SURREAL NUMBERS, DERIVATIONS AND TRANSSERIES

ALESSANDRO BERARDUCCI AND VINCENZO MANTOVA

Abstract. Several authors have conjectured that Conway's field of surreal numbers, equipped with the exponential function of Kruskal and Gonshor, can be described as a field of transseries and admits a compatible differential structure of Hardy-type. In this paper we give a complete positive solution to both problems. We also show that with this new differential structure, the surreal numbers are Liouville closed, namely the derivation is surjective.

1. Introduction

Conway's class "**No**" of surreal numbers is a remarkable mathematical structure introduced in [Con76]. Besides being a *universal domain* for ordered fields (in the sense that every ordered field whose domain is a set can be embedded in **No**), it admits an exponential function $\exp : \mathbf{No} \to \mathbf{No}$ [Gon86] and an interpretation of the real analytic functions restricted to finite numbers [All87], making it, thanks to the results of [Res93, vdDMM94], into a model of the theory of the field of real numbers endowed with the exponential function and all the real analytic functions restricted to a compact box [vdDE01].

It has been suggested that **No** could be equipped with a derivation compatible with exp and with its natural structure of generalized power series field. One would like such a derivation to formally behave as the natural derivation on the germs at infinity of functions $f : \mathbb{R} \to \mathbb{R}$ belonging to a "Hardy field" [Bou76, Ros83, Mil12]. This can be given a precise meaning through the notion of H-field [AvdD02, AvdD05], a formal algebraic counterpart of the notion of Hardy field.

A related conjecture is that **No** can be viewed as a universal domain for various generalized power series fields equipped with an exponential function, including Écalle's field of *transseries* [É92] (introduced in connection with Dulac's problem) and its variants, such as the *logarithmic-exponential series* of L. van den Dries, A. Macintyre and D. Marker [vdDMM97, vdDMM01], the *exponential-logarithmic series* of S. Kuhlmann [Kuh00], and the transseries of J. van der Hoeven [vdH97, vdH06] and M. Schmeling [Sch01]. Referring to logarithmic-exponential series, in [vdDMM01] the authors say that "There are also potential connections with the theory of surreal numbers of Conway and Kruskal, and super exact asymptotics".

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Some years later, a more precise formulation was given in [vdH06]: "We expect that it is actually possible to construct isomorphisms between the class of surreal numbers and the class of generalized transseries of the reals with so called transfinite iterators of the exponential function and nested transseries. A start of this project has been carried out in collaboration with my former student M. Schmeling [Sch01]. If this project could be completed, this would lead to a remarkable correspondence between growth-rate functions and numbers." Further steps in this direction were taken by S. Kuhlmann and M. Matusinski in [KM11, KM12] leading to the explicit conjecture that **No** is a field of "exponential-logarithmic transseries" [KM15, Conj. 5.2] and can be equipped with a "Hardy-type series derivation" [Mat14, p. 368].

In this paper we give a complete solution to the above problems showing that the surreal numbers have a natural transseries structure in the sense of [Sch01, Def. 2.2.1] (although not in the sense of [KM15]) and finding a compatible Hardy-type derivation.

We expect that these results will lead to a considerable simplification of the treatment of transseries (which will be investigated in a forthcoming paper) and thus provide a valuable tool for the study of the asymptotic behavior of functions. In the light of the model completeness conjectures of [AvdDvdH13], we also expect that **No**, equipped with this new differential structure, is an elementary extension of the field of the logarithmic-exponential series.

In order to describe the results in some detail, we recall that the surreal numbers can be represented as binary sequences of transfinite ordinal length, so that one can endow **No** with a natural tree-like well-founded partial order $<_s$ called "simplicity relation". Another very useful representation describes surreal numbers as infinite sums $\sum_{x \in \mathbf{No}} a_x \omega^x$, where $x \mapsto \omega^x$ is Conway's omega-function, $a_x \in \mathbb{R}$ for all x, and the support $\{\omega^x : a_x \neq 0\}$ is a reverse well-ordered set, namely every non-empty subset has a maximum. In other words, **No** coincides with the *Hahn field* $\mathbb{R}((\omega^{\mathbf{No}}))$ with coefficients in $\mathbb{R}$ and monomial group $(\omega^{\mathbf{No}}, \cdot)$ (see Subsection 2.3). In particular, we have a well defined notion of infinite "summable" families in **No**. In this paper we prove:

Theorem A (6.30). *Conway's field of surreal numbers* **No** *admits a derivation* $D : \mathbf{No} \to \mathbf{No}$ *satisfying the following properties:*

(1) *Leibniz' rule:* $D(xy) = xD(y) + yD(x)$*;*
(2) *strong additivity:* $D\left(\sum_{i \in I} x_i\right) = \sum_{i \in I} D(x_i)$ *if* $(x_i : i \in I)$ *is summable;*
(3) *compatibility with exponentiation:* $D(\exp(x)) = \exp(x)D(x)$*;*
(4) *constant field* $\mathbb{R}$*:* $\ker(D) = \mathbb{R}$*;*
(5) *H-field: if* $x > \mathbb{N}$ *then* $D(x) > 0$*.*

We call **surreal derivation** any function $D : \mathbf{No} \to \mathbf{No}$ satisfying properties (1)-(5) in Theorem A. We show in fact that there are several surreal derivations, among which a "simplest" one $\partial : \mathbf{No} \to \mathbf{No}$. We can prove that the simplest derivation ∂ satisfies additional good properties, such as $\partial(\omega) = 1$ and the existence of anti-derivatives.

Theorem B (7.7). *The surreal numbers equipped with* ∂ *are a Liouville closed H-field with small derivation in the sense of* [AvdD02, p. 3]*, namely,* ∂ *is surjective and sends infinitesimals to infinitesimals.*

In the course of the proof, we also discover that **No** is a field of transseries as anticipated in [vdH06].

Theorem C (8.10). **No** *is a field of transseries in the sense of* [Sch01, Def. 2.2.1].

As an application of the above results, we note that the existence of surreal derivations yields an immediate proof that **No** satisfies the statement of Schanuel's conjecture "modulo $\mathbb{R}$", thanks to Ax's theorem [Ax71], similarly to what was observed in [KMS13] for various fields of transseries (Corollary 6.34). Since **No** is a monster model of the theory of $\mathbb{R}_{\exp}$, the same statement follows for every elementary extension of $\mathbb{R}_{\exp}$. It is actually known that any model of the theory of $\mathbb{R}_{\exp}$ satisfies an even stronger Schanuel type statement "modulo $\mathrm{dcl}(\emptyset)$" (see [JW08] and [Kir10]).

The strategy to prove the existence of a surreal derivation D is the following. Let $\mathbb{J} \subset \mathbf{No}$ be the subring of the **purely infinite numbers**, consisting of the surreal numbers $\sum_{x\in\mathbf{No}} a_x\omega^x$ having only infinite monomials ω^x in their support (namely, $x > 0$ whenever $a_x \neq 0$). It is known that

$$\omega^{\mathbf{No}} = \exp(\mathbb{J}),$$

so we can write $\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}})) = \mathbb{R}((\exp(\mathbb{J})))$. In other words, every surreal number can be written in the form $\sum_{\gamma\in\mathbb{J}} r_\gamma \exp(\gamma)$. We baptize this "Ressayre form" in honor of J.-P. Ressayre, who showed in [Res93] that every "real closed exponential field" admits a similar representation. A surreal derivation must satisfy

$$D\left(\sum_{\gamma\in\mathbb{J}} r_\gamma \exp(\gamma)\right) = \sum_{\gamma\in\mathbb{J}} r_\gamma D\left(\exp(\gamma)\right) = \sum_{\gamma\in\mathbb{J}} r_\gamma \exp(\gamma) D(\gamma). \tag{1.1}$$

Using the displayed equation, the problem of defining D is reduced to the problem of defining $D(\gamma)$ for $\gamma \in \mathbb{J}$.

The iteration of this procedure is not sufficient by itself to find a definition of D. For instance, the above equation gives almost no information on the values of D on the subclass $\mathbb{L}$ of the **log-atomic numbers**, namely the elements $\lambda \in \mathbf{No}$ such that all the iterated logarithms $\log_n(\lambda)$ are of the form $\exp(\gamma)$ for some $\gamma \in \mathbb{J}$. Indeed, for $\lambda \in \mathbb{L}$, the above equation reduces merely to $D(\exp(\lambda)) = \exp(\lambda)D(\lambda)$, and it is easy to see that this condition is not sufficient for a map $D_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}$ to extend to a surreal derivation.

As pointed out in the work of S. Kuhlmann and M. Matusinski [KM11, KM15], the class $\mathbb{L}$ of log-atomic numbers is crucial for defining a derivation, so we should first give some details about the relationship between $\mathbb{L}$ and **No**. On the face of the definition it is not immediate that $\mathbb{L}$ is non-empty, but it can be shown that $\omega \in \mathbb{L}$, and more generally that every "ε-number" (see [Gon86] for a definition) belongs to $\mathbb{L}$. In fact, in [KM15] there is an explicit parametrization of a class of log-atomic numbers, properly including the ε-numbers, called "κ-numbers". In the same paper it is conjectured that the κ-numbers generate $\mathbb{L}$ under application of log and exp. However, we will show that the class $\mathbb{L}$ is even larger (Proposition 5.24) and we shall provide an explicit parametrization of the whole of $\mathbb{L}$ (Corollary 5.17). It turns out that log-atomic numbers can be seen as the natural representatives of certain equivalence classes (Definition 5.2) which are similar but finer than those in [KM15], and correspond exactly to the "levels" of an Hardy field [Ros87, MM97], except that in our case there are uncountably many levels (actually a proper class of them).

Once $\mathbb{L}$ is understood, consider the smallest subfield $\mathbb{R}\langle\mathbb{L}\rangle$ of $\mathbf{No}$ containing $\mathbb{R}\cup\mathbb{L}$ and closed under taking exp, log and infinite sums. We shall see that $\mathbb{R}\langle\mathbb{L}\rangle$ is the largest subfield of $\mathbf{No}$ satisfying axiom ELT4 (Proposition 8.6), proposed in [KM15, Def. 5.1] as part of a general notion of transseries and satisfied by the logarithmic-exponential series of [vdDMM97] and the exponential-logarithmic series of [Kuh00]. Clearly, any derivation on $\mathbb{R}\langle\mathbb{L}\rangle$ satisfying (1)-(5) (as in Theorem A) is uniquely determined by its restriction to $\mathbb{L}$. A natural question is now whether $\mathbb{R}\langle\mathbb{L}\rangle = \mathbf{No}$. This is equivalent to the first part of Conjecture 5.2 in [KM15]. However, we shall prove that axiom ELT4 fails in the surreal numbers, and therefore the inclusion $\mathbb{R}\langle\mathbb{L}\rangle \subseteq \mathbf{No}$ is strict (see Theorem 8.7).

Despite the fact that $\mathbb{L}$ does not generate $\mathbf{No}$ under exp, log and infinite sums, a fundamental issue in our construction is understanding how a surreal derivation should behave on $\mathbb{L}$. One can verify that if a map $D_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}$ extends to a surreal derivation, then necessarily $D_{\mathbb{L}}(\lambda) > 0$ for all $\lambda \in \mathbb{L}$, and moreover

$$|\log(D_{\mathbb{L}}(\lambda)) - \log(D_{\mathbb{L}}(\mu))| < \frac{1}{n}|\lambda - \mu| \tag{1.2}$$

for all $\lambda, \mu \in \mathbb{L}$ and all $n \in \mathbb{N}$. This inequality plays a crucial role in this paper, and it can be proved to hold for the natural derivation on any Hardy field closed under log, provided λ, μ and $|\lambda - \mu|$ are positive infinite.

We start our construction by defining a "pre-derivation" $D_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}$ satisfying (1.2) and $D_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda)D_{\mathbb{L}}(\lambda)$ for all $\lambda \in \mathbb{L}$. It turns out that the *simplest* pre-derivation, which we call $\partial_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}$, can be calculated by a rather explicit formula. For this, we need a bit of notation involving a subclass of the κ-numbers of [KM15]. For $\alpha \in \mathbf{On}$ (where $\mathbf{On}$ is the class of all ordinal numbers) define inductively $\kappa_{-\alpha} \in \mathbf{No}$ as the simplest positive infinite surreal number less than $\log_n(\kappa_{-\beta})$ for all $n \in \mathbb{N}$ and $\beta < \alpha$. With this notation we have (see Definition 6.7)

$$\partial_{\mathbb{L}}(\lambda) = \exp\left(- \sum_{\substack{\alpha \in \mathbf{On} \\ \exists n\,:\, \exp_n(\kappa_{-\alpha}) > \lambda}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} \log_i(\lambda) \right).$$

(For the sake of exposition, we shall use the above formula as definition of $\partial_{\mathbb{L}}$, and only at the end of the paper we shall prove that it is the simplest pre-derivation; see Theorem 9.6.)

Once $\partial_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}$ is given, we can use (1.1) to give a tentative definition of a surreal derivation $\partial : \mathbf{No} \to \mathbf{No}$ extending $\partial_{\mathbb{L}}$. We adopt the same formalism used by Schmeling in [Sch01]. First of all, we recall Schmeling's notion of "path" . Given $x = \sum_{\gamma \in \mathbb{J}} r_\gamma \exp(\gamma) \in \mathbf{No}$, a **path** of x is a function $P : \mathbb{N} \to \mathbf{No}$ such that

- $P(0)$ is a term of x, namely $P(0) = r_\gamma \exp(\gamma)$ for some $r_\gamma \neq 0$;
- if $P(n) = r\exp(\eta)$, then $P(n+1)$ is a term of η.

For the moment, we restrict our attention to $\mathbb{R}\langle\mathbb{L}\rangle$. As we already mentioned, $\mathbb{R}\langle\mathbb{L}\rangle$ satisfies the axiom ELT4 of [KM15], which can be paraphrased as saying that for all $x \in \mathbb{R}\langle\mathbb{L}\rangle$, every path P of x enters $\mathbb{L}$, namely there is $n = n_P \in \mathbb{N}$ such that $P(n) \in \mathbb{L}$. Let $\mathcal{P}(x)$ be the set of all paths of x. Iterating (1.1), we immediately see that our desired surreal derivation ∂ extending $\partial_{\mathbb{L}}$ must satisfy

$$\partial(x) = \sum_{P \in \mathcal{P}(x)} \prod_{i < n_P} P(i) \cdot \partial_{\mathbb{L}}(P(n_P)), \tag{1.3}$$

provided the terms on right-hand side are "summable" in the sense of the Hahn field structure of **No**.

Guided by this observation, we use the right-hand side of (1.3) as the *definition* of $\partial(x)$ for $x \in \mathbb{R}\langle\mathbb{L}\rangle$. For general $x \in \mathbf{No}$, we define $\partial(x)$ using the same equation, but discarding the paths P of x that never enter $\mathbb{L}$. Our problem is now reduced to showing that the above sum is indeed summable, so that $\partial(x)$ is well-defined.

A formally similar problem was tackled by Schmeling in [Sch01] in order to extend derivations on transseries fields to their exponential extensions. For our problem, we use some of his techniques, even though our starting function $\partial_{\mathbb{L}}$ is not a derivation, and **No** cannot be seen as an exponential extension. Most importantly, however, we need to verify that **No** is a field of transseries.

The key step is proving the existence of a suitable ordinal valued function $\mathrm{NR} : \mathbf{No} \to \mathbf{On}$, which we call "nested truncation rank" (Theorem 4.24, Definition 4.25). Using the inequality (1.2) and the fact that the values of $\partial_{\mathbb{L}}$ are monomials, we are able to prove by induction on the rank that the terms in (1.3) are summable, and therefore ∂ is well-defined. It is then easy to prove that ∂ is also a surreal derivation, proving Theorem A. Theorem C also follows easily from the existence of the rank.

The study of the rank requires an in-depth investigation of the behavior of exp with respect to the simplicity relation $<_s$; this is non-trivial, as in general it is not true that if x is simpler than y then $\exp(x)$ is simpler than $\exp(y)$ (e.g., ω is simpler than $\log(\omega)$, but $\exp(\omega)$ is not simpler than $\exp(\log(\omega)) = \omega$). To carry out this analysis, we provide a short characterization of exp which may be of independent interest (Theorem 3.8), and prove that surreal numbers simplify under some natural operations that we call "nested truncations" (see Definition 4.3 and Theorem 4.24). For instance, the classical truncation of a series to one of its initial segments is a special case of nested truncation. Moreover, if γ is a nested truncation of some $\delta \in \mathbb{J}$, then $\exp(\gamma)$ is also a nested truncation of $\exp(\delta)$.

Since simplicity is well-founded, nested truncations are well-founded as well. We then define the rank $\mathrm{NR} : \mathbf{No} \to \mathbf{On}$ as the foundation of rank of nested truncations (Definition 4.25). Thus in particular the rank strictly decreases under non-trivial nested truncations. The properties of the rank are crucial to show that ∂ is well-defined. For instance, the numbers of rank 0 are exactly the elements of $\pm\mathbb{L}^{\pm 1} \cup \mathbb{R}$, on which ∂ can be easily calculated using $\partial_{\mathbb{L}}$. On the other hand, if $\gamma \in \mathbb{J}$, then γ and $\exp(\gamma)$ have the same rank.

The existence of the rank NR is essentially equivalent to the fact that **No** has a suitable structure of field of transseries as it was variously conjectured, so it is worth to discuss the critical axioms in some detail. We have already commented on the fact that $\mathbb{R}\langle\mathbb{L}\rangle$ is the largest subfield of **No** satisfying axiom ELT4 of [KM15], but that unfortunately $\mathbb{R}\langle\mathbb{L}\rangle \subsetneq \mathbf{No}$. On the other hand, **No** satisfies a similar but weaker axiom isolated under the name "T4" in [Sch01], where it is given as part of an axiomatization of a more general notion of transseries inspired by the nested expressions of [vdH97]. In the context of the surreal numbers, T4 reads as follows:

T4. for all sequences of monomials $\mathfrak{m}_i \in \exp(\mathbb{J})$, with $i \in \mathbb{N}$, such that

$$\mathfrak{m}_i = \exp(\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1})$$

where $r_{i+1} \in \mathbb{R}^*$, $\gamma_{i+1}, \delta_{i+1} \in \mathbb{J}$, and $\mathrm{S}(\gamma_{i+1}) > \mathrm{S}(r_{i+1}\mathfrak{m}_{i+1}) > \mathrm{S}(\delta_{i+1})$ (where S denotes the support), there exists $k \in \mathbb{N}$ such that $r_{i+1} = \pm 1$ and $\delta_{i+1} = 0$ for all $i \geq k$.

Note that the sequence $P(n) := r_n\mathfrak{m}_n$ (with $r_0 := 1$) is a path of $\mathfrak{m}_0$, according to the definition given above. The condition $\delta_{i+1} = 0$ states that a path must stop bifurcating on the "right" from a certain point on. (The aforementioned axiom ELT4 further prescribes that $\gamma_{i+1} = 0$, in which case the path eventually stops bifurcating on both sides, and therefore enters $\mathbb{L}$.) An immediate consequence of the existence of the rank NR is that T4 holds in the surreal numbers, from which Theorem C follows.

The proof of Theorem B, namely that ∂ is surjective, is based on other techniques, and it relies on some further properties of the function ∂. Indeed, to prove Theorem B we first verify that ∂ satisfies the hypothesis of a theorem of Rosenlicht [Ros83], so that we may establish the existence of asymptotic integrals (Proposition 7.4), and then we iterate asymptotic integration transfinitely many times and prove, using a version of Fodor's lemma, that the procedure eventually yields an integral. We shall also give an example of a rather natural surreal derivation that is not surjective (see Definition 6.6).

The structure of the paper is reflected in the following table of contents.

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2. Surreal numbers

We assume some familiarity with the ordered field of surreal numbers (see [Con76, Gon86]) which we denote by $\mathbf{No}$. In this section we give a brief presentation of the basic definitions and results, and we fix the notations that will be used in the rest of the paper.

2.1. Order and simplicity.

The usual definition of the class $\mathbf{No}$ of surreal numbers is by transfinite recursion, as in [Con76]. However, it is also possible to give a more concrete equivalent definition, as in [Gon86].

The domain of $\mathbf{No}$ is the *class* $\mathbf{No} = 2^{<\mathbf{On}}$ of all binary sequences of some ordinal length $\alpha \in \mathbf{On}$, namely the functions of the form $s : \alpha \to 2 = \{0,1\}$ (Gonshor writes "$-,+$" instead of "$0,1$"). The **length** (also called **birthday**) of a surreal number s is the ordinal number $\alpha = \mathrm{dom}(s)$. Note that $\mathbf{No}$ is not a set but a proper class, and all the relations and functions we shall define on $\mathbf{No}$ are going to be class-relations and class-functions, usually constructed by transfinite induction.

We say that $x \in \mathbf{No}$ is **simpler** than $y \in \mathbf{No}$ if $x \subseteq y$, i.e., if x is an initial segment of y as a binary sequence; we shall write $x \leq_s y$ when this is the case. We say that x is **strictly simpler** than y, written $x <_s y$, if $x \leq_s y$ and $x \neq y$. Note that $\leq_s$ is well-founded, and the empty sequence, which will play the role of the number zero, is simpler than any other surreal number. Moreover, the simplicity relation is a binary tree-like partial order on $\mathbf{No}$, with the immediate successors of a node $x \in \mathbf{No}$ being the sequences $x^\frown 0$ and $x^\frown 1$ obtained by appending 0 or 1 at the end of the binary sequence x.

We can introduce a total order $<$ on $\mathbf{No}$ in the following way. Writing $x^\frown y$ for the concatenation of binary sequences, we stipulate that $x^\frown 0 < x < x^\frown 1$ and more generally $x^\frown 0^\frown u < x < x^\frown 1^\frown v$ for every u, v. This defines a total order on $\mathbf{No}$ which coincides with the lexicographic order on sequences of the same length.

We say that a *subclass* C of $\mathbf{No}$ is **convex** if whenever $x < y$ are in C, every surreal number z such that $x < z < y$ is also in C. It is easy to see that every non-empty convex class contains a simplest number (given by the intersection $\bigcap C$).

Given two *sets* $A \subseteq \mathbf{No}$ and $B \subseteq \mathbf{No}$ with $A < B$ (meaning that $a < b$ for all $a \in A$ and $b \in B$), the *class*

$$(A; B) := \{y \in \mathbf{No} \,:\, A < y < B\}$$

is non-empty and convex and therefore it contains a simplest number x which is denoted by

$$x = A \mid B.$$

Such a pair $A \mid B$ is called a **representation of** x, and we call $(A;B)$ the **associated convex class**.

Every surreal number x has several different representations $x = A \mid B = A' \mid B'$. For instance, if A is cofinal with A' and B is coinitial with B', then clearly $(A;B) = (A';B')$. In this situation, we shall say that $A \mid B = A' \mid B'$ **by cofinality**. Note that it may well happen that $A \mid B = A' \mid B'$ even if A is not cofinal with A' or B is not coinitial with B', because two distinct convex classes may still have the same simplest number in common. The **canonical** representation $x = A \mid B$ is the unique one such that $A \cup B$ is exactly the set of all surreal numbers strictly simpler than x.

Remark 2.1. By definition, if $x = A \mid B$ and $A < y < B$, then $x \leq_s y$.

The converse does not always hold.

Definition 2.2. We call a representation $x = A \mid B$ **simple** if $x \leq_s y$ implies $A < y < B$.

In other words, a representation is simple when the associated convex class $(A;B)$ is as large as possible. An example of simple representation is the canonical one. In fact, we have the following.

Proposition 2.3. *Let* $c, x, y \in \mathbf{No}$. *We have:*

1. *if* $c <_s x \leq_s y$, *then* $c < x$ *if and only if* $c < y$;
2. *if* $x = A \mid B$, *and* $A \cup B$ *contains only numbers strictly simpler than* x, *then* $A \mid B$ *is simple; in particular, every surreal number admits a simple representation (for instance the canonical one).*

Proof. Point (1) follows at once from the definition of $<$. For (2), let $x = A \mid B$ be as in the hypothesis and let $c \in A \cup B$. We have $c <_s x$. If we now assume $x \leq_s y$ then, by (1), $c < x \leftrightarrow c < y$. Since $c \in A \cup B$ was arbitrary, we obtain $A < y < B$, and therefore $A \mid B$ is simple. □

2.2. **Field operations.** We can define ring operations $+, \cdot$ on $\mathbf{No}$ by induction on simplicity as follows:

$$x + y := \{x' + y,\ x + y'\} \mid \{x'' + y,\ x + y''\}$$

where x', x'' range over the elements simpler than x with $x' < x < x''$ and similarly for y, y', y'', or in other words, when $x = \{x'\} \mid \{x''\}$ and $y = \{y'\} \mid \{y''\}$ are the canonical representations of x and y respectively.

The definition of the product is slightly more complicated:

$$xy := \{x'y + xy' - x'y',\ x''y + xy'' - x''y''\} \mid \{x'y + xy'' - x'y'',\ x''y + xy' - x''y'\}.$$

The first expression in the left bracket ensures $x'y + xy' - x'y' < xy$, namely $(x - x')(y - y') > 0$. The mnemonic rule for the other expressions can be obtained in a similar way.

Remark 2.4. The definitions of sum and product are **uniform** in the sense of [Gon86, p. 15], namely the equations that define $x + y$ and xy continue to hold if we choose arbitrary representations $x = A \mid B$ and $y = C \mid D$ (not necessarily the canonical ones) and we let x', x'', y', y'' range over A, B, C, D respectively.

