$  small enough that

$$2^{2s} (1 + \frac{1}{2}c\theta)^{-1} + \eta \leq 2^{-2s},$$

to get

$$y_{n+2m} \leq 2^{-2s(k+1)}, \quad n \in \{n_k - 1, n_k\}, \quad n + 2m \geq n_{k+1}.$$

Since for all  $j \geq n_{k+1}$  there exist  $n$  and  $m$  such that  $n_k - 1 \leq n \leq n_k$  and  $j = n + 2m \geq n_{k+1}$ , we have proved

$$\sup_{j \geq n_{k+1}} y_j \leq 2^{-2s(k+1)},$$

closing the induction.  $\square$

LEMMA 11. *Suppose  $g$  is non-decreasing and  $\sum_{n \geq 1} g_n^{-1} = \infty$ . Let  $n_0 \geq 1$  and  $\theta > 0$  be constant. If  $(n_k)_{k \geq 0}$  is as in Definition 9 then there exist infinitely many  $k \geq 1$  such that  $n_{k+1} = n_k + 2$ .*

PROOF. Suppose that there exists a non-negative integer  $r$  such that  $n_{k+1} \geq n_k + 3$  for all  $k \geq r$ . By the definition of the sequence  $(n_k)_{k \geq 0}$ , we know that for  $k \geq r$ ,

$$\sum_{j=n_k+2}^{n_{k+1}-1} g_j^{-1} < 2^{-sk} \theta.$$

Summing on  $k$  we obtain

$$\sum_{k \geq r} \sum_{j=n_k+2}^{n_{k+1}-1} g_j^{-1} < \infty,$$

hence since  $\sum_{j \geq n_r} g_j^{-1} = \infty$ ,

$$(14) \quad \sum_{k \geq r} (g_{n_k}^{-1} + g_{n_{k+1}}^{-1}) = \infty.$$

But  $g_{n_{k+1}}^{-1} \leq g_{n_k}^{-1} \leq g_{n_{k-1}+2}^{-1} \leq 2^{-s(k-1)}\theta$ , which is in contradiction with (14). Hence there exist infinitely many  $k$  such that  $n_{k+1} = n_k + 2$ .  $\square$

LEMMA 12. *Let  $x \in H^s$  for some  $s > 0$  and let  $y$  be the bounding sequence for  $x$ . Suppose that  $g_n$  is non-decreasing,  $g_n 2^{-sn}$  is eventually non-increasing and that  $\sum_{n \geq 1} g_n^{-1} = \infty$ . Then*

$$\sum_{n \geq 1} 2^{2sn} y_n < \infty.$$

PROOF. Let us recall the recursion (6) that defines the bounding sequence  $y$ ,

$$(15) \quad y_{n+2} := c_{n+2} y_n + f_{n+2}, \quad n \geq 1,$$

where

$$c_n := C_n(y_{n-1}^{1/2}) := \left( 1 + \frac{1}{\frac{1}{2} g_n \|\phi\|_{\ell^\infty} y_{n-1}^{1/2}} \right)^{-1}, \quad n \geq 3,$$

and where  $f_n := F_n(0)$ ,  $n \geq 3$ . Since  $x \in H^s$ , it is immediate that

$$(16) \quad \sum_{n \geq 1} 2^{2sn} f_n < \infty,$$

so our strategy will be to show that  $c_n \rightarrow 0$  as  $n \rightarrow \infty$ .

By Lemma 10 there exist  $n_0$  and  $\theta$  such that the sequence  $(n_i)_{i \geq 1}$  of Definition 9 satisfies

$$(17) \quad \sup_{j \geq n_i} y_j \leq 2^{-2si}, \quad i \geq 0.$$

*A fortiori* these inequalities hold also if we take larger values for  $n_0$  and  $\theta$ , so let  $\theta$  be large enough to verify inequality (20) below and let  $n_0$  be large enough that:

- (1)  $f_n \leq \frac{1}{2} 2^{-2sn}$  for  $n \geq n_0$  (a consequence of (16));
- (2)  $n \mapsto g_n 2^{-sn}$  is non-increasing for  $n \geq n_0$ .

By Lemma 11 there exists  $k$  such that  $n_{k+1} = 2 + n_k$ , that is,  $g_{n_{k+2}}^{-1} \geq 2^{-sk}\theta$  hence

$$(18) \quad g_{n_k+m} \leq \frac{1}{2^{2s}\theta} 2^{s(k+m)}, \quad m \geq 2.$$

We have all the ingredients to prove the following inequality:

$$(19) \quad y_{n_k+m} \leq 2^{2s} 2^{-2s(k+m)}, \quad m \geq 0.$$

Let us proceed by induction on  $m$ . The initial steps for  $m = 0$  and  $m = 1$  follow immediately from (17) with  $i = k$ .

