

**LOW ENERGY SOLUTIONS FOR SINGULARLY PERTURBED
COUPLED NONLINEAR SYSTEMS ON A RIEMANNIAN
MANIFOLD WITH BOUNDARY.**

MARCO GHIMENTI AND ANNA MARIA MICHELETTI

ABSTRACT. Let (M, g) be a smooth, compact Riemannian manifold with smooth boundary, with $n = \dim M = 2, 3$. We suppose the boundary ∂M to be a smooth submanifold of M with dimension $n - 1$. We consider a singularly perturbed nonlinear system, namely Klein-Gordon-Maxwell-Proca system, or Klein-Gordon-Maxwell system or Schrödinger-Maxwell system on M . We prove that the number of low energy solutions, when the perturbation parameter is small, depends on the topological properties of the boundary ∂M , by means of the Lusternik Schnirelmann category. Also, these solutions have a unique maximum point that lies on the boundary.

1. INTRODUCTION

Let (M, g) be a smooth, compact Riemannian manifold with smooth boundary, with $n = \dim M = 2, 3$. We suppose the boundary ∂M to be a smooth submanifold of M with dimension $n - 1$.

We consider the following singularly perturbed electrostatic Klein-Gordon-Maxwell-Proca (shortly KGMP) system on M with Neumann boundary condition

$$(1) \quad \begin{cases} -\varepsilon^2 \Delta_g u + au = |u|^{p-2}u + \omega^2(qv - 1)^2u & \text{in } M \\ -\Delta_g v + (1 + q^2 u^2)v = qu^2 & \text{in } M \\ \frac{\partial u}{\partial \nu} = 0, \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial M \end{cases}$$

Here $\varepsilon > 0$, $a > 0$, $q > 0$, $\omega \in (-\sqrt{a}, \sqrt{a})$ and $4 \leq p < 2^*$ being $2^* = 6$ for $n = 3$ or $2^* = +\infty$ for $n = 2$.

The Neumann condition for the function u is interesting since it shows how the topological properties of the boundary influence the number of solutions of (1). Moreover from a physical viewpoint, give a Neumann condition for the second function v corresponds to fix the electrical field on ∂M which is a natural condition (for a more detailed discussion on this topic, we refer to [8, 10]).

The study of KGMP systems recently has known a rise of interest in the mathematical community. In [13, 14, 15] equation (1) has been studied on a Riemannian boundaryless manifold M . A similar problem has been considered in a flat domain

2010 *Mathematics Subject Classification.* 35J60, 35J20, 53C80, 81V10.

Key words and phrases. Riemannian manifolds with boundary, Klein-Gordon-Maxwell systems, Schrödinger-Maxwell systems, Lusternik Schnirelmann category, one peak solutions.

The authors were partially supported by 2014 GNAMPA project: “Equazioni di campo non-lineari: solitoni e dispersione”.

Ω by D'Aprile and Wei [5, 6]. In the context of flat domains, moreover, many authors have dealt with Klein Gordon Maxwell systems without singular perturbation in the Laplacian term [1, 2, 3, 7, 9, 16].

In this paper we prove the following result.

Theorem 1. *For ε small enough the KGMP system (1) has at least $\text{cat}\partial M$ non constant distinct solutions $(u_\varepsilon, v_\varepsilon)$ with low energy. Here $\text{cat}\partial M$ is the Lusternik Schnirelmann category. Moreover the functions u_ε have a unique maximum point $P_\varepsilon \in \partial M$ and $u_\varepsilon = Z_{\varepsilon, P_\varepsilon} + \Psi_\varepsilon$ where $Z_{\varepsilon, P_\varepsilon}$ is defined in (6) and $\|\Psi_\varepsilon\|_{L^\infty(M)} \rightarrow 0$.*

Remark 2. We notice that the same result can be obtained verbatim for the electrostatic Klein-Gordon-Maxwell (shortly KGM) system with Neumann/Dirichlet boundary condition,

$$(2) \quad \begin{cases} -\varepsilon^2 \Delta_g u + au = |u|^{p-2}u + \omega^2(qv-1)^2u & \text{in } M \\ -\Delta_g v + q^2 u^2 v = qu^2 & \text{in } M \\ \frac{\partial u}{\partial \nu} = 0, v = 0 & \text{on } \partial M \end{cases}$$

and for the Schroedinger-Maxwell system with Neumann/Dirichlet boundary condition, for $\varepsilon > 0$, $a > 0$, $q > 0$, $\omega \in \mathbb{R}$ and $4 < p < 2^*$

$$(3) \quad \begin{cases} -\varepsilon^2 \Delta_g u + au + \omega uv = |u|^{p-2}u & \text{in } M \\ -\Delta_g v = qu^2 & \text{in } M \\ \frac{\partial u}{\partial \nu} = 0, v = 0 & \text{on } \partial M \end{cases}$$

We explicitly treat systems (1) and (2) in the paper, pointing out the differences in the proofs whenever necessary. For system (3) the estimates are easier and left to the reader. We just mention that we have to rule out the case $p = 4$ in order to have a smooth Nehari manifold (cfr. section 3)

Remark 3. The result of this paper relies on the topology of the boundary ∂M . In a forthcoming paper the authors will point out how the geometry of ∂M affects the number of one peaked solutions.

The paper is structured as follows: in Section 2 some basic concepts are recalled and it is introduced the variational structure of the problem. The Nehari manifold that is a natural constraint for the variational problem is introduced in Section 3. Section 4 contains the lines of the proof of Theorem 1, while in sections 5, 6 and 7 the steps of the proof are explained in full details. The profile description is contained in Section 8. Some technical result is postponed in Section 9 to do not overload the presentation of the results.

2. PRELIMINARIES

We recall some well know result on Riemannian manifold with boundary. At first we introduce a coordinates system for a neighborhood of the boundary ∂M .

If ξ belongs to ∂M , let $\bar{y} = (y_1, \dots, y_{n-1})$ be Riemannian normal coordinates on the $n-1$ manifold ∂M at the point ξ . For a point $x \in M$ close to ξ , there exists a unique $\bar{x} \in \partial M$ such that $d_g(x, \partial M) = d_g(x, \bar{x})$. We set $\bar{y}(x) \in \mathbb{R}^{n-1}$ the normal coordinates for \bar{x} and $y_n(x) = d_g(x, \partial M)$. Then we define a chart $\Psi_\xi^\partial : \mathbb{R}_+^n \rightarrow M$ such that $(\bar{y}(x), y_n(x)) = \left(\Psi_\xi^\partial\right)^{-1}(x)$. These coordinates are called *Fermi coordinates* at $\xi \in \partial M$.

We note by d_g^∂ and \exp^∂ respectively the geodesic distance and the exponential map on by ∂M .

We define the following neighborhood of a point $\xi \in \partial M$

$$I_\xi(\rho, R) = \{x \in M : y_n = d_g(x, \partial M) < \rho \text{ and } |\bar{y}| = d_g^\partial(\exp_q^\partial(\bar{y}(x)), \xi) < R\}.$$

where $R, \rho > 0$ are smaller than the injectivity radius of M . Often we will denote $I_\xi(R) = I_\xi(R, R)$ and, if no ambiguity is present, we simply use I_ξ for $I_\xi(R, \rho)$ or for $I_\xi(R)$.

Let $\mathbb{R}_+^n = \{y = (\bar{y}, y_n) : \bar{y} \in \mathbb{R}^{n-1}, y_n \geq 0\}$. It is well known that there exists a least energy solution $V \in H^1(\mathbb{R}_+^n)$ of the equation

$$(4) \quad \begin{cases} -\Delta V + (a - \omega^2)V = |V|^{p-2}V, & V > 0 \quad \text{on } \mathbb{R}_+^n \\ \frac{\partial V}{\partial y_n}|_{(\bar{y}, 0)} = 0. \end{cases}$$

We remark that, set U the least energy solution of

$$(5) \quad \begin{cases} -\Delta U + (a - \omega^2)U = |U|^{p-2}U, & U > 0 \quad \text{on } \mathbb{R}^n \\ U \in H^1(\mathbb{R}^n) \end{cases}$$

which is radially symmetric, we have that $V = U|_{y_n \geq 0}$.

Set $V_\varepsilon(y) = V(\frac{y}{\varepsilon})$, and fixed $\xi \in \partial M$ we define the function $Z_{\varepsilon, \xi}(x)$ as

$$(6) \quad Z_{\varepsilon, \xi}(x) = \begin{cases} V_\varepsilon(y(x)) \chi_R(|\bar{y}(x)|) \chi_\rho(y_n(x)) & \text{if } x \in I_\xi \\ 0 & \text{otherwise} \end{cases}$$

where $\chi_T : \mathbb{R}^+ \rightarrow [0, 1]$ is a smooth cut off function such that $\chi_T(s) \equiv 1$ for $0 \leq s \leq T/2$, $\chi_T(s) \equiv 0$ for $s \geq T$ and $|\chi_T'(s)| \leq 1/T$.

We endow $H_g^1(M)$ with the scalar product and norm

$$\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^n} \int_M \varepsilon^2 \nabla_g u \nabla_g v + (a - \omega^2) u v d\mu_g; \quad \|u\|_\varepsilon = \langle u, u \rangle_\varepsilon^{1/2}.$$

We call H_ε the space H_g^1 equipped with the norm $\|\cdot\|_\varepsilon$. We also define L_ε^p as the space $L_g^p(M)$ endowed with the norm $\|u\|_{\varepsilon, p} = \frac{1}{\varepsilon^n} \left(\int_M u^p d\mu_g \right)^{1/p}$. We also use the obvious notation $H_{0, \varepsilon}$ for the space $H_{0, g}^1$ with the norm $\|\cdot\|_\varepsilon$, where H_g^1 (resp. $H_{0, g}^1$) is the closure of $C^\infty(M)$ (resp. $C_0^\infty(M)$) with respect to the norm $\int_M |\nabla_g u|^2 + u^2$ (resp. $\int_M |\nabla_g u|^2$).

2.1. The function ψ . First of all, we reduce the system to a single equation. In order to overcome the problems given by the competition between u and v , using an idea of Benci and Fortunato [2], we introduce the map ψ defined by the equation

$$(7) \quad \begin{cases} -\Delta_g \psi + (1 + q^2 u^2) \psi = q u^2 & \text{in } M \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial M \end{cases}$$

in case of Neumann boundary condition or by

$$(8) \quad \begin{cases} -\Delta_g \psi + q u^2 \psi = q u^2 & \text{in } M \\ \psi = 0 & \text{on } \partial M \end{cases}$$

in case of Dirichlet boundary condition.