It is well know that these operations, together with the order, make **No** into an ordered field, which is in fact a real closed field (see [Gon86, Thm. 5.10]). In particular, we have a unique embedding of the rational numbers in **No**, so we can identify $\mathbb{Q}$ with a subfield of **No**. The subgroup of the **dyadic rationals** $\frac{m}{2^n} \in \mathbb{Q}$, with $m, n \in \mathbb{N}$, correspond exactly to the surreal numbers $s : k \to \{0, 1\}$ of finite ordinal length $k \in \mathbb{N}$.

The **real numbers** $\mathbb{R}$ can be isomorphically identified with a subfield of **No** by sending $z \in \mathbb{R}$ to the number $A \mid B$ where $A \subseteq \mathbf{No}$ is the set of rationals $< z$ and $B \subseteq \mathbf{No}$ is the set of rationals $> z$. This turns out to be a homomorphism, and therefore it agrees with the previous definition for $z \in \mathbb{Q}$. We may thus write $\mathbb{Q} \subset \mathbb{R} \subset \mathbf{No}$. Note that the length of a real number is at most ω (the least infinite ordinal). There are however surreal numbers of length ω which are not real numbers.

The **ordinal numbers** can be identified with a subclass of **No** by sending the ordinal α to the sequence $s : \alpha \to \{0, 1\}$ with constant value 1. Under this identification, the ring operations of **No**, when restricted to the ordinals $\mathbf{On} \subset \mathbf{No}$, coincide with the Hessenberg sum and product of ordinal numbers. On the other hand, the sequence $s : \alpha \to \{0, 1\}$ with constant value 0 corresponds to the additive inverse of the ordinal α, namely $-\alpha$. We remark that $x \in \mathbf{On}$ if and only if x admits a representation of the form $x = A \mid B$ with B empty, and similarly $x \in -\mathbf{On}$ if and only if we can write $x = A \mid B$ with A empty.

Note that under the above identification of $\mathbb{Q}$ as a subfield of **No**, the natural numbers $\mathbb{N} \subseteq \mathbb{Q}$ are exactly the finite ordinals.

2.3. **Hahn fields.** Let Γ be an ordered abelian group, written multiplicatively. We recall the definition of the **Hahn field** $\mathbb{R}((\Gamma))$ with coefficients in $\mathbb{R}$ and "monomials" in Γ. The domain of $\mathbb{R}((\Gamma))$ consists of all the functions $f : \Gamma \to \mathbb{R}$ whose **support** $\mathrm{S}(f) := \{\mathfrak{m} \in \Gamma \ : \ f(\mathfrak{m}) \neq 0\}$ is a **reverse well-ordered** subset of Γ, namely every non-empty subset of Γ has a maximum (when Γ is a proper class, we still require that $\mathrm{S}(f)$ is a set). For each f which is not identically 0, $\mathrm{S}(f)$ has a maximum element $\mathfrak{m}$; if $f(\mathfrak{m}) > 0$, we say that f is positive. For later reference, given $\mathfrak{m} \in \mathfrak{M}$, the **truncation** of f at $\mathfrak{m}$ is the function $f|\mathfrak{m} : \mathfrak{M} \to \mathbb{R}$ which coincides with f on arguments $> \mathfrak{m}$ and is zero on arguments $\leq \mathfrak{m}$.

The addition of two elements $f, g \in \mathbb{R}((\Gamma))$ is defined as

$$(f + g)(\mathfrak{m}) := f(\mathfrak{m}) + g(\mathfrak{m}),$$

and the multiplication is given by

$$(f \cdot g)(\mathfrak{m}) := \sum_{\mathfrak{n}+\mathfrak{o}=\mathfrak{m}} f(\mathfrak{n}) g(\mathfrak{o}).$$

Note that since the supports are reverse well-ordered, only finitely many terms of the latter sum can be non-zero, hence the multiplication is well defined. These operations make $\mathbb{R}((\Gamma))$ into an ordered field (which is real closed when Γ is divisible).

It is well known that **No** can be endowed with a Hahn field structure. To this aim recall that two non-zero surreal numbers $x, y \in \mathbf{No}^*$ are in the same **archimedean class** if each of them is bounded in modulus by an integer multiple of the other, namely $|x| \leq k|y|$ and $|y| \leq k|x|$ for some $k \in \mathbb{N}$.

A positive surreal number $x \in \mathbf{No}^*$ is called a **monomial** if it is the simplest positive element in its archimedean class. The class $\mathfrak{M} \subset \mathbf{No}^*$ of all monomials is a group under multiplication (see Fact 2.17). A **term** is a non-zero real number $r \in \mathbb{R}^*$ multiplied by a monomial; we denote by $\mathbb{R}^*\mathfrak{M}$ the class of all terms.

One of Conway's remarkable insights is that we can identify **No** with $\mathbb{R}((\mathfrak{M}))$ in the following way. Given $f \in \mathbb{R}((\mathfrak{M}))$, write $f_\mathfrak{m}$ for the real number $f(\mathfrak{m})$. Note that $f_\mathfrak{m}\mathfrak{m} = f(\mathfrak{m})\mathfrak{m} \in \mathbb{R}\mathfrak{M}$ is a well defined element of **No**. We define the map $\sum : \mathbb{R}((\mathfrak{M})) \to \mathbf{No}$ by induction on the order type of the support.

Definition 2.5. Let $f \in \mathbb{R}((\mathfrak{M}))$.

(1) If the support of f is empty, we define $\sum f := 0 \in \mathbf{No}$.
(2) If the support of f contains a smallest monomial $\mathfrak{n}$, we define

$$\sum f := \sum f|\mathfrak{n} + f_\mathfrak{n}\mathfrak{n}.$$

(3) If the support of f is non-empty and has no smallest monomial, we define

$$\sum f := \left\{\sum f|\mathfrak{m} + q'\mathfrak{m}\right\} \mid \left\{\sum f|\mathfrak{m} + q''\mathfrak{m}\right\}$$

where $\mathfrak{m}$ varies in $\mathrm{S}(f)$ and q', q'' varies among the rational numbers such that $q' < f_\mathfrak{m} < q''$.

Note that in (3), for $\sum f$ to be well defined, one needs to show by a simultaneous induction that each number on the left-hand side is smaller than each number on the right-hand side (see [Gon86, p. 59] for a detailed argument).

By [Gon86, Lemma 5.3], if $\mathrm{S}(f)$ contains a smallest element $\mathfrak{m}$, then $\sum f = \sum f|\mathfrak{m} + f_\mathfrak{m}\mathfrak{m}$ can be characterized as the simplest surreal number such that, for every $q', q'' \in \mathbb{Q}$ with $q' < f_\mathfrak{m} < q''$, we have $\sum f|\mathfrak{m} + q'\mathfrak{m} < \sum f < \sum f|\mathfrak{m} + q''\mathfrak{m}$. It then follows that the three cases in Definition 2.5 can be subsumed under a single equation, as follows.

Proposition 2.6. *For every $f \in \mathbb{R}((\mathfrak{M}))$ we have*

$$\sum f = \left\{\sum f|\mathfrak{m} + q'\mathfrak{m}\right\} \mid \left\{\sum f|\mathfrak{m} + q''\mathfrak{m}\right\}$$

where $\mathfrak{m}$ varies in $\mathrm{S}(f)$ and q', q'' varies among the rational numbers such that $q' < f_\mathfrak{m} < q''$.

This also holds when $\mathrm{S}(f) = \emptyset$; indeed, in this case $\sum f$ is just the simplest surreal numbers satisfying the empty set of inequalities, hence $\sum f = 0$. We could in fact take the above equation as the definition of $\sum f$, but then it would be more difficult to verify that $\sum(f+g) = \sum f + \sum g$.

As a matter of notations, we write $\sum_{\mathfrak{m}\in\mathfrak{M}} f_\mathfrak{m}\mathfrak{m}$ for $\sum f$, namely we think of $\sum f$ as a decreasing formal infinite sum of terms $f_\mathfrak{m}\mathfrak{m}$ with reverse well-ordered support. It turns out that the map $\sum : \mathbb{R}((\mathfrak{M})) \to \mathbf{No}$ is an isomorphism of ordered fields (in particular it is surjective), so we can identify $f \in \mathbb{R}((\mathfrak{M}))$ with $\sum f = \sum_\mathfrak{m} f_\mathfrak{m}\mathfrak{m} \in \mathbf{No}$ and write

$$\mathbf{No} = \mathbb{R}((\mathfrak{M})).$$

A short proof of surjectivity is in Conway's book [Con76, pp. 32-33], which however should be integrated with some details that can be found in Gonshor (in particular [Gon86, Lemmas 5.2 and 5.3]). We remark that Conway and Gonshor prove the result in the opposite direction, by defining a map $\mathbf{No} \to \mathbb{R}((\mathfrak{M}))$ which is the inverse of our $\sum : \mathbb{R}((\mathfrak{M})) \to \mathbf{No}$. For a full proof see [Gon86, Thm. 5.6].

Under the identification $f = \sum f$ we can drop the summation sign in Proposition 2.6. For instance, when f is a single monomial $\mathfrak{m}$ the equation reduces to

$$\mathfrak{m} = \{q'\mathfrak{m}\} \mid \{q''\mathfrak{m}\}$$

where q', q'' range over the rational numbers with $q' < 1 < q''$.

The identification $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$ makes it possible to extend to $\mathbf{No}$ the various notions that are given on Hahn fields:

Definition 2.7. Let $x \in \mathbf{No}$ and write $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}}\mathfrak{m}$.

(1) The **support** $\mathrm{S}(x)$ of x is the support of the corresponding element of $\mathbb{R}((\mathfrak{M}))$, namely $\mathrm{S}(x) := \{\mathfrak{m} \in \mathfrak{M} : x_{\mathfrak{m}} \neq 0\}$.
(2) The **terms** of x are the numbers in the set $\{x_{\mathfrak{m}}\mathfrak{m} : x_{\mathfrak{m}} \neq 0\} \subset \mathbb{R}^*\mathfrak{M}$.
(3) The **coefficient** of $\mathfrak{m}$ in x is $x_{\mathfrak{m}}$.
(4) The **leading monomial** of x is the maximum monomial in $\mathrm{S}(x)$.
(5) The **leading term** of x is the leading monomial multiplied by its coefficient.
(6) Given $\mathfrak{n} \in \mathfrak{M}$, the **truncation** of x at $\mathfrak{n}$ is the number $x|\mathfrak{n} := \sum_{\mathfrak{m}>\mathfrak{n}} x_{\mathfrak{m}}\mathfrak{m}$. If $y \in \mathbf{No}$ is a truncation of x, we write $y \trianglelefteq x$, and $y \triangleleft x$ if moreover $x \neq y$.

The relation $\trianglelefteq$ is clearly a partial order with a tree-like structure, and it is actually a weakening of the simplicity relation.

Proposition 2.8. *If $x \trianglelefteq y$ then $x \leq_s y$.*

In [Gon86, Thm. 5.12] this statement obtained as a corollary of an explicit calculation of the binary sequence corresponding to an infinite sum, but it can also be immediately deduced from Proposition 2.6. We include the proof to illustrate the method, as it will be applied again in the sequel.

Proof of Proposition 2.8. Given $x \in \mathbf{No}$, by Proposition 2.6 we can write $x = A \mid B$ where

$$A = \{x|\mathfrak{n} + q'\mathfrak{n}\}, \quad B = \{x|\mathfrak{n} + q''\mathfrak{n}\}$$

with $\mathfrak{n}$ varying in $\mathrm{S}(x)$, and q', q'' varying among the rational numbers such that $q' < x_{\mathfrak{n}} < q''$. Similarly, $y = A' \mid B'$ where

$$A' = \{y|\mathfrak{n} + q'\mathfrak{n}\}, \quad B' = \{y|\mathfrak{n} + q''\mathfrak{n}\}$$

with $\mathfrak{n} \in \mathrm{S}(y)$ and $q' < y_{\mathfrak{n}} < q''$. Since $x \trianglelefteq y$ we have $\mathrm{S}(x) \subseteq \mathrm{S}(y)$, and for every $\mathfrak{n} \in \mathrm{S}(x)$ we have $x|\mathfrak{n} = y|\mathfrak{n}$ and $x_{\mathfrak{n}} = y_{\mathfrak{n}}$. It follows that $A \subseteq A'$ and $B \subseteq B'$, hence $x \leq_s y$. □

2.4. **Summability.** The identification $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$ makes it possible to extend to $\mathbf{No}$ the notion of infinite sum.

Definition 2.9. Let I be a set and $(x_i : i \in I)$ be an indexed family of surreal numbers. We say that $(x_i : i \in I)$ is **summable** if $\bigcup_i \mathrm{S}(x_i)$ is reverse well-ordered and if for each $\mathfrak{m} \in \bigcup_i \mathrm{S}(x_i)$ there are only finitely many $i \in I$ such that $\mathfrak{m} \in \mathrm{S}(x_i)$.

When $(x_i : i \in I)$ is summable, we define its **sum** $y := \sum_{i \in I} x_i$ as the unique surreal number such that $\mathrm{S}(y) \subseteq \bigcup_i \mathrm{S}(x_i)$ and, for every $\mathfrak{m} \in \mathfrak{M}$,

$$y_{\mathfrak{m}} = \left(\sum_{i \in I} x_i\right)_{\mathfrak{m}} = \sum_{i \in I} (x_i)_{\mathfrak{m}}.$$

Note that by assumption for each $\mathfrak{m}$ there are finitely many i such that $(x_i)_{\mathfrak{m}} \neq 0$, so that each $y_{\mathfrak{m}}$ is a well defined real number.

The result is coherent with our previous definitions: if $x \in \mathbf{No}$, the family $(x_{\mathfrak{m}}\mathfrak{m} : \mathfrak{m} \in \mathrm{S}(x))$ is obviously summable, and its sum $\sum_{\mathfrak{m}} x_{\mathfrak{m}}\mathfrak{m}$ in the just defined sense is exactly x.

Remark 2.10. The following criterion for summability follows at once from the definition: $(x_i : i \in I)$ is summable if and only if there are no injective sequence $n \mapsto i_n \in I$ and monomials $\mathfrak{m}_n \in \mathrm{S}(x_{i_n})$ such that $\mathfrak{m}_n \leq \mathfrak{m}_{n+1}$ for every $n \in \mathbb{N}$.

Remark 2.11. It can be verified that infinite sums are infinitely associative and distributive over products, see [Gon86] for the details.

Definition 2.12. A function $F : \mathbf{No} \to \mathbf{No}$ is **strongly linear** if for all $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}}\mathfrak{m}$ we have $F(x) = \sum_{\mathfrak{m}} x_{\mathfrak{m}} F(\mathfrak{m})$ (in particular, $(x_{\mathfrak{m}} F(\mathfrak{m}) : \mathfrak{m} \in \mathfrak{M})$ is summable).

Proposition 2.13. *If $F : \mathbf{No} \to \mathbf{No}$ is strongly linear, then for any summable $(x_i : i \in I)$ the family $(F(x_i) : i \in I)$ is summable and*

$$F\left(\sum_{i \in I} x_i\right) = \sum_{i \in I} F(x_i).$$

Proof. We have

$$F\left(\sum_{i\in I} x_i\right) = F\left(\sum_{\mathfrak{m}\in\mathfrak{M}} \left(\sum_{i\in I} x_i\right)_{\mathfrak{m}} \mathfrak{m}\right) = \sum_{\mathfrak{m}\in\mathfrak{M}} \sum_{i\in I} (x_i)_{\mathfrak{m}} F(\mathfrak{m}) = \sum_{i\in I} F(x_i). \quad \square$$

2.5. Purely infinite numbers.

We use Hardy's notation "$\preceq$" for the **dominance** relation.

Definition 2.14. Given $x, y \in \mathbf{No}$ we write

- $x \preceq y$ if $|x| \leq k|y|$ for some $k \in \mathbb{N}$;
- $x \prec y$ if $|x| < \frac{1}{k}|y|$ for all positive $k \in \mathbb{N}$;
- $x \asymp y$ if $\frac{1}{k}|y| \leq |x| \leq k|y|$ for some positive $k \in \mathbb{N}$;
- $x \sim y$ if $x - y \prec x$ (equivalently $\left|1 - \frac{y}{x}\right| \prec 1$ when $x \neq 0$).

We say that $x \in \mathbf{No}$ is **finite** if $x \preceq 1$. We say that x is **infinitesimal** if $x \prec 1$. We shall denote the class of all infinitesimal numbers by $o(1)$. In general, we denote by $o(x)$ the class of all $y \in \mathbf{No}$ such that $\frac{y}{x}$ is infinitesimal, namely such that $y \prec x$. Note that $x \asymp y$ if and only if x and y are in the same archimedean class.

We say that $x \in \mathbf{No}$ is **purely infinite** if all the monomials $\mathfrak{m} \in \mathfrak{M}$ in its support are infinite (or equivalently $\mathrm{S}(x) > 1$). The class of purely infinite numbers of $\mathbf{No}$ shall be denoted by $\mathbb{J}$. We have a direct sum decomposition

$$\mathbf{No} = \mathbb{J} + \mathbb{R} + o(1)$$

as a real vector space.

The **surreal integers** are the numbers in $\mathbb{J} + \mathbb{Z}$. They coincide with Conway's "omnific integers", namely the numbers x such that $x = \{x - 1\} \mid \{x + 1\}$.

2.6. **The omega-map.** Another remarkable feature of surreal numbers is that the class $\mathfrak{M}$ of monomials can be parametrized in a rather canonical way by the surreal numbers themselves.

Definition 2.15 ([Con76, p. 31])**.** Given $x \in \mathbf{No}$, we let

$$\omega^x := \{0, k\omega^{x'}\} \mid \left\{\frac{1}{2^k}\omega^{x''}\right\}$$

where k ranges in the natural numbers, and x', x'' range over the surreal numbers strictly simpler than x and such that $x' < x < x''$.

As in the case of the ring operations, the above definition is uniform, namely if $x = A \mid B$ is any representation of x, the equation in the definition of ω^x remains true if we let x', x'' range over A, B respectively. The following remark follows at once from the fact that for $x \leq_s y$ the convex class associated to the above representation of ω^x includes the convex class associated to the corresponding representation of ω^y.

Remark 2.16. If $x \leq_s y$, then $\omega^x \leq_s \omega^y$.

Fact 2.17 ([Con76, Thms. 19 and 20], [Gon86, Thms. 5.3 and 5.4])**.** *The map $x \mapsto \omega^x$ is an isomorphism from $(\mathbf{No}, +, <)$ to $(\mathfrak{M}, \cdot, <)$. In particular, ω^x is the simplest positive representative of its archimedean class, $\omega^0 = 1$ and $\omega^{x+y} = \omega^x \cdot \omega^y$.*

From the equalities $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$ and $\mathfrak{M} = \omega^{\mathbf{No}}$ we obtain

$$\mathbf{No} = \mathbb{R}((\omega^{\mathbf{No}})),$$

namely every surreal number x can be written uniquely in the form

$$x = \sum_{y \in \mathbf{No}} a_y \omega^y$$

where $a_y \in \mathbb{R}$ and $a_y \neq 0$ if and only if ω^y is in the support of x. This representation is called the **normal form** of x and it coincides with Cantor's normal form when $x \in \mathbf{On} \subset \mathbf{No}$.

3. Exponentiation

3.1. **Gonshor's exponentiation.** The surreal numbers admit a well behaved exponential function defined as follows.

Definition 3.1 ([Gon86, p. 145])**.** Let $x = \{x'\} \mid \{x''\}$ be the canonical representation of x. We define inductively

$$\exp(x) := \{0, \exp(x') \cdot [x - x']_n, \exp(x'')[x - x'']_{2n+1}\} \mid \left\{\frac{\exp(x'')}{[x'' - x]_n}, \frac{\exp(x')}{[x' - x]_{2n+1}}\right\},$$

where n ranges in $\mathbb{N}$ and

$$[x]_n = 1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!},$$

with the further convention that the expressions containing terms of the form $[y]_{2n+1}$ are to be considered only when $[y]_{2n+1} > 0$.

It can be shown that the function exp is a surjective homomorphism $\exp : (\mathbf{No}, +) \to (\mathbf{No}^{>0}, \cdot)$ extending exp on $\mathbb{R}$ that makes $(\mathbf{No}, +, \cdot, \exp)$ into an elementary extension of $(\mathbb{R}, +, \cdot, \exp)$ (see [vdDMM94, Cor. 2.11, Cor. 4.6], [vdDE01] and [Res93]).

We recall here a list of properties that were proved in [Gon86]. We shall use them to give an alternative characterization of the function exp.

Fact 3.2. *The function* exp *has the following properties.*

(1) *Definition 3.1 is uniform* [Gon86, Cor. 10.1].
(2) *The restriction of* exp *to* $\mathbb{R} \subseteq \mathbf{No}$ *is the real exponential function* [Gon86, Thm. 10.2].
(3) *If* $\varepsilon \prec 1$*, then the sequence* $(\frac{\varepsilon^n}{n!} : n \in \mathbb{N})$ *is summable and* $\exp(\varepsilon) = \sum_n \frac{\varepsilon^n}{n!}$ [Gon86, Thm. 10.3].
(4) *The function* exp *is an isomorphism from* $(\mathbf{No}, +, <)$ *to* $(\mathbf{No}^{>0}, \cdot, <)$*. In particular,* $\exp(0) = 1$ *and* $\exp(x + y) = \exp(x) \cdot \exp(y)$ *for every* $x, y \in \mathbf{No}$ [Gon86, Cor. 10.1].
(5) *If* $x > 0$*, then* $\exp(\omega^x) = \omega^{\omega^{g(x)}}$*, where* $g : \mathbf{No}^{>0} \to \mathbf{No}$ *is defined by*
$$g(x) := \{c(x), g(x')\} \,|\, \{g(x'')\}\,,$$
and $c(x)$ *is the unique number such that* $\omega^{c(x)}$ *and* x *are in the same archimedean class* [Gon86, Thm. 10.11].
(6) *If* $x = \sum_y a_y \omega^y$ *is purely infinite, then* [Gon86, Thm. 10.13]
$$\exp\left(\sum_y a_y \omega^y\right) = \omega^{\sum_y a_y \omega^{g(y)}}.$$

Definition 3.3. Let $\log : \mathbf{No}^{>0} \to \mathbf{No}$ (called **logarithm**) be the inverse of exp.

We let $\exp_n$ and $\log_n$ be the n-fold iterated compositions of exp and log with themselves.

Remark 3.4. One can easily verify that if $\varepsilon \prec 1$ we must have
$$\log(1 + \varepsilon) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\varepsilon^n}{n}.$$

In the sequel we shall make repeated use of the fact that exp grows more than any polynomial. In particular we have:

Remark 3.5. If $x > \mathbb{N}$, then $\exp(x) \succ x^n$ for every $n \in \mathbb{N}$.

3.2. Ressayre representation.

By [Gon86, Thms. 10.8, 10.9], the monomials are the image under exp of the purely infinite surreal numbers:
$$\exp(\mathbb{J}) = \mathfrak{M} = \omega^{\mathbf{No}}.$$
Since $\mathbf{No} = \mathbb{R}((\mathfrak{M})) = \mathbb{R}((\omega^{\mathbf{No}}))$, it then follows that $\mathbf{No} = \mathbb{R}((\exp(\mathbb{J})))$ as well, namely every surreal number $x \in \mathbf{No}$ can be written uniquely in the form
$$x = \sum_{\gamma \in \mathbb{J}} r_\gamma \exp(\gamma)$$
where $r_\gamma \neq 0$ if and only if $\exp(\gamma) \in \mathrm{S}(x)$. We call this the **Ressayre form** of $x \in \mathbf{No}$. Note that, unlike the case of normal form $x = \sum_{y \in \mathbf{No}} a_y \omega^y$, the summation is indexed by elements of $\mathbb{J}$.

Definition 3.6. Given a non-zero number $x = \sum_{\gamma\in\mathbb{J}} r_\gamma \exp(\gamma)$ we define $\ell(x) \in \mathbb{J}$ as the maximal γ such that $r_\gamma \neq 0$, or in other words, as the logarithm γ of the largest monomial $\mathfrak{m} = \exp(\gamma)$ in its support.

It is easy to verify that $\ell : \mathbf{No} \setminus \{0\} \to \mathbb{J}$ satisfies:

(1) $\ell(x+y) \leq \max\{\ell(x), \ell(y)\}$, with the equality holding if $\ell(x) \neq \ell(y)$;
(2) $\ell(xy) = \ell(x) + \ell(y)$.

This makes $-\ell$ into a Krull valuation. We call $\ell(x)$ the **ℓ-value** of x.

Remark 3.7. Given $x, y \in \mathbf{No}^*$ we have

- $x \preceq y$ if and only if $\ell(x) \leq \ell(y)$,
- $x \prec y$ if and only if $\ell(x) < \ell(y)$,
- $x \asymp y$ if and only if $\ell(x) = \ell(y)$,
- $x \sim y$ if and only if $\ell(x-y) < \ell(x)$.

Moreover, if $x \not\asymp 1$ then $\ell(x) \sim \log(x)$.

3.3. A characterization of exp.

In order to understand the interaction between exp and the simplicity relation $<_s$, we first give a rather short characterization of exp. We start with Gonshor's description Fact 3.2 and we further simplify it by dropping any references to the omega-map or to the function g.