For the inductive step, suppose the inequality (19) is true up to  $m+1$ . By (18) and the inductive hypothesis,

$$c_{n_k+m+2} = \left( 1 + \frac{1}{\frac{1}{2}g_{n_k+m+2}\|\phi\|_{\ell^\infty}y_{n_k+m+1}^{1/2}} \right)^{-1} \leq \left( 1 + \frac{2\theta}{\|\phi\|_{\ell^\infty}} \right)^{-1} \leq \frac{1}{2}2^{-4s},$$

if we choose  $\theta$  large enough that

$$(20) \quad \left( 1 + \frac{2\theta}{\|\phi\|_{\ell^\infty}} \right)^{-1} \leq \frac{1}{2}2^{-4s}.$$

Moreover, since  $n_k \geq k-1$ , we have

$$f_{n_k+m+2} \leq \frac{1}{2}2^{-2s(n_k+m+2)} \leq \frac{1}{2}2^{2s}2^{-2s(k+m+2)},$$

hence

$$y_{n_k+m+2} = c_{n_k+m+2}y_{n_k+m} + f_{n_k+m+2} \leq 2^{2s}2^{-2s(k+m+2)},$$

thus closing the induction.

Inequality (19) says us that  $y_n \rightarrow 0$  at least as fast as  $2^{-2sn}$ . To get  $\sum_n 2^{2sn}y_n < \infty$  we need a little bit more. We proved above that for any  $\theta$  large enough there exists  $n_k$  such that

$$\sup_{j \geq n_k} c_j \leq \left( 1 + \frac{2\theta}{\|\phi\|_{\ell^\infty}} \right)^{-1}.$$

By the arbitrariness of  $\theta$ ,  $\lim_{n \rightarrow \infty} c_n = 0$ . This together with (15) and (16) proves that

$$\sum_{n \geq 1} 2^{2sn}y_n < \infty. \quad \square$$

### 3.2. Global existence, uniqueness and regularity.

**THEOREM 13.** *Let  $x \in H^s$  for some  $s > \frac{\beta}{3}$ . Suppose that  $g_n \in H^{-s}$  is non-decreasing,  $g_n 2^{-sn}$  is eventually non-increasing and that  $\sum_{n \geq 1} g_n^{-1} = \infty$ . Then there exists a solution in the class  $L^\infty([0, \infty); H^s)$ . This solution is unique in the class  $L^3_{\text{loc}}([0, \infty); W^{\beta/3, 3})$ .*

**PROOF.** Let  $T > 0$  be the maximal time of existence in  $H^s$  of the solution provided by Proposition 4. In particular,  $X \in L^\infty([0, t]; H^s)$  for all  $t < T$  and, since  $s > \beta/3$ ,  $X \in L^3([0, t]; W^{\beta/3, 3})$  for all  $t < T$ . Hence, by Proposition 3,  $X$  satisfies the energy equality on  $[0, t]$ . Lemma 7 applies, so if  $y$  denotes the bounding sequence for  $x$ , we have

$$(21) \quad X_n^2(t) \leq y_n, \quad n \geq 1, \quad t \in [0, T].$$

By Lemma 12

$$\sup_{t \in [0, T]} \|X(t)\|_{H^s}^2 \leq \sum_{n \geq 1} 2^{2sn}y_n < \infty.$$

If  $T = \infty$  we just proved  $X \in L^\infty([0, \infty); H^s)$ . Suppose by contradiction that  $T < \infty$ . Then the bound in (21) can be extended to  $t \in [0, T]$  by the continuity of  $X_n$  hence again by Lemma 12,  $X(T) \in H^s$  and it would be possible to apply Proposition 4, in contradiction with the maximality of  $T$ .

Finally we turn to uniqueness in  $L^3_{\text{loc}}([0, \infty); W^{\beta/3, 3})$ . By Proposition 3, Lemma 7 and Lemma 12, if  $X$  is a solution of class  $L^3_{\text{loc}}([0, \infty); W^{\beta/3, 3})$ , then  $X$  is also of class  $L^\infty_{\text{loc}}([0, \infty); H^s)$ , hence by Proposition 4 it is unique.  $\square$

**3.3. Additional remarks.** The last part of the paper is devoted to some final remarks about our results. They have been collected here in order to give a more complete understanding of the problem, while focusing, in the main body of the paper, on the assumptions corresponding to those of [13].