In what follows we call $H = H_g^1$ for the Neumann problem and $H = H_{0, g}^1$ for the Dirichlet problem. Thus with abuse of language we will say that $\psi : H \rightarrow H$ in both (7) and (8). Moreover, from standard variational arguments, it is easy to see that ψ is well-defined in H and it holds

$$(9) \quad 0 \leq \psi(u) \leq 1/q$$

for all $u \in H$.

Lemma 4. *The map $\psi : H \rightarrow H$ is C^2 and its differential $\psi'(u)[h] = V_u[h]$ at u is the map defined by*

$$(10) \quad -\Delta_g V_u[h] + (1 + q^2 u^2) V_u[h] = 2qu(1 - q\psi(u))h \text{ for all } h \in H.$$

in case of Neumann boundary condition or

$$(11) \quad -\Delta_g V_u[h] + q^2 u^2 V_u[h] = 2qu(1 - q\psi(u))h \text{ for all } h \in H.$$

in case of Dirichlet boundary condition.

Also, we have

$$0 \leq \psi'(u)[u] \leq \frac{2}{q}.$$

Finally, the second derivative $(h, k) \rightarrow \psi''(u)[h, k] = T_u(h, k)$ is the map defined by the equation

$$-\Delta_g T_u(h, k) + (1 + q^2 u^2) T_u(h, k) = -2q^2 u(kV_u(h) + hV_u(k)) + 2q(1 - q\psi(u))hk$$

in case of Neumann boundary condition or

$$-\Delta_g T_u(h, k) + q^2 u^2 T_u(h, k) = -2q^2 u(kV_u(h) + hV_u(k)) + 2q(1 - q\psi(u))hk$$

in case of Dirichlet boundary condition.

Lemma 5. *The map $\Theta : H \rightarrow \mathbb{R}$ given by*

$$\Theta(u) = \frac{1}{2} \int_M (1 - q\psi(u)) u^2 d\mu_g$$

is C^2 and

$$\Theta'(u)[h] = \int_M (1 - q\psi(u))^2 u h d\mu_g$$

for any $u, h \in H$

For the proofs of these results we refer to [11], in which the case of KGMP is treated. For KGM systems, the proof is identical.

Now, we introduce the functionals $I_\varepsilon, J_\varepsilon, G_\varepsilon : H \rightarrow \mathbb{R}$

$$(12) \quad I_\varepsilon(u) = J_\varepsilon(u) + \frac{\omega^2}{2} G_\varepsilon(u),$$

where

$$(13) \quad J_\varepsilon(u) := \frac{1}{2\varepsilon^n} \int_M [\varepsilon^2 |\nabla_g u|^2 + (a - \omega^2) u^2] d\mu_g - \frac{1}{p\varepsilon^n} \int_M (u^+)^p d\mu_g$$

and

$$(14) \quad G_\varepsilon(u) := \frac{1}{\varepsilon^n} q \int_M \psi(u) u^2 d\mu_g.$$

By Lemma 5 we deduce that

$$(15) \quad \frac{1}{2} G'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^n} \int_M [2q\psi(u) - q^2 \psi^2(u)] u \varphi d\mu_g.$$

If $u \in H$ is a critical point of I_ε then the pair $(u, \psi(u))$ is the desired solution of Problem (1) or (2).

3. THE NEHARI MANIFOLD

It is well known that a critical point of the free functional $I_\varepsilon(u)$ can be found as a critical point constrained on the natural constraint

$$\mathcal{N}_\varepsilon = \{u \in H \setminus \{0\} : I'_\varepsilon(u)u = 0\}.$$

We want to prove that the Nehari manifold \mathcal{N}_ε is a C^2 manifold when $p \geq 4$. (Here is the only point in which for Schroedinger Maxwell systems we require $p > 4$).

Lemma 6. *It holds that*

- (1) \mathcal{N}_ε is a C^2 manifold and $\inf_{\mathcal{N}_\varepsilon} \|u\|_\varepsilon > 0$.
- (2) It holds the Palais-Smale condition for the functional $I_{\varepsilon|\mathcal{N}_\varepsilon}$ on \mathcal{N}_ε and for the functional $I_{\varepsilon|H}$ on H .
- (3) For all $u \in H$ such that $|u^+|_{\varepsilon,p} = 1$ there exists a unique positive number $t_\varepsilon = t_\varepsilon(u)$ such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$. Moreover $t_\varepsilon(u)$ depends continuously on u , provided that $u^+ \not\equiv 0$.
- (4) $\lim_{\varepsilon \rightarrow 0} t_\varepsilon(Z_\varepsilon, \xi) = 1$ uniformly with respect to $\xi \in \partial M$

The proof of this lemma is postponed in the appendix.

Remark 7. We notice that, if $u \in \mathcal{N}_\varepsilon$, then

$$\begin{aligned} I_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_\varepsilon^2 + \left(\frac{1}{2} - \frac{2}{p}\right) \frac{\omega^2 q}{\varepsilon^n} \int_M u^2 \psi(u) d\mu_g + \frac{\omega^2 q^2}{\varepsilon^n p} \int_M u^2 \psi^2(u) d\mu_g \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) |u^+|_{p,\varepsilon}^p + \frac{1}{2} \frac{\omega^2 q^2}{\varepsilon^n} \int_M u^2 \psi^2(u) d\mu_g - \frac{1}{2} \frac{\omega^2 q}{\varepsilon^n} \int_M u^2 \psi(u) d\mu_g \end{aligned}$$

Definition 8. We define

$$m_\varepsilon := \inf \{I_\varepsilon(u) : u \in \mathcal{N}_\varepsilon\}.$$

4. STRATEGY OF THE PROOF OF THEOREM 1

We sketch the proof of our main result. First of all, since the functional $I_\varepsilon \in C^2$ is bounded below and satisfies PS condition on the manifold \mathcal{N}_ε , we have, by well known Lusternik Schnirelmann theorem, that I_ε has at least $\text{cat} I_\varepsilon^d$ critical points in the sublevel

$$I_\varepsilon^d = \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \leq d\}.$$

We prove that, for ε and δ small enough, it holds

$$(16) \quad \text{cat} \partial M \leq \text{cat} \left(\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon^+ + \delta} \right)$$

where $m_\varepsilon^+ \in \mathbb{R}$ will be defined in Section 5 (Proposition 9)

To get (16) we build two continuous operators

$$\begin{aligned} \Phi_\varepsilon &: \partial M \rightarrow \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon^+ + \delta} \\ \beta &: \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon^+ + \delta} \rightarrow (\partial M)_{2\rho} \end{aligned}$$

where $(\partial M)_{2\rho} = \{x \in \mathbb{R}^N : d(x, \partial M) < 2\rho\}$ with ρ small enough in order to have $\text{cat} \partial M \leq \text{cat}(\partial M)_{2\rho}$.

We build these operators Φ_ε and β such that $\beta \circ \Phi_\varepsilon : \partial M \rightarrow (\partial M)_{2\rho}$ is homotopic to the immersion $i : \partial M \rightarrow (\partial M)_{2\rho}$. Thus, by the properties of Lusternik Schinirelmann category we obtain (16). Then applying the above mentioned Lusternik Schnirelmann theorem we obtain the proof of our main result.

5. THE MAP Φ_ε

We define a function

$$\begin{aligned}\Phi_\varepsilon &: \partial M \rightarrow \mathcal{N}_\varepsilon \\ \Phi_\varepsilon(q) &= t_\varepsilon(Z_{\varepsilon,\xi})Z_{\varepsilon,\xi}\end{aligned}$$

Proposition 9. *For any $\varepsilon > 0$ the application $\Phi_\varepsilon : \partial M \rightarrow \mathcal{N}_\varepsilon$ is continuous. Moreover, for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that, if $\varepsilon < \varepsilon_0$ then*

$$\Phi_\varepsilon(\xi) \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta} \text{ for all } \xi \in \partial M$$

being

$$\begin{aligned}m_\varepsilon^+ &= \inf \{E^+(v) : v \in \mathcal{N}(E^+)\} \\ E^+(v) &= \int_{\mathbb{R}_+^n} \frac{1}{2} |\nabla v|^2 + \frac{(a - \omega^2)}{2} |v|^2 - \frac{1}{p} |v^+|^p dx; \\ \mathcal{N}(E^+) &= \{v \in H^1(\mathbb{R}_+^n) \setminus \{0\} : E^+(v)v = 0\};\end{aligned}$$

Proof. The continuity follows directly by the continuity of t_ε . For the second claim, we observe that

$$I_\varepsilon(t_\varepsilon(Z_{\varepsilon,\xi})Z_{\varepsilon,\xi}) = \frac{1}{2} t_\varepsilon^2 \|Z_{\varepsilon,\xi}\|_\varepsilon^2 - \frac{1}{p} t_\varepsilon^p |Z_{\varepsilon,\xi}|_{\varepsilon,p}^p + \frac{1}{\varepsilon^n} q t_\varepsilon^2 \int_M \psi(t_\varepsilon Z_{\varepsilon,\xi}) Z_{\varepsilon,\xi} d\mu_g$$

In light of Lemma 6, claim 4, we have that $t_\varepsilon(Z_{\varepsilon,\xi}) \rightarrow 1$ as $\varepsilon \rightarrow 0$, uniformly with respect to $\xi \in \partial M$. Moreover, since $t_\varepsilon(Z_{\varepsilon,\xi}) \rightarrow 1$ and by (42) have, uniformly with respect to ξ ,

$$\frac{1}{\varepsilon^n} q t_\varepsilon^2 \int_M \psi(t_\varepsilon Z_{\varepsilon,\xi}) Z_{\varepsilon,\xi} d\mu_g \rightarrow 0$$

Finally, by Remark 21, we get

$$(17) \quad \lim_{\varepsilon \rightarrow 0} I_\varepsilon(t_\varepsilon(Z_{\varepsilon,q})Z_{\varepsilon,q}) = \frac{1}{2} \int_{\mathbb{R}_+^n} |\nabla V|^2 + (a - \omega^2)V^2 dy - \frac{1}{p} \int_{\mathbb{R}_+^n} V^p dy = m_\varepsilon^+$$

uniformly with respect to $q \in \partial M$. \square

Remark 10. By Proposition 9, given δ , we have that $\mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta} \neq \emptyset$ for ε small enough. Moreover we have

$$\limsup_{\varepsilon \rightarrow 0} m_\varepsilon \leq m_\varepsilon^+.$$

6. CONCENTRATION RESULTS

For any $\varepsilon > 0$ we can construct a finite closed partition $\mathcal{P}^\varepsilon = \{P_j^\varepsilon\}_{j \in \Lambda_\varepsilon}$ of M such that