Theorem 3.8. *The function* $\exp : \mathbf{No} \to \mathbf{No}$ *is uniquely determined by the following properties:*

(1) *if* $\mathfrak{m} \in \mathfrak{M}^{>1}$ *is an infinite monomial, then*
$$\exp(\mathfrak{m}) = \left\{\mathfrak{m}^k,\, \exp(\mathfrak{m}')^k\right\} \mid \left\{\exp(\mathfrak{m}'')^{\frac{1}{k}}\right\}$$
where k ranges in the positive integers and $\mathfrak{m}', \mathfrak{m}''$ range in the set of monomials simpler than $\mathfrak{m}$ and such that $\mathfrak{m}' < \mathfrak{m} < \mathfrak{m}''$;

(2) *if* $\gamma = \sum_{\mathfrak{m}\in\mathfrak{M}} \gamma_\mathfrak{m} \mathfrak{m} \in \mathbb{J}$ *is a purely infinite surreal number, then*
$$\exp(\gamma) = \left\{0,\, \exp(\gamma|\mathfrak{m}) \exp(\mathfrak{m})^{q'}\right\} \mid \left\{\exp(\gamma|\mathfrak{m}) \exp(\mathfrak{m})^{q''}\right\}$$
where $\mathfrak{m}$ ranges in $\mathrm{S}(\gamma)$ and q', q'' range among the rational numbers such that $q' < \gamma_\mathfrak{m} < q''$;

(3) *if* $x = \gamma + r + \varepsilon$*, where* $\gamma \in \mathbb{J}$*,* $r \in \mathbb{R}$ *and* $\varepsilon \in o(1)$*, then*
$$\exp(\gamma + r + \varepsilon) = \exp(\gamma) \cdot \exp(r) \cdot \left(\sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\right),$$
where $\exp(r)$ *is the value of the real exponential function at* r*.*

Point (1) is the minimum requirement that ensures that the $\exp_{\restriction\mathfrak{M}^{>1}}$ is increasing and grows more than any power function. Point (2) shows that, for $\gamma \in \mathbb{J}$, $\exp(\gamma)$ is the simplest element satisfying some natural inequalities determined by the values of exp on the truncations of γ. Point (3) just says that the behavior on the finite numbers is the one given by the classical Taylor expansion of exp.

Proof. (1) Let $x \in \mathbf{No}^{>0}$ be such that $\mathfrak{m} = \omega^x$. By Fact 3.2 we have $\exp(\mathfrak{m}) = \exp(\omega^x) = \omega^{\omega^{g(x)}}$. Since $y = g(x) = \{c(x),\, g(x')\} \mid \{g(x'')\}$ we have
$$\omega^{g(x)} = \left\{0,\, k\omega^{c(x)},\, k\omega^{g(x')}\right\} \mid \left\{\frac{1}{k}\omega^{g(x'')}\right\},$$

where k ranges in $\mathbb{N}^{>0}$. Computing ω^y with $y = \omega^{g(x)}$ we obtain

$$\omega^{\omega^{g(x)}} = \left\{0,\ j\omega^0,\ j\omega^{k\omega^{c(x)}},\ j\omega^{k\omega^{g(x')}}\right\} \mid \left\{\frac{1}{j}\omega^{\frac{1}{k}\omega^{g(x'')}}\right\}$$

where j and k range in $\mathbb{N}^{>0}$.

By cofinality, since k varies over the positive integers, we can drop the coefficients j, $\frac{1}{j}$ and the first two expressions; moreover, we note that $\{k\omega^{c(x)} : k \in \mathbb{N}^{>0}\}$ is cofinal with $\{kx : k \in \mathbb{N}^{>0}\}$. We deduce that

$$\omega^{\omega^{g(x)}} = \left\{\omega^{kx},\ \omega^{k\omega^{g(x')}}\right\} \mid \left\{\omega^{\frac{1}{k}\omega^{g(x'')}}\right\}.$$

Now, recalling that $\omega^{ky} = (\omega^y)^k$ and $\omega^{\omega^{g(y)}} = \exp(\omega^y)$, we get

$$\exp(\mathfrak{m}) = \omega^{\omega^{g(x)}} = \left\{\mathfrak{m}^k,\ \exp(\omega^{x'})^k\right\} \mid \left\{\exp(\omega^{x''})^{1/k}\right\}.$$

Finally, by Remark 2.16, we note that the monomials $\mathfrak{m}', \mathfrak{m}''$ simpler than $\mathfrak{m}$ with $\mathfrak{m}' < \mathfrak{m} < \mathfrak{m}''$ are exactly those of the form $\omega^{x'}$, $\omega^{x''}$ with x', x'' simpler than x and such that $x' < x < x''$, and we are done.

(2) Given a purely infinite surreal number $\gamma = \sum_{\mathfrak{m}\in\mathfrak{M}} \gamma_{\mathfrak{m}}\mathfrak{m} = \sum_{y\in\mathbf{No}} a_y\omega^y$, let $G(\gamma) := \sum_y a_y\omega^{g(y)}$. By Fact 3.2, we have $\exp(\gamma) = \omega^{G(\gamma)}$. Note that G is strictly increasing, strongly linear, surjective, and sends monomials to monomials. In particular, for all $\mathfrak{n} \in \mathfrak{M}$, $G(\gamma)|\mathfrak{n} = G(\gamma|\mathfrak{m})$ where $\mathfrak{m} = G^{-1}(\mathfrak{n})$. By Proposition 2.6 it follows that

$$G(\gamma) = \{G(\gamma)|G(\mathfrak{m}) + q'G(\mathfrak{m})\} \mid \{G(\gamma)|G(\mathfrak{m}) + q''G(\mathfrak{m})\}$$

where $\mathfrak{m}$ ranges in $\mathrm{S}(\gamma)$ and q', q'' range among the rational numbers such that $q' < \gamma_{\mathfrak{m}} < q''$.

By definition of ω^y, setting $y = G(\gamma)$, we obtain

$$\omega^{G(\gamma)} = \left\{0,\ k\omega^{G(\gamma)|G(\mathfrak{m})+q'G(\mathfrak{m})}\right\} \mid \left\{\frac{1}{k}\omega^{G(\gamma)|G(\mathfrak{m})+q''G(\mathfrak{m})}\right\}$$

with k ranging in $\mathbb{N}^{>0}$ and $\mathfrak{m}, q', q''$ as above. By cofinality, we can drop k:

$$\omega^{G(\gamma)} = \left\{0,\ \omega^{G(\gamma)|G(\mathfrak{m})+q'G(\mathfrak{m})}\right\} \mid \left\{\omega^{G(\gamma)|G(\mathfrak{m})+q''G(\mathfrak{m})}\right\}.$$

Since $\omega^{G(\gamma)|G(\mathfrak{m})+qG(\mathfrak{m})} = \omega^{G(\gamma|\mathfrak{m})}\left(\omega^{G(\mathfrak{m})}\right)^q = \exp(\gamma|\mathfrak{m})\exp(\mathfrak{m})^q$, we get

$$\exp(\gamma) = \omega^{G(\gamma)} = \{0,\ \exp(\gamma|\mathfrak{n})\exp(\mathfrak{m})^{q'}\} \mid \{\exp(\gamma|\mathfrak{m})\exp(\mathfrak{m})^{q''}\},$$

as desired.

Part (3) follows easily from Fact 3.2. □

4. Nested truncations

Unlike the omega-map, the function exp is not monotone with respect to simplicity, as $x \leq_s y$ does not always imply $\exp(x) \leq_s \exp(y)$; for instance, we have $\omega <_s \log(\omega)$ while $\exp(\log(\omega)) = \omega <_s \exp(\omega)$. However, under some additional assumptions exp does preserve simplicity. For example, exp preserves simplicity if we know that x is a truncation of y (this is well known, although it seems to have never been stated in this form).

Proposition 4.1. *Let $\gamma, \delta \in \mathbb{J}$. If $\gamma \trianglelefteq \delta$, then $\exp(\gamma) \leq_s \exp(\delta)$.*

Proof. The argument is similar to the one for Proposition 2.8, so we will be brief. As in Theorem 3.8(2), we can write $\exp(\gamma) = A \mid B$ where

$$A = \left\{0,\, \exp(\gamma|\mathfrak{m})\exp(\mathfrak{m})^{q'}\right\}, \quad B = \left\{\exp(\gamma|\mathfrak{m})\exp(\mathfrak{m})^{q''}\right\},$$

with $\mathfrak{m} \in \mathrm{S}(\gamma)$ and $q' < \gamma_{\mathfrak{m}} < q''$. Similarly, $\exp(\delta) = A' \mid B'$ where

$$A' = \left\{0,\, \exp(\delta|\mathfrak{m})\exp(\mathfrak{m})^{q'}\right\}, \quad B' = \left\{\exp(\delta|\mathfrak{m})\exp(\mathfrak{m})^{q''}\right\},$$

with $\mathfrak{m} \in \mathrm{S}(\delta)$ and $q' < \delta_{\mathfrak{m}} < q''$. Since $\gamma \trianglelefteq \delta$, we have $A \subseteq A'$ and $B \subseteq B'$, and therefore $\exp(\gamma) \leq_s \exp(\delta)$, as desired. □

The above proposition is far from being sufficient for our purposes. For instance, since $\exp(\gamma) \ntrianglelefteq \exp(\delta)$ for $\gamma \neq \delta$ in $\mathbb{J}$, we cannot iterate it to compare $\exp(\exp(\gamma))$ and $\exp(\exp(\delta))$. We remedy to this problem by defining a more powerful notion of "nested truncation".

Definition 4.2. We say that a sum $x_1 + x_2 + \cdots + x_n$ of surreal numbers is in **standard form** if $\mathrm{S}(x_1) > \mathrm{S}(x_2) > \cdots > \mathrm{S}(x_n)$.

Given $x \in \mathbf{No}^*$, we let $\operatorname{sign}(x) := 1$ if $x > 0$ and $\operatorname{sign}(x) := -1$ if $x < 0$.

Definition 4.3. For $n \in \mathbb{N}$, we define $\blacktriangleleft\!\!\!\!\!\__n$ on $\mathbf{No}^*$ inductively as follows:

(1) $x \,\underline{\blacktriangleleft}_0\, y$ if $x \trianglelefteq y$;
(2) $x \,\underline{\blacktriangleleft}_{n+1}\, y$ if there are $\gamma \,\underline{\blacktriangleleft}_n\, \delta$ in $\mathbb{J}^*$, $z, w \in \mathbf{No}$ and $r \in \mathbb{R}^*$ such that

$$x = z + \operatorname{sign}(r)\exp(\gamma), \quad y = z + r\exp(\delta) + w,$$

where both sums are in standard form.

We say that $x \,\underline{\blacktriangleleft}\, y$, or that x is a **nested truncation** of y, if $x \,\underline{\blacktriangleleft}_n\, y$ for some $n \in \mathbb{N}$. We write $x \blacktriangleleft y$ if $x \,\underline{\blacktriangleleft}\, y$ and $x \neq y$.

Remark 4.4. Note that even if $z + r\exp(\delta) + w$ is in standard form, and $\gamma \,\underline{\blacktriangleleft}\, \delta$, the sum $z + \operatorname{sign}(r)\exp(\gamma)$ may well not be in standard form.

Remark 4.5. For all $x, y \in \mathbf{No}^*$, if $x \,\underline{\blacktriangleleft}\, y$, then $x > 0$ if and only if $y > 0$.

Nested truncations behave rather similarly to truncations. First of all, like $\trianglelefteq$, the relation $\underline{\blacktriangleleft}$ is a partial order.

Proposition 4.6. *The relation* $\underline{\blacktriangleleft}$ *is a partial order on* $\mathbf{No}^*$.

Proof. Reflexivity is trivial since $x \,\underline{\blacktriangleleft}_0\, x$ always holds.

For antisymmetry, assume $x \,\underline{\blacktriangleleft}_n\, y$ and $y \,\underline{\blacktriangleleft}\, x$ for some n. We prove by induction on n that $x = y$. Note first that the supports of x and y must clearly have the same order type. In the case $n = 0$, since $x \trianglelefteq y$, this immediately implies that $x = y$. If $n > 0$, write $x = z + \operatorname{sign}(r)\exp(\gamma)$ and $y = z + r\exp(\delta) + w$ in standard form with $\gamma \,\underline{\blacktriangleleft}_{n-1}\, \delta$. The observation on the order type immediately implies that $w = 0$, and since $y \,\underline{\blacktriangleleft}\, x$, we immediately get that $r = \operatorname{sign}(r) = \pm 1$ and $\delta \,\underline{\blacktriangleleft}\, \gamma$. By the inductive hypothesis we obtain $\gamma = \delta$, hence $x = y$.

For transitivity we distinguish two cases. If $x \,\underline{\blacktriangleleft}_n\, y \,\underline{\blacktriangleleft}_0\, u$, then trivially we have $x \,\underline{\blacktriangleleft}_n\, u$. For the remaining case $x \,\underline{\blacktriangleleft}_n\, y$ and $y \,\underline{\blacktriangleleft}_{m+1}\, u$, we reason by induction on n to prove that $x \,\underline{\blacktriangleleft}_n\, u$. Write $y = z + \operatorname{sign}(r)\exp(\gamma)$ and $u = z + r\exp(\delta) + w$ in standard form with $\gamma \,\underline{\blacktriangleleft}_m\, \delta$. If $x \,\underline{\blacktriangleleft}_n\, z$, since $z \,\underline{\blacktriangleleft}_0\, u$ it follows that $x \,\underline{\blacktriangleleft}_n\, u$, and we are done. If $x \,\not\underline{\blacktriangleleft}_n\, z$, then we must have $x = z + \operatorname{sign}(r)\exp(\gamma')$ in standard form with $\gamma' \,\underline{\blacktriangleleft}_{n-1}\, \gamma$. By inductive hypothesis, this implies that $\gamma' \,\underline{\blacktriangleleft}_{n-1}\, \delta$, which means that $x \,\underline{\blacktriangleleft}_n\, u$, as desired. □

Remark 4.7. The relation $\underline{\blacktriangleleft}$ is the smallest transitive one such that:

(1) for all $x, y \in \mathbf{No}^*$, $x \trianglelefteq y$ implies $x \underline{\blacktriangleleft} y$;
(2) for all $\gamma, \delta \in \mathbb{J}^*$, $z \in \mathbf{No}$ and $r \in \mathbb{R}^*$, $\gamma \underline{\blacktriangleleft} \delta$ implies $z + \operatorname{sign}(r)\exp(\gamma) \underline{\blacktriangleleft} z + r\exp(\delta)$, provided both sums are in standard form.

Moreover, the two relations share the following convexity property.

Proposition 4.8. *For all $x \in \mathbf{No}$, the class $\{y \in \mathbf{No} : x \trianglelefteq y\}$ is convex.*

Proof. Let $x \in \mathbf{No}$ be given and let u be a number in the convex hull of $\{y \in \mathbf{No} : x \trianglelefteq y\}$. This easily implies that there exists $y \in \mathbf{No}$ with $x \trianglelefteq y$ such that u is between x and y, and in particular $|u - x| \leq |y - x|$. Since $x \trianglelefteq y$, we have $|y - x| \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{S}(x)$, which implies that $|u - x| \prec \mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{S}(x)$. Therefore, $x \trianglelefteq u$, as desired. □

Proposition 4.9. *For all $x \in \mathbf{No}^*$, the class $\{y \in \mathbf{No}^* : x \underline{\blacktriangleleft} y\}$ is convex.*

Proof. Let $x \in \mathbf{No}^*$ be given and let u be a number in the convex hull of $\{y \in \mathbf{No}^* : x \underline{\blacktriangleleft} y\}$. This easily implies that there exist $y \in \mathbf{No}^*$ and $n \in \mathbb{N}$ with $x \underline{\blacktriangleleft}_n y$ such that u is between x and y. We reason by induction on n to prove that $x \underline{\blacktriangleleft}_n u$.

If $n = 0$, the conclusion follows trivially from Proposition 4.8.

If $n > 0$, there are $\gamma \underline{\blacktriangleleft}_{n-1} \delta \in \mathbb{J}^*$, $z, w \in \mathbf{No}$, $r \in \mathbb{R}^*$ such that

$$x = z + \operatorname{sign}(r)\exp(\gamma), \quad y = z + r\exp(\delta) + w,$$

with both sums in standard form. Since $z \trianglelefteq x, y$, we have that $z \trianglelefteq u$ as well by Proposition 4.8. Therefore, there are $\delta' \in \mathbb{J}$, $w' \in \mathbf{No}$, $r' \in \mathbb{R}$ such that

$$u = z + r'\exp(\delta') + w'$$

is in standard form. Since u is between x and y, we clearly have $\operatorname{sign}(r') = \operatorname{sign}(r)$ and δ' between δ and γ. By Remark 4.5 we have $\delta' \neq 0$. By induction, we deduce that $\gamma \underline{\blacktriangleleft}_{n-1} \delta'$, whence $x \underline{\blacktriangleleft}_n z$, as desired. □

Note also that the ring $\mathbb{J}$, which is closed under $\trianglelefteq$, is also closed under $\underline{\blacktriangleleft}$.

Proposition 4.10. *For all $x \in \mathbf{No}^*$ and $\delta \in \mathbb{J}^*$, if $x \underline{\blacktriangleleft} \delta$, then $x \in \mathbb{J}^*$.*

Proof. If $x \trianglelefteq \delta$ the conclusion is trivial. If $x \underline{\blacktriangleleft}_{n+1} \delta$, there are $\gamma \underline{\blacktriangleleft}_n \delta' \in \mathbb{J}^*$, $z, w \in \mathbf{No}$ and $r \in \mathbb{R}^*$ such that

$$x = z + \operatorname{sign}(r)\exp(\gamma), \quad \delta = z + r\exp(\delta') + w.$$

Note that $\delta' > 0$ since $\delta \in \mathbb{J}$. By Remark 4.5 we must have $\gamma > 0$. Moreover, we clearly have $z \in \mathbb{J}$. It follows that $x \in \mathbb{J}^*$, as desired. □

Finally, just like $x \trianglelefteq y$ implies $x \leq_s y$ (Proposition 2.8), also $x \underline{\blacktriangleleft} y$ implies $x \leq_s y$ (Theorem 4.24); in particular, the relation $\underline{\blacktriangleleft}$ is well-founded. This implies that $\underline{\blacktriangleleft}$ has an associated ordinal rank which is crucial for our inductive proofs. The rest of the section is devoted to the proof of the fact that $\underline{\blacktriangleleft}$ is well-founded.

4.1. **Products and inverses of monomials.** We first establish a few formulas for products and inverses of monomials in the case we are working with representations of a special type.

Definition 4.11. Let $x = A \mid B$ with $x > 0$. We say that $A \mid B$ is a **monomial representation** if for all $y \in \mathbf{No}$ and $k \in \mathbb{N}^{>0}$, we have $A < y < B$ if and only if $A < ky < B$. Equivalently, $A \mid B$ is monomial if

(1) for every $a \in A$ there is $a' \in A$ such that $2a < a'$;
(2) for every $b \in B$ there is $b' \in B$ such that $b' < \frac{1}{2}b$.

As the name suggests, monomial representations define monomials.

Proposition 4.12. *If $x \in \mathbf{No}^{>0}$ admits a monomial representation then $x \in \mathfrak{M}$.*

Proof. Suppose that $x = A \mid B$ is a monomial representation of x. Clearly, $A < y < B$ for every positive number y such that $y \asymp x$, and in particular, $A < \mathfrak{m} < B$ for the unique monomial $\mathfrak{m} \in \mathfrak{M}$ such that $\mathfrak{m} \asymp x$. Since a monomial is the simplest positive element of its archimedean class, we have $\mathfrak{m} \leq_s x$. But $x \leq_s \mathfrak{m}$ holds as well, since x is the simplest number such that $A < x < B$, and therefore $x = \mathfrak{m} \in \mathfrak{M}$. □

Conversely, all monomials admit monomial representations. In fact, we shall prove that all simple representations of monomials are monomial.

Lemma 4.13. *Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ with $\mathfrak{m} <_s \mathfrak{n}$.*
(1) *If $\mathfrak{m} < \mathfrak{n}$, then $k\mathfrak{m} <_s \mathfrak{n}$ for all $k \in \mathbb{N}$.*
(2) *If $\mathfrak{m} > \mathfrak{n}$, then $\frac{1}{2^k}\mathfrak{m} <_s \mathfrak{n}$ for all $k \in \mathbb{N}$.*

Proof. (1) Let $\mathfrak{m} < \mathfrak{n}$ be given. We wish to prove that $k\mathfrak{m} <_s \mathfrak{n}$. We recall that $k = \{0, 1, \ldots, k-1\} \mid \emptyset$. By definition of product, we have $k\mathfrak{m} = A \mid B$ with

$$\begin{aligned} A &= \{k'\mathfrak{m} + k\mathfrak{m}' - k'\mathfrak{m}'\}, \\ B &= \{k'\mathfrak{m} + k\mathfrak{m}'' - k'\mathfrak{m}''\}, \end{aligned}$$

where k' ranges in $\{0, 1, \ldots, k-1\}$.

By Remark 2.1, it suffices to check that $A < \mathfrak{n} < B$. We can easily verify that

$$k'\mathfrak{m} + k\mathfrak{m}' - k'\mathfrak{m}' = k'\mathfrak{m} + (k - k')\mathfrak{m}' < k\mathfrak{m} < \mathfrak{n},$$

and therefore $A < \mathfrak{n}$. We also note that

$$k'\mathfrak{m} + k\mathfrak{m}'' - k'\mathfrak{m}'' = k'\mathfrak{m} + (k - k')\mathfrak{m}'' \geq \mathfrak{m}''.$$

But $\mathfrak{m}'' > \mathfrak{m}$ and $\mathfrak{m}'' <_s \mathfrak{m} <_s \mathfrak{n}$, hence by Proposition 2.3 $\mathfrak{m}'' > \mathfrak{n}$. Therefore, $A < \mathfrak{n} < B$, hence $k\mathfrak{m} <_s \mathfrak{n}$, as desired.

(2) Let $\mathfrak{m} > \mathfrak{n}$ be given. We recall that $\frac{1}{2^k} = \{0\} \mid \left\{\frac{1}{2^{k'}}\right\}$ for $k' = 0, 1, \ldots, k-1$, hence $\frac{1}{2^k}\mathfrak{m} = A \mid B$ with

$$\begin{aligned} A &= \left\{\frac{1}{2^k}\mathfrak{m}',\ \frac{1}{2^{k'}}\mathfrak{m} + \frac{1}{2^k}\mathfrak{m}'' - \frac{1}{2^{k'}}\mathfrak{m}''\right\}, \\ B &= \left\{\frac{1}{2^k}\mathfrak{m}'',\ \frac{1}{2^{k'}}\mathfrak{m} + \frac{1}{2^k}\mathfrak{m}' - \frac{1}{2^{k'}}\mathfrak{m}'\right\}. \end{aligned}$$

Again, it suffices to verify that $A < \mathfrak{n} < B$. We compare each expression in the above brackets with $\mathfrak{n}$.

- We have $\mathfrak{m}' <_s \mathfrak{m} <_s \mathfrak{n}$ and $\mathfrak{m}' < \mathfrak{m}$. By Proposition 2.3 it follows that $\mathfrak{m}' < \mathfrak{n}$ and *a fortiori* $\frac{1}{2^k}\mathfrak{m}' < \mathfrak{n}$.
- If $k > 0$, the expression $\frac{1}{2^{k'}}\mathfrak{m} + \frac{1}{2^k}\mathfrak{m}'' - \frac{1}{2^{k'}}\mathfrak{m}''$ is negative (so in particular $< \mathfrak{n}$) because $\mathfrak{m} \prec \mathfrak{m}''$ and $k' < k$. If $k = 0$, the expression can be dropped since k' ranges in the empty set.
- Since $\mathfrak{m}'' <_s \mathfrak{m} <_s \mathfrak{n}$ and $\mathfrak{m}'' > \mathfrak{m}$, we obtain $\mathfrak{n} < \mathfrak{m}''$ by Proposition 2.3. Moreover, since $\mathfrak{m}'' <_s \mathfrak{n}$ and $\mathfrak{n}$ is the simplest positive number in its archimedean class, $\mathfrak{m}''$ must belong to a different archimedean class, and therefore $\mathfrak{n} \prec \mathfrak{m}''$. It follows that $\mathfrak{n} < \frac{1}{2^k}\mathfrak{m}''$, as desired.

- If $k > 0$, then $\frac{1}{2^{k'}}\mathfrak{m} + \frac{1}{2^k}\mathfrak{m}' - \frac{1}{2^{k'}}\mathfrak{m}' \geq \frac{1}{2^k}\mathfrak{m} > \mathfrak{n}$. If $k = 0$, the expression can be dropped since k' ranges in the empty set.

We have thus proved that $A < \mathfrak{n} < B$, whence $\frac{1}{2^k}\mathfrak{m} <_s \mathfrak{n}$, as desired. □

Corollary 4.14. *Let $x <_s \mathfrak{m} \leq_s z$, with $\mathfrak{m} \in \mathfrak{M}$. Then $x \prec \mathfrak{m}$ if and only if $x \prec z$.*

Proof. By Proposition 2.3, $x < \mathfrak{m}$ if and only if $x < z$. Excluding the trivial cases, we may assume that $x \neq 0$. Moreover, recall that since $\mathfrak{m} > 0$ and $x <_s \mathfrak{m}$ we have $x > 0$, and since $\mathfrak{m}$ is the simplest number in its archimedean class we have $x \not\asymp \mathfrak{m}$.

Let $\mathfrak{n}$ be the unique monomial such that $\mathfrak{n} \asymp x$; since $x > 0$ we have $\mathfrak{n} \leq_s x$ and therefore $\mathfrak{n} <_s \mathfrak{m}$. We apply Lemma 4.13 to $\mathfrak{m}$ and $\mathfrak{n}$, distinguishing two cases.

