HYPERNATURAL NUMBERS AS ULTRAFILTERS

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Abstract. In this paper we present a use of nonstandard methods in the theory of ultrafilters and in related applications to combinatorics of numbers.

1. Introduction.

Ultrafilters are really peculiar and multifaced mathematical objects, whose study turned out a fascinating and often elusive subject. Researchers may have diverse intuitions about ultrafilters, but they seem to agree on the subtlety of this concept; *e.g.*, read the following quotations: *"The space $\beta\omega$ is a monster having three heads"* (J. van Mill [41]); *"...the somewhat esoteric, but fascinating and very useful object $\beta\mathbb{N}$"* (V. Bergelson [5]).

The notion of ultrafilter can be formulated in diverse languages of mathematics: in set theory, ultrafilters are maximal families of sets that are closed under supersets and intersections; in measure theory, they are described as $\{0,1\}$-valued finitely additive measures defined on the family of all subsets of a given space; in algebra, they exactly correspond to maximal ideals in rings of functions $\mathbb{F}^I$ where I is a set and $\mathbb{F}$ is a field. Ultrafilters and the corresponding construction of ultraproduct are a common tool in mathematical logic, but they also have many applications in other fields of mathematics, most notably in topology (the notion of limit along an ultrafilter, the Stone-Čech compactification βX of a discrete space X, *etc.*), and in Banach spaces (the so-called *ultraproduct technique*).

In 1975, F. Galvin and S. Glazer found a beautiful ultrafilter proof of *Hindman's theorem*, namely the property that for every finite partition of the natural numbers $\mathbb{N} = C_1 \cup \ldots \cup C_r$, there exists an infinite set X and a piece C_i such that all sums of distinct elements from X belong to C_i. Since this time, ultrafilters on $\mathbb{N}$ have been successfully used also in combinatorial number theory and in Ramsey theory. The key fact is that the compact space $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$ can be equipped

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with a pseudo-sum operation, so that the resulting structure $(\beta\mathbb{N}, \oplus)$ is a compact topological left semigroup. Such a space satisfies really intriguing properties that have direct applications in the study of structural properties of sets of integers (See the monography [27], where the extensive research originated from that approach is surveyed.)

Nonstandard analysis and ultrafilters are intimately connected. In one direction, ultrapowers are the basic ingredient for the usual constructions of models of nonstandard analysis since W.A.J. Luxemburg's lecture notes [39] of 1962. Actually, by a classic result of H.J. Keisler, the models of nonstandard analysis are characterized up to isomorphisms as *limit ultrapowers*, a class of elementary submodels of ultrapowers which correspond to direct limits of ultrapowers (see [32] and [11, §4.4]).

In the other direction, the idea that elements of a nonstandard extension ${}^*X$ correspond to ultrafilters on X goes back to the golden years of nonstandard analysis, starting from the seminal paper [40] by W.A.J. Luxemburg appeared in 1969. This idea was then systematically pursued by C. Puritz in [43] and by G. Cherlin and J. Hirschfeld in [12]. In those papers, as well as in Puritz' follow-up [44], new results about the Rudin-Keisler ordering were proved by nonstandard methods, along with new characterizations of special ultrafilters, such as P-points and selective ultrafilters. (See also [42], where the study of similar properties as in Puritz' papers was continued.) In [7], A. Blass pushed that approach further and provided a comprehensive treatment of ultrafilter properties as reflected by the nonstandard numbers of the associated ultrapowers.

Several years later, J. Hirschfeld [28] showed that hypernatural numbers can also be used as a convenient tool to investigate certain Ramsey-like properties. In the last years, a new nonstandard technique based on the use of iterated hyper-extensions has been developed to study partition regularity of equations (see [18, 36]).

This paper aims at providing a self-contained introduction to a nonstandard theory of ultrafilters; several examples are also included to illustrate the use of such a theory in applications.

For gentle introductions to ultrafilters, see the papers [33, 22]; a comprehensive reference is the monography [14]. Recent surveys on applications of ultrafilters across mathematics can be found in the book [1]. As for nonstandard analysis, a short but rigorous presentation can be found in [11, §4.4]; organic expositions covering virtually all aspects of nonstandard methods are provided in the books [1, 35, 16]. We remark that here we adopt the so-called *external* approach, based on the existence of a *star-map* $*$ that associates an *hyper-extension* (or nonstandard extension) ${}^*A$ to each object A under study, and satisfies the *transfer principle*. This is to be confronted to the *internal* viewpoint as formalized by E. Nelson's *Internal Set Theory* IST or by K. Hrbáček's *Nonstandard Set Theories*. (See [31] for a thorough treatise of nonstandard set theories.)