- P_j^ε is closed for every j and $P_j^\varepsilon \cap P_k^\varepsilon \subset \partial P_j^\varepsilon \cap \partial P_k^\varepsilon$ for $j \neq k$;
- $K_1 \varepsilon \leq d_j^\varepsilon \leq K_2 \varepsilon$, where d_j^ε is the diameter of P_j^ε and $c_1 \varepsilon^n \leq \mu_g(P_j^\varepsilon) \leq c_2 \varepsilon^n$;
- for any j there exists an open set $I_j^\varepsilon \supset P_j^\varepsilon$ such that, if $P_j^\varepsilon \cap \partial M = \emptyset$, then $d_g(I_j^\varepsilon, \partial M) > K\varepsilon/2$, while, if $P_j^\varepsilon \cap \partial M \neq \emptyset$, then $I_j^\varepsilon \subset \{x \in M : d_g(x, \partial M) \leq \frac{3}{2}K\varepsilon\}$;
- there exists a finite number $\nu(M) \in \mathbb{N}$ such that every $x \in M$ is contained in at most $\nu(M)$ sets I_j^ε , where $\nu(M)$ does not depends on ε .

By compactness of M such a partition exists, at least for small ε . In the following we will choose always $\varepsilon_0(\delta)$ sufficiently small in order to have this partition.

Lemma 11. *There exists a constant $\gamma > 0$ such that, for any fixed $\delta > 0$ and for any $\varepsilon \in (0, \varepsilon_0(\delta))$, where $\varepsilon_0(\delta)$ is as in Proposition 9, given any partition \mathcal{P}^ε of M as above, and any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta}$, there exists a set $P_j^\varepsilon \subset \mathcal{P}^\varepsilon$ such that*

$$\frac{1}{\varepsilon^n} \int_{P_j^\varepsilon} |u^+|^p d\mu_g \geq \gamma > 0.$$

Proof. By Remark 10 we have that $\mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta} \neq \emptyset$. For any function $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + \delta}$ we denote by u_j^+ the restriction of u^+ to the set P_j^ε . Then we can write

$$\begin{aligned} \|u\|_\varepsilon^2 &= \frac{1}{\varepsilon^n} \int_M (u^+)^p d\mu_g - \frac{q\omega^2}{\varepsilon^3} \int_M (2 - q\psi(u)) \psi(u) u^2 d\mu_g \\ &\leq \frac{1}{\varepsilon^n} \int_M (u^+)^p d\mu_g = \frac{1}{\varepsilon^n} \sum_j \int_M (u_j^+)^p d\mu_g = \\ &= \sum_j \frac{|u_j^+|_p^{p-2}}{\varepsilon^{\frac{n(p-2)}{p}}} \frac{|u_j^+|_p^2}{\varepsilon^{\frac{2n}{p}}} \leq \max_j \left\{ \frac{|u_j^+|_p^{p-2}}{\varepsilon^{\frac{n(p-2)}{p}}} \right\} \sum_j \frac{|u_j^+|_p^2}{\varepsilon^{\frac{2n}{p}}}. \end{aligned}$$

Then the proof follows exactly as in [12], Lemma 5.1. \square

Remark 12. Fixed δ and ε , we recall that the Ekeland variational principle states that, for any $u \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\varepsilon^+ + 2\delta}$ there exists $u_\delta \in \mathcal{N}_\varepsilon$ such that

$$I_\varepsilon(u_\delta) < I_\varepsilon(u), \quad \|u_\delta - u\|_\varepsilon < 4\sqrt{\delta};$$

$$\left| (I_\varepsilon|_{\mathcal{N}_\varepsilon})'(u_\delta)[\varphi] \right| < \sqrt{\delta} \|\varphi\|_\varepsilon.$$

Moreover, since a Palais Smale sequence for $I_\varepsilon|_{\mathcal{N}_\varepsilon}$ is indeed a PS sequence for the free functional we have also that

$$|I'_\varepsilon(u_\delta)[\varphi]| < \sqrt{\delta} \|\varphi\|_\varepsilon.$$

Proposition 13. *For all $\eta \in (0, 1)$ there exists a $\delta_0 < m_\varepsilon^+$ such that for any $\delta \in (0, \delta_0)$ for any $\varepsilon \in (0, \varepsilon_0(\delta))$ (as in Prop. 9) and for any function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon^+ + \delta}$ we can find a point $\xi = \xi(u) \in \partial M$ for which*

$$(18) \quad \left(\frac{1}{2} - \frac{1}{p} \right) \frac{1}{\varepsilon^n} \int_{I_\xi(\rho, R)} |u^+|^p d\mu_g \geq (1 - \eta) m_\varepsilon^+$$

Proof. We first prove this property for $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon^+ + \delta} \cap I_\varepsilon^{m_\varepsilon^+ + 2\delta}$.

Assume, by contradiction, that there exists $\eta \in (0, 1)$, two sequences of vanishing real numbers $\{\delta_k\}_k$ and $\{\varepsilon_k\}_k$ and a sequence of functions $\{u_k\}_k \subset \mathcal{N}_{\varepsilon_k} \cap I_{\varepsilon_k}^{m_{\varepsilon_k}^+ + \delta_k} \cap I_{\varepsilon_k}^{m_{\varepsilon_k}^+ + 2\delta_k}$ such that, for any $\xi \in \partial M$ it holds

$$(19) \quad \left(\frac{1}{2} - \frac{1}{p} \right) \frac{1}{\varepsilon_k^n} \int_{I_\xi(\rho, R)} |u_k^+|^p d\mu_g < (1 - \eta) m_{\varepsilon_k}^+.$$

By Remark 12 we can assume

$$J'_{\varepsilon_k}(u_k)[\varphi] \leq \sqrt{\delta_k} \|\varphi\|_{\varepsilon_k} \quad \text{for all } \varphi \in H_g^1(M).$$

By Lemma 11 there exists a set $P_k^{\varepsilon_k} \in \mathcal{P}_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon_k^n} \int_{P_k^{\varepsilon_k}} |u_k^+|^p d\mu_g \geq \gamma > 0.$$

we have to examine two cases: either there exists a subsequence $P_{i_k}^{\varepsilon_{i_k}}$ such that $P_{i_k}^{\varepsilon_{i_k}} \cap \partial M \neq \emptyset$, or there exists a subsequence $P_{i_k}^{\varepsilon_{i_k}}$ such that $P_{i_k}^{\varepsilon_{i_k}} \cap \partial M = \emptyset$. For simplicity we write simply P_k for $P_{i_k}^{\varepsilon_{i_k}}$.

First case: $P_k \cap \partial M \neq \emptyset$. We choose a point ξ_k interior to $P_k \cap \partial M$. We have the Fermi coordinates $\Psi_{\xi_k}^\partial : B_{n-1}(0, R) \times [0, R] \rightarrow M$, $\Psi_{\xi_k}^\partial(\bar{y}, y_n) = (\bar{x}, x_n) = x$, being $B_{n-1}(0, R) = \{\bar{y} \in \mathbb{R}^{n-1}, |\bar{y}| < R\}$. In what follows we simply call

$$B(R)_+ := B_{n-1}(0, R) \times [0, R]$$

We consider the function $w_k : \mathbb{R}_+^n \rightarrow \mathbb{R}$ defined by

$$u_k(\Psi_{\xi_k}^\partial(\bar{y}, y_n)) \chi_R(|\bar{y}|) \chi_R(y_n) = u_k(\Psi_{\xi_k}^\partial(\varepsilon_k \bar{z}, \varepsilon z_n)) \chi_R(|\varepsilon_k \bar{z}|) \chi_R(\varepsilon z_n) = w_k(\bar{z}, z_n).$$

It is clear that $w_k \in H^1(\mathbb{R}_+^n)$ with $w_k(\bar{z}, z_n) = 0$ when $|\bar{z}| = 0, R/\varepsilon_k$ or $z_n = R/\varepsilon_k$. We now show some properties of the function w_k .

It is easy to see (cfr. [12], Prop. 5.3) that $\{w_k\}_k$ is bounded in $H^1(\mathbb{R}_+^n)$. Then there exists $w \in H^1(\mathbb{R}_+^n)$ such that w_k converges to w weakly in $H^1(\mathbb{R}_+^n)$ and strongly in $L_{\text{loc}}^p(\mathbb{R}_+^n)$.

We claim that the limit function w is a weak solution of

$$\begin{cases} -\Delta w + (a - \omega^2)w = (w^+)^{p-1} & \text{in } \mathbb{R}_+^n; \\ \frac{\partial w}{\partial \nu} = 0 & \text{for } y = (\bar{y}, 0); \end{cases}$$

First, for any $f \in C_0^\infty(\mathbb{R}_+^n)$ we define on the manifold M the function

$$f_k(x) := f\left(\frac{1}{\varepsilon_k} (\Psi_{\xi_k}^\partial)^{-1}(x)\right) = f(z) \text{ where } x = \Psi_{\xi_k}^\partial(\varepsilon_k z).$$

We notice that for every $f \in C_0^\infty(\mathbb{R}^3)$, there exists k such that $\text{supp } f \subset B(0, R/2\varepsilon_k)$. Thus, $\text{supp } f_k \subset I_{\xi_k}(R/2)$.