If $x \prec \mathfrak{m}$ we have $\mathfrak{n} \prec \mathfrak{m}$, and in particular $\mathfrak{n} < \mathfrak{m}$. Let k be any integer such that $k\mathfrak{n} > x$. By Lemma 4.13(1), we have $k\mathfrak{n} <_s \mathfrak{m}$. From $k\mathfrak{n} <_s \mathfrak{m} \leq_s z$ and $k\mathfrak{n} < \mathfrak{m}$, we obtain $k\mathfrak{n} < z$ by Proposition 2.3. Since $x < k\mathfrak{n} < z$ and k is arbitrarily large, we conclude that $x \prec z$, as desired.

If $x \succ \mathfrak{m}$ we proceed similarly, applying Lemma 4.13(2) to $\mathfrak{m}$ and $\frac{1}{2^k}\mathfrak{n}$. □

Corollary 4.15. *The representation $\omega^x = \{0, k\omega^{x'}\} \mid \left\{\frac{1}{2^k}\omega^{x''}\right\}$ of Definition 2.15 is simple.*

Proof. By Remark 2.16, $\omega^{x'}, \omega^{x''} <_s \omega^x$, hence by Lemma 4.13 $k\omega^{x'}, \frac{1}{2^k}\omega^{x''} <_s \omega^x$ for every $k \in \mathbb{N}$. Therefore, by Proposition 2.3 the representation is simple. □

Proposition 4.16. *For all $x \in \mathfrak{M}$, any simple representation of x is a monomial representation.*

Proof. Suppose that $x \in \mathfrak{M}$ and let $y \in \mathbf{No}$ be such that $x = \omega^y$. The representation of ω^y given by Definition 2.15 is clearly monomial, and it is also simple by Corollary 4.15. Since all simple representations $A \mid B$ define the same convex class $(A; B) = \{z \in \mathbf{No} : A < z < B\}$, all simple representations are monomial, as desired. □

Thanks to the above observation, we can find simplified formulas for the product of two monomials and for the inverse of a monomial.

Proposition 4.17. *Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$. If $\mathfrak{m} = \{\mathfrak{m}'\} \mid \{\mathfrak{m}''\}$ and $\mathfrak{n} = \{\mathfrak{n}'\} \mid \{\mathfrak{n}''\}$ are two monomial representations, then $\mathfrak{m}\mathfrak{n} = \{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'\} \mid \{\mathfrak{m}\mathfrak{n}'', \mathfrak{m}''\mathfrak{n}\}$.*

Proof. Since $\mathfrak{m}, \mathfrak{n} > 0$, we may discard the expressions $\mathfrak{m}', \mathfrak{n}'$ that are strictly less than 0 from the representations of $\mathfrak{m}$ and $\mathfrak{n}$. Since the representations are monomial, we then obtain $\mathfrak{m}' \prec \mathfrak{m} \prec \mathfrak{m}''$ and $\mathfrak{n}' \prec \mathfrak{n} \prec \mathfrak{n}''$. It follows immediately that $\{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'\} < \mathfrak{m}\mathfrak{n} < \{\mathfrak{m}\mathfrak{n}'', \mathfrak{m}''\mathfrak{n}\}$. Let now $y \in \mathbf{No}$ be a number such that $\{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'\} < y < \{\mathfrak{m}\mathfrak{n}'', \mathfrak{m}''\mathfrak{n}\}$. We need to prove that $\mathfrak{m}\mathfrak{n} \leq_s y$.

By definition of product, we have $\mathfrak{m}\mathfrak{n} = A \mid B$ with

$$\begin{aligned} A &= \{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}' - \mathfrak{m}'\mathfrak{n}', \mathfrak{m}''\mathfrak{n} + \mathfrak{m}\mathfrak{n}'' - \mathfrak{m}''\mathfrak{n}''\}, \\ B &= \{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'' - \mathfrak{m}'\mathfrak{n}'', \mathfrak{m}''\mathfrak{n} + \mathfrak{m}\mathfrak{n}' - \mathfrak{m}''\mathfrak{n}'\}. \end{aligned}$$

By Proposition 2.3, it suffices to show that $A < y < B$. The inequality $A < y$ follows immediately from the assumption $\{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'\} < y$, as the first expression in A is smaller than $\mathfrak{m}'\mathfrak{n}+\mathfrak{m}\mathfrak{n}'$ and the second one is negative. For $y < B$, observe that, since the given representations of $\mathfrak{n}, \mathfrak{m}$ are monomial, the assumption $y < \{\mathfrak{m}\mathfrak{n}'', \mathfrak{m}''\mathfrak{n}\}$ implies $y < \left\{\frac{1}{k}\mathfrak{m}\mathfrak{n}'', \frac{1}{k}\mathfrak{m}''\mathfrak{n}\right\}$ for any positive $k \in \mathbb{N}$. The inequality $y < B$ then follows easily from the fact that each element of B is dominated by an expression of the form $\mathfrak{m}\mathfrak{n}''$ or $\mathfrak{m}''\mathfrak{n}$. □

Proposition 4.18. *If $\mathfrak{m} = \{\mathfrak{m}'\} \mid \{\mathfrak{m}''\}$ is a monomial representation, then*

$$\mathfrak{m}^{-1} = \left\{0,\, (\mathfrak{m}'')^{-1}\right\} \mid \left\{(\mathfrak{m}')^{-1}\right\}$$

where $(\mathfrak{m}')^{-1}$ is only taken when $\mathfrak{m}' > 0$.

Proof. Let

$$\mathfrak{n} := \left\{0,\, (\mathfrak{m}'')^{-1}\right\} \mid \left\{(\mathfrak{m}')^{-1}\right\},$$

where $(\mathfrak{m}')^{-1}$ is only taken when $\mathfrak{m}' > 0$. We need to prove that $\mathfrak{m}\mathfrak{n} = 1$. We note that the above representation of $\mathfrak{n}$ is monomial, and therefore $\mathfrak{n} \in \mathfrak{M}$ by Proposition 4.16. By Proposition 4.17, $\mathfrak{m}\mathfrak{n} = A \mid B$ where

$$\begin{aligned} A &= \left\{\mathfrak{m}'\mathfrak{n},\, \mathfrak{m}'\mathfrak{n} + \mathfrak{m}\,(\mathfrak{m}'')^{-1}\right\} \\ B &= \left\{\mathfrak{m}\,(\mathfrak{m}')^{-1},\, \mathfrak{m}''\mathfrak{n}\right\}, \end{aligned}$$

and $(\mathfrak{m}')^{-1}$ is only taken when $\mathfrak{m}' > 0$. Using $(\mathfrak{m}'')^{-1} < \mathfrak{n} < (\mathfrak{m}')^{-1}$, it is easy to verify that $A < 1 < B$. Since A contains at least one non-negative number of the form $\mathfrak{m}'\mathfrak{n}$, it follows that $1 = A \mid B = \mathfrak{m}\mathfrak{n}$, as desired. □

Corollary 4.19. *If $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ and $\mathfrak{m} \leq_s \mathfrak{n}$, then $\mathfrak{m}^{-1} \leq_s \mathfrak{n}^{-1}$.*

Proof. Take a simple monomial representation $\mathfrak{m} = \{\mathfrak{m}'\} \mid \{\mathfrak{m}''\}$, which exists by Proposition 4.16. Since $\mathfrak{m} \leq_s \mathfrak{n}$ we have $\{\mathfrak{m}'\} < \mathfrak{n} < \{\mathfrak{m}''\}$. This immediately implies that

$$\left\{0, (\mathfrak{m}'')^{-1}\right\} < \mathfrak{n}^{-1} < \left\{(\mathfrak{m}')^{-1}\right\}$$

when $\mathfrak{m}' > 0$, and therefore $\mathfrak{m}^{-1} \leq_s \mathfrak{n}^{-1}$ by Proposition 4.18. □

4.2. **Simplicity.** Using the tools of the above subsection, we can prove several results regarding the interaction between exp and $\leq_s$.

We start by generalizing the implication $x \trianglelefteq y \to x \leq_s y$ to sums in standard form.

Proposition 4.20. *Let $x, y, z \in \mathbf{No}$. If both $z + x$ and $z + y$ are in standard form and $x \leq_s y$, then $z + x \leq_s z + y$.*

Proof. Let $z = \{z'\} \mid \{z''\}$ and $x = \{x'\} \mid \{x''\}$ be canonical, and in particular simple, representations. By definition of sum, we have $z + x = A \mid B$ where

$$A = \{z' + x,\, z + x'\}, \quad B = \{z'' + x,\, z + x''\}.$$

It suffices to verify that $A < z + y < B$. Since $x \leq_s y$ and $x = \{x'\} \mid \{x''\}$ is simple, we have $x' < y < x''$. After adding z on all sides, we get $z + x' < z + y < z + x''$.

It remains to show that $z' + x < z + y < z'' + x$. Let $\tilde{z}$ be either z' or z''. Since $\tilde{z} <_s z$, we have $z \ntrianglelefteq \tilde{z}$ by Proposition 2.8, and therefore there is a monomial $\mathfrak{m} \in \mathrm{S}(z)$ such that $\mathfrak{m} \preceq \tilde{z} - z$. Since $z + x$ and $z + y$ are in standard form, we know that $x, y \prec \mathfrak{n}$ for all $\mathfrak{n} \in \mathrm{S}(z)$, and in particular $x, y \prec \mathfrak{m}$. It follows that $x - y \prec \mathfrak{m} \preceq \tilde{z} - z$, which implies that $|x - y| < |\tilde{z} - z|$. If $\tilde{z} = z'$, we get $x - y < z - z'$, or in other words, $z' + x < z + y$. If $\tilde{z} = z''$, we get $y - x < z'' - z$, or in other words, $z + y < z'' + x$.

Therefore, $A < z + y < B$, which implies $z + x \leq_s z + y$, as desired. □

A similar statement holds when taking the exponential of sums expressed in standard form, as follows.

Proposition 4.21. *Let $\gamma, \delta, \eta \in \mathbb{J}$. If $\eta + \gamma$ and $\eta + \delta$ are in standard form and $\exp(\gamma) \leq_s \exp(\delta)$, then $\exp(\eta + \gamma) \leq_s \exp(\eta + \delta)$.*

Proof. Let $\mathfrak{m} = \exp(\eta)$, $\mathfrak{n} = \exp(\gamma)$ and $\mathfrak{o} = \exp(\delta)$. Our hypothesis says that $\mathfrak{n} \leq_s \mathfrak{o}$ and $\mathrm{S}(\ell(\mathfrak{m})) \succ \ell(\mathfrak{n}), \ell(\mathfrak{o})$, and we must prove that $\mathfrak{mn} \leq_s \mathfrak{mo}$.

Consider the two canonical representations $\mathfrak{m} = \{\mathfrak{m}'\} \mid \{\mathfrak{m}''\}$, $\mathfrak{n} = \{\mathfrak{n}'\} \mid \{\mathfrak{n}''\}$. Recall that since $\mathfrak{m}$, $\mathfrak{n}$ are monomials we must have $\mathfrak{m}' \prec \mathfrak{m} \prec \mathfrak{m}''$ and $\mathfrak{n}' \prec \mathfrak{n} \prec \mathfrak{n}''$. Moreover, since the representations are canonical, they are simple, and by Proposition 4.16 they are monomial.

By Proposition 4.17, $\exp(\eta + \gamma) = \mathfrak{mn} = A \mid B$ with

$$\begin{aligned} A &= \{\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}'\}, \\ B &= \{\mathfrak{m}\mathfrak{n}'', \mathfrak{m}''\mathfrak{n}\}. \end{aligned}$$

It now suffices to prove that $A < \mathfrak{mo} = \exp(\eta + \delta) < B$, namely $\mathfrak{m}'\mathfrak{n} + \mathfrak{m}\mathfrak{n}' < \mathfrak{mo}$, $\mathfrak{mo} < \mathfrak{m}\mathfrak{n}''$ and $\mathfrak{mo} < \mathfrak{m}''\mathfrak{n}$. Simplifying further, it suffices to show that $\mathfrak{n}' \prec \mathfrak{o} \prec \mathfrak{n}''$ and $\mathfrak{m}'\mathfrak{n} \prec \mathfrak{mo} \prec \mathfrak{m}''\mathfrak{n}$.

Since $\mathfrak{n}', \mathfrak{n}'' <_s \mathfrak{n} \leq_s \mathfrak{o}$, the inequality $\mathfrak{n}' \prec \mathfrak{o} \prec \mathfrak{n}''$ follows immediately from Corollary 4.14 and $\mathfrak{n}' \prec \mathfrak{n} \prec \mathfrak{n}''$. For $\mathfrak{m}'\mathfrak{n} \prec \mathfrak{mo} \prec \mathfrak{m}''\mathfrak{n}$, we note that it is equivalent to saying that $\ell(\mathfrak{m}'\mathfrak{n}) < \ell(\mathfrak{mo}) < \ell(\mathfrak{m}''\mathfrak{n})$. Rearranging the terms, we wish to prove that $\ell(\mathfrak{m}') - \ell(\mathfrak{m}) < \ell(\mathfrak{o}) - \ell(\mathfrak{n}) < \ell(\mathfrak{m}'') - \ell(\mathfrak{m})$.

Let $\tilde{\mathfrak{m}}$ be either $\mathfrak{m}'$ or $\mathfrak{m}''$. If $\ell(\mathfrak{m}) \trianglelefteq \ell(\tilde{\mathfrak{m}})$, then by Proposition 4.1 $\mathfrak{m} = \exp(\ell(\mathfrak{m})) \leq_s \exp(\ell(\tilde{\mathfrak{m}})) = \tilde{\mathfrak{m}}$, contradicting $\tilde{\mathfrak{m}} <_s \mathfrak{m}$. Therefore, we have $\ell(\mathfrak{m}) \ntrianglelefteq \ell(\tilde{\mathfrak{m}})$, or in other words, there is a monomial $\mathfrak{p} \in \mathrm{S}(\ell(\mathfrak{m}))$ such that $\mathfrak{p} \preceq \ell(\tilde{\mathfrak{m}}) - \ell(\mathfrak{m})$. From the hypothesis $\mathrm{S}(\ell(\mathfrak{m})) \succ \ell(\mathfrak{n}), \ell(\mathfrak{o})$ we obtain $\mathrm{S}(\ell(\mathfrak{m})) \succ \ell(\mathfrak{o}) - \ell(\mathfrak{n})$, and in particular $\mathfrak{p} \succ \ell(\mathfrak{o}) - \ell(\mathfrak{n})$. Therefore, $\ell(\tilde{\mathfrak{m}}) - \ell(\mathfrak{m}) \succ \ell(\mathfrak{o}) - \ell(\mathfrak{n})$, from which it follows that $|\ell(\tilde{\mathfrak{m}}) - \ell(\mathfrak{m})| > |\ell(\mathfrak{o}) - \ell(\mathfrak{n})|$. Recalling that $\tilde{\mathfrak{m}}$ is one of $\mathfrak{m}'$ or $\mathfrak{m}''$, this easily implies $\ell(\mathfrak{m}') - \ell(\mathfrak{m}) < \ell(\mathfrak{o}) - \ell(\mathfrak{n}) < \ell(\mathfrak{m}'') - \ell(\mathfrak{m})$, as desired. □

Moreover, exp preserves simplicity under suitable assumptions.

Proposition 4.22. *Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}^{>1}$ be such that $\mathfrak{m} \leq_s \mathfrak{n}$ and $\log(\mathfrak{m}) \prec \mathfrak{n}$. Then $\exp(\mathfrak{m}) \leq_s \exp(\mathfrak{n})$ and $\exp(-\mathfrak{m}) \leq_s \exp(-\mathfrak{n})$.*

Proof. It suffices to show that $\exp(\mathfrak{m}) \leq_s \exp(\mathfrak{n})$, as the second part then follows from Corollary 4.19. By Theorem 3.8(1), we have $\exp(\mathfrak{m}) = A \mid B$ with

$$A = \left\{\mathfrak{m}^k, \exp(\mathfrak{m}')^k\right\}, \; B = \left\{\exp(\mathfrak{m}'')^{\frac{1}{k}}\right\},$$

where $\mathfrak{m}', \mathfrak{m}''$ run over the infinite monomials simpler than $\mathfrak{m}$ and such that $\mathfrak{m}' < \mathfrak{m} < \mathfrak{m}''$, and k runs over the positive integers. For the conclusion, it suffices to verify that $A < \exp(\mathfrak{n}) < B$.

Since $\mathfrak{m}' <_s \mathfrak{m} \leq_s \mathfrak{n}$ and $\mathfrak{m}' < \mathfrak{m}$, we have $\mathfrak{m}' < \mathfrak{n}$. It follows that $k\mathfrak{m}' < \mathfrak{n}$ for all k (since $\mathfrak{m}', \mathfrak{n}$ are monomials), and therefore $\exp(\mathfrak{m}')^k = \exp(k\mathfrak{m}') < \exp(\mathfrak{n})$. Similarly, since $\mathfrak{m}'' <_s \mathfrak{m} \leq_s \mathfrak{n}$ and $\mathfrak{m}'' > \mathfrak{m}$, we have $\mathfrak{m}'' > \mathfrak{n}$. This implies that $\frac{1}{k}\mathfrak{m}'' > \mathfrak{n}$ for all k, and therefore $\exp(\mathfrak{n}) < \exp(\mathfrak{m}'')^{\frac{1}{k}}$.

Finally, since $\log(\mathfrak{m}) \prec \mathfrak{n}$ and $\log(\mathfrak{m}), \mathfrak{n} > 0$ we have that $k \log(\mathfrak{m}) < \mathfrak{n}$ for all $k \in \mathbb{N}$. In particular, $\exp(k \log(\mathfrak{m})) = \mathfrak{m}^k < \exp(\mathfrak{n})$ for all $k \in \mathbb{N}$. Therefore, $A < \exp(\mathfrak{n}) < B$, as desired. □

Proposition 4.23. *If $\gamma \in \mathbb{J}^*$ and $\mathfrak{m} \in \mathfrak{M}^{>1}$ is the leading monomial of γ, then* $\exp(\operatorname{sign}(\gamma)\mathfrak{m}) \leq_s \exp(\gamma)$.

Proof. By Corollary 4.19, it suffices to prove the case where $\gamma > 0$.

Call $\exp(\mathfrak{m}) = A \mid B$ the representation given by Theorem 3.8(1). Since clearly $\mathfrak{m} \asymp \gamma$ and $\gamma > 0$, there is some positive $k \in \mathbb{N}$ such that $\frac{1}{k}\mathfrak{m} \leq \gamma \leq k\mathfrak{m}$, hence $\exp(\mathfrak{m})^{\frac{1}{k}} \leq \exp(\gamma) \leq \exp(\mathfrak{m})^k$. It is now easy to verify that $A < \exp(\mathfrak{m})^{\frac{1}{k}} \leq \exp(\gamma) \leq \exp(\mathfrak{m})^k < B$, and therefore that $\exp(\mathfrak{m}) \leq_s \exp(\gamma)$, as desired. □

Putting all of the above results together, we are finally able to prove that $x \trianglelefteq y$ implies $x \leq_s y$.

Theorem 4.24. *For all $x, y \in \mathbf{No}^*$, if $x \trianglelefteq y$ then $x \leq_s y$. Therefore, the relation $\trianglelefteq$ is well founded.*

Proof. By definition there is some $n \in \mathbb{N}$ such that $x \trianglelefteq_n y$. We prove the conclusion by induction on n. At the same time, we also prove that if we further assume $x \succ 1$ then $\log|x| \prec y$.

First, assume $n = 0$, so that $x \unlhd y$. It immediately follows that $x \leq_s y$ by Proposition 2.8. Moreover, we have $x \asymp y$; it follows that if $x \succ 1$, then $\log|x| \prec x \asymp y$, as desired.

Now assume that $n > 0$. We first prove the case $x = \exp(\gamma), y = \exp(\delta) \in \mathfrak{M}$. By assumption we must have $\gamma \trianglelefteq_{n-1} \delta$.

If $n - 1 = 0$, namely $\gamma \unlhd \delta$, then $\exp(\gamma) \leq_s \exp(\delta)$ follows from Proposition 4.1; moreover, if $\exp(\gamma) \succ 1$ then $\log(\exp(\gamma)) = \gamma \asymp \delta \prec \exp(\delta)$, as desired.

If $n - 1 > 0$, we can write $\gamma = z' + \operatorname{sign}(r')\exp(\gamma')$, $\delta = z' + r'\exp(\delta') + w'$ in standard form with $\gamma' \trianglelefteq_{n-2} \delta'$, and necessarily $\gamma', \delta' > 0$. By inductive hypothesis, we get $\exp(\gamma') \leq_s \exp(\delta')$ and $\log(\exp(\gamma')) = \gamma' \prec \exp(\delta')$. Combining Corollary 4.19, Proposition 4.22, Proposition 4.23 and Proposition 4.1 we get

$$\exp(\operatorname{sign}(r')\exp(\gamma')) \leq_s \exp(\operatorname{sign}(r')\exp(\delta')) \leq_s \\ \leq_s \exp(r'\exp(\delta')) \leq_s \exp(r'\exp(\delta') + w').$$

By Proposition 4.21, we deduce that

$$\exp(\gamma) = \exp(z' + \operatorname{sign}(r')\exp(\gamma')) \leq_s \exp(z' + r'\exp(\delta') + w') = \exp(\delta),$$

namely $x \leq_s y$. Finally, if $\exp(\gamma) \succ 1$ we have $\gamma, \delta > 0$. If $z' \neq 0$, then $\log(\exp(\gamma)) = \gamma \asymp \delta \prec \exp(\delta)$. If $z' = 0$, we recall that $\gamma' \prec \exp(\delta') \asymp r'\exp(\delta') + w'$; it follows that $\log(\exp(\gamma)) = \gamma \asymp \exp(\gamma') \prec \exp(r'\exp(\delta') + w') = \exp(\delta)$. In both cases we obtain $\log|x| \prec y$, as desired.

For general x and y, we must have $x = z + \operatorname{sign}(r)\exp(\gamma)$, $y = z + r\exp(\delta) + w$ in standard form, with $\gamma \trianglelefteq_{n-1} \delta$. Note in particular that $\exp(\gamma) \trianglelefteq_n \exp(\delta)$, and by the previous argument, $\exp(\gamma) \leq_s \exp(\delta)$. By Proposition 2.8 we get $\operatorname{sign}(r)\exp(\gamma) \leq_s \operatorname{sign}(r)\exp(\delta) \leq_s r\exp(\delta) \leq_s r\exp(\delta) + w$. By Proposition 4.20 we get

$$x = z + \operatorname{sign}(r)\exp(\gamma) \leq_s z + r\exp(\delta) + w = y.$$

Finally, assume $x \succ 1$. If $z \neq 0$, then $x \asymp y$, hence $\log|x| \asymp \log|y| \prec y$. If $z = 0$, then $\log|x| \sim \ell(|x|) = \gamma$ and $\gamma > 0$; by inductive hypothesis, we have $\log(\gamma) \prec \delta = \ell(|y|) \sim \log|y|$ and therefore $\log|x| \sim \gamma \prec \exp(\log|y|) \asymp y$, as desired. □

4.3. **The nested truncation rank.** We now use the well-foundedness of $\underline{\blacktriangleleft}$ to define an appropriate notion of rank with ordinal values. We shall see in Section 8 that the existence of this rank is essentially equivalent to saying that **No** is a field of transseries in the sense of Schmeling.

Definition 4.25. For all $x \in \mathbf{No}$, the **nested truncation rank** $\mathrm{NR}(x)$ of x is the foundation rank of $\underline{\blacktriangleleft}$, namely, $\mathrm{NR}(x) := \sup\{\mathrm{NR}(y) + 1 : y \blacktriangleleft x\} \in \mathbf{On}$.

Proposition 4.26. *We have the following:*

(1) *for any $\gamma \in \mathbb{J}$, $\mathrm{NR}(\pm\exp(\gamma)) = \mathrm{NR}(\gamma)$;*
(2) *if $r\mathfrak{m} = r\exp(\gamma)$ is a term of x, then $\mathrm{NR}(x) \geq \mathrm{NR}(\gamma)$; if $\mathfrak{m}$ is not minimal in $\mathrm{S}(x)$, or if $\gamma \neq 0$ and $r \neq \pm 1$, then $\mathrm{NR}(x) > \mathrm{NR}(\gamma)$.*

Proof. (1) By definition, $y \blacktriangleleft \pm\exp(\gamma)$ if and only if $x = \pm\exp(\delta)$ for some $\delta \in \mathbb{J}^*$ with $\delta \blacktriangleleft \gamma$. By Proposition 4.10, the only numbers $x \in \mathbf{No}^*$ such that $x \blacktriangleleft \gamma$ are in $\mathbb{J}^*$. It follows easily that $\mathrm{NR}(\pm\exp(\gamma)) = \mathrm{NR}(\gamma)$.

(2) Let x be a surreal number and $r\mathfrak{m} = r\exp(\gamma)$ be one of its terms. We reason by induction on $\alpha := \mathrm{NR}(x)$.

If $\mathfrak{m}$ is not minimal in $\mathrm{S}(x)$, then there exists $y \in \mathbf{No}^*$ such that $y \triangleleft x$ and $r\mathfrak{m}$ is a term of y. By definition of the rank, we have $\mathrm{NR}(y) < \alpha$, hence $\mathrm{NR}(\gamma) < \alpha$ by inductive hypothesis, as desired. Suppose now that $\mathfrak{m}$ is minimal, which means that we can write $x = z + r\exp(\gamma)$ in standard form for some $z \in \mathbf{No}$. If $\gamma = 0$, then clearly $\mathrm{NR}(\gamma) = 0 \leq \mathrm{NR}(x)$, so we may assume that $\gamma \neq 0$.

If $r \neq \pm 1$, since $\gamma \neq 0$ we have $x' := z + \mathrm{sign}(r)\exp(\gamma) \blacktriangleleft x$. By definition, it follows that $\mathrm{NR}(x') < \alpha$, and by inductive hypothesis we must have $\mathrm{NR}(\gamma) \leq \mathrm{NR}(x') < \alpha$, as desired.