Let us recall here the saturation property. A family $\mathcal{F}$ has the *finite intersection property* (FIP for short) if $A_1 \cap \ldots \cap A_n \neq \emptyset$ for any choice of finitely many elements $A_i \in \mathcal{F}$.

Definition 1.1. Let κ be an infinite cardinal. A model of nonstandard analysis is *κ-saturated* if it satisfies the property:

- Let $\mathcal{F}$ be a family of internal sets with cardinality $|\mathcal{F}| < \kappa$. If $\mathcal{F}$ has the FIP then $\bigcap_{A \in \mathcal{F}} A \neq \emptyset$.

When *κ-saturation* holds, then every infinite internal set A has a cardinality $|A| \geq \kappa$. Indeed, the family of internal sets $\{A \setminus \{a\} \mid a \in A\}$ has the FIP, and has the same cardinality as A. If by contradiction $|A| < \kappa$, then by *κ-saturation* we would obtain $\bigcap_{a \in A} A \setminus \{a\} \neq \emptyset$, which is absurd.

With the exceptions of Sections 3 and 4, throughout this paper we will work in a fixed $\mathfrak{c}^+$-saturated model of nonstandard analysis, where $\mathfrak{c}$ is the cardinality of the *continuum*. (We recall that κ^+ denotes the successor cardinal of κ. So, κ^+-saturation applies to families $|\mathcal{F}| \leq \kappa$.) In consequence, our hypernatural numbers will have cardinality $|{}^*\mathbb{N}| \geq \mathfrak{c}^+$.

2. The u-equivalence

There is a canonical way of associating an ultrafilter on $\mathbb{N}$ to each hypernatural number.

Definition 2.1. The *ultrafilter generated* by a hypernatural number $\alpha \in {}^*\mathbb{N}$ is the family

$$\mathfrak{U}_\alpha = \{X \subseteq \mathbb{N} \mid \alpha \in {}^*X\}.$$

It is easily verified that $\mathfrak{U}_\alpha$ actually satisfies the properties of ultrafilter. Notice that $\mathfrak{U}_\alpha$ is principal if and only if $\alpha \in \mathbb{N}$ is finite.

Definition 2.2. We say that $\alpha, \beta \in {}^*\mathbb{N}$ are *u-equivalent*, and write $\alpha \underset{u}{\sim} \beta$, if they generate the same ultrafilter, *i.e.* $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$. The equivalence classes $u(\alpha) = \{\beta \mid \beta \underset{u}{\sim} \alpha\}$ are called *u-monads*.

Notice that α and β are u-equivalent if and only if they cannot be separated by any hyper-extension, *i.e.* if $\alpha \in {}^*X \Leftrightarrow \beta \in {}^*X$ for every

$X \subseteq \mathbb{N}$. In consequence, the equivalence classes $u(\alpha)$ are characterized as follows:

$$u(\alpha) \;=\; \bigcap\{{}^*X \mid X \in \mathfrak{U}_\alpha\}.$$

(The notion of *filter monad* $\mu(\mathcal{F}) = \bigcap\{{}^*F \mid F \in \mathcal{F}\}$ of a filter $\mathcal{F}$ was first introduced by W.A.J. Luxemburg in [40].)

For every ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the family $\{{}^*X \mid X \in \mathcal{U}\}$ is a family of cardinality $\mathfrak{c}$ with the FIP and so, by $\mathfrak{c}^+$-*saturation*, there exist hypernatural numbers $\alpha \in {}^*\mathbb{N}$ such that $\mathfrak{U}_\alpha = \mathcal{U}$. (Actually, the $\mathfrak{c}^+$-*enlargement* property suffices: see Definition 4.1.) In consequence,

$$\beta\mathbb{N} \;=\; \{\mathfrak{U}_\alpha \mid \alpha \in \mathbb{N}\}.$$

Thus one can identify $\beta\mathbb{N}$ with the quotient set ${}^*\mathbb{N}/\underset{u}{\sim}$ of the u-monads.

Example 2.3. Let $f : \mathbb{N} \to \mathbb{R}$ be bounded. If $\alpha \underset{u}{\sim} \beta$ are u-equivalent then ${}^*f(\alpha) \approx {}^*f(\beta)$ are at infinitesimal distance.
To see this, for every real number $r \in \mathbb{R}$ consider the set

$$\Gamma(r) \;=\; \{n \in \mathbb{N} \mid f(n) < r\}.$$

Then, by the hypothesis, one has $\alpha \in {}^*\Gamma(r) \Leftrightarrow \beta \in {}^*\Gamma(r)$, *i.e.* ${}^*f(\alpha) < r \Leftrightarrow {}^*f(\beta) < r$. As this holds for all $r \in \mathbb{R}$, it follows that the bounded hyperreal numbers ${}^*f(\alpha) \approx {}^*f(\beta)$ are infinitely close.
(This example was suggested to the author by E. Gordon.)