Moreover, we have $\|f_k\|_{\varepsilon_k} \leq C\|f\|_{H^1(\mathbb{R}^3)}$, thus, by Ekeland principle we have

$$(20) \quad |I'_{\varepsilon_k}(u_k)[f_k]| \leq \sigma_k \|f_k\|_{\varepsilon_k} \rightarrow 0 \text{ while } k \rightarrow \infty.$$

On the other hand we have

$$(21) \quad \begin{aligned} I'_\varepsilon(u_k)[f_k] &= \frac{1}{\varepsilon_k^n} \int_M \varepsilon_k^2 \nabla_g u_k \nabla_g f_k + a u_k f_k - (u_k^+)^{p-1} f_k - \omega^2 (1 - q\psi(u_k))^2 u_k f_k d\mu_g \\ &= \langle u_k, f_k \rangle_{\varepsilon_k} - \frac{1}{\varepsilon_k^n} \int_M (u_k^+)^{p-1} f_k d\mu_g + \frac{q\omega^2}{\varepsilon_k^3} \int_M (2 - q\psi(u_k)) \psi(u_k) u_k f_k d\mu_g \\ &= \int_{T_k} \left[\sum_{ij} g_{\xi_k}^{ij}(\varepsilon_k z) \partial_{z_i} w_k(z) \partial_{z_j} f(z) + (a - \omega^2) w_k(z) f(z) \right] |g_{\xi_k}(\varepsilon z)|^{1/2} dz \\ &\quad - \int_{T_k} (w_k^+(z))^{p-1} f(z) |g_{\xi_k}(\varepsilon z)|^{1/2} dz \\ &\quad + q\omega^2 \int_{T_k} (2 - q\tilde{\psi}_k(z)) \tilde{\psi}_k(z) w_k(z) f(z) |g_{\xi_k}(\varepsilon z)|^{1/2} dz \end{aligned}$$

Here $T_k = B(R/2\varepsilon_k)_+ \cap \text{supp} f$ and $\psi(u_k)(x) := \psi_k(x) = \psi_k(\Psi_{\xi_k}^\partial(\varepsilon_k z)) := \tilde{\psi}_k(z)$ where $x \in I_{\xi_k}(R)$ and $z \in B(R/\varepsilon_k)_+$. Since $\text{supp} f_k \subset I_{\xi_k}(R/2)$, for KGMP systems, by (1) we have

$$\begin{aligned} 0 &= \int_M \nabla_g \psi(u_k) \nabla_g f_k + (1 + q^2 u_k) \psi(u_k) f_k - q u_k^2 f_k d\mu_g \\ &= \frac{\varepsilon_k^3}{\varepsilon_k^2} \int_{T_k} \sum_{ij} g_{q_k}^{ij}(\varepsilon_k z) \partial_{z_i} \tilde{\psi}_k(z) \partial_{z_j} f(z) |g_{q_k}(\varepsilon z)|^{1/2} dz \\ &\quad + \varepsilon_k^3 \int_{T_k} (1 + q^2 w_k(z)) \tilde{\psi}_k(z) f(z) |g_{q_k}(\varepsilon z)|^{1/2} dz \\ &\quad - \varepsilon_k^3 \int_{T_k} q w_k^2(z) f(z) |g_{q_k}(\varepsilon z)|^{1/2} dz, \end{aligned}$$

The above equation holds for KGMP systems but the analogous for KGM systems is obvious. We have

$$\begin{aligned} (22) \quad & - \int_{T_k} \sum_{ij} g_{\xi_k}^{ij}(\varepsilon_k z) \partial_{z_i} \tilde{\psi}_k(z) \partial_{z_j} f(z) |g_{\xi_k}(\varepsilon z)|^{1/2} dz = \\ & = \varepsilon_k^2 \int_{T_k} \left((1 + q^2 w_k(z)) \tilde{\psi}_k(z) - q w_k^2(z) \right) f(z) |g_{\xi_k}(\varepsilon z)|^{1/2} dz \end{aligned}$$

Arguing as in Lemma 19 we have that

$$\begin{aligned} c \int_{B(R/\varepsilon_k)_+} |\nabla \tilde{\psi}_k(z)|^2 dz &\leq \frac{\varepsilon_k^2}{\varepsilon_k^n} \int_M |\nabla_g \psi_k|^2 d\mu_g \leq \frac{1}{\varepsilon_k^n} q \int_M u_k^2 \psi_k \\ &\leq \frac{1}{\varepsilon_k^n} \int u_k^2 \leq \|u_k\|_{\varepsilon_k}^2 \leq C \end{aligned}$$

where $c, C > 0$ are suitable constants. Moreover, by Lemma 19

$$\begin{aligned} c_1 \int_{B(0, R/\varepsilon_k)} |\tilde{\psi}_k(z)|^2 dz &\leq \frac{1}{\varepsilon_k^n} \int_M \psi_k^2 d\mu_g \leq \frac{1}{\varepsilon_k^n} \|\psi_k\|_{H_g^1}^2 \leq c_2 \frac{1}{\varepsilon_k^n} |u_k|_{4, g}^4 \\ &\leq c_2 |u_k|_{4, \varepsilon}^4 \leq C \end{aligned}$$

where $c_1, c_2, C > 0$ are suitable constants. Conclucing, we have that $\|\tilde{\psi}_k\|_{H^1(B(R/\varepsilon_k)_+)}$ is bounded, and then also $\|\chi_{R/\varepsilon_k}(z) \tilde{\psi}_k(z)\|_{H^1(\mathbb{R}_+^n)}$ is bounded. So, there exists a $\bar{\psi} \in H^1(\mathbb{R}_+^n)$ such that $\bar{\psi}_k(z) := \chi_{R/\varepsilon_k}(z) \tilde{\psi}_k(z) \rightarrow \bar{\psi}$ weakly in $H^1(\mathbb{R}_+^n)$ and strongly in $L_{\text{loc}}^p(\mathbb{R}_+^n)$ for any $2 \leq p < 2^*$.

By (22) we have

$$\begin{aligned} & - \int_{\mathbb{R}_+^n} \sum_{ij} g_{\xi_k}^{ij}(\varepsilon_k z) \partial_{z_i} \bar{\psi}_k(z) \partial_{z_j} f(z) |g_{\xi_k}(\varepsilon z)|^{1/2} dz = \\ & = \varepsilon_k^2 \int_{\mathbb{R}_+^n} \left((1 + q^2 w_k(z)) \bar{\psi}_k(z) - q w_k^2(z) \right) f(z) |g_{\xi_k}(\varepsilon z)|^{1/2} dz \end{aligned}$$

and, using that $g_k^{ij}(\varepsilon z) = \delta_{ij} + O(\varepsilon_k |z|)$ and that $|g_q(\varepsilon z)|^{1/2} = 1 + O(\varepsilon_k |z|)$ we get

$$\int_{\mathbb{R}_+^n} \nabla \bar{\psi}_k(z) \nabla f(z) dz = O(\varepsilon_k).$$

Thus, the function $\bar{\psi} \in H^1(\mathbb{R}_+^n)$ is a weak solution of $-\Delta \bar{\psi} = 0$, so $\bar{\psi} = 0$.

At this point, arguing as above we have

$$(23) \quad \begin{aligned} \frac{1}{\varepsilon_k^n} \int_M (2 - q\psi(u_k)) \psi(u_k) u_k f_k d\mu_g &= \frac{1}{\varepsilon_k^n} \int_{I_{\xi_k}(R/2)} (2 - q\psi(u_k)) \psi(u_k) u_k f_k d\mu_g = \\ &= \int_{\text{supp}f} (2 - q\bar{\psi}_k) \bar{\psi}_k w_k f |g_{\xi_k}(\varepsilon z)|^{1/2} dz \rightarrow 0 \end{aligned}$$

while $k \rightarrow \infty$ because $\bar{\psi}_k \rightarrow 0$ strongly in $L^p_{\text{loc}}(\mathbb{R}_+^n)$ for any $2 \leq p < 2^*$. Thus, by (23), (20) and (21) and because $w_k \rightarrow w$ in H^1 we deduce that, for any $f \in C_0^\infty(\mathbb{R}^3)$, it holds

$$\int_{\mathbb{R}_+^n} \nabla w \nabla f + (a - \omega^2) w f - (w^+)^{p-1} f = 0.$$

Thus, w is a weak solution of $-\Delta w + (a - \omega^2)w = w^{p-1}$ on \mathbb{R}_+^n with Neumann boundary condition. Since $u_k \in \mathcal{N}_{\varepsilon_k} \cap I_{\varepsilon_k}^{m_e^+ + \delta_k}$ we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|u_k\|_{\varepsilon_k}^2 \leq I_{\varepsilon_k}(u_k) \leq m_e^+ + \delta_k,$$

hence

$$(24) \quad \|w\|_a^2 \leq \liminf_k \|w_k\|_a^2 \leq \frac{2p}{p-2} m_e^+$$

where $\|w\|_a^2 = \frac{1}{2} \int_{\mathbb{R}_+^n} |\nabla w|^2 + (a - \omega^2)w^2$. Set

$$\mathcal{N}_\infty = \{v \in H^1(\mathbb{R}_+^n) \setminus \{0\} : \|v\|_a^2 = |v|_p^p\},$$

we have that $w \in \mathcal{N}_\infty \cup \{0\}$. Since $P_k \cap \partial M \neq \emptyset$, we can choose $T > 0$ such that

$$P_k \subset I_{\xi_k}(\varepsilon_k T, \varepsilon_k T) \text{ for } k \text{ large enough.}$$

for $\xi_k \in P_k \cap \partial M$. By definition of w_k and by Lemma 11 there exist a ξ_k such that, for k large enough

$$(25) \quad \begin{aligned} \|w_k^+\|_{L^p(B_{n-1}(0,T) \times [0,T])} &= \int_{B_{n-1}(0,T) \times [0,T]} |\chi_R(\varepsilon_k |\bar{z}|) \chi_\rho(\varepsilon_k z_n) u_k^+(\psi_{q_k}^\partial(\varepsilon_k z))|^p dz = \\ &= \frac{1}{\varepsilon_k^n} \int_{B_{n-1}(0, \varepsilon_k T) \times [0, \varepsilon_k T]} |u_k^+(\psi_{q_k}^\partial(y))|^p dy \geq \\ &\geq \frac{c}{\varepsilon_k^n} \int_{B_{n-1}(0, \varepsilon_k T) \times [0, \varepsilon_k T]} |u_k^+(\psi_{q_k}^\partial(y))|^p |g_{q_k}(y)|^{1/2} dy = \\ &\geq \frac{c}{\varepsilon_k^n} \int_{I_{q_k}(\varepsilon_k T, \varepsilon_k T)} |u_k^+|^p d\mu_g \geq c\gamma > 0. \end{aligned}$$

Since w_k converge strongly to w in $L^p(B_{n-1}(0, T) \times [0, T])$, we have $w \neq 0$, so $w \in \mathcal{N}_\infty$. Hence, by (24) we obtain that

$$(26) \quad \|w\|_a^2 = |w|_p^p = \frac{2p}{p-2} m_e^+$$

and that $w_k \rightarrow w$ strongly in $H^1(\mathbb{R}_+^n)$. From this we derive the contradiction. Indeed, since $|g_q(\varepsilon_k z)|^{1/2} = 1 + O(\varepsilon_k |z|)$, fixed T , by (18), for k large it holds

$$(27) \quad \int_{B(T)_+} (w_k^+)^p dz \leq \left(1 - \frac{\eta}{2}\right) \frac{2p}{p-2} m_\infty.$$

Moreover, by (26) there exists a $T > 0$ such that $\int_{B(T)_+} w^p dz > (1 - \frac{\eta}{8}) \frac{2p}{p-2} m_\infty$ and, since $w_k \rightarrow w$ strongly in $L^p_{\text{loc}}(\mathbb{R}^n_+)$, $\int_{B(T)_+} (w_k^+)^p dz > (1 - \frac{\eta}{4}) \frac{2p}{p-2} m_\infty$, that contradicts (27).