If $r = \pm 1$, suppose by contradiction that $\mathrm{NR}(\gamma) \geq \alpha + 1$. Then there exists $\delta \blacktriangleleft \gamma$ such that $\mathrm{NR}(\delta) \geq \alpha$. By Proposition 4.10, we have $\delta \in \mathbb{J}^*$. Let $x' := z + r\exp(\delta)$. We claim that $x' \blacktriangleleft x$, hence $\mathrm{NR}(x') < \alpha$. Since $r = \pm 1$, we only have to prove that $z + r\exp(\delta)$ is in standard form.

Suppose by contradiction that this is not the case. This means that there is a monomial $\exp(\eta)$ in the support of z such that $\delta \geq \eta$, while by hypothesis we have $\eta > \gamma$. By Proposition 4.9 it follows that $\delta \,\underline{\blacktriangleleft}\, \eta$, hence $\mathrm{NR}(\eta) \geq \mathrm{NR}(\delta) \geq \alpha$. On the other hand, we must have $z \neq 0$, hence $z \blacktriangleleft x$, and therefore $\mathrm{NR}(z) < \alpha$. By inductive hypothesis, we get $\mathrm{NR}(\eta) \leq \mathrm{NR}(z) < \alpha$, a contradiction.

Since now $\mathrm{NR}(x') < \alpha$, by inductive hypothesis we have $\mathrm{NR}(\delta) \leq \mathrm{NR}(x') < \alpha$, another contradiction. Therefore, $\mathrm{NR}(\gamma) \leq \alpha$, as desired. □

5. Log-atomic numbers

As anticipated in the introduction, a crucial subclass of **No** is the one of log-atomic numbers. We defined them as follows.

Definition 5.1. A positive infinite surreal number $x \in \mathbf{No}$ is **log-atomic** if for every $n \in \mathbb{N}$, $\log_n(x)$ is an infinite monomial. We call $\mathbb{L}$ the class of all log-atomic numbers.

Note that $\mathbb{L} \subset \mathfrak{M}^{>1}$. It turns out that the log-atomic numbers are the natural representatives of a certain equivalence relation, similarly to how the monomials are the natural representatives of the archimedean equivalence $\asymp$.

5.1. **Levels.** We first define an appropriate notion of magnitude, which is weaker than the dominance relation $\preceq$.

Definition 5.2. Given two elements $x, y \in \mathbf{No}$ with $x, y > \mathbb{N}$, we write

(1) $x \preceq^L y$ if $x \leq \exp_h(k \log_h(y))$ for some $h, k \in \mathbb{N}$ (equivalently, $\log_h(x) \preceq \log_h(y)$ for some $h \in \mathbb{N}$);
(2) $x \prec^L y$ if $x < \exp_h\left(\frac{1}{k}\log_h(y)\right)$ for all $h, k \in \mathbb{N}$ with $k > 0$ (equivalently, $\log_h(x) \prec \log_h(y)$ for all $h \in \mathbb{N}$);
(3) $x \asymp^L y$ if $\exp_h\left(\frac{1}{k}\log_h(y)\right) \leq x \leq \exp_h(k\log_h(y))$ for some $h, k \in \mathbb{N}$ with $k > 0$ (equivalently, $\log_h(x) \asymp \log_h(y)$ for some $h \in \mathbb{N}$).

We call **level of** x the class $[x] := \{y \in \mathbf{No} : y > \mathbb{N}, y \asymp^L x\}$.

Proposition 5.3. *The relation $\asymp^L$ is an equivalence relation. Moreover, $x \asymp^L y$ if and only if there exists $n \in \mathbb{N}$ such that $\log_n(x) \sim \log_n(y)$.*

Proof. Since $\log_h(x) \asymp \log_h(y)$ implies $\log_k(x) \asymp \log_k(y)$ for all $k \geq h$, we immediately get that $\asymp^L$ is an equivalence relation. Moreover, if $\log_h(x) \asymp \log_h(y)$, then clearly $\log_{h+1}(x) \sim \log_{h+1}(y)$, as desired. □

Remark 5.4. The equivalence relation $\asymp^L$ generalizes the notion of level in Hardy fields as in [Ros87, MM97], whence the name. While in those papers the only levels under consideration are given by $\log_n(x)$ and $\exp_n(x)$, the surreal numbers have uncountably many levels, and in fact a proper class of them.

Proposition 5.5. *Each level $[x]$ is a union of positive parts of archimedean classes and $\preceq^L$ induces a total order on levels.*

The proof is trivial and left to the reader.

We can verify that $\mathbb{L}$ is a class of representatives for the equivalence relation $\asymp^L$. For instance, any two distinct log-atomic numbers have necessarily different levels.

Proposition 5.6. *Let $\mu, \lambda \in \mathbb{L}$. If $\mu < \lambda$, then $\mu \prec^L \lambda$.*

Proof. Suppose by contradiction that $\mu \succeq^L \lambda$ and $\mu < \lambda$. Then $\mu \asymp^L \lambda$, so there exists some $n \in \mathbb{N}$ such that $\log_n(\mu) \sim \log_n(\lambda)$. Since $\log_n(\mu)$ and $\log_n(\lambda)$ are both monomials, we obtain $\log_n(\mu) = \log_n(\lambda)$, hence $\lambda = \mu$, a contradiction. □

On the other hand, any positive infinite surreal number has the same level of some log-atomic number.

Lemma 5.7. *Let $x, y > \mathbb{N}$. If $x \trianglelefteq y$, then $x \asymp^L y$.*

Proof. Let $n \in \mathbb{N}$ be such that $x \trianglelefteq_n y$. We claim that $\log_n(x) \asymp \log_n(y)$, hence by definition $x \asymp^L y$. We work by induction on n.

If $x \trianglelefteq_0 y$, then obviously $x \asymp y$.

If $x \trianglelefteq_{n+1} y$, write $x = z + \operatorname{sign}(r)\exp(\gamma)$, $y = z + r\exp(\delta) + w$ in standard form with $\gamma \trianglelefteq_n \delta$. If $z \neq 0$ then again $x \asymp y \asymp z$. If $z = 0$, then $\log(x) \sim \ell(x) = \gamma$ and $\log(y) \sim \ell(x) = \delta$ by Remark 3.7. By inductive hypothesis, $\log_{n-1}(\gamma) \asymp \log_{n-1}(\delta)$. This immediately implies that $\log_n(x) \sim \log_{n-1}(\gamma) \asymp \log_{n-1}(\delta) \sim \log_n(y)$, as desired. □

Proposition 5.8. *If $x > \mathbb{N}$, there exists $\lambda \in \mathbb{L}$ such that $\lambda \trianglelefteq x$, and therefore such that $\lambda \asymp^L x$.*

Proof. Suppose by contradiction that there exists a counterexample $x > \mathbb{N}$ such that $\lambda \ntrianglelefteq x$ for all $\lambda \in \mathbb{L}$. By Theorem 4.24, we may assume that x is a minimal counterexample with respect to $\trianglelefteq$ (for instance, we may take x of minimal simplicity). Note that obviously $x \notin \mathbb{L}$.

Since x is positive infinite, we may write $x = r_0\mathfrak{m}_0 + \delta_0$ and then inductively

$$\mathfrak{m}_n =: \exp(r_{n+1}\mathfrak{m}_{n+1} + \delta_{n+1})$$

with $r_n \in \mathbb{R}^{>0}$, $\mathfrak{m}_n \in \mathfrak{M}^{>1}$, $\delta_n \in \mathbb{J}$ and $r_n\mathfrak{m}_n + \delta_n$ in standard form (in other words, $r_{n+1}\mathfrak{m}_{n+1}$ is the leading term of $\log(\mathfrak{m}_n)$). We claim that $r_n = 1$ and $\delta_n = 0$ for all $n \in \mathbb{N}$; this implies that $\log_n(x) = \mathfrak{m}_{n+1} \in \mathfrak{M}$ and therefore that x is log-atomic. We reason by induction on n.

For $n = 0$, it suffices to note that $\mathfrak{m}_0 \trianglelefteq x$. Clearly, $\lambda \ntrianglelefteq \mathfrak{m}_0$ for all $\lambda \in \mathbb{L}$. By minimality of x, we must have $x = \mathfrak{m}_0$ and in particular $r_0 = 1$ and $\delta_0 = 0$.

If $n > 0$, assume that for all $m < n$ we have $r_m = 1$ and $\delta_m = 0$. In particular, we must have $x = \exp_n(r_n\mathfrak{m}_n + \delta_n)$. It follows immediately that $\exp_n(\mathfrak{m}_n) \trianglelefteq x$, hence $\lambda \ntrianglelefteq \exp_n(\mathfrak{m}_n)$ for all $\lambda \in \mathbb{L}$. By minimality of x, we must have $r_n = 1$ and $\delta_n = 0$, as desired.

Finally, since $r_n = 1$ and $\delta_n = 0$ for all $n \in \mathbb{N}$, we have $\lambda = \exp_n(\mathfrak{m}_n)$ for all $n \in \mathbb{N}$, and therefore $\log_n(x) = \mathfrak{m}_n \in \mathfrak{M}^{>1}$. This means that x is log-atomic, a contradiction. ☐

Corollary 5.9. *$\mathbb{L}$ is a class of representatives for $\asymp^L$. Moreover, for each $\lambda \in \mathbb{L}$, λ is the simplest number in its level.*

Proof. Let $[x]$ be a given level. By Proposition 5.8, there exists $\lambda \in \mathbb{L}$ such that $\lambda \trianglelefteq x$ and $\lambda \asymp^L x$, and by Proposition 5.6, λ is also unique. By Theorem 4.24, we also have $\lambda \leq_s x$. This shows that λ is the simplest number in $[x]$, as desired. ☐

Corollary 5.10. *For all $x \in \mathbf{No}$, $\mathrm{NR}(x) = 0$ if and only if either $x \in \mathbb{R}$ or $x = \pm\lambda^{\pm 1}$ for some $\lambda \in \mathbb{L}$.*

Proof. It is easy to see that $\mathrm{NR}(r) = \mathrm{NR}(\pm\lambda^{\pm 1}) = 0$ for all $r \in \mathbb{R}$ and $\lambda \in \mathbb{L}$.

Conversely, suppose that x satisfies $\mathrm{NR}(x) = 0$. Let $r\exp(\gamma)$ be the leading term of x. By Proposition 4.26, we must have $\mathrm{NR}(\gamma) \leq \mathrm{NR}(x)$; since this forces $\mathrm{NR}(\gamma) = \mathrm{NR}(x) = 0$, we must also have that $\exp(\gamma)$ is minimal in $\mathrm{S}(x)$, which implies $x = r\exp(\gamma)$.

If $\gamma = 0$, then $x = r \in \mathbb{R}$, and we are done. If $\gamma \neq 0$, then $r = \pm 1$. Since $|\gamma| > \mathbb{N}$, by Proposition 5.8 there is $\mu \in \mathbb{L}$ such that $\mu \trianglelefteq |\gamma|$. Since $\mathrm{NR}(\gamma) = \mathrm{NR}(-\gamma) = 0$, we must have $\gamma = \pm\mu$. Letting $\lambda := \exp(\mu)$, it follows that $x = \pm\lambda^{\pm 1}$, as desired. ☐

Corollary 5.11. *For any $x \in \mathbf{No}$ such that $\ell(x) \neq 0$, there is $n \in \mathbb{N}$ such that $\ell_n(x) \in \mathbb{L}$, where $\ell_n = \ell \circ \dots \circ \ell$ is the n-fold composition of ℓ with itself.*

Proof. Note first that $\ell_2(x) > \mathbb{N}$ for any $x \in \mathbf{No}$ such that $\ell(x) \neq 0$. By Corollary 5.9, there is $\lambda \in \mathbb{L}$ such that $\ell_2(x) \asymp^L \lambda$, whence $\log_n(\ell_2(x)) \sim \log_n(\lambda) = \ell_n(\lambda)$ for some $n \in \mathbb{N}$. Since $\ell(y) \sim \log(y)$ for all $y > \mathbb{N}$, we get $\ell_{n+2}(x) \sim \ell_n(\lambda)$. But then $\ell_{n+3}(x) = \ell_{n+1}(\lambda) \in \mathbb{L}$, as desired. ☐

5.2. **Parametrizing the levels.** Mimicking the definition of the omega-map, there is a natural way of defining a function $\lambda : \mathbf{No} \to \mathbf{No}$ whose values are the simplest representatives for the $\asymp^L$-equivalence classes.

Definition 5.12. Let $x \in \mathbf{No}$ and let $x = \{x'\} \mid \{x''\}$ be its canonical representation. We define

$$\lambda_x := \{k, \exp_h(k \log_h(\lambda_{x'}))\} \mid \left\{\exp_h\left(\frac{1}{k}\log_h(\lambda_{x''})\right)\right\}$$

where h, k range in $\mathbb{N}^{>0}$.

Proposition 5.13. *The function $x \mapsto \lambda_x$ is well defined, increasing, and if $x < y$ then $\lambda_x \prec^L \lambda_y$.*

Proof. By abuse of notation we say "λ_x is well defined" if there exists a (necessarily unique) function $z \mapsto \lambda_z$ defined for all $z \leq_s x$ which satisfies the equation in Definition 5.12 on its domain of definition. Obviously, if λ_x is well defined, then λ_z is well defined for all $z \leq_s x$. Note that if $x = \{x'\} \mid \{x''\}$ is a canonical representation and λ_x is well defined, then

$$\{k, \exp_h(k \log_h(\lambda_{x'}))\} < \lambda_x < \left\{\exp_h\left(\frac{1}{k}\log_h(\lambda_{x''})\right)\right\}$$

for all k, h, and therefore $\lambda_x > \mathbb{N}$ and $\lambda_{x'} \prec^L \lambda_x \prec^L \lambda_{x''}$. It follows that if λ_x and λ_y are both well defined and $y <_s x$, then

$$x < y \iff \lambda_x \prec^L \lambda_y.$$

The above equivalence then holds even without the assumption $y <_s x$, since we can always find some z between x and y with $z \leq_s x, y$.

To prove that λ_x is well defined for every x we proceed by induction on simplicity. Consider the canonical representation $x = \{x'\} \mid \{x''\}$. By inductive hypothesis we can assume that $\lambda_{x'}$, $\lambda_{x''}$ are all well defined and therefore, by the above arguments, $\lambda_{x'} \prec^L \lambda_{x''}$. This easily implies that the convex class associated to the definition of λ_x is non-empty, and therefore λ_x is also well defined. □

Remark 5.14. It is immediate to see that $x \leq_s y$ if and only if $\lambda_x \leq_s \lambda_y$.

Corollary 5.15. *The definition of λ_x is uniform.*

Proof. The uniformity follows easily from the fact that if $x < y$ then $\lambda_x \prec^L \lambda_y$. □

In the same way one proves that every surreal number x is in the same archimedean class of some ω^y, we can prove that every x is in the same level of some λ_y.

Proposition 5.16. *For every $x \in \mathbf{No}$ with $x > \mathbb{N}$ there is a (unique) $y \in \mathbf{No}$ such that $x \asymp^L \lambda_y$ and $\lambda_y \leq_s x$. In particular, λ_y is the simplest number in its level.*

Proof. Since x is positive infinite, its canonical representation is of the form $x = \mathbb{N} \cup A \mid B$, where A (if non-empty) is greater than $\mathbb{N}$. By induction on simplicity, we can assume that every element $c \in A \cup B$ is in the same level of some $\lambda_z \leq_s c$. Define $F = \{z : (\exists a \in A)(a \asymp^L \lambda_z)\}$ and $G = \{z : (\exists b \in B)(b \asymp^L \lambda_z)\}$. Note that F and G are sets. We distinguish a few cases.

If $F \not< G$, there are $z \in F, w \in G$ with $z \geq w$, whence $\lambda_z \geq \lambda_w$. Let $a \in A$ and $b \in B$ be such that $a \asymp^L \lambda_z \leq_s a$ and $b \asymp^L \lambda_w \leq_s b$. Since $a < x < b$ and the levels are convex, we immediately get that $\lambda_z = \lambda_w$ (so $z = w$) and $x \asymp^L a \asymp^L \lambda_z$. In particular, $x \asymp^L \lambda_z \leq_s a <_s x$, and we are done.

Suppose now that $F < G$. If x is equivalent to λ_y for some $y \in F \cup G$, we are done, so assume otherwise. We must have $[\lambda_{y'}] < x < [\lambda_{y''}]$ for every $y' \in F$, $y'' \in G$. By inductive hypothesis, we have $A \subset \bigcup_{y' \in F}[\lambda_{y'}]$ and $B \subset \bigcup_{y'' \in G}[\lambda_{y''}]$. Therefore, if we let $y := F \mid G$, we have $x = \mathbb{N} \cup A \mid B = \mathbb{N} \cup \bigcup_{y' \in F} [\lambda_{y'}] \mid \bigcup_{y'' \in G} [\lambda_{y''}] = \lambda_y$, and we are done. □

Corollary 5.17. *We have* $\mathbb{L} = \lambda_{\mathbf{No}} = \{\lambda_x : x \in \mathbf{No}\}$.

Proof. By Corollary 5.9 and Proposition 5.16, both classes $\mathbb{L}$ and $\lambda_{\mathbf{No}}$ are exactly the class of the simplest numbers in each level, and therefore they are equal. □

Remark 5.18. It is easy to verify that $\lambda_0 = \omega$ and $\lambda_1 = \exp(\omega)$.

Indeed, by definition we have $\lambda_0 = \{k\} \mid \emptyset$ and $\lambda_1 = \{k, \exp_h(k \log_h(\omega))\} \mid \emptyset$ for h, k ranging in $\mathbb{N}$. It clearly follows that $\lambda_0 = \omega$. For λ_1, note that $\exp(\omega)$ is log-atomic and $\exp(\omega) \succ^L \omega$, hence $\exp(\omega) > \exp_h(k \log_h(\omega))$ for all $h, k \in \mathbb{N}$. It follows that $\lambda_1 \leq_s \exp(\omega)$. On the other hand, by Theorem 3.8 we have $\exp(\omega) = \left\{\omega^k\right\} \mid \emptyset$ for k ranging in $\mathbb{N}$. Since $\lambda_1 > \omega^k = \exp(k \log(\omega))$ for all $k \in \mathbb{N}$, we get $\exp(\omega) \leq_s \lambda_1$, hence $\lambda_1 = \exp(\omega)$, as desired.

With a similar argument, one can verify that $\lambda_n = \exp_n(\omega)$ and $\lambda_{-n} = \log_n(\omega)$ for all $n \in \mathbb{N}$. It follows, for instance, that there exist several log-atomic numbers between ω and $\exp(\omega)$, such as $\lambda_{\frac{1}{2}}$, and in particular several levels between $[\omega]$ and $[\exp(\omega)]$.

5.3. **κ-numbers.** We recall here the notion of κ-numbers defined in [KM15, Def. 3.1]. They can be defined using again an appropriate notion of magnitude.

Definition 5.19. Given two elements $x, y \in \mathbf{No}$ with $x, y > \mathbb{N}$, we write

1. $x \preceq^K y$ if $x \leq \exp_h(y)$ for some $h \in \mathbb{N}$;
2. $x \prec^K y$ if $x < \log_h(y)$ for all $h \in \mathbb{N}$;
3. $x \asymp^K y$ if $\log_h(y) \leq x \leq \exp_h(y)$ for some $h \in \mathbb{N}$.

It is easy to verify that the relation $\asymp^K$ is an equivalence relation, and that $\preceq^K$ induces a total order on its equivalence classes. The following proposition is also easy, and its proof is left to the reader.

Proposition 5.20. *For all $x, y \in \mathbf{No}$ with $x, y > \mathbb{N}$, $x \asymp^L y$ implies $x \asymp^K y$.*

The κ-numbers are a natural class of representatives for the $\asymp^K$-classes.

Definition 5.21 ([KM15, Def. 3.1]). Let $x \in \mathbf{No}$ and let $x = \{x'\} \mid \{x''\}$ be its canonical representation. We define

$$\kappa_x := \{\exp_n(0), \exp_n(\kappa_{x'})\} \mid \{\log_n(\kappa_{x''})\},$$

where n runs in $\mathbb{N}$. We call $\kappa_{\mathbf{No}}$ the class of the numbers of the form κ_x.

Remark 5.22. It can be easily verified that $\kappa_0 = \omega$ and $\kappa_1 = \varepsilon_0$, where ε_0 is the least ordinal such that $\omega^{\varepsilon_0} = \varepsilon_0$; see [KM15, Ex. 3.3].

One can verify that this definition is uniform, and again that $x \leq_s y$ if and only if $\kappa_x \leq_s \kappa_y$. One can also see that for every $x > \mathbb{N}$ there exists a $\kappa_y \leq_s x$ such that $\kappa_y \asymp^K x$. In particular, κ_y is the simplest number in its $\asymp^K$-equivalence class. We refer to [KM15] for more details.

It is proved in [KM15] that $\log_n(\kappa_x)$ is always of the form ω^{ω^y}, and in particular belongs to $\mathfrak{M}$, showing that the κ-numbers are log-atomic. We rephrase their statement as follows, and we give an extremely short proof exploiting the relationship between $\asymp^L$ and $\asymp^K$.

Theorem 5.23 ([KM15, Thm. 4.3]). $\kappa_{\mathbf{No}} \subseteq \mathbb{L}$.

Proof. Since κ_x is the simplest number in its $\asymp^K$-equivalence class, by Proposition 5.20 it must also be the simplest number in its level. Therefore, by Corollary 5.9, $\kappa_x = \lambda$ for some $\lambda \in \mathbb{L}$, as desired. □

It was conjectured in [KM15, Conj. 5.2] that $\kappa_{\mathbf{No}}$ generates all the log-atomic numbers by iterated applications of exp and log. However, we can exhibit numbers in $\mathbb{L}$ that are not of this form.

Proposition 5.24. *There are numbers in $\mathbb{L}$ that cannot be obtained from numbers in $\kappa_{\mathbf{No}}$ by finitely many applications of* exp *and* log*.*

Proof. As seen in Remark 5.18, there are log-atomic numbers between ω and $\exp(\omega)$, such as $\lambda_{\frac{1}{2}}$. On the other hand, it is easy to verify that no number of the form $\log_n(\kappa)$ or $\exp_n(\kappa)$, with $n \in \mathbb{N}$ and $\kappa \in \kappa_{\mathbf{No}}$, lies between ω and $\exp(\omega)$. Indeed, this is trivial if $\kappa = \omega$, while if $\kappa' < \omega < \kappa''$ we have $\kappa' \prec^K \omega \prec^K \kappa''$, and in particular $\exp_n(\kappa') < \omega < \exp(\omega) < \log_n(\kappa'')$ for all $n \in \mathbb{N}$. Therefore, $\lambda_{\frac{1}{2}} \neq \exp_n(\kappa)$ and $\lambda_{\frac{1}{2}} \neq \log_n(\kappa)$ for all $n \in \mathbb{N}$ and $\kappa \in \kappa_{\mathbf{No}}$, as desired. □

For our construction, the κ-numbers that matter are actually the ones of the form $\kappa_{-\alpha}$ for $\alpha \in \mathbf{On}$.

Remark 5.25. If $\alpha \in \mathbf{On}$ then

$$\kappa_{-\alpha} = \mathbb{N} \mid \{\log_n(\kappa_{-\beta}) : n \in \mathbb{N}, \beta < \alpha\},$$

namely $\kappa_{-\alpha}$ is the simplest positive infinite number less than $\log_n(\kappa_{-\beta})$ for all $n \in \mathbb{N}$ and $\beta < \alpha$. Moreover, if $\beta < \alpha$, then $\kappa_{-\beta} <_s \kappa_{-\alpha}$ and of course $\kappa_{-\alpha} < \kappa_{-\beta}$.

Proposition 5.26. *The sequence $(\kappa_{-\alpha} : \alpha \in \mathbf{On})$ is decreasing and coinitial in the positive infinite numbers, namely every positive infinite number is greater than some $\kappa_{-\alpha}$. In particular, $\mathbb{L}$ is coinitial in the positive infinite numbers.*

Proof. Let $x > \mathbb{N}$. We know that $x \asymp^K \kappa_y$ for some y. It now suffices to note that there exists an $\alpha \in \mathbf{On}$ such that $-\alpha < y$, and therefore $\kappa_{-\alpha} \prec^K \kappa_y \asymp^K x$. In particular, $\kappa_{-\alpha} < x$, as desired. □

6. Surreal derivations

6.1. **Derivations.** We begin with our definition of surreal derivation. It is the specialization to surreal numbers of other notions which have been defined by several authors in the context of H-fields or transseries.

Definition 6.1. A **surreal derivation** is a function $D : \mathbf{No} \to \mathbf{No}$ satisfying the following properties:

1. Leibniz rule: $D(xy) = xD(y) + yD(x)$;
2. strong additivity: $D\left(\sum_{i\in I} x_i\right) = \sum_{i\in I} D(x_i)$ if $(x_i : i \in I)$ is summable;
3. compatibility with exponentiation: $D(\exp(x)) = \exp(x)D(x)$;
4. constant field $\mathbb{R}$: $\ker(D) = \mathbb{R}$;
5. H-field: if $x > \mathbb{N}$ then $D(x) > 0$.

Conditions (4) and (5), together with the fact that D is a derivation, make the pair $(\mathbf{No}, D)$ into an H-field, the abstract counterpart of the notion of Hardy field. Note that in the definition of H-field in [AvdD02] one also requires that if $|x| \leq c$ for some $c \in \ker(D)$, then there is some $d \in \ker(D)$ such that $|x - d| < c$ for every positive $c \in \ker(D)$, which is always true when $\ker(D) = \mathbb{R}$.