3.3.1. *A useful generalization.* The results presented in the previous sections allow for more general coefficients  $\phi$ . Namely, assume that

$$(22) \quad \phi_n = \phi_n(t, X_{n-m}, X_{n-m+1}, \dots, X_{n+m}),$$

for all  $n \geq 1$ , where  $m \geq 1$  is a fixed integer. For convenience we set  $X_{-m} = X_{-m+1} = \dots = X_0 = 0$ . Assume moreover that the functions  $(\phi_n)_{n \geq 1}$  are uniformly bounded and uniformly Lipschitz. This ensures that the local existence and uniqueness theorem (Proposition 4) still holds. In Proposition 3 and Lemma 7 we only use the uniform boundedness, while lemmas 8, 10-12 deal only with bounding sequences.

The above model has a nice application to the averaged Navier-Stokes system studied by Tao in [14]. By making a special average on the transport of the NS equations, the author derives a vector-valued dyadic system, very similar to (1) but with four components for each  $n$ . A general version of this averaged system is

$$(23) \quad \begin{cases} X'_{1,n} = -\frac{k_n^\alpha}{g_n} X_{1,n} + k_n^\gamma (-C_1 X_{3,n} X_{4,n} - C_2 X_{1,n} X_{2,n} - C_3 X_{1,n} X_{3,n} + C_4 X_{4,n-1}^2), \\ X'_{2,n} = -\frac{k_n^\alpha}{g_n} X_{2,n} + k_n^\gamma (C_2 X_{1,n}^2 - C_5 X_{3,n}^2), \\ X'_{3,n} = -\frac{k_n^\alpha}{g_n} X_{3,n} + k_n^\gamma (C_3 X_{1,n}^2 + C_5 X_{2,n} X_{3,n}), \\ X'_{4,n} = -\frac{k_n^\alpha}{g_n} X_{4,n} + k_n^\gamma C_1 X_{1,n} X_{3,n} - k_{n+1}^\gamma C_4 X_{4,n} X_{1,n+1}, \\ X_{\cdot,n}(0) = x_{\cdot,n}, \end{cases}$$

for all  $t > 0$  and  $n \geq 1$ .

Here  $X = (X_{i,n})_{i \in \{1,2,3,4\}, n \geq 1}$  is a family of real functions,  $X_{i,n} : [0, \infty) \rightarrow \mathbb{R}$ ;  $X_{\cdot,0} \equiv 0$ ;  $x = (x_{i,n})_{i \in \{1,2,3,4\}, n \geq 1}$  is the given initial condition,  $k_n = 2^{\beta n}$  with  $\beta > 0$ , and  $C_1, \dots, C_5$  are five real constants.

In the framework of Navier-Stokes equations the constants  $\alpha = 2$  and  $\gamma = \frac{5}{2}$  give a strictly supercritical regime. In [14] the author shows that this system with a suitable initial condition develops a singularity. For the system (23) the critical regime is for  $\alpha = \gamma$  and  $g \equiv 1$  (it is the regime in which the transport effects are of the same order of the dissipative effect) whereas the logarithmically supercritical regime conjectured in [13] is given by  $\alpha = \gamma$  and  $g$  such that  $\sum_n g_n^{-1} = \infty$ .

The latter case can be included in our model (1) with general coefficients (22). Indeed, by summing up the components  $X_n^2 := \sum_{i=1}^4 X_{i,n}^2$  one gets

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} X_n^2 &= -\frac{k_n^\alpha}{g_n} X_n^2 + C_4 (k_n^\gamma X_{4,n-1}^2 X_{1,n} - k_{n+1}^\gamma X_{4,n}^2 X_{1,n+1}) \\ &= -\frac{k_n^\alpha}{g_n} X_n^2 + \phi_n k_n^\gamma X_{n-1}^2 X_n - \phi_{n+1} k_{n+1}^\gamma X_n^2 X_{n+1} \end{aligned}$$

and this can be reduced to the system (1), when  $\alpha = \gamma$  by a suitable choice of  $\beta$  and appropriate functions  $\phi_n(t)$  depending on  $n$  and  $t$  and uniformly bounded.

3.3.2. *Conditions for smoothing.* Here we study the smoothing effect of the dissipative part. We work under the assumptions of Theorem 5.

The linear operator. Consider the system  $Z'_n = -\frac{k_n}{g_n} Z_n$ ,  $n \geq 1$ , the linear part of (1).

LEMMA 14. Assume additionally that  $\frac{ng_n}{k_n} \rightarrow 0$ . If  $x \in \ell^2$  and  $Z$  is the solution starting at  $x$ , then  $Z(t) \in L^\infty([\epsilon, \infty); H^s)$  for every  $\epsilon > 0$  and every  $s > 0$ .