Second case: $P_k^\varepsilon \cap \partial M = \emptyset$. In this case we choose a point ξ_k interior to P_k^ε and we consider the normal coordinates at ξ_k . We set $w_k(z)$ as

$$u_k(x) \chi_R(\exp_{\xi_k}^{-1}(x)) = u_k(\exp_{\xi_k}(y)) \chi_R(y) = u_k(\exp_{\xi_k}(\varepsilon_k z)) \chi_R(\varepsilon_k z) = w_k(z).$$

Arguing as in the previous case, we can establish that w_k is bounded in $H^1(\mathbb{R}^n)$ and converges to some $w \in H^1(\mathbb{R}^n)$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^p_{\text{loc}}(\mathbb{R}^n)$. Moreover $w \neq 0$ and is a solution of $-\Delta w + (a - \omega^2)w = w^{p-1}$ in \mathbb{R}^n . Thus $\|w\|_a^2 = |w|_p^p = 2 \frac{2p}{p-2} m_e^+$ and $w_k \rightarrow w$ strongly in $H^1(\mathbb{R}^n)$ and from this follows the contradiction.

Conclusion: We have proved the claim for $u_k \in \mathcal{N}_{\varepsilon_k} \cap I_{\varepsilon_k}^{m_e^+ + \delta_k} \cap I_{\varepsilon_k}^{m_e + 2\delta_k}$. We prove now the claim in the general case. For u_k it holds

$$\begin{aligned} I_{\varepsilon_k}(u_k) &= \left(\frac{1}{2} - \frac{1}{p} \right) |u_k^+|_{p, \varepsilon_k}^p + \frac{1}{2} \frac{\omega^2 q^2}{\varepsilon_k^{n_i}} \int_M u_k^2 \psi^2(u_k) d\mu_g - \frac{1}{2} \frac{\omega^2 q}{\varepsilon_k^n} \int_M u_k^2 \psi(u_k) d\mu_g \\ &\geq (1 - \eta) m_e^+ - \frac{1}{2} \frac{\omega^2 q}{\varepsilon_k^3} \int_M u_k^2 \psi(u_k) d\mu_g \end{aligned}$$

By compactness of M there exists $\xi_1, \dots, \xi_m \in M \setminus \partial M$ and $\xi_{m+1}, \dots, \xi_l \in \partial M$ such that

$$\frac{1}{\varepsilon_k^n} \int_M u_k^2 \psi(u_k) d\mu_g \leq \sum_{i=1}^m \frac{1}{\varepsilon_k^n} \int_{B_g(\xi_i, r)} u_k^2 \psi(u_k) d\mu_g + \sum_{i=m+1}^l \frac{1}{\varepsilon_k^n} \int_{I_{\xi_i}(r)} u_k^2 \psi(u_k) d\mu_g$$

For any ξ_i , $i = 1, \dots, m$, arguing as above, we can introduce two sequences of functions w_k^i and $\bar{\psi}_k^i$ such that $w_k^i \rightarrow w^i$, strongly in $H^1(\mathbb{R}^n)$, w^i solution of $-\Delta w + (a - \omega^2)w = w^{p-1}$, and that $\bar{\psi}_k^i \rightarrow 0$ strongly in $L^p_{\text{loc}}(\mathbb{R}^n)$ for any $2 \leq p < 2^*$. We thus have that, for any ξ^i

$$\frac{1}{\varepsilon_k^n} \int_{B_g(\xi^i, r)} u_k^2 \psi(u_k) d\mu_g \leq \int_{\mathbb{R}^n} (w_k^i)^2 \bar{\psi}_k^i dx \rightarrow 0.$$

It follows identically, for $i = m+1, \dots, l$,

$$\frac{1}{\varepsilon_k^n} \int_{I_{\xi^i}(r)} u_k^2 \psi(u_k) d\mu_g \leq \int_{\mathbb{R}^n_+} (w_k^i)^2 \bar{\psi}_k^i dx \rightarrow 0.$$

Thus $\limsup_k m_{\varepsilon_k} \geq m_e^+$, and, in light of Remark 10, $\lim_k m_{\varepsilon_k} = m_e^+$. Hence, when ε, δ are small enough, we have $\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_e^+ + \delta} \subset \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_e + 2\delta}$ and the general claim follows. \square

7. THE MAP β

For any $u \in \mathcal{N}_\varepsilon$ with we can define its center of mass as a point $\beta(u) \in \mathbb{R}^N$ by

$$(28) \quad \beta(u) = \frac{\int_M x |u^+(x)|^p d\mu_g}{\int_M |u^+(x)|^p d\mu_g}.$$

The application is well defined on \mathcal{N}_ε , since $u \in \mathcal{N}_\varepsilon$ implies $u^+ \neq 0$ (it follows immediatly by Lemma 6). In the following we will show that if $u \in \mathcal{N}_\varepsilon \cap J^{m_e^+ + \delta}$

then $\beta(u)$ belong to a tubular neighborhood of ∂M , provided ε and δ sufficiently small.

Proposition 14. *For any $u \in \mathcal{N}_\varepsilon \cap J^{m_\varepsilon^+ + \delta}$, with ε and δ small enough, it holds*

$$\beta(u) \in (\partial M)_{3\rho},$$

being $(\partial M)_r = \{x \in \mathbb{R}^N \mid d(x, \partial M) < r\}$ a neighborhood of ∂M in the space \mathbb{R}^N where the manifold M is embedded. Moreover the composition

$$\beta \circ \Phi_\varepsilon : \partial M \rightarrow (\partial M)_{3\rho}$$

is well defined and homotopic to the identity of ∂M .

Proof. Since $m_\varepsilon \rightarrow m_\varepsilon^+$ and by Proposition 13 we get that for any $u \in \mathcal{N}_\varepsilon \cap J^{m_\varepsilon^+ + \delta}$ there exists $\xi \in \partial M$ such that

$$(29) \quad (1 - \eta)m_\varepsilon^+ \leq \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} |u^+|_{L^p(I_\xi(\rho, R))}^p.$$

Since $u \in \mathcal{N}_\varepsilon \cap J^{m_\varepsilon^+ + \delta}$ we have

$$\begin{aligned} m_\varepsilon^+ + \delta &\geq I_\varepsilon(u) = \left(\frac{p-2}{2p}\right) |u^+|_{p, \varepsilon}^p + \frac{\omega^2 q^2}{2\varepsilon^n} \int_M u^2 \psi^2(u) d\mu_g - \frac{\omega^2 q}{2\varepsilon^n} \int_M u^2 \psi(u) d\mu_g \geq \\ &\geq \left(\frac{p-2}{2p}\right) |u^+|_{p, \varepsilon}^p - \frac{\omega^2 q}{2\varepsilon^n} \int_M u^2 \psi(u) d\mu_g \end{aligned}$$

Now, arguing as in Lemma 19 we have that, by Holder inequality that $\|\psi(u)\|_H \leq \left(\int_M u^{12/5}\right)^{5/6}$, and, in the same way, that

$$\begin{aligned} \frac{1}{\varepsilon^n} \int_M \psi(u) u^2 &\leq \frac{1}{\varepsilon^n} \|\psi\|_H \left(\int_M u^{12/5}\right)^{5/6} \leq C \frac{1}{\varepsilon^n} \left(\int_M u^{12/5}\right)^{5/3} \\ &\leq C \varepsilon^{\frac{2}{3}n} |u|_{12/5, \varepsilon}^4 \leq C \varepsilon^{\frac{2}{3}n} \|u\|_\varepsilon^4 \leq C \varepsilon^{\frac{2}{3}n}, \end{aligned}$$

since $\|u\|_\varepsilon$ is bounded because $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty + \delta}$.

So, provided we choose $\varepsilon(\delta_0)$ small enough, we have

$$(30) \quad \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^n} |u^+|_{p, g}^p < m_\varepsilon^+ + 2\delta.$$

By (29) and (30) we get

$$\int_{I_\xi(\rho, R)} \frac{|u^+|^p}{|u^+|_{p, g}^p} d\mu_g \geq \frac{1 - \eta}{1 + \frac{2\delta}{m_\varepsilon^+}}.$$

By definition of β we have

$$\begin{aligned} |\beta(u) - q| &\leq \left| \int_{I_\xi(\rho, R)} (x - q) \frac{|u^+|^p}{|u^+|_{p, g}^p} d\mu_g \right| + \left| \int_{M \setminus I_\xi(\rho, R)} (x - q) \frac{|u^+|^p}{|u^+|_{p, g}^p} d\mu_g \right| \leq \\ &\leq 2\rho + D \left(1 - \frac{1 - \eta}{1 + \frac{\delta}{m_\varepsilon^+}}\right), \end{aligned}$$

where D is the diameter of the manifold M as a subset of \mathbb{R}^n . Here we supposed, without loss of generality that $R < \rho$. Choosing η and δ small enough we get the first claim. The second claim is standard. \square

8. PROFILE DESCRIPTION

Let u_ε a low energy solution. By regularity theory (see [4, Th. 1]) we can prove that $u_\varepsilon \in C^\infty(\bar{M})$. So there exists at least one maximum point of u_ε on M . We can prove that, for ε small, u_ε has a unique local maximum point $P_\varepsilon \in \partial M$ and we can describe the profile of u_ε .