Remark 6.2. When x is infinitesimal, point (3) follows from (1) and (2). Indeed, if x is infinitesimal we have

$$D(\exp(x)) = D\left(1 + x + \frac{x^2}{2!} + \dots\right).$$

By strong additivity and the Leibniz rule we get

$$D(\exp(x)) = D(x) + xD(x) + \frac{x^2}{2!}D(x) + \dots = \exp(x)D(x).$$

Remark 6.3. By points (2) and (5), if $x, y > \mathbb{N}$ and $x \succ y$ then $D(x) > D(y) > 0$.

Before embarking on the construction of a surreal derivation, we recall a few properties that can be easily derived from the above axioms. We remark that these properties hold in any Hardy field closed under the functions exp and log.

Proposition 6.4. *Let D be a surreal derivation and let $x, y \in \mathbf{No}$. We have:*

1. *if $1 \not\asymp x \succ y$, then $D(x) \succ D(y)$;*
2. *if $1 \not\asymp x \sim y$, then $D(x) \sim D(y)$;*
3. *if $1 \not\asymp x \asymp y$, then $D(x) \asymp D(y)$.*

Proof. Without loss of generality, we may assume that $x, y > 0$.

(1) For all $r \in \mathbb{R}$ we have $x - ry \asymp x$ and $x - ry > 0$.

If $x \asymp x - ry \succ 1$, then $D(x - ry) = D(x) - rD(y) > 0$. Therefore, $|D(x)| > |r| \cdot |D(y)|$ for all $r \in \mathbb{R}$, hence $D(x) \succ D(y)$.

If $x \asymp x - ry \prec 1$, then $\frac{1}{x-ry} \succ 1$, hence $D\left(\frac{1}{x-ry}\right) = -\frac{D(x)-rD(y)}{(x-ry)^2} > 0$, i.e., $D(x) < rD(y)$. Therefore, $|D(x)| > |r| \cdot |D(y)|$ for all $r \in \mathbb{R}$, hence $D(x) \succ D(y)$.

(2) We have $x - y \prec x$. By (1), it follows that $D(x) \succ D(x - y) = D(x) - D(y)$, which means $D(x) \sim D(y)$, as desired.

(3) The conclusion is trivial if $x = y = 0$, so assume $x \neq 0$. The assumption $x \asymp y$ then implies that $x \sim ry$ for some $r \in \mathbb{R}^*$. By (2) we have $D(x) \sim D(ry) = rD(y) \asymp D(y)$, as desired. □

The following proposition will play a crucial role in the sequel.

Proposition 6.5. *Given $x, y \in \mathbf{No}$, if $x, y, x - y$ are positive infinite, then*

$$\log|D(x)| - \log|D(y)| \prec x - y \preceq \max\{x, y\}.$$

Proof. Since $x, y, (x - y) > \mathbb{N}$, we have $D(x), D(y), D(x - y) > 0$, or in other words $D(x) > D(y) > 0$. Moreover, for every $r \in \mathbb{R}^{>0}$ we have $\exp(r(x - y)) > \mathbb{N}$, namely $\exp(rx) \succ \exp(ry)$. Taking the inverses we get $\exp(-rx) \prec \exp(-ry)$.

Since $y \succ 1$, we have $\exp(-ry) \prec 1$, and another application of Proposition 6.4 yields $D(\exp(-rx)) \prec D(\exp(-ry))$. In particular, $|D(\exp(-rx))| < |D(\exp(-ry))|$; since D is compatible with exp, we get

$$\exp(-rx) \cdot r \cdot |D(x)| < \exp(-ry) \cdot r \cdot |D(y)|.$$

Taking the logarithms on both sides and rearranging the summands, we get

$$\log|D(x)| - \log|D(y)| < r(x - y).$$

Since this holds for an arbitrary $r \in \mathbb{R}^{>0}$, and $\log|D(x)| > \log|D(y)|$, we obtain

$$\log|D(x)| - \log|D(y)| \prec x - y \preceq \max\{x, y\},$$

as desired. □

6.2. **Derivatives of log-atomic numbers.** As anticipated in the introduction, we shall construct a surreal derivation by first giving its values on the class $\mathbb{L}$. Clearly, if we want to take a function $D : \mathbb{L} \to \mathbf{No}^{>0}$ and extend it to a surreal derivation, it must at least satisfy the inequality of Proposition 6.5.

With some heuristics, it is not difficult to find a map $\partial'_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}^{>0}$ satisfying the inequalities of Proposition 6.5 and compatible with exp. If $\lambda \in \mathbb{L}$, we note that we must have $\partial'_{\mathbb{L}}(\lambda) = \lambda \cdot \partial'_{\mathbb{L}}(\log \lambda)$. Iterating, we obtain $\partial'_{\mathbb{L}}(\lambda) = \lambda \cdot \log(\lambda) \cdot \log_2(\lambda) \cdot \ldots \cdot \log_{i-1}(\lambda) \cdot \partial'_{\mathbb{L}}(\log_i(\lambda))$, or equivalently $\partial'_{\mathbb{L}}(\lambda) = \exp(\log(\lambda) + \log_2(\lambda) + \log_3(\lambda) + \ldots + \log_i(\lambda)) \cdot \partial'_{\mathbb{L}}(\log_i(\lambda))$. This suggests the following definition.

Definition 6.6. If $\lambda \in \mathbb{L}$, we let

$$\partial'_{\mathbb{L}}(\lambda) := \exp\left(\sum_{i=1}^{\infty} \log_i(\lambda)\right).$$

It is an easy exercise to check that $\partial'_{\mathbb{L}}$ does satisfy the inequalities of Proposition 6.5. It can be further shown that $\partial'_{\mathbb{L}}$ extends to a surreal derivation $\partial' : \mathbf{No} \to \mathbf{No}$ (using Theorem 6.32). However, this derivation is not the "simplest" possible one with respect to the simplicity relation, and moreover, its behavior is not really nice; for instance, there is no $x \in \mathbf{No}$ such that $\partial'(x) = 1$.

The simplest function $\partial_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}^{>0}$ satisfying the inequalities of Proposition 6.5 is given by a similar but different formula involving the subclass of the κ-numbers (those with indexes of the form $-\alpha$ where α is an ordinal, see Definition 5.21 and Remark 5.25). We postpone to Section 9 the proof that $\partial_{\mathbb{L}}$ is indeed the simplest one.

Definition 6.7. If $\lambda \in \mathbb{L}$, we let

$$\partial_{\mathbb{L}}(\lambda) := \exp\left(-\sum_{\substack{\alpha \in \mathbf{On} \\ \kappa_{-\alpha} \succeq^K \lambda}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} \log_i(\lambda)\right).$$

Remark 6.8. Note that the sequence $(\kappa_{-\alpha})_{\alpha \in \mathbf{On}}$ is decreasing (see Remark 5.25), so its largest possible value is $\kappa_0 = \omega$. It follows that if $\lambda \succ^K \omega$ we have

$$\partial_{\mathbb{L}}(\lambda) = \exp\left(\sum_{i=1}^{\infty} \log_i(\lambda)\right) = \partial'_{\mathbb{L}}(\lambda).$$

Another special case is when $\lambda = \kappa_{-\alpha}$ for some $\alpha \in \mathbf{On}$. In this case the terms of the form $\log_i(\lambda)$ cancel out and the formula specializes to:

$$\partial_{\mathbb{L}}(\kappa_{-\alpha}) = \exp\left(-\sum_{\substack{\beta \in \mathbf{On} \\ \beta < \alpha}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\beta})\right).$$

In particular, $\partial_{\mathbb{L}}(\omega) = \partial_{\mathbb{L}}(\kappa_0) = 1$.

Before proving that $\partial_{\mathbb{L}} : \mathbf{No} \to \mathbf{No}$ extends to a surreal derivation $\partial : \mathbf{No} \to \mathbf{No}$, let us first verify that the necessary condition given by Proposition 6.5 is met.

Proposition 6.9. *For all $\lambda, \mu \in \mathbb{L}$, $\log(\partial_{\mathbb{L}}(\lambda)) - \log(\partial_{\mathbb{L}}(\mu)) \prec \max\{\lambda, \mu\}$.*

Proof. Without loss of generality we may assume that $\mu < \lambda$. Clearly, the inequality $\kappa_{-\alpha} \succeq^K \lambda$ implies $\kappa_{-\alpha} \succeq^K \mu$, hence

$$\log(\partial_{\mathbb{L}}(\lambda)) - \log(\partial_{\mathbb{L}}(\mu)) = \sum_{\substack{\alpha \in \mathbf{On} \\ \mu \preceq^K \kappa_{-\alpha} \prec^K \lambda}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} (\log_i(\lambda) - \log_i(\mu)).$$

It follows that

$$\log(\partial_{\mathbb{L}}(\lambda)) - \log(\partial_{\mathbb{L}}(\mu)) \preceq \max_{i \geq 1} \left\{ \log_i(\lambda), \log_i(\mu), \max_{\substack{\alpha \in \mathbf{On} \\ \mu \preceq^K \kappa_{-\alpha} \prec^K \lambda}} \{\log_i(\kappa_{-\alpha})\} \right\}.$$

However, if $\kappa_{-\alpha} \prec^K \lambda$, then $\log_i(\kappa_{-\alpha}) \prec \lambda$ for all $i \geq 1$, and moreover $\log_i(\mu) \prec \mu \prec \lambda$ and $\log_i(\lambda) \prec \lambda$ for all $i \geq 1$. Therefore, the right hand side of the above inequality is $\prec \lambda = \max\{\lambda, \mu\}$, as desired. □

Proposition 6.10. *For all* $\lambda \in \mathbb{L}$, $\partial_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda)\partial_{\mathbb{L}}(\lambda)$.

Proof. Let $\lambda \in \mathbb{L}$. Clearly, $\kappa_{-\alpha} \succeq^K \lambda$ if and only if $\kappa_{-\alpha} \succeq^K \exp(\lambda)$. Therefore,

$$\partial_{\mathbb{L}}(\exp(\lambda)) = \exp\left(- \sum_{\substack{\alpha \in \mathbf{On} \\ \kappa_{-\alpha} \succeq^K \exp(\lambda)}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} \log_i(\exp(\lambda)) \right) =$$

$$= \exp\left(- \sum_{\substack{\alpha \in \mathbf{On} \\ \kappa_{-\alpha} \succeq^K \lambda}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} \log_i(\lambda) + \lambda \right) = \exp(\lambda)\partial_{\mathbb{L}}(\lambda). \quad \square$$

The proof that $\partial_{\mathbb{L}}$ extends to a surreal derivation is done by induction on the rank NR, and uses ideas from [Sch01]. In the proof we shall not use the actual definition of $\partial_{\mathbb{L}}$, but only the fact that $\partial_{\mathbb{L}}$ satisfies the inequalities of Proposition 6.5, is compatible with exp, and takes values in $\mathbb{R}^*\mathfrak{M}$.

6.3. **Path-derivatives.** We consider the following sequences of terms, as in [Sch01].

Definition 6.11. We call **path** a sequence of terms $P : \mathbb{N} \to \mathbb{R}^*\mathfrak{M}$ such that $P(i+1)$ is a term of $\ell(P(i))$ for all $i \in \mathbb{N}$. We call $\mathcal{P}(x)$ the set of paths such that $P(0)$ is a term of x.

Note that $P(0) \notin \mathbb{R}$ (as otherwise there would be no possible value for $P(1)$) and $P(i+1) \in \mathbb{R}^*\mathfrak{M}^{>1}$ for every i (because $\mathbb{J} \cap \mathbb{R}^*\mathfrak{M} = \mathbb{R}^*\mathfrak{M}^{>1}$).

Remark 6.12. If $\lambda \in \mathbb{L}$ there exists a unique path P such that $P(0) = \lambda$, and for that path $P(i) = \log_i(\lambda) \in \mathbb{L}$ for all i.

Definition 6.13. Given a path $P : \mathbb{N} \to \mathbb{R}^*\mathfrak{M}$ we define its **path-derivative** $\partial_{\mathcal{P}}(P) \in \mathfrak{M}$ as follows:

(1) if there is $k \in \mathbb{N}$ such that $P(k) \in \mathbb{L}$, we let $\partial_{\mathcal{P}}(P) := \prod_{i<k} P(i) \cdot \partial_{\mathbb{L}}(P(k))$;
(2) if $P(i) \notin \mathbb{L}$ for all $i \in \mathbb{N}$, then $\partial_{\mathcal{P}}(P) := 0$.

Note that the value of $\partial_{\mathcal{P}}(P)$ does not depend on the choice of k in (1), since $\partial_{\mathbb{L}}(P(i)) = P(i) \cdot \partial_{\mathbb{L}}(P(i+1))$ whenever $P(i) \in \mathbb{L}$, thanks to the fact that $\partial_{\mathbb{L}}$ is compatible with exp. Therefore, if $P(i) \in \mathbb{L}$ then for every $k \geq i$ we have

$$\partial_{\mathcal{P}}(P) = P(0) \cdot P(1) \cdot \ldots \cdot P(k-1) \cdot \partial_{\mathbb{L}}(P(k)).$$

We now wish to define $\partial(x)$ as the sum of all the path-derivatives of the paths in $\mathcal{P}(x)$. Indeed, we can prove that the family $(\partial_{\mathcal{P}}(P) : P \in \mathcal{P}(x))$ is summable.

Lemma 6.14. *If P is a path, then $1 \prec P(i+1) \preceq \log(|P(i)|) \prec P(i)$ for all $i > 0$.*

Proof. Trivial, since $P(i) \in \mathbb{J}$ for all $i > 0$. □

Lemma 6.15. *If $t \preceq u$ are in $\mathbb{R}^*\mathfrak{M}$, and t' is a term of $\ell(t)$ but not of $\ell(u)$, then $(t')^n \prec \frac{u}{t}$ for all $n \in \mathbb{N}$.*

Proof. We need to prove $n \cdot \ell(t') < \ell(u) - \ell(t)$. The hypothesis on t' implies that $\ell(u) \neq \ell(t)$ and that $r \cdot t'$ is a term of $\ell(u) - \ell(t)$ for some $r \in \mathbb{R}^*$. In particular, $t' \preceq \ell(u) - \ell(t)$. Now observe that $\ell(u) - \ell(t)$ is positive and belongs to $\mathbb{J}$, hence $\ell(u) - \ell(t) > \mathbb{N}$. Since $\exp(x) > x^n$ for all $x > \mathbb{N}$ and $n \in \mathbb{N}$, we have

$$(t')^n \preceq (\ell(u) - \ell(t))^n \prec \exp(\ell(u) - \ell(t)) \asymp \frac{u}{t}$$

for all $n \in \mathbb{N}$, as desired. □

Lemma 6.16. *Let P, Q be two paths such that $\partial_{\mathcal{P}}(P), \partial_{\mathcal{P}}(Q) \neq 0$.*

If $P(0) \preceq Q(0)$ and $P(1)^n \prec \frac{Q(0)}{P(0)}$ for all $n \in \mathbb{N}$, then $\partial_{\mathcal{P}}(P) \prec \partial_{\mathcal{P}}(Q)$.

More generally, suppose that there exists i such that

(1) *for all $j \leq i$, $P(j) \preceq Q(j)$;*
(2) *$P(i+1)^n \prec \frac{Q(i)}{P(i)}$ for all $n \in \mathbb{N}$.*

Then $\partial_{\mathcal{P}}(P) \prec \partial_{\mathcal{P}}(Q)$.

Proof. We prove the first part, as the second then follows easily.

For the sake of notation, write $x_j := P(j)$ and $y_j := Q(j)$. Let $k > 1$ be such that $x_k, y_k \in \mathbb{L}$. We need to prove that

$$\partial_{\mathcal{P}}(P) = x_0 \cdot x_1 \cdot \ldots \cdot x_{k-1} \cdot \partial_{\mathbb{L}}(x_k) \prec y_0 \cdot y_1 \cdot \ldots \cdot y_{k-1} \cdot \partial_{\mathbb{L}}(y_k) = \partial_{\mathcal{P}}(Q).$$

We observe that $y_2, \ldots, y_{k-1} \in \mathbb{J}$ are infinite. Therefore, it suffices to prove the stronger inequality

$$x_0 \cdot x_1 \cdot \ldots \cdot x_{k-1} \cdot \partial_{\mathbb{L}}(x_k) \prec y_0 \cdot y_1 \cdot \partial_{\mathbb{L}}(y_k),$$

or equivalently,

$$x_1 \cdot \ldots \cdot x_{k-1} \cdot \frac{\partial_{\mathbb{L}}(x_k)}{\partial_{\mathbb{L}}(y_k)} \prec \frac{y_0 y_1}{x_0}.$$

By Lemma 6.14, $1 \prec x_k \prec \ldots \prec x_2 \preceq \log|x_1| \prec x_1$, and similarly $y_k \preceq \log|y_1| \prec y_1$. By Proposition 6.9 we have

$$\log(\partial_{\mathbb{L}}(x_k)) - \log(\partial_{\mathbb{L}}(y_k)) \prec \max\{x_k, y_k\} \preceq \max\{\log|x_1|, \log|y_1|\}.$$

In particular, $\frac{\partial_{\mathbb{L}}(x_k)}{\partial_{\mathbb{L}}(y_k)} \leq \max\{|x_1|, |y_1|\}$. By the hypothesis on $x_1 = P(1)$ we get

$$\left| x_1 \cdot \ldots \cdot x_{k-1} \cdot \frac{\partial_{\mathbb{L}}(x_k)}{\partial_{\mathbb{L}}(y_k)} \right| \leq |x_1|^{k-1} \cdot \max\{|x_1|, |y_1|\} \leq |x_1|^k \cdot |y_1| \prec \frac{y_0 y_1}{x_0},$$

reaching the desired conclusion. □

Corollary 6.17. *Let P, Q be two paths such that $\partial_{\mathcal{P}}(P), \partial_{\mathcal{P}}(Q) \neq 0$. Suppose that there exists $i \in \mathbb{N}$ such that:*

(1) *for all $j \leq i$, $P(j) \preceq Q(j)$;*
(2) *$P(i+1)$ is not a term of $\ell(Q(i))$.*

Then $\partial_{\mathcal{P}}(P) \prec \partial_{\mathcal{P}}(Q)$.

Proof. By Lemma 6.15 we have $P(i+1)^n \prec \frac{Q(i)}{P(i)}$ for all $n \in \mathbb{N}$. It then follows from Lemma 6.16 that $\partial_{\mathcal{P}}(P) \prec \partial_{\mathcal{P}}(Q)$, as desired. □

Lemma 6.18. *Given* $P \in \mathcal{P}(x)$, *we have* $\mathrm{NR}(P(0)) \leq \mathrm{NR}(x)$, *and if the equality holds then the support of* x *has a minimum* $\mathfrak{m}$ *and* $P(0) = \pm\mathfrak{m}$.

Moreover, for all $i \in \mathbb{N}$ *we have* $\mathrm{NR}(P(i+1)) \leq \mathrm{NR}(P(i))$, *and if the equality holds then the support of* $\ell(P(i))$ *has a minimum* $\mathfrak{m}$ *and* $P(i+1) = \pm\mathfrak{m}$.

Proof. Immediate by Proposition 4.26. □

Corollary 6.19. *For all* $x \in \mathbf{No}$, *there is at most one path* $P \in \mathcal{P}(x)$ *such that* $\mathrm{NR}(P(i)) = \mathrm{NR}(x)$ *for all* $i \in \mathbb{N}$.

Proposition 6.20. *For all* $x \in \mathbf{No}$, *the family* $(\partial_{\mathcal{P}}(P) : P \in \mathcal{P}(x))$ *is summable.*

Proof. We need to prove that no sequence of distinct paths $(P_j)_{j\in\mathbb{N}}$ in $\mathcal{P}(x)$ is such that $\partial_{\mathcal{P}}(P_0) \preceq \partial_{\mathcal{P}}(P_1) \preceq \dots$. Suppose by contradiction that such a sequence exists. Since the paths are distinct, there exists a minimum integer m such that $P_j(m) \neq P_k(m)$ for some j, k, and clearly $P_j(i) = P_0(i)$ for all $i < m$ and j.

Let $\alpha := \mathrm{NR}(x)$. We work by primary induction on α and secondary induction on m to reach a contradiction. Let $r\exp(\gamma)$ be the term of maximum ℓ-value among $\{P_j(0) : j \in \mathbb{N}\}$. Note that if $\mathrm{NR}(\gamma) = \alpha$, then by Lemma 6.18 $r\exp(\gamma)$ is also the term of *minimum* ℓ-value, hence $P_j(0) = P_0(0)$ for all j, and therefore $m > 0$; otherwise, we must have $\mathrm{NR}(\gamma) < \alpha$. After extracting a subsequence, we may assume that $r\exp(\gamma) = P_0(0) \succeq P_1(0) \succeq \dots$.

Now, if $P_j(1)$ is not a term of $\gamma = \ell(P_0(0))$ for some $j \in \mathbb{N}$, then by Corollary 6.17 we get $\partial_{\mathcal{P}}(P_0) \succ \partial_{\mathcal{P}}(P_j)$, a contradiction. Therefore, $P_j(1)$ is a term of γ for all $j \in \mathbb{N}$. Consider the paths P'_j defined by $P'_j(i) := P_j(i+1)$ for $i \in \mathbb{N}$ and let m' be the minimum integer such that $P'_j(m') \neq P'_k(m')$ for some j, k. Clearly, if $m > 0$, then $m' = m - 1$.

Note that $P'_j \in \mathcal{P}(\gamma)$ for all $j \in \mathbb{N}$. By the equality

$$\partial_{\mathcal{P}}(P_j) = P_j(0) \cdot \partial_{\mathcal{P}}(P'_j)$$

and $P_0(0) \succeq P_1(0) \succeq \dots$, it follows that $\partial_{\mathcal{P}}(P'_0) \preceq \partial_{\mathcal{P}}(P'_1) \preceq \dots$. Since we have either $\mathrm{NR}(\gamma) < \alpha$, or $\mathrm{NR}(\gamma) = \alpha$ and $m' < m$, this contradicts the inductive hypothesis that no such sequences exist.

Therefore, $(\partial_{\mathcal{P}}(P) : P \in \mathcal{P}(x))$ is summable, as desired. □

6.4. **A surreal derivation.** Thanks to Proposition 6.20, we can finally define $\partial : \mathbf{No} \to \mathbf{No}$ by summing all the path-derivatives.

Definition 6.21. We define $\partial : \mathbf{No} \to \mathbf{No}$ by

$$\partial(x) := \sum_{P \in \mathcal{P}(x)} \partial_{\mathcal{P}}(P).$$

We claim that $\partial : \mathbf{No} \to \mathbf{No}$ is indeed a surreal derivation.

Definition 6.22. Given $x \in \mathbf{No} \setminus \mathbb{R}$, its **dominant path** is the path $Q \in \mathcal{P}(x)$ such that $Q(0)$ is the term of maximum non-zero ℓ-value of x and $Q(i+1)$ is the leading term of $\ell(Q(i))$ for all $i \in \mathbb{N}$.

Lemma 6.23. *If $x \in \mathbf{No} \setminus \mathbb{R}$ and Q is the dominant path of x, then $\partial_{\mathcal{P}}(Q) \neq 0$ and $\partial_{\mathcal{P}}(Q)$ is the leading term of $\partial(x)$.*

Proof. Let $Q \in \mathcal{P}(x)$ be the dominant path of x. Without loss of generality, we may assume that $x \not\asymp 1$ (if $x \asymp 1$, it suffices to subtract the leading real number), so that $Q(0)$ is the leading term of x, and $Q(i+1)$ is the leading term of $\ell(Q(i))$. Letting ℓ_i be the i-fold composition $\ell \circ \ldots \circ \ell$, it follows that $\ell(Q(i)) = \ell_{i+1}(x)$ for all $i \in \mathbb{N}$. By Corollary 5.11, there exists $k \in \mathbb{N}$ such that $Q(k) \in \mathbb{L}$, and therefore $\partial_{\mathcal{P}}(Q) \neq 0$. Let $P \in \mathcal{P}(x)$ be any other path different from P and such that $\partial_{\mathcal{P}}(P) \neq 0$.

We distinguish two cases. Suppose first that $P(i+1)$ is a term of $\ell(Q(i))$ for all i. By definition of Q, this clearly implies that $Q(i) \succeq P(i)$ for all $i \in \mathbb{N}$. Moreover, we must have $Q(i) = P(i)$ for all $i > k$. Since $Q \neq P$, there must be an $i \leq k$ such that $Q(i) \succ P(i)$, and it follows immediately that $\partial_{\mathcal{P}}(Q) \succ \partial_{\mathcal{P}}(P)$.

In the other case, take the minimal $j \in \mathbb{N}$ such that $P(j+1)$ is not a term of $\ell(Q(j))$. By definition of Q, we have $Q(i) \succeq P(i)$ for all $j \leq i$. By Corollary 6.17, we have $\partial_{\mathcal{P}}(Q) \succ \partial_{\mathcal{P}}(P)$ in this case as well. Since $\partial(x) = \sum_{P \in \mathcal{P}(x)} \partial_{\mathcal{P}}(P)$ by definition, we get that $\partial_{\mathcal{P}}(Q)$ is the leading term of $\partial(x)$, as desired. □

Corollary 6.24. *For all $x \in \mathbf{No}$, $\partial(x) = 0$ if and only if $x \in \mathbb{R}$.*

Proof. By Lemma 6.23, if $x \notin \mathbb{R}$ then $\partial(x) \neq 0$. Conversely, if $x \in \mathbb{R}$, then $\mathcal{P}(x) = \emptyset$, whence $\partial(x) = 0$, as desired. □

Corollary 6.25. *If $x > \mathbb{N}$ then $\partial(x) > 0$.*

Proof. By Lemma 6.23, it suffices to prove that if $x > \mathbb{N}$ and P is the dominant path of x, then $\partial_{\mathcal{P}}(P) > 0$.