PROOF. Clearly  $Z_n(t) = x_n \exp(-\frac{k_n}{g_n}t)$  and  $\sup_n(2^{sn}Z_n(t)) < \infty$  for all  $s > 0$ ,  $t > 0$  if and only if  $\frac{ng_n}{k_n} \rightarrow 0$ .  $\square$

REMARK 15. If  $\frac{ng_n}{k_n} \not\rightarrow 0$ , the linear dissipation may not have a smoothing effect. Indeed, it is easy to construct a counterexample. Choose  $n_1 \geq 1$  and set  $n_{p+1} = n_p 2^{\beta k_{n_p}/n_p}$ ,  $p \geq 1$ ,  $g_{n_p} = k_{n_p}/n_p$ , and define  $g_n = g_{n_p}$  for  $n_p \leq n \leq m_p$ , and  $g_n = g_{n_p} k_n/k_{m_p}$  for  $m_p + 1 \leq n < n_{p+1}$ , where  $m_p = n_p + k_{n_p}/n_p$ . It is easy to verify that  $(g_n)_{n \geq 1}$  satisfies our standing assumptions and that there are sequences  $(x_n)_{n \geq 1} \in \ell^2$  such that the corresponding solution  $Z$  is not smooth.

Smoothing by dissipation. We now analyse the smoothing effect for the non-linear equation. Our final result is the following.

THEOREM 16. Assume additionally that  $\frac{ng_n}{k_n} \rightarrow 0$ . If  $s > \beta$  and  $X$  is a solution such that  $X(0) \in H^s$  and  $X \in L^\infty([0, T]; H^s)$ , then  $X \in L^\infty_{\text{loc}}((0, T]; H^s)$  for every  $s > 0$ .

The theorem follows immediately from the following lemma.

LEMMA 17. Under the same assumptions of the previous theorem, let  $s_1 > \beta$  and  $s_2 \in (s_1, 2s_1 - \beta)$ . If  $X$  is a solution such that  $X(0) \in H^{s_1}$  and  $X \in L^\infty([0, T]; H^{s_1})$ , then  $X \in L^\infty_{\text{loc}}((0, T]; H^{s_2})$ . More precisely, there is a non-decreasing upper semi-continuous function  $\varphi : (0, \infty) \rightarrow \mathbf{R}$  such that  $\varphi$  is continuous in 0 with  $\varphi(0) = 0$ , and

$$\sup_{t \in [0, T]} (\varphi(t) \|X(t)\|_{H^{s_2}}) < \infty.$$

PROOF. We have that

$$2^{s_2 n} X_n(t) = 2^{s_2 n} e^{-\frac{k_n}{g_n}t} X_n(0) + 2^{s_2 n} \int_0^t e^{-\frac{k_n}{g_n}(t-s)} (\phi_{n-1} k_{n-1} X_{n-1}^2 - \phi_n k_n X_n X_{n+1}) ds,$$

and consider the two terms on the right hand side separately.

For the non-linear term, we use the inequality  $|X_n(t)| \leq 2^{-s_1 n} \|X\|_{L^\infty(H^{s_1})}$  to get

$$(24) \quad \begin{aligned} & \left| 2^{s_2 n} \int_0^t e^{-\frac{k_n}{g_n}(t-s)} (\phi_{n-1} k_{n-1} X_{n-1}^2 - \phi_n k_n X_n X_{n+1}) ds \right| \leq \\ & \leq c \|X\|_{L^\infty(H^{s_1})}^2 \|\phi\|_{\ell^\infty} 2^{n(s_2 - 2s_1)} g_n \leq c \|X\|_{L^\infty(H^{s_1})}^2 \|\phi\|_{\ell^\infty} 2^{n(s_2 - 2s_1 + \beta)} \in \ell^2, \end{aligned}$$

since  $g_n \leq ck_n$  and, by the choice of  $s_2$ ,  $s_2 - 2s_1 + \beta < 0$ .

For the term with the initial condition we notice that

$$2^{s_2 n} e^{-\frac{k_n}{g_n}t} |X_n(0)| = 2^{(s_2 - s_1)n} e^{-\frac{k_n}{g_n}t} (2^{s_1 n} |X_n(0)|) \leq \psi(t) (2^{s_1 n} |X_n(0)|) \in \ell^2,$$

where  $\psi(t) = \sup_n (2^{(s_2 - s_1)n} \exp(-\frac{k_n}{g_n}t))$ . It is easy to check that  $\psi$  is non-increasing, lower semi-continuous and  $\psi(t) \uparrow \infty$  as  $t \downarrow 0$ . Choose  $\varphi = 1/\psi$  to conclude the proof.  $\square$

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