Lemma 15. *Let $(u_\varepsilon, \psi(u_\varepsilon))$ be solution of (2) such that $I_\varepsilon(u_\varepsilon) \leq m_\varepsilon^+ + \delta < 2m_\varepsilon^+$. Then, for ε small, u_ε is not constant on M .*

Proof. At first we notice that if u_ε is constant, also $\psi(u_\varepsilon)$ is constant. Moreover, by (2) the values of u_ε and $\psi(u_\varepsilon)$ depend only on a, ω, q and p . Let $u_\varepsilon = u_0$ and $\psi(u_\varepsilon) = \psi_0$. Immediately we have

$$I_\varepsilon(u_\varepsilon) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{1}{\varepsilon^3} \int_M (a - \omega^2) u_0^2 d\mu_g + \left(\frac{1}{2} - \frac{2}{p}\right) \frac{\omega^2 q}{\varepsilon^3} \int_M u_0^2 \psi_0 d\mu_g + \frac{\omega^2 q^2}{\varepsilon^3 p} \int_M u_0^2 \psi_0^2 d\mu_g \rightarrow +\infty$$

which leads us to a contradiction. \square

Since u_ε is not constant and continuous on \bar{M} , then there exists at least a maximum point $P \in \bar{M}$. Proceeding as in [13], it is easy to see that if $P \in M \setminus \partial M$ then $I_\varepsilon(u_\varepsilon) \geq m_\infty = 2m_\varepsilon^+$ where

$$m_\infty = \inf \{E(v) : v \in \mathcal{N}(E)\} = E(U) \text{ with } U \text{ defined in (5)}$$

$$E(v) = \int_{\mathbb{R}^n} \frac{1}{2} |\nabla v|^2 + \frac{(a - \omega^2)}{2} |v|^2 - \frac{1}{p} |v^+|^p dx;$$

$$\mathcal{N}(E) = \{v \in H^1(\mathbb{R}^n) \setminus \{0\} : E(v)v = 0\}.$$

This implies that $P \in \partial M$. Now, since u_ε is regular and $\frac{\partial u}{\partial \nu} = 0$ on ∂M , P is also a critical point for $u_\varepsilon|_{\partial M}$ and $\Delta_g u_\varepsilon(x_0) \leq 0$. We have the following result.

Lemma 16. *Let $P \in \partial M$ be a maximum point for u_ε solution of (2). Then*

$$(31) \quad (u_\varepsilon(P))^{p-2} > a - \omega^2$$

Proof. We have just pointed out that $\Delta_g u_\varepsilon(P) \leq 0$. Then

$$0 \geq \varepsilon^2 \Delta_g u_\varepsilon(P) = u_\varepsilon(P) \left[a - (u_\varepsilon(P))^{p-2} - \omega^2 (q\psi(u_\varepsilon)(P) - 1)^2 \right]$$

and, since $|q\psi(u_\varepsilon) - 1| < 1$,

$$a \leq (u_\varepsilon(P))^{p-2} + \omega^2 (q\psi(u_\varepsilon)(P) - 1)^2 \leq (u_\varepsilon(P))^{p-2} + \omega^2.$$

This ends the proof. \square

Lemma 17. *Let u_ε be a solution of (2) such that $I_\varepsilon(u_\varepsilon) \leq m_\varepsilon^+ + \delta < 2m_\varepsilon^+$. Then, when ε is sufficiently small, u_ε has a unique maximum point $P \in \partial M$.*

Proof. We argue by contradiction. Suppose that u_ε has two maximum points $P_\varepsilon^1, P_\varepsilon^2 \in \partial M$. We first prove that $d_g(P_\varepsilon^1, P_\varepsilon^2) \rightarrow 0$.

Otherwise, we can find a sequence of vanishing positive numbers ε_j and for each ε_j a solution u_{ε_j} with (at least) two maximum points $P_{\varepsilon_j}^1 \rightarrow P^1$ and $P_{\varepsilon_j}^2 \rightarrow P^2$ as $j \rightarrow \infty$ with $P^1 \neq P^2$.

We define $Q_{\varepsilon_j}^i \in \mathbb{R}^{n-1}$ such that

$$P_{\varepsilon_j}^i = \exp_{P^i}^{\partial}(Q_{\varepsilon_j}^i) \quad i = 1, 2.$$

and we can define a sequence v_j^1 as

$$v_j^1(z) = \begin{cases} u_{\varepsilon_j} \left(\psi_{P^1}^{\partial}(Q_{\varepsilon_j}^1 + \varepsilon_j z) \right) & \text{for } z_n \geq 0 \\ u_{\varepsilon_j} \left(\psi_{P^1}^{\partial}(Q_{\varepsilon_j}^1 + \varepsilon_j z^\tau) \right) & \text{for } z_n < 0 \end{cases}$$

where $z^\tau = (z_1, \dots, z_{n-1}, -z_n)$, and $z \in \mathbb{R}^n$ sufficiently small such that the Fermi coordinates $\psi_{P^1}^{\partial}$ are well defined. In the same way we define v_j^2 . At this point we can proceed as in [13] and we can prove that for any bounded set B eventually $v_j^i \in C^2(B)$ and $v_j^i \xrightarrow{j} U$ in $C^2(B)$, where U is the positive, radially symmetric least energy solution of (5). Now choose \bar{R} such that

$$\int_{B(0, \bar{R})} |\nabla U|^2 + (a - \omega^2)U^2 > \frac{2p}{p-2} \cdot \frac{m_\infty + 2\delta}{2}.$$

For ε_j sufficiently small, we have that $\varepsilon_j \bar{R} \leq \frac{d_g(P^1, P^2)}{2}$, thus

$$\begin{aligned} 2I_{\varepsilon_j}(u_{\varepsilon_j}) &\geq 2 \left(\frac{1}{2} - \frac{1}{p} \right) \|u_{\varepsilon_j}\|_{\varepsilon_j}^2 \\ &\geq 2 \left(\frac{1}{2} - \frac{1}{p} \right) \frac{1}{\varepsilon_j^n} \int_{I_{P^1}(\varepsilon_j \bar{R}) \cup I_{P^2}(\varepsilon_j \bar{R})} \varepsilon^2 |\nabla_g u_{\varepsilon_j}|^2 + (a - \omega^2) u_{\varepsilon_j}^2 \\ &\geq 2 \left(\frac{1}{2} - \frac{1}{p} \right) \int_{B(0, \bar{R}) \cap z_n \geq 0} |\nabla v_j^1(z)|^2 + (a - \omega^2) (v_j^1)^2 \\ (32) \quad &+ 2 \left(\frac{1}{2} - \frac{1}{p} \right) \int_{B(0, \bar{R}) \cap z_n < 0} |\nabla v_j^2(z)|^2 + (a - \omega^2) |v_j^2|^2 + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{p} \right) \int_{B(0, \bar{R})} |\nabla v_j^1(z)|^2 + (a - \omega^2) |v_j^1|^2 \\ &+ \left(\frac{1}{2} - \frac{1}{p} \right) \int_{B(0, \bar{R})} |\nabla v_j^2(z)|^2 + (a - \omega^2) |v_j^2|^2 + o(1) \\ &\rightarrow 2 \left(\frac{1}{2} - \frac{1}{p} \right) \int_{B(0, \bar{R})} |\nabla U|^2 + (a - \omega^2) U^2 > m_\infty + 2\delta \end{aligned}$$

and thus $I_{\varepsilon_j}(u_{\varepsilon_j}) > m_e^+ + 2\delta$ that is a contradiction.

Now we have that $d_g(P_\varepsilon^1, P_\varepsilon^2) \rightarrow 0$. With the same technique we can prove also that

$$(33) \quad \lim_{j \rightarrow \infty} \frac{1}{\varepsilon_j} d_g(P_{\varepsilon_j}^1, P_{\varepsilon_j}^2) = 0$$

To conclude the proof we have to show that (33) raises to a contradiction. In fact suppose that $d_g(P_{\varepsilon_j}^1, P_{\varepsilon_j}^2) \leq c\varepsilon_j$ for some $c > 0$ and consider the sequence of functions

$$(34) \quad w_{\varepsilon_j} = \begin{cases} u_{\varepsilon_j} \left(\psi_{P^1}^{\partial}(Q_{\varepsilon_j}^1 + \varepsilon_j z) \right) & \text{for } z_n \geq 0 \\ u_{\varepsilon_j} \left(\psi_{P^1}^{\partial}(Q_{\varepsilon_j}^1 + \varepsilon_j z^\tau) \right) & \text{for } z_n < 0 \end{cases} \quad \text{with } |z| \leq c.$$

For any j , w_{ε_j} has two maximum points in $B(0, c)$. Moreover, we can argue, as in the previous steps, that $w_{\varepsilon_j} \rightarrow U$ in $C^2(B(0, c))$ and this is a contradiction. \square

Lemma 18. *Write $u_\varepsilon = Z_{\varepsilon, P_\varepsilon} + \Psi_\varepsilon$ where $Z_{\varepsilon, P_\varepsilon}$ is defined in (6) and $P_\varepsilon \in \partial M$ is the unique maximum point. It holds that $\|\Psi_\varepsilon\|_{L^\infty(M)} \rightarrow 0$.*

Proof. By the C^2 convergence proved in Lemma 17 we have that, given $\rho > 0$, and defined w_ε as in (34), we get, as before,

$$2\|u_\varepsilon - Z_{\varepsilon, P_\varepsilon}\|_{C^0(I_{P_\varepsilon}(\varepsilon\rho))} = \|w_\varepsilon(z) - U(z)\|_{C^0(B(0, \rho))} + o(1) \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Moreover, since u_ε has a unique maximum point by Lemma 17, we have that, for any $\rho > 0$,

$$\max_{x \in M \setminus I_{P_\varepsilon}(\varepsilon\rho)} u_\varepsilon(x) = \max_{x \in \partial I_{P_\varepsilon}(\varepsilon\rho)} u_\varepsilon(x) = \max_{|z|=\rho} U(z) + \sigma(\varepsilon) \leq ce^{-\alpha\rho} + \sigma_1(\varepsilon)$$

for some constant $c, \alpha > 0$ and for some $\sigma_1(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$. This proves the claim. \square

9. PROOF OF TECHNICAL RESULTS

Here we collect some technical result which has been used in the proof of the main result.