By definition, $\partial_{\mathcal{P}}(P) = P(0) \cdot P(1) \cdot \ldots \cdot P(k-1) \cdot \partial_{\mathbb{L}}(P(k))$, where k is such that $P(k) \in \mathbb{L}$. We can easily prove by induction that $P(i) > \mathbb{N}$ for all i. Clearly $P(0) > \mathbb{N}$ holds by assumption. If $P(i) > \mathbb{N}$, then $\ell(P(i)) > 0$; since $P(i+1)$ is the leading term of $\ell(P(i))$, we must have $P(i+1) > 0$ as well. Since $P(i+1) \in \mathbb{J}$, it follows that $P(i+1) > \mathbb{N}$, concluding the induction. Moreover, $\partial_{\mathbb{L}}(P(k)) > 0$, since $\partial_{\mathbb{L}}$ takes only positive values. Therefore, $\partial_{\mathcal{P}}(P) > 0$, as desired. □

Proposition 6.26. *The function ∂ is strongly linear, hence strongly additive.*

Proof. It suffices to note that if $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}} \mathfrak{m}$ then

$$\partial(x) = \sum_{P \in \mathcal{P}(x)} \partial_{\mathcal{P}}(P) = \sum_{\mathfrak{m} \in \mathrm{S}(x)} \sum_{P \in \mathcal{P}(\mathfrak{m})} x_{\mathfrak{m}} \partial_{\mathcal{P}}(P) = \sum_{\mathfrak{m}} x_{\mathfrak{m}} \partial(\mathfrak{m}).$$

By Remark 2.10 it follows that ∂ is strongly additive. □

Proposition 6.27. *For all $\gamma \in \mathbb{J}$, $\partial(\exp(\gamma)) = \exp(\gamma)\partial(\gamma)$.*

Proof. Let $\gamma \in \mathbb{J}$. Consider the bijection $\mathcal{P}(\exp(\gamma)) \to \mathcal{P}(\gamma)$ sending $P \in \mathcal{P}(\exp(\gamma))$ to the path $P' \in \mathcal{P}(\gamma)$ defined by $P'(i) := P(i+1)$ for $i \in \mathbb{N}$. Recall that by definition $\partial_{\mathcal{P}}(P) = \exp(\gamma) \cdot \partial_{\mathcal{P}}(P')$. We thus obtain

$$\partial(\exp(\gamma)) = \sum_{P \in \mathcal{P}(\exp(\gamma))} \partial_{\mathcal{P}}(P) = \exp(\gamma) \sum_{P \in \mathcal{P}(\gamma)} \partial_{\mathcal{P}}(P) = \exp(\gamma)\partial(\gamma). \qquad \square$$

Proposition 6.28. *For all $x, y \in \mathbf{No}$, $\partial(xy) = x\partial(y) + y\partial(x)$.*

Proof. We first prove the conclusion on $\mathfrak{M}$. Let $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$ and write $\mathfrak{m} = \exp(\gamma)$, $\mathfrak{n} = \exp(\delta)$ with $\gamma, \delta \in \mathbb{J}$. By Proposition 6.27, we get $\partial(\mathfrak{m}) = \exp(\gamma)\partial(\gamma)$, $\partial(\mathfrak{n}) = \exp(\delta)\partial(\delta)$ and $\partial(\mathfrak{mn}) = \exp(\gamma+\delta)\partial(\gamma+\delta)$. By Proposition 6.26, we conclude $\partial(\mathfrak{mn}) = \mathfrak{m}\partial(\mathfrak{n}) + \partial(\mathfrak{m})\mathfrak{n}$.

For the general case, let $x, y \in \mathbf{No}$ and write $x = \sum_{\mathfrak{m}} x_{\mathfrak{m}}\mathfrak{m}$ and $y = \sum_{\mathfrak{n}} y_{\mathfrak{n}}\mathfrak{n}$. By Proposition 6.26 again, $\partial(xy) = \partial(\sum_{\mathfrak{m},\mathfrak{n}} x_{\mathfrak{m}} y_{\mathfrak{n}} \mathfrak{mn}) = \sum_{\mathfrak{m},\mathfrak{n}} x_{\mathfrak{m}} y_{\mathfrak{n}} \partial(\mathfrak{mn}) = \sum_{\mathfrak{m},\mathfrak{n}} (x_{\mathfrak{m}}\mathfrak{m} \cdot y_{\mathfrak{n}}\partial(\mathfrak{n}) + x_{\mathfrak{m}}\partial(\mathfrak{m}) \cdot y_{\mathfrak{n}}\mathfrak{n}) = x\partial(y) + y\partial(x)$, as desired. □

Corollary 6.29. *For all $x \in \mathbf{No}$, $\partial(\exp(x)) = \exp(x)\partial(x)$.*

Proof. Let $x \in \mathbf{No}$. Write $x = \gamma + r + \varepsilon$ with $\gamma \in \mathbb{J}$, $r \in \mathbb{R}$, $\varepsilon \in o(1)$. Since ε is infinitesimal, we can apply the strong additivity (Proposition 6.26) and the Leibniz rule (Proposition 6.28) as in Remark 6.2 to obtain $\partial(\exp(\varepsilon)) = \exp(\varepsilon)\partial(\varepsilon)$. Since $\gamma \in \mathbb{J}$, we have $\partial(\exp(\gamma)) = \exp(\gamma)\partial(\gamma)$ by Proposition 6.27. By Corollary 6.24, we also have $\partial(\exp(r)) = 0 = \exp(r)\partial(r)$. By Leibniz' rule (Proposition 6.28) applied to the product $\exp(\gamma)\exp(r)\exp(\varepsilon) = \exp(x)$ we conclude that $\partial(\exp(x)) = \exp(x)\partial(x)$, as desired. □

Therefore, ∂ is a surreal derivation.

Theorem 6.30. *The function $\partial : \mathbf{No} \to \mathbf{No}$ is a surreal derivation extending $\partial_{\mathbb{L}}$.*

Proof. The function ∂ satisfies Leibniz' rule by Proposition 6.28, strong additivity by Corollary 6.25, it is compatible with exponentiation by Corollary 6.29, its kernel is $\mathbb{R}$ by Corollary 6.24, and it is an H-field derivation by Proposition 6.26. □

Remark 6.31. Note that the restriction of $\partial : \mathbf{No} \to \mathbf{No}$ to $\mathbb{L}$, namely the map $\partial_{\mathbb{L}}$, takes values in the subfield $\mathbb{R}\langle\mathbb{L}\rangle$ of $\mathbf{No}$. Since ∂ is calculated using finite products and infinite sums, we can easily verify that $\partial(\mathbb{R}\langle\mathbb{L}\rangle) \subseteq \mathbb{R}\langle\mathbb{L}\rangle$. Therefore, the restriction $\partial_{\restriction \mathbb{R}\langle\mathbb{L}\rangle}$ induces a structure of H-field on $\mathbb{R}\langle\mathbb{L}\rangle$.

In more generality, with the same proof we obtain:

Theorem 6.32. *Let $D : \mathbb{L} \to \mathbf{No}^{>0}$ be a map such that:*

1. *for all $\lambda, \mu \in \mathbb{L}$, $\log(D(\lambda)) - \log(D(\mu)) \prec \max\{\lambda, \mu\}$;*
2. *for all $\lambda \in \mathbb{L}$, $D(\exp(\lambda)) = \exp(\lambda)D(\lambda)$;*
3. *$D(\mathbb{L}) \subset \mathbb{R}^*\mathfrak{M}$.*

Then D extends to a surreal derivation on $\mathbf{No}$.

Once we have a derivation, we can apply Ax's theorem to deduce some transcendence results. If V is a $\mathbb{Q}$-vector space and W is a subspace of V, we say that a set $H \subset V$ is *$\mathbb{Q}$-linearly independent modulo W* if its projection to the quotient V/W is $\mathbb{Q}$-linearly independent.

Theorem 6.33 ([Ax71])**.** *Let (K, D) be a differential field. If $x_1, \ldots, x_n, y_1, \ldots, y_n$ are such that $D(x_i) = D(y_i)/y_i$ for $i = 1, \ldots, n$, and if $x_1, \ldots, x_n$ are $\mathbb{Q}$-linearly independent modulo $\ker(D)$, then*

$$\text{tr.deg.}_{\ker(D)}(x_1, \ldots, x_n, y_1, \ldots, y_n) \geq n + 1.$$

In our case, it suffices to take $(\mathbf{No}, \partial)$ as differential field and $y_i = \exp(x_i)$ to deduce the following corollary.

Corollary 6.34. *If $x_1, \ldots, x_n \in \mathbf{No}$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{R}$, then*

$$\text{tr.deg.}_{\mathbb{R}}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n + 1.$$

We remark that this is just a special case of a much more general statement regarding all models of the theory of $\mathbb{R}_{\exp}$. We recall the general version for completeness.

Theorem 6.35 ([JW08], [Kir10])**.** *Let R_E be a model of the theory of $\mathbb{R}_{\exp}$. If $x_1, \ldots, x_n \in R$ are $\mathbb{Q}$-linearly independent modulo $\mathrm{dcl}(\emptyset)$, then*

$$\mathrm{tr.deg.}_{\mathrm{dcl}(\emptyset)}(x_1, \ldots, x_n, E(x_1), \ldots, E(x_n)) \geq n + k$$

where k is the exponential transcendence degree of $x_1, \ldots, x_n$ over $\mathrm{dcl}(\emptyset)$.

The above statement can be proved by noting that the definable closure operator coincides with the exponential-algebraic closure [JW08, Thm. 4.2] and that the above Schanuel type statement holds modulo the exponential-algebraic closure of the empty set [Kir10, Thm. 1.2].

7. Integration

We can easily prove that the derivation ∂ of Definition 6.21 is surjective, or in other words, every surreal number has an integral. Our proof is based on a theorem of Rosenlicht that links the existence of integrals to the values of the logarithmic derivative [Ros83].

We quote here the relevant theorem. Let K be a Hardy field. If $f \in K$, we denote by f' its derivative, and we let v be the Archimedean valuation on K. Recall that $f \sim g$ means $v(f-g) > v(g)$.

Fact 7.1 ([Ros83, Thm. 1])**.** *Let K be a Hardy field and consider the set of valuations $\Psi := \{v(f'/f) : f \in K,\ v(f) \neq 0\}$. If $f \in K^*$ is such that $v(f) \neq \sup \Psi$, then there exists $u_0 \in K^*$ with $v(u_0) \neq 0$ such that whenever $u \in K^*$ and $|v(u_0)| \geq |v(u)| > 0$ we have*

$$\left(f \cdot \frac{fu/u'}{(fu/u')'}\right)' \sim f.$$

The result of Rosenlicht shows that every $f \in K^*$ with $f \neq \sup \Psi$ has an asymptotic integral, i.e., a function g whose derivative g' is asymptotic to f. In particular, if $\sup \Psi$ does not exist, then *every* $f \in K^*$ has an asymptotic integral. The proof is purely algebraic and holds more generally in the context of H-fields, and in particular it holds for the surreal numbers **No** equipped with our derivation $\partial : \mathbf{No} \to \mathbf{No}$ and the valuation $-\ell$. To be able to apply Rosenlicht's result, the first step is to check whether $\{\ell(\partial(x)/x) : x \in \mathbf{No},\ \ell(x) \neq 0\}$ has an infimum.

Proposition 7.2. *The class $\Psi_{\mathbb{L}} := \{\ell(\partial(\lambda)/\lambda) : \lambda \in \mathbb{L}\}$ has no infimum in $\mathbb{J}$.*

Proof. Since $\partial(\lambda)/\lambda = \partial(\log(\lambda))$ and $\mathbb{L} = \log(\mathbb{L})$, we have that $\Psi_{\mathbb{L}} = \{\ell(\partial(\lambda)) : \lambda \in \mathbb{L}\}$. Moreover, note that the sequence $y(\alpha) := \ell(\partial(\kappa_{-\alpha}))$ is co-initial in $\Psi_{\mathbb{L}}$ by Proposition 5.26 and Remark 6.3, so it suffices to prove that the class $\{y(\alpha) : \alpha \in \mathbf{On}\}$ has no infimum in $\mathbb{J}$. Recall that we have $y(\alpha) = \log(\partial(\kappa_{-\alpha})) = -\sum_{\beta<\alpha} \sum_{i=1}^{\infty} \log_i(\kappa_{-\beta})$ and observe that if $\beta < \alpha$, then $y(\beta) \lhd y(\alpha)$ and $y(\beta) > y(\alpha)$.

Let $x < y(\alpha)$ for all $\alpha \in \mathbf{On}$, with $x \in \mathbb{J}$. We must show that x is not an infimum of $\{y(\alpha) : \alpha \in \mathbf{On}\}$ in $\mathbb{J}$. Since the supports $\mathrm{S}(y(\alpha))$ are increasing in α, their intersection with $\mathrm{S}(x)$ must stabilize, namely there are $A \subseteq \mathrm{S}(x)$ and $\gamma \in \mathbf{On}$ such that $\mathrm{S}(y(\alpha)) \cap \mathrm{S}(x) = A$ for all $\alpha \geq \gamma$. Let $\mathfrak{m}$ be the maximal monomial such that $x_{\mathfrak{m}} \neq y(\gamma)_{\mathfrak{m}}$. For all $\alpha \geq \gamma$, by construction of γ, and since $y(\gamma) \unlhd y(\alpha)$, the same

$\mathfrak{m}$ is also the maximal monomial such that $x_{\mathfrak{m}} \neq y(\alpha)_{\mathfrak{m}}$, and $y(\alpha)_{\mathfrak{m}} = y(\gamma)_{\mathfrak{m}}$. Since $x < y(\gamma)$ we must have $x_{\mathfrak{m}} < y(\gamma)_{\mathfrak{m}}$. Now take any $x' \in \mathbb{J}$ such that $x'|\mathfrak{m} = x|\mathfrak{m}$ and $x_{\mathfrak{m}} < x'_{\mathfrak{m}} < y(\gamma)_{\mathfrak{m}}$. Then $x < x' < y(\alpha)$ for all $\alpha \geq \gamma$, and therefore for all α. This means that x is not an infimum of $\{y(\alpha) : \alpha \in \mathbf{On}\}$ in $\mathbb{J}$, as desired. □

In fact, the same proof also shows that $\Psi_{\mathbb{L}}$ has no infimum even in $\mathbf{No}$.

Corollary 7.3. *The class* $\Psi := \{\ell(\partial(x)/x) : x \in \mathbf{No},\ \ell(x) \neq 0\}$ *has no infimum in* $\mathbb{J}$.

Proof. We have $\partial(x)/x = \partial(\log |x|)$. Moreover, $\ell(x) \neq 0$ if and only if $\log |x| > \mathbb{N}$. Since $\log(\mathbf{No}^{>0}) = \mathbf{No}$, we have that $\Psi = \{\ell(\partial(x)) : x > \mathbb{N}\}$. Since $\mathbb{L}$ is co-initial with all the infinite positive elements of $\mathbf{No}$ (see Proposition 5.26) then $\Psi_{\mathbb{L}} = \{\ell(\partial(\lambda)) : \lambda \in \mathbb{L}\}$ is co-initial with Ψ by Remark 6.3. But $\Psi_{\mathbb{L}}$ has no infimum in $\mathbb{J}$ by Proposition 7.2, so Ψ does not have it either. □

We can now apply [Ros83, Thm. 1] to show that every surreal number has an asymptotic integral, namely, for every $x \in \mathbf{No}^*$ there is a $y \in \mathbf{No}^*$ such that $x \sim \partial(y)$. For later convenience, we construct an asymptotic integral y belonging to $\mathbb{R}^*\mathfrak{M}^{\neq 1}$.

Proposition 7.4. *There is a class function* $A : \mathbf{No}^* \to \mathbb{R}^*\mathfrak{M}^{\neq 1}$ *such that* $x \sim \partial(A(x))$ *for all* $x \in \mathbf{No}^*$.

Proof. We define the function $A : \mathbf{No}^* \to \mathbb{R}^*\mathfrak{M}^{\neq 1}$ as follows. Let $x \in \mathbf{No}^*$. By Corollary 7.3, $\ell(x)$ is not an infimum for Ψ. Therefore, by [Ros83, Thm. 1] applied to $(\mathbf{No}, \partial)$, there is $u_0 \in \mathbf{No}$ with $\ell(u_0) \neq 0$ such that for any $u \in \mathbf{No}$ with $0 < |\ell(u)| \leq |\ell(u_0)|$ we have

$$x \sim \partial\left(x \cdot \frac{(xu/\partial(u))}{\partial(xu/\partial(u))}\right).$$

This gives us an asymptotic integral $y := x \cdot \frac{(xu/\partial(u))}{\partial(xu/\partial(u))}$ of x which in fact depends on the choice of u. For the sake of definiteness, we choose $u := \kappa_{-\alpha}$ with α minimal (such an α always exists, since the elements of $\kappa_{\mathbf{No}}$ are co-initial in the positive infinite numbers by Proposition 5.26).

We make a minor adjustment to obtain an asymptotic integral belonging to $\mathbb{R}^*\mathfrak{M}^{\neq 1}$. Let $r \in \mathbb{R}$ be the coefficient of the monomial 1 in y, so that $y - r \not\asymp 1$, and define $A(x)$ as the leading term of $(y - r)$. Clearly $A(x) \sim y - r$, hence $\partial(A(x)) \sim \partial(y - r) = \partial(y) \sim x$, while $A(x) \in \mathbb{R}^*\mathfrak{M}^{\neq 1}$, as desired. □

Using the above observation, one could try to use [Kuh11, Thm. 47] to obtain actual integrals; however, one should adapt the notion of "spherically complete" to the *class* $\mathbf{No}$ and verify that the proof goes through. This argument is rather delicate, as the field $\mathbf{No}$, with the valuation $-\ell$, may not be spherically complete if seen from a more powerful model of set theory, for instance when using an inaccessible cardinal.

For the sake of completeness, we give a different self-contained argument. In order to find a solution to the differential equation $\partial(y) = x$ for a given x, we simply iterate the above procedure for finding an asymptotic integral, and we verify that the procedure converges using a specialized version of Fodor's lemma.

Lemma 7.5 (Specialized Fodor's lemma)**.** *Let $f : \mathbf{On} \setminus \{\emptyset\} \to \mathbf{On}$ be a class function such that $f(\alpha) < \alpha$ for all $\alpha \in \mathbf{On} \setminus \{\emptyset\}$. Then there exists $\beta \in \mathbf{On}$ such that $f^{-1}(\beta)$ is a proper class.*

Proof. Suppose by contradiction that for each $\beta \in \mathbf{On}$ the class $f^{-1}(\beta)$ is a set. We define the following class function $g : \mathbf{On} \to \mathbf{On}$ by induction: given $\alpha \in \mathbf{On}$, we let $g(\alpha)$ be the minimum ordinal strictly greater than all the elements of $f^{-1}(\beta) \cup \{g(\beta)\}$ for $\beta < \alpha$. This is a strictly increasing continuous function $g : \mathbf{On} \to \mathbf{On}$. As is well known, the ordinal $\alpha_0 := \sup_{n<\omega} g^{(n)}(0)$ satisfies $g(\alpha_0) = \alpha_0$. By definition, we have $\alpha_0 = g(\alpha_0) > f^{-1}(\beta)$ for all $\beta < \alpha_0$, and in particular $f(\alpha_0) \neq \beta$ for all $\beta < \alpha_0$. Therefore, $f(\alpha_0) \geq \alpha_0$, contradicting the hypothesis. □

Proposition 7.6. *The surreal derivation $\partial : \mathbf{No} \to \mathbf{No}$ is surjective.*

Proof. Clearly, $\partial(0) = 0$, so 0 is in the image of ∂.

Now take a surreal number $x \in \mathbf{No}^*$. We define inductively a sequence of terms $t_\alpha \in \mathbb{R}^*\mathfrak{M}^{\neq 1}$ as follows. We start with $t_0 := A(x)$. If t_β has been defined for every $\beta < \alpha$, and $x \neq \sum_{\beta<\alpha} \partial(t_\beta)$, we define

$$t_\alpha := A\left(x - \sum_{\beta<\alpha} \partial(t_\beta)\right),$$

otherwise we stop. We claim that $\ell(t_\beta)$ is strictly decreasing for all $\beta < \alpha$, so that $\sum_{\beta<\alpha} t_\beta$ is a surreal number and $\sum_{\beta<\alpha} \partial(t_\beta)$ is its derivative, ensuring that t_α is well defined. In fact, we may assume by induction that $\ell(t_\beta)$ is strictly decreasing and we only need to check that $\ell(t_\alpha) < \ell(t_\beta)$, i.e. $t_\alpha \prec t_\beta$, for all $\beta < \alpha$.

Note that by construction, $t_\beta \not\asymp 1$ for all $\beta < \alpha$, and therefore, by Proposition 6.4, we have that $\ell(\partial(t_\beta))$ is strictly decreasing for $\beta < \alpha$. Now fix $\gamma < \alpha$. By definition of asymptotic integral,

$$\partial(t_\alpha) \sim x - \sum_{\beta<\alpha} \partial(t_\beta) \preceq \max\left\{\left|x - \sum_{\beta\leq\gamma} \partial(t_\beta)\right|, \left|\sum_{\gamma<\beta<\alpha} \partial(t_\beta)\right|\right\}.$$

Note that this is true even if the last sum is empty, namely when $\alpha = \gamma + 1$. Since $\partial(t_\beta) \prec \partial(t_\gamma)$ for all $\gamma < \beta < \alpha$, then $\sum_{\gamma<\beta<\alpha} \partial(t_\beta) \prec \partial(t_\gamma)$. Moreover, again by definition of asymptotic integral, $x - \sum_{\beta\leq\gamma} \partial(t_\beta) = \left(x - \sum_{\beta<\gamma} \partial(t_\beta)\right) - \partial(t_\gamma) \prec \partial(t_\gamma)$. Therefore, $\partial(t_\alpha) \prec \partial(t_\gamma)$, and by Proposition 6.4 we get $t_\alpha \prec t_\gamma$, as desired.

We now claim that there is an α such that $x = \sum_{\beta<\alpha} \partial(t_\beta)$. Suppose by contradiction that $x \neq \sum_{\beta<\alpha} \partial(t_\beta)$ for all $\alpha \in \mathbf{On}$. Let $\mathfrak{m}_\alpha$ be the leading monomial of $x - \sum_{\beta<\alpha} \partial(t_\beta)$. Recall that by construction $\mathfrak{m}_\alpha \asymp x - \sum_{\beta<\alpha} \partial(t_\beta) \sim \partial(t_\alpha)$; since $\ell(\partial(t_\alpha))$ is strictly decreasing, the sequence $\mathfrak{m}_\alpha$ is strictly decreasing as well, and in particular injective. Let $f : \mathbf{On} \to \mathbf{On}$ be the class function that sends α to the minimum $\beta \in \mathbf{On}$ such that $\mathfrak{m}_\alpha \in \mathrm{S}(\partial(t_\beta)) \cup \mathrm{S}(x)$; clearly, such a β always exists and it must be strictly less than α.

Since $f(\alpha) < \alpha$ for all $\alpha \in \mathbf{On}$, by Lemma 7.5 there exists a $\beta \in \mathbf{On}$ such that $f^{-1}(\beta)$ is a proper class. However, by definition of f the class $\{\mathfrak{m}_\alpha : \alpha \in f^{-1}(\beta)\}$ is actually a subset of $\mathrm{S}(\partial(t_\beta)) \cup \mathrm{S}(x)$. Since the map $\alpha \mapsto \mathfrak{m}_\alpha$ is injective, this implies that $\mathrm{S}(\partial(t_\beta)) \cup \mathrm{S}(x)$ contains a proper class, a contradiction.

Therefore, for some α we have $x = \sum_{\beta<\alpha} \partial(t_\beta) = \partial\left(\sum_{\beta<\alpha} t_\beta\right)$, as desired. □

Theorem 7.7. *The differential field* $(\mathbf{No}, \partial)$ *is a Liouville closed* H*-field with small derivation in the sense of* [AvdD02, p. 3].

Proof. By Proposition 7.6, the function ∂ is surjective. In particular, the differential equations $\partial(x) = y$ and $\partial(x)/x = \partial(\log|x|) = y$ always have solution in $\mathbf{No}$, and therefore $(\mathbf{No}, \partial)$ is Liouville-closed. Moreover, since $\partial(\omega) = 1$, we have that if $x \prec 1$ then $\partial(x) \prec \partial(\omega) = 1$, which means by definition that the derivation is small, as desired. □

Remark 7.8. Note that the conclusion of Corollary 7.3 applies to $\mathbb{R}\langle\mathbb{L}\rangle$ as well. Since the remaining construction is done using just field operations and infinite sums, we can easily verify that $\mathbb{R}\langle\mathbb{L}\rangle$, equipped with the derivation $\partial_{\restriction \mathbb{R}\langle\mathbb{L}\rangle}$ (see Remark 6.31), is Liouville-closed as well.

8. Transseries

As anticipated in Section 4, the fact that the nested truncation $\trianglelefteq$ is well-founded is related in an essential way to the structure of $\mathbf{No}$ as a field of transseries. We discuss here in which sense $\mathbf{No}$ can be seen as a field of transseries and compare the result to a previous conjecture.

8.1. Axiom ELT4 of [KM15].

We mentioned in the introduction that a rather natural object to consider is the smallest subfield of $\mathbf{No}$ containing $\mathbb{L}$ and closed under some natural operations.