Proof of Lemma 6. If $u \in \mathcal{N}_\varepsilon$, by (15), we have

$$\begin{aligned} 0 = N_\varepsilon(u) &= \|u\|_\varepsilon^2 - |u^+|_{\varepsilon, p}^p + \frac{q\omega^2}{\varepsilon^n} \int_M (2 - q\psi(u)) \psi(u) u^2 d\mu_g \\ (35) \quad &= \|u\|_\varepsilon^2 - |u^+|_{\varepsilon, p}^p + \frac{q\omega^2}{2\varepsilon^n} \int_M (2\psi(u) + \psi'(u)[u]) u^2 d\mu_g. \end{aligned}$$

The functional N_ε is of class C^2 for $2 < p < 2^*$ because ψ is of class C^2 . Also, for $4 \leq p < 2^*$ we have $N'_\varepsilon(u)[u] < 0$ for all $u \in \mathcal{N}_\varepsilon$. In fact by (35) we have

$$\begin{aligned} N'_\varepsilon(u)[u] &= 2\|u\|_\varepsilon^2 - p|u^+|_{\varepsilon, p}^p + \frac{q\omega^2}{\varepsilon^n} \int_M (2 - q\psi(u)) \psi'(u)[u] u^2 d\mu_g \\ &\quad + \frac{2q\omega^2}{\varepsilon^n} \int_M (2 - q\psi(u)) \psi(u) u^2 d\mu_g - \frac{q^2\omega^2}{\varepsilon^n} \int_M \psi'(u)[u] \psi(u) u^2 d\mu_g = \\ &= (2 - p)\|u\|_\varepsilon^2 + \frac{q\omega^2}{\varepsilon^n} \int_M [4 - p - 2q\psi(u)] \psi(u) u^2 d\mu_g \\ (36) \quad &\quad + \frac{q\omega^2}{\varepsilon^n} \int_M \left[2 - \frac{p}{2} - 2q\psi(u)\right] \psi'(u)[u] u^2 d\mu_g < 0 \text{ for } p \geq 4, \end{aligned}$$

thus \mathcal{N}_ε is a C^2 manifold.

Now, assume by contradiction that there exists a sequence $\{u_k\}_k \in \mathcal{N}_\varepsilon$ with $\|u_k\|_\varepsilon \rightarrow 0$ while $k \rightarrow +\infty$. Thus, using that $N_\varepsilon(u) = 0$ and that $0 \leq \psi(u_k) \leq 1/q$ we have

$$\|u_k\|_\varepsilon^2 \leq \|u_k\|_\varepsilon^2 + \frac{q\omega^2}{\varepsilon^n} \int_M [2 - q\psi(u_k)] u_k^2 \psi(u_k) d\mu_g = |u_k^+|_{\varepsilon, p}^p \leq C\|u_k\|_\varepsilon^p,$$

so $1 \leq C\|u_k\|_\varepsilon^{p-2} \rightarrow 0$ that gives us a contradiction, so claim 1 is proved.

To prove claim 2, first, we show that if $\{u_k\}_k \in \mathcal{N}_\varepsilon$ is a Palais-Smale sequence for the functional I_ε constrained on \mathcal{N}_ε , then $\{u_k\}_k$ is a Palais-Smale sequence for the free functional I_ε on H_ε

Indeed, let $\{u_k\}_k \in \mathcal{N}_\varepsilon$ such that

$$I_\varepsilon(u_k) \rightarrow c \\ |I'_\varepsilon(u_k)[\varphi] - \lambda_k N'(u_k)[\varphi]| \leq \sigma_k \|\varphi\|_\varepsilon \quad \text{with } \sigma_k \rightarrow 0$$

In particular $I'_\varepsilon(u_k) \left[\frac{u_k}{\|u_k\|_\varepsilon} \right] - \lambda_k N'(u_k) \left[\frac{u_k}{\|u_k\|_\varepsilon} \right] \rightarrow 0$. Thus, since $u_k \in \mathcal{N}_\varepsilon$,

$$\lambda_k N'(u_k) \left[\frac{u_k}{\|u_k\|_\varepsilon} \right] \rightarrow 0.$$

By (36), if $\inf |\lambda_k| \neq 0$, we have that $\|u_k\|_\varepsilon \rightarrow 0$ that contradicts Lemma 6. Thus $\lambda_k \rightarrow 0$. Moreover, since

$$I_\varepsilon(u_k) = \left(\frac{1}{2} - \frac{1}{p} \right) \|u_k\|_\varepsilon^2 + \left(\frac{1}{2} - \frac{2}{p} \right) \frac{\omega^2 q}{\varepsilon^3} \int_M u_k^2 \psi_k d\mu_g + \frac{\omega^2 q^2}{\varepsilon^{2p}} \int_M u_k^2 \psi_k^2 d\mu_g \rightarrow c,$$

we have that $\|u_n\|_\varepsilon$ is bounded. By Remark 20 we have that $|N'(u_n)[\varphi]| \leq c\|\varphi\|_\varepsilon$. Thus $\{u_k\}_k$ is a PS sequence for the free functional I_ε .

To conclude the proof of claim 2, we prove that I_ε satisfies the PS condition on the whole space H_ε . Let $\{u_k\}_k \in H_\varepsilon$ such that

$$I_\varepsilon(u_k) \rightarrow c \quad |I'_\varepsilon(u_k)[\varphi]| \leq \sigma_k \|\varphi\|_\varepsilon \quad \text{where } \sigma_k \rightarrow 0$$

We have that $\|u_k\|_\varepsilon$ is bounded. Indeed, by contradiction, suppose $\|u_n\|_\varepsilon \rightarrow \infty$. Then, by PS hypothesis

$$\frac{pI_\varepsilon(u_k) - I'_\varepsilon(u_k)[u_k]}{\|u_k\|_\varepsilon} = \\ \left(\frac{p}{2} - 1 \right) \|u_k\|_\varepsilon + \frac{q\omega^2}{\varepsilon^n} \int_M \left[\frac{p}{2} - 2 + q\psi(u_k) \right] \frac{u_k^2 \psi(u_k)}{\|u_k\|_\varepsilon} d\mu_g \rightarrow 0$$

Since $p \geq 4$ and $\psi(u_n) \geq 0$ this leads to a contradiction. At this point, up to subsequence $u_k \rightarrow u$ in H_ε , then by Lemma 19 we have, up to subsequence, $\psi(u_k) := \psi_k \rightarrow \bar{\psi} = \psi(u)$.

We have that

$$u_k - i_\varepsilon^*[(u_k^+)^{p-1}] - \omega^2 q i_\varepsilon^* [(q\psi_k^2 - 2\psi_k) u_k] \rightarrow 0$$

where the operator $i_\varepsilon^* : L_g^{p'}, |\cdot|_{\varepsilon, p'} \rightarrow H_\varepsilon$ is the adjoint operator of the immersion operator $i_\varepsilon : H_\varepsilon \rightarrow L_g^p, |\cdot|_{\varepsilon, p}$. Since $u_k \rightarrow u$ in $L^{p'}$, to get H_g^1 strong convergence of $\{u_k\}_k$ it is sufficient to show that $(q\psi_k^2 - 2\psi_k) u_n \rightarrow (q\bar{\psi}^2 - 2\bar{\psi}) u$ in $L_g^{p'}$. We have

$$(37) \quad |\psi_k u_k - \bar{\psi} u|_{p', g} \leq |(\psi_k - \bar{\psi}) u|_{p', g} + |\psi_k (u_k - u)|_{p', g}.$$

and

$$(38) \quad |\psi_k^2 u_k - \bar{\psi}^2 u|_{p', g} \leq |(\psi_k^2 - \bar{\psi}^2) u|_{p', g} + |\psi_k^2 (u_k - u)|_{p', g}.$$

For the first term of (37) we have, by Holder inequality

$$\int_M |\psi_k - \bar{\psi}|^{\frac{p}{p-1}} |u|^{\frac{p}{p-1}} \leq \left(\int_M |\psi_k - \bar{\psi}|^p \right)^{\frac{1}{p-1}} \left(\int_M |u|^{\frac{p}{p-2}} \right)^{\frac{p-2}{p-1}} \rightarrow 0,$$

and for the other terms we proceed in the same way.

To prove claim 3, define, for $t > 0$

$$H(t) = I_\varepsilon(tu) = \frac{1}{2} t^2 \|u\|_\varepsilon^2 + \frac{q\omega^2}{2\varepsilon^n} t^2 \int_M \psi(tu) u^2 d\mu_g - \frac{t^p}{p}.$$

Thus, by (15)

$$\begin{aligned}
 H'(t) &= t \left(\|u\|_\varepsilon^2 + \frac{q\omega^2}{2\varepsilon^n} \int_M [2 - q\psi(tu)] \psi(tu) u^2 d\mu_g - t^{p-2} \right) \\
 (39) \quad &= t \left(\|u\|_\varepsilon^2 + \frac{q\omega^2}{\varepsilon^n} \int_M \psi(tu) u^2 d\mu_g + \frac{q\omega^2}{2\varepsilon^n} t \int_M \psi'(tu) [u] u^2 d\mu_g - t^{p-2} \right)
 \end{aligned}$$

$$\begin{aligned}
 H''(t) &= \|u\|_\varepsilon^2 + \frac{q\omega^2}{2\varepsilon^n} \int_M [2 - q\psi(tu)] \psi(tu) u^2 d\mu_g \\
 (40) \quad &+ \frac{q\omega^2}{\varepsilon^n} t \int_M [1 - q\psi(tu)] \psi'(tu) [u] u^2 d\mu_g - (p-1)t^{p-2}
 \end{aligned}$$

By (39) there exists $t_\varepsilon > 0$ such that $H'(t_\varepsilon) = 0$, because, for small t , $H'(t) > 0$ and, since $p \geq 4$, it holds $H'(t) < 0$ for t large. Moreover,

$$t_\varepsilon^{p-2} = \|u\|_\varepsilon^2 + \frac{q\omega^2}{\varepsilon^n} \int_M \psi(t_\varepsilon u) u^2 d\mu_g + \frac{q\omega^2}{2\varepsilon^n} t_\varepsilon \int_M \psi'(t_\varepsilon u) [u] u^2 d\mu_g$$

then, by Lemma 4

$$\begin{aligned}
 H''(t_\varepsilon) &= (2-p)\|u\|_\varepsilon^2 + \frac{q\omega^2}{\varepsilon^n} \int_M \left[2 - p - \frac{q}{2}\psi(t_\varepsilon u) \right] \psi(t_\varepsilon u) u^2 d\mu_g \\
 &+ \frac{q\omega^2}{2\varepsilon^n} \int_M [3 - p - 2q\psi(t_\varepsilon u)] \psi'(t_\varepsilon u) [t_\varepsilon u] u^2 d\mu_g < 0,
 \end{aligned}$$

so t_ε is unique. The continuity of t_ε is standard.