Definition 8.1. We call $\mathbb{R}\langle\mathbb{L}\rangle$ the smallest subfield of $\mathbf{No}$ containing $\mathbb{R}(\mathbb{L})$ and closed under infinite sums, exponentiation and logarithm.

A natural question is whether $\mathbf{No} = \mathbb{R}\langle\mathbb{L}\rangle$; we can see that this is equivalent to the first part of Conjecture 5.2 in [KM15]. However, we can verify that the inclusion is strict. To prove this, we characterize $\mathbb{R}\langle\mathbb{L}\rangle$ in terms of paths.

Proposition 8.2. *For all* $x \in \mathbf{No}$*,* $x \in \mathbb{R}\langle\mathbb{L}\rangle$ *if and only if for every path* $P \in \mathcal{P}(x)$ *there exists* i *such that* $P(i) \in \mathbb{L}$*.*

Proof. Let $\mathbb{F}$ be the class of all $x \in \mathbf{No}$ such that for every $P \in \mathcal{P}(x)$ there exists i such that $P(i) \in \mathbb{L}$. If $x \notin \mathbb{R}\langle\mathbb{L}\rangle$ then clearly there is some term $r\exp(\gamma)$ in x with $\gamma \notin \mathbb{R}\langle\mathbb{L}\rangle$. Iterating this procedure we produce an infinite path $P \in \mathcal{P}(x)$ with $P(0) = r\exp(\gamma)$ and $P(i) \notin \mathbb{R}\langle\mathbb{L}\rangle$ for every $i \in \mathbb{N}$. In particular, $P(i) \notin \mathbb{L}$ for all i, hence $x \notin \mathbb{F}$. Since x was arbitrary, we have proved $\mathbb{F} \subseteq \mathbb{R}\langle\mathbb{L}\rangle$.

For the other inclusion, it is enough to observe that $\mathbb{F}$ is a field containing $\mathbb{R} \cup \mathbb{L}$ and closed under infinite sums, exp and log. The verification is easy once we recall that when x is finite $\exp(x)$ and $\log(1 + x)$ are given by power series expansions. The details are as follows:

1. $\mathbb{F}$ is clearly closed under infinite sums and contains $\mathbb{R} \cup \mathbb{L}$;
2. for $\gamma \in \mathbb{J}$, we have $\gamma \in \mathbb{F}$ if and only if $r\exp(\pm\gamma) \in \mathbb{F}$ for all $r \in \mathbb{R}^*$;
3. using (2), if $t, u \in \mathbb{R}^*\mathfrak{M} \cap \mathbb{F}$, then $t \cdot u \in \mathbb{R}^*\mathfrak{M} \cap \mathbb{F}$ and $t^{-1} \in \mathbb{R}^*\mathfrak{M} \cap \mathbb{F}$;
4. by infinite distributivity, if $x, y \in \mathbb{F}$, then $x \cdot y \in \mathbb{F}$;
5. expanding the definitions of exp and log (see Theorem 3.8 and Remark 3.4) and using the above (1)-(4), if $x \in \mathbb{F}$ then $\exp(x)$ and $\log(x)$ are in $\mathbb{F}$.

Therefore, $\mathbb{R}\langle\mathbb{L}\rangle \subseteq \mathbb{F}$, hence $\mathbb{F} = \mathbb{R}\langle\mathbb{L}\rangle$, as desired. □

The above proposition shows that $\mathbb{R}\langle\mathbb{L}\rangle$ is a "field of exponential-logarithmic transseries" in the sense of [KM15, Def. 5.1]. We omit here the full definition of exponential-logarithmic transseries and just recall their main defining property.

Definition 8.3. Let $\mathbb{F}$ be a subfield of **No**. Following [MR93] we say that $\mathbb{F}$ is **truncation closed** if for every $f \in \mathbb{F}$ and $\mathfrak{m} \in \mathfrak{M}$ we have $f|\mathfrak{m} \in \mathbb{F}$.

The following definition is a slight variation of [KM15, Def. 5.1].

Definition 8.4 ([KM15, Def. 5.1])**.** A truncation closed subfield $\mathbb{F}$ of **No** closed under log satisfies ELT4 if and only if the following holds:

ELT4. For all sequences of monomials $\mathfrak{m}_i \in \mathfrak{M} \cap \mathbb{F}$, with $i \in \mathbb{N}$, such that

$$\mathfrak{m}_i = \exp(\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1})$$

where $r_{i+1} \in \mathbb{R}^*$, $\gamma_{i+1}, \delta_{i+1} \in \mathbb{J}$, and $\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1}$ is in standard form, there is $k \in \mathbb{N}$ such that $r_{i+1} = 1$ and $\gamma_{i+1} = \delta_{i+1} = 0$ for all $i \geq k$.

Remark 8.5. Note that ELT4 implies that the sequence $(\mathfrak{m}_i)$ eventually satisfies $\mathfrak{m}_i \in \mathbb{L}$. We can rephrase this in term of paths: a truncation closed subfield $\mathbb{F}$ of **No** closed under log satisfies ELT4 if and only if for every $x \in \mathbb{F}$ and every path $P \in \mathcal{P}(x)$ there exists k such that $P(k+1) \in \mathbb{L}$.

Proposition 8.6. *$\mathbb{R}\langle\mathbb{L}\rangle$ is the largest truncation closed subfield of* **No** *closed under* log *and satisfying ELT4.*

Proof. By Remark 8.5 and Proposition 8.2, $\mathbb{R}\langle\mathbb{L}\rangle$ satisfies ELT4 and every other truncation closed subfield $\mathbb{F}$ of **No** closed under log and satisfying ELT4 is included in $\mathbb{R}\langle\mathbb{L}\rangle$. □

In [KM15, Conj. 5.2] it was conjectured that **No** satisfies ELT4, which is equivalent to saying that $\mathbb{R}\langle\mathbb{L}\rangle = \mathbf{No}$. However, this is not the case.

Theorem 8.7. *We have $\mathbb{R}\langle\mathbb{L}\rangle \subsetneq \mathbf{No}$.*

Proof. Let $(\mathfrak{m}_i)$ be a sequence of monomials in $\mathfrak{M}^{>1}$ such that $\mathfrak{m}_{i+1} \prec \log(\mathfrak{m}_i)$.

For $i \in \mathbb{N}$, let C_i be the non-empty convex class defined by

$$C_i := \exp(\mathfrak{m}_1 + \exp(\mathfrak{m}_2 + \ldots + \exp(\mathfrak{m}_i + o(\mathfrak{m}_i))\ldots)).$$

Since $\mathfrak{m}_{i+1} \prec \log(\mathfrak{m}_i)$, we have $\ell(\exp(\mathfrak{m}_{i+1} + o(\mathfrak{m}_{i+1}))) < 2\mathfrak{m}_{i+1} \prec \log(\mathfrak{m}_i) \asymp \ell(\mathfrak{m}_i)$, and in particular $\exp(\mathfrak{m}_{i+1} + o(\mathfrak{m}_{i+1})) \subseteq o(\mathfrak{m}_i)$. Therefore, $C_{i+1} \subseteq C_i$. By the saturation properties of surreal numbers, the intersection $\bigcap_i C_i$ is non empty.

Let $x \in \bigcap_i C_i$. We can write, for every $i \in \mathbb{N}$,

$$x = x_0 = \exp(\mathfrak{m}_1 + \exp(\mathfrak{m}_2 + \ldots + \exp(\mathfrak{m}_i + x_i)\ldots))$$

where $x_i \prec \mathfrak{m}_i$ for $i > 0$. By construction we have $x_i = \exp(\mathfrak{m}_{i+1} + x_{i+1})$. Note, however, that this may not be the Ressayre representation of x_i, as x_{i+1} is not necessarily in $\mathbb{J}$.

Write $x_i = \gamma_i + r_i + \varepsilon_i$, with $\gamma_i \in \mathbb{J}$, $r_i \in \mathbb{R}$, $\varepsilon_i \in o(1)$. By the assumption $x_i \prec \mathfrak{m}_i$ we get $\gamma_i \prec \mathfrak{m}_i$. Moreover, since $x_i > \exp(\frac{1}{2}\mathfrak{m}_{i+1}) \succ 1$, we have $\gamma_i \neq 0$.

Now define $P(i)$ as the leading term of γ_i for $i \in \mathbb{N}$. We claim that P in a path in $\mathcal{P}(x)$. Recall that if $y = \gamma + r + \varepsilon$, with $\gamma \in \mathbb{J}$, $r \in \mathbb{R}$, $\varepsilon \in o(1)$, then $\ell(\exp(y)) = \gamma$. It follows that

$$\ell(P(i)) = \ell(\gamma_i) = \ell(x_i) = \ell(\exp(\mathfrak{m}_{i+1} + \gamma_{i+1} + r_{i+1} + \varepsilon_{i+1})) = \mathfrak{m}_{i+1} + \gamma_{i+1},$$

Since $P(i+1)$ is a term of γ_{i+1}, it is also a term of $\ell(P(i))$, hence P is a path, and clearly $P \in \mathcal{P}(x)$.

Since $\ell(P(i)) = \ell(\gamma_i) = \mathfrak{m}_{i+1} + \gamma_{i+1}$, where $\gamma_{i+1} \neq 0$, we have $P(i) \notin \mathfrak{M} \supset \mathbb{L}$ for all i. By Proposition 8.2 we have that $x \notin \mathbb{R}\langle\mathbb{L}\rangle$, and therefore $\mathbb{R}\langle\mathbb{L}\rangle \subsetneq \mathbf{No}$. □

Despite the fact that some paths may not end in $\mathbb{L}$, recall that if $x \in \mathbf{No} \setminus \mathbb{R}$ and P is its dominant path, then there exists i such that $P(i) \in \mathbb{L}$ (see Corollary 5.11 or Lemma 6.23).

8.2. **Axiom T4 of [Sch01].** We have seen that axiom ELT4 fails in **No**. However, as we prove in this section, **No** satisfies a weaker axiom called "T4" in [Sch01, Def. 2.2.1]. In fact, we shall see that T4 is essentially equivalent to the fact that the relation $\trianglelefteq$ of nested truncation is well-founded. This will show that **No** is a field of transseries as axiomatized by Schmeling.

We recall the definition of transseries in [Sch01]. One starts with an ordered field C equipped with an increasing homomorphism $\exp : C \to C$, with $\exp(x) \geq 1+x$ for all $x \in C$ and $\mathrm{Im}(\exp) = C^{>0}$. Given an ordered group Γ and a partial increasing homomorphism $\exp : C((\Gamma)) \to C((\Gamma))$ extending the one on C, we say that $C((\Gamma)))$ is a field of transseries if

- **T1.** $\mathrm{Im}(\exp) = C((\Gamma))^{>0}$;
- **T2.** $\Gamma \subseteq \exp(C((\Gamma^{>0})))$;
- **T3.** $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ for all $x \in C((\Gamma^{<0}))$;
- **T4.** for all sequences of monomials $\mathfrak{m}_i \in \Gamma$, with $i \in \mathbb{N}$, such that

$$\mathfrak{m}_i = \exp(\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1})$$

where $r_{i+1} \in C^*$, $\gamma_{i+1}, \delta_{i+1} \in C((\Gamma^{>0}))$, and $\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1}$ is in standard form, there is $k \in \mathbb{N}$ such that $r_{i+1} = \pm 1$ and $\delta_{i+1} = 0$ for $i \geq k$.

If we take $C = \mathbb{R}$ equipped with the classical function exp, and $\Gamma = \mathfrak{M} = \exp(\mathbb{J})$, then $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$ is clearly a model of T1-T3. Axiom T4 is related to the nested truncation $\trianglelefteq$: it is not difficult to see that assuming T4 one can easily deduce that $\trianglelefteq$ is well-founded. We shall now verify that since $\trianglelefteq$ is well-founded (Theorem 4.24), axiom T4 holds on **No**, thereby proving that **No** is a field of transseries.

Definition 8.8. Consider a path P and write

$$P(i) = r_i \exp(\gamma_{i+1} + P(i+1) + \delta_{i+1})$$

where $0 \neq r_i \in \mathbb{R}$, $\gamma_{i+1}, \delta_{i+1} \in \mathbb{J}$, and $\gamma_{i+1} + P(i+1) + \delta_{i+1}$ is in standard form.

We say that P **satisfies T4** if there exists $k \in \mathbb{N}$ such that $r_{i+1} = \pm 1$ and $\delta_{i+1} = 0$ for all $i \geq k$, otherwise we say that P **refutes T4**.

We say that $x \in$ **No satisfies T4** if all paths in $\mathcal{P}(x)$ satisfy T4, otherwise we say that x **refutes T4**.

Clearly, T4 is equivalent to saying that every $x \in \mathbf{No}$ satisfies T4. If we translate Proposition 4.26 in terms of paths, we get the following.

Lemma 8.9. *Let $x \in \mathbf{No}$ and $P \in \mathcal{P}(x)$. If $\mathrm{NR}(P(i)) = \mathrm{NR}(x)$ for all $i \in \mathbb{N}$, then P satisfies T4.*

Proof. It follows at once from Lemma 6.18. □

We can now prove that T4 holds on **No**.

Theorem 8.10. *Axiom T4 of* [Sch01, Def. 2.2.1] *holds in* **No** *(with* $C = \mathbb{R}$ *and* $\Gamma = \mathfrak{M}$*), hence* **No** *is a field of transseries in the sense of that paper.*

Proof. We prove that all $x \in \mathbf{No}$ satisfy T4. Let $x \in \mathbf{No}$, and assume by induction that y satisfies T4 for all $y \in \mathbf{No}$ with $\mathrm{NR}(y) < \alpha := \mathrm{NR}(x)$.

Let $P \in \mathcal{P}(x)$ be any path. If $\mathrm{NR}(P(j)) < \alpha$ for some $j \in \mathbb{N}$, then by inductive hypothesis the path $i \mapsto P(j+i)$ in $\mathcal{P}(P(j))$ satisfies T4, and in particular P itself satisfies T4. On the other hand, if $\mathrm{NR}(P(j)) = \alpha$ for all $j \in \mathbb{N}$, then P satisfies T4 by Lemma 8.9. Since P was an arbitrary path, x satisfies T4, as desired. □

9. Pre-derivations

The purpose of this section is to show that $\partial_{\mathbb{L}}$ (Definition 6.7) is the simplest function (in the sense of 9.6) with positive values satisfying the inequalities of Proposition 6.5. As anticipated in the introduction, we call such functions "pre-derivations".

Definition 9.1. A **pre-derivation** is a map $D_{\mathbb{L}} : \mathbb{L} \to \mathbb{R}^{>0}\mathfrak{M}$ such that

$$\log\left(D_{\mathbb{L}}(\lambda)\right) - \log\left(D_{\mathbb{L}}(\mu)\right) \prec \max\{\lambda, \mu\}$$

and $D_{\mathbb{L}}(\exp(\lambda)) = \exp(\lambda) D_{\mathbb{L}}(\lambda)$ for all $\lambda, \mu \in \mathbb{L}$.

By Theorem 6.32, any pre-derivation can be extended to a surreal derivation. We can verify that $\partial_{\mathbb{L}}$ has an inductive definition that involves a variant of the inequalities of Definition 9.1, and as a corollary, that $\partial_{\mathbb{L}}$ is the simplest pre-derivation. We first note that pre-derivations must satisfy the following condition.

Proposition 9.2. *If* $D_{\mathbb{L}}$ *is a pre-derivation, then*

$$\log\left(\frac{D_{\mathbb{L}}(\lambda)}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) - \log\left(\frac{D_{\mathbb{L}}(\mu)}{\prod_{i=0}^{m-1}\log_i(\mu)}\right) \prec \max\{\log_l(\lambda), \log_m(\mu)\}$$

for all $\lambda, \mu \in \mathbb{L}$ *and* $l, m \in \mathbb{N}$.

Proof. The conclusion follows trivially from

$$\log(D_{\mathbb{L}}(\log_l(\lambda))) - \log(D_{\mathbb{L}}(\log_m(\mu))) \prec \max\{\log_l(\lambda), \log_m(\mu)\}$$

since $\frac{D_{\mathbb{L}}(\lambda)}{\prod_{i=0}^{l-1}\log_i(\lambda)} = D_{\mathbb{L}}(\log_l(\lambda))$ and $\frac{D_{\mathbb{L}}(\mu)}{\prod_{i=0}^{m-1}\log_i(\mu)} = D_{\mathbb{L}}(\log_m(\mu))$. □

We now use the above inequalities to give an inductive definition for $\partial_{\mathbb{L}}$.

Lemma 9.3. *Let* $x \in \mathbf{No}$ *be such that* $x > \mathbb{N}$*. If* $\alpha \in \mathbf{On}$ *is the minimum ordinal such that* $\kappa_{-\alpha} \preceq^K x$*, then* $\kappa_{-\alpha} \leq_s x$*.*

Proof. Let $z \in \mathbf{No}$ be the unique number such that $x \asymp^K \kappa_{-z}$. It follows that $\kappa_{-z} \leq_s x$. Now note that $\alpha \in \mathbf{On}$ is necessarily the minimum ordinal such that $-\alpha \leq -z$. Since the representation $-\alpha = \emptyset \mid \{-\beta : \beta < \alpha\}$ is simple, and $-z < -\beta$ for all $\beta < \alpha$, we have $-\alpha \leq_s -z$. It follows that $\kappa_{-\alpha} \leq_s \kappa_{-z} \leq_s x$, as desired. □

Lemma 9.4. *For all* $\lambda \in \mathbb{L}$, $\partial_{\mathbb{L}}(\lambda)$ *is the simplest number* $x \in \mathbf{No}^{>0}$ *such that*

$$\log\left(\frac{x}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) - \log\left(\frac{\partial_{\mathbb{L}}(\mu)}{\prod_{i=0}^{m-1}\log_i(\mu)}\right) \prec \max\{\log_l(\lambda), \log_m(\mu)\}$$

for all $\mu <_s \lambda$ *and* $l, m \in \mathbb{N}$.

Proof. Let $\lambda \in \mathbb{L}$ and let $x \in \mathbf{No}^{>0}$ be a number satisfying the above inequalities. We need to prove that $\partial_{\mathbb{L}}(\lambda) \leq_s x$. If $\lambda = \omega$, then $\partial_{\mathbb{L}}(\lambda) = 1$, and we already know that $1 \leq_s x$ since $x > 0$. For arbitrary λ, we claim that $\log(\partial_{\mathbb{L}}(\lambda)) \trianglelefteq \log(x)$. When $\lambda \neq \omega$, this clearly implies that $\partial_{\mathbb{L}}(\lambda) \blacktriangleleft x$, hence $\partial_{\mathbb{L}}(\lambda) \leq_s x$ by Theorem 4.24.

Since $\partial_{\mathbb{L}}$ is a pre-derivation, we have

$$\log\left(\frac{\partial_{\mathbb{L}}(\lambda)}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) - \log\left(\frac{\partial_{\mathbb{L}}(\mu)}{\prod_{i=0}^{m-1}\log_i(\mu)}\right) \prec \max\{\log_l(\lambda), \log_m(\mu)\}$$

for all $l, m \in \mathbb{N}$ and $\mu \in \mathbb{L}$. It follows that

$$\log\left(\frac{x}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) - \log\left(\frac{\partial_{\mathbb{L}}(\lambda)}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) \prec \max\{\log_l(\lambda), \log_m(\mu)\}$$

for all $l, m \in \mathbb{N}$ and $\mu <_s \lambda$. Expanding the two logarithms, we get

$$\log(x) - \log(\partial_{\mathbb{L}}(\lambda)) \prec \max\{\log_l(\lambda), \log_m(\mu)\} \tag{9.1}$$

for all $l, m \in \mathbb{N}$ and $\mu <_s \lambda$.

In order to prove $\log(\partial_{\mathbb{L}}(\lambda)) \trianglelefteq \log(x)$, let $\mathfrak{m}$ be a monomial in the support of $\log(\partial_{\mathbb{L}}(\lambda))$. We need to prove that

$$\log(x) - \log(\partial_{\mathbb{L}}(\lambda)) \prec \mathfrak{m}.$$

Let α be the minimum ordinal such that $\kappa_{-\alpha} \preceq^K \lambda$. By Lemma 9.3, we have $\kappa_{-\alpha} \leq_s \lambda$. Note moreover that $\kappa_{-\alpha} \preceq^K \lambda \prec^K \kappa_{-\beta}$ for all $\beta < \alpha$. We distinguish two cases.

If $\lambda = \kappa_{-\alpha}$, then $\log(\partial_{\mathbb{L}}(\lambda)) = -\sum_{\beta<\alpha}\sum_{i=1}^{\infty}\log_i(\kappa_{-\beta})$. Therefore, $\mathfrak{m}$ is of the form $\log_i(\kappa_{-\beta})$ for some $\beta < \alpha$ and $i \in \mathbb{N}$. Note that $\kappa_{-\beta} <_s \kappa_{-\alpha} = \lambda$ and $\kappa_{-\beta} \succ^K \kappa_{-\alpha}$. It follows that $\log_l(\lambda) < \log_i(\kappa_{-\beta})$ for all $l \in \mathbb{N}$. Taking $\mu = \kappa_{-\beta}$ and $m = i$ in (9.1), we get $\log(x) - \log(\partial_{\mathbb{L}}(\lambda)) \prec \log_i(\kappa_{-\beta}) = \mathfrak{m}$, as desired.

If $\lambda \neq \kappa_{-\alpha}$, then $\log(\partial_{\mathbb{L}}(\lambda)) = -\sum_{\kappa_{-\beta}\succeq^K\lambda}\sum_{i=1}^{\infty}\log_i(\kappa_{-\beta}) + \sum_{i=1}^{\infty}\log_i(\lambda)$, and $\kappa_{-\alpha} <_s \lambda$. By the choice of α we also have $\kappa_{-\alpha} \preceq^K \lambda$, which means that for all $l \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\log_m(\kappa_{-\alpha}) < \log_l(\lambda)$. Since $\kappa_{-\alpha} <_s \lambda$, we can take $\mu = \kappa_{-\alpha}$ in (9.1) and deduce that for all $l \in \mathbb{N}$ we have

$$\log(x) - \log(\partial_{\mathbb{L}}(\lambda)) \prec \log_l(\lambda).$$

If $\mathfrak{m} = \log_l(\lambda)$ for some $l \in \mathbb{N}$, we are done. If $\mathfrak{m} = \log_i(\kappa_{-\beta})$ for some $i \in \mathbb{N}$ and some $\kappa_{-\beta} \succeq^K \lambda$, then there exists l such that $\log_l(\lambda) < \log_i(\kappa_{-\beta})$, and therefore

$$\log(x) - \log(\partial_{\mathbb{L}}(\lambda)) \prec \log_l(\lambda) \prec \log_i(\kappa_{-\beta}) = \mathfrak{m},$$

as desired. □

Remark 9.5. Lemma 9.4 shows that one can *define* inductively $\partial_{\mathbb{L}}(\lambda)$ as the simplest $x \in \mathbf{No}^{>0}$ satisfying the inequalities of the lemma. However, the fact that x is the *simplest* such number is not essential, and other choices of $x \in \mathbb{R}^*\mathfrak{M}$ satisfying the same inequalities are possible and lead to other surreal derivations.

Theorem 9.6. *Let $D_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}^{>0}$ be a pre-derivation. If $\lambda \in \mathbb{L}$ is a number of minimal simplicity such that $D_{\mathbb{L}}(\lambda) \neq \partial_{\mathbb{L}}(\lambda)$, then $\partial_{\mathbb{L}}(\lambda) <_s D_{\mathbb{L}}(\lambda)$.*

Proof. Let $\lambda \in \mathbb{L}$ is a number of minimal simplicity such that $D_{\mathbb{L}}(\lambda) \neq \partial_{\mathbb{L}}(\lambda)$. By assumption, $D_{\mathbb{L}}(\mu) = \partial_{\mathbb{L}}(\mu)$ for all $\mu <_s \lambda$. Since $D_{\mathbb{L}}$ is a pre-derivation, by Proposition 9.2 it follows that for all $\mu <_s \lambda$ and $l, m \in \mathbb{N}$ we have

$$\log\left(\frac{D_{\mathbb{L}}(\lambda)}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) - \log\left(\frac{\partial_{\mathbb{L}}(\mu)}{\prod_{i=0}^{m-1}\log_i(\mu)}\right) \prec \max\left\{\log_l(\lambda), \log_m(\mu)\right\}.$$

By Lemma 9.4, this implies that $\partial_{\mathbb{L}}(\lambda) <_s D_{\mathbb{L}}(\lambda)$, as desired. □

Remark 9.7. A similar argument can be applied to the function $\partial'_{\mathbb{L}}$ of Definition 6.6 to prove that for all $\lambda \in \mathbb{L}$, $\partial'_{\mathbb{L}}(\lambda)$ is the simplest *infinite* number $x \in \mathbf{No}^{>0}$ such that for all $\mu <_s \lambda$ and $l, m \in \mathbb{N}$ we have

$$\log\left(\frac{x}{\prod_{i=0}^{l-1}\log_i(\lambda)}\right) - \log\left(\frac{\partial'_{\mathbb{L}}(\mu)}{\prod_{i=0}^{m-1}\log_i(\mu)}\right) \prec \max\left\{\log_l(\lambda), \log_m(\mu)\right\}.$$

In particular, $\partial'_{\mathbb{L}}$ is the simplest pre-derivation with only infinite values.

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Università di Pisa, Dipartimento di Matematica, Largo Bruno Pontecorvo 5, 56127 Pisa, PI, Italy
E-mail address: berardu@dm.unipi.it

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, PI, Italy.
E-mail address: vincenzo.mantova@sns.it