We now prove the last claim. We have

$$\begin{aligned}
 (41) \quad t_\varepsilon^{p-2} |Z_{\varepsilon, \xi}|_{\varepsilon, p}^p &= \|Z_{\varepsilon, \xi}\|_\varepsilon^2 + \frac{q\omega^2}{\varepsilon^n} \int_M \psi(t_\varepsilon Z_{\varepsilon, \xi}) Z_{\varepsilon, \xi}^2 d\mu_g \\
 &- \frac{q^2\omega^2}{2\varepsilon^n} \int_M \psi^2(t_\varepsilon Z_{\varepsilon, \xi}) Z_{\varepsilon, \xi}^2 d\mu_g
 \end{aligned}$$

where $t_\varepsilon = t_\varepsilon(Z_{\varepsilon, q})$. It holds

$$(42) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n t_\varepsilon^2} \int_M \psi(t_\varepsilon Z_{\varepsilon, \xi}) Z_{\varepsilon, \xi}^2 d\mu_g = 0$$

$$(43) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^n t_\varepsilon^4} \int_M \psi^2(t_\varepsilon Z_{\varepsilon, \xi}) Z_{\varepsilon, \xi}^2 d\mu_g = 0$$

In fact, set $\psi(t_\varepsilon Z_{\varepsilon, \xi}) := \psi_\varepsilon$. We have, by Remark 21 and by definition of ψ_ε ,

$$\begin{aligned}
 \|\psi_\varepsilon\|_H^2 &\leq \|\psi_\varepsilon\|_H^2 + q^2 \int_M \psi_\varepsilon^2 t_\varepsilon^2 Z_{\varepsilon, \xi}^2 d\mu_g = t_\varepsilon^2 q \int_M Z_{\varepsilon, \xi}^2 \psi_\varepsilon d\mu_g \leq \\
 &\leq ct_\varepsilon^2 |\psi_\varepsilon|_{6, g} \left(\int_M Z_{\varepsilon, \xi}^{12/5} d\mu_g \right)^{5/6} \leq ct_\varepsilon^2 \|\psi_\varepsilon\|_{H_g^1} \varepsilon^{\frac{5n}{6}}.
 \end{aligned}$$

Moreover

$$\frac{1}{\varepsilon^n} \int_M \psi_\varepsilon Z_{\varepsilon, \xi}^2 d\mu_g \leq \frac{1}{\varepsilon^n} \|\psi_\varepsilon\|_{H_g^1} \left(\int_M Z_{\varepsilon, \xi}^{12/5} d\mu_g \right)^{5/6} \leq ct_\varepsilon^2 \frac{1}{\varepsilon^n} \varepsilon^{\frac{10n}{6}} = ct_\varepsilon^2 \varepsilon^{\frac{2n}{3}},$$

and

$$\frac{1}{\varepsilon^n} \int_M \psi_\varepsilon^2 Z_{\varepsilon, \xi}^2 d\mu_g \leq \frac{1}{\varepsilon^n} \left(\int_M \psi_\varepsilon^6 d\mu_g \right)^{1/3} \left(\int_M Z_{\varepsilon, \xi}^3 d\mu_g \right)^{2/3} \leq \frac{1}{\varepsilon^n} \|\psi_\varepsilon\|_{H_g^1}^2 \varepsilon^{\frac{2n}{3}} \leq t_\varepsilon^4 \varepsilon^{\frac{4n}{3}}.$$

This proves (42) and (43). For any sequence $\varepsilon_k \rightarrow 0$, by (41), (42) and (43) and by Remark 21 we have that t_{ε_k} is bounded. Then, up to subsequences $t_{\varepsilon_k} \rightarrow \bar{t}$. By (41) and Remark 21 we have $\bar{t}^{p-2}|V|_p^p = \int_{\mathbb{R}_+^n} |\nabla V|^2 + (a - \omega^2)V^2 dx$. By (4) we complete the proof. \square

Lemma 19. *Let $u_k \rightharpoonup u$ in $H_g^1(M)$. Then, up to subsequence, $\psi(u_k) \rightharpoonup \psi(u)$ in $H_g^1(M)$.*

Proof. We set $\psi_k := \psi(u_k)$. By (7), it holds

$$\|\psi_k\|_{H_g^1}^2 \leq \|\psi_k\|_{H_g^1}^2 + \int_M q^2 u_k^2 \psi_k^2 d\mu_g = q \int_M u_k^2 \psi_k d\mu_g \leq c \|u_k\|_{L_g^4}^2 \|\psi_k\|_{H_g^1}$$

then $\|\psi_k\|_{H_g^1} \leq c \|u_k\|_{L_g^4}^2$, thus $\|\psi_k\|_{H_g^1}$ is bounded and, up to subsequence, $\psi_k \rightharpoonup \bar{\psi}$ in $H_g^1(M)$. We recall that ψ_k solves (7), thus passing to the limit we have that $\bar{\psi}$ also solves (7). Since (7) admits a unique solution, we get $\bar{\psi} = \psi(u)$. If $\psi(u_k)$ solves (8) the proof follows in the same way if we use on $H_{0,g}^1$ the equivalent norm $\|u\|_{H_{0,g}^1} = \|\nabla u\|_{L_g^2}$. \square

Remark 20. We have that $\|V_u(h)\|_H \leq c|h|_{3,g}|u|_{3,g}$. In fact, by Lemma 4

$$\begin{aligned} \|V_u(h)\|_H^2 &\leq \|V_u(h)\|_H^2 + \int_M q^2 u^2 V_u^2(h) d\mu_g \leq \\ &\leq \int_M 2qu(1 - q\psi(u))hV_u(h) d\mu_g \leq c\|V_u(h)\|_H |h|_{3,g}|u|_{3,g}. \end{aligned}$$

Remark 21. the following limits hold uniformly with respect to $q \in \partial M$.

$$(44) \quad \lim_{\varepsilon \rightarrow 0} \|Z_{\varepsilon,\xi}\|_{2,\varepsilon}^2 = \int_{\mathbb{R}_+^n} V^2(y) dy$$

$$(45) \quad \lim_{\varepsilon \rightarrow 0} \|Z_{\varepsilon,\xi}\|_{p,\varepsilon}^p = \int_{\mathbb{R}_+^n} V^p(y) dy$$

$$(46) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \|\nabla Z_{\varepsilon,\xi}\|_{2,\varepsilon}^2 = \int_{\mathbb{R}_+^n} |\nabla V|^2(y) dy$$

REFERENCES

- [1] Antonio Azzollini and Alessio Pomponio, *Ground state solutions for the nonlinear Klein-gordon-maxwell equations*, Topol. Methods Nonlinear Anal. **35** (2010), 33–42.
- [2] Vieri Benci and Donato Fortunato, *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Rev. Math. Phys. **14** (2002), no. 4, 409–420. MR 1901222 (2003f:35079)
- [3] Daniele Cassani, *Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations*, Nonlinear Anal. **58** (2004), no. 7-8, 733–747. MR 2085333 (2005f:35297)
- [4] Pascal Cherrier, *Problèmes de Neumann non linéaires sur les variétés riemanniennes*, J. Funct. Anal. **57** (1984), no. 2, 154–206. MR 749522 (86c:58154)
- [5] Tea D'Aprile and Juncheng Wei, *Layered solutions for a semilinear elliptic system in a ball*, J. Differential Equations **226** (2006), 269–294.
- [6] Tea D'Aprile and Juncheng Wei, *Clustered solutions around harmonic centers to a coupled elliptic system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **226** (2007), 605–628.
- [7] Teresa D'Aprile and Dimitri Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), no. 5, 893–906. MR 2099569 (2005h:35319)

- [8] P. d’Avenia, L. Pisani, and G. Siciliano, *Dirichlet and Neumann problems for Klein-Gordon-Maxwell systems*, *Nonlinear Anal.* **71** (2009), no. 12, e1985–e1995. MR 2671970 (2011e:35317)
- [9] Pietro d’Avenia and Lorenzo Pisani, *Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations*, *Electron. J. Differential Equations* (2002), No. 26, 13. MR 1884995 (2003e:78006)
- [10] Pietro d’Avenia, Lorenzo Pisani, and Gaetano Siciliano, *Klein-Gordon-Maxwell systems in a bounded domain*, *Discrete Contin. Dyn. Syst.* **26** (2010), no. 1, 135–149. MR 2552782 (2010j:35528)
- [11] Olivier Druet and Emmanuel Hebey, *Existence and a priori bounds for electrostatic Klein-Gordon-Maxwell systems in fully inhomogeneous spaces*, *Commun. Contemp. Math.* **12** (2010), no. 5, 831–869. MR 2733200 (2012j:58018)
- [12] Marco Ghimenti and Anna M. Micheletti, *Positive solutions of singularly perturbed nonlinear elliptic problem on Riemannian manifolds with boundary*, *Topol. Methods Nonlinear Anal.* **35** (2010), no. 2, 319–337. MR 2676820 (2011i:58028)
- [13] Marco Ghimenti and Anna Maria Micheletti, *Number and profile of low energy solutions for singularly perturbed Klein-Gordon-Maxwell systems on a Riemannian manifold*, *J. Differential Equations* **256** (2014), no. 7, 2502–2525. MR 3160452
- [14] Emmanuel Hebey and Trong Tuong Truong, *Static Klein-Gordon-Maxwell-Proca systems in 4-dimensional closed manifolds*, *J. Reine Angew. Math.* **667** (2012), 221–248. MR 2929678
- [15] Emmanuel Hebey and Juncheng Wei, *Resonant states for the static Klein-Gordon-Maxwell-Proca system*, *Math. Res. Lett.* **19** (2012), no. 4, 953–967. MR 3008428
- [16] Dimitri Mugnai, *Coupled klein-gordon and born-infeld-type equations: Looking for solitary waves*, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **460** (2004), 1519–1527.

(Marco Ghimenti) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, VIA F. BUONARROTI 1/C, 56127 PISA, ITALY

E-mail address: marco.ghimenti@dma.unipi.it.

(Anna Maria Micheletti) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, VIA F. BUONARROTI 1/C, 56127 PISA, ITALY

E-mail address: a.micheletti@dma.unipi.it.