Proposition 2.4. ${}^*f\,(u(\alpha)) = u\,({}^*f(\alpha))$. *Indeed:*

1. *If* $\alpha \underset{u}{\sim} \beta$ *then* ${}^*f(\alpha) \underset{u}{\sim} {}^*f(\beta)$.
2. *If* ${}^*f(\alpha) \underset{u}{\sim} \gamma$ *then* $\gamma = {}^*f(\beta)$ *for some* $\beta \underset{u}{\sim} \alpha$.

Proof. (1). For every $A \subseteq \mathbb{N}$, one has the following chain of equivalences:

$$\begin{gathered} {}^*f(\alpha) \in {}^*A \;\Leftrightarrow\; \alpha \in {}^*\{n \mid f(n) \in A\} \;\Leftrightarrow \\ \Leftrightarrow\; \beta \in {}^*\{n \mid f(n) \in A\} \;\Leftrightarrow\; {}^*f(\beta) \in {}^*A. \end{gathered}$$

(2). For every $A \subseteq \mathbb{N}$, $\alpha \in {}^*A \Rightarrow {}^*f(\alpha) \in {}^*(f(A)) \Leftrightarrow \gamma \in {}^*(f(A))$, *i.e.* $\gamma = {}^*f(\beta)$ for some $\beta \in {}^*A$. But then the family of internal sets

$$\{{}^*f^{-1}(\gamma) \cap {}^*A \mid \alpha \in {}^*A\}$$

has the finite intersection property. By $\mathfrak{c}^+$-saturation, there exists an element β in the intersection of that family. Clearly, ${}^*f(\beta) = \gamma$ and $\beta \underset{u}{\sim} \alpha$. □

Before starting to develop our nonstandard theory, let us consider a well-known combinatorial property which constitutes a fundamental preliminary step in the theory of ultrafilters. The proof given below consists of two steps: we first show a finite version of the desired property, and then use a non-principal ultrafilter to obtain the global version. Although the result is well-known, this particular argument seems to be new in the literature.

Lemma 2.5. *Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n) \neq n$ for all n. Then there exists a 3-coloring $\chi : \mathbb{N} \to \{1, 2, 3\}$ such that $\chi(n) \neq \chi(f(n))$ for all n.*

Proof. We begin by showing the following "finite approximation" to the desired result.

- *For every finite $F \subset \mathbb{N}$ there exists $\chi_F : F \to \{1, 2, 3\}$ such that $\chi_F(x) \neq \chi_F(f(x))$ whenever both x and $f(x)$ belong to F.*

We proceed by induction on the cardinality of F. The basis is trivial, because if $|F| = 1$ then it is never the case that both $x, f(x) \in F$. For the inductive step, notice that by the *pigeonhole principle* there must be at least one element $\overline{x} \in F$ which is the image under f of at most one element in F, *i.e.* $|\{y \in F \mid f(y) = \overline{x}\}| \leq 1$. Now let $F' = F \setminus \{\overline{x}\}$ and let $\chi' : F' \to \{1, 2, 3\}$ be a 3-coloring as given by the inductive hypothesis. We want to extend χ' to a 3-coloring χ of F. To this end, define $\chi(\overline{x})$ in such a way that $\chi(\overline{x}) \neq \chi'(f(\overline{x}))$ if $f(\overline{x}) \in F$, and $\chi(\overline{x}) \neq \chi'(y)$ if $f(y) = \overline{x}$. This is always possible because there is at most one such element y, and because we have 3 colors at disposal.

We now have to glue together the finite 3-colorings so as to obtain a 3-coloring of the whole set $\mathbb{N}$. (Of course, this cannot be done directly, because two 3-colorings do not necessarily agree on the intersection of their domains.) One possible way is the following. For every $n \in \mathbb{N}$, fix a 3-coloring $\chi_n : \{1, \ldots, n\} \to \{1, 2, 3\}$ such that $\chi_n(x) \neq \chi_n(f(x))$ whenever both $x, f(x) \in \{1, \ldots, n\}$. Then pick any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define the map $\chi : \mathbb{N} \to \{1, 2, 3\}$ by putting

$$\chi(k) = i \iff \Gamma_i(k) = \{n \geq k \mid \chi_n(k) = i\} \in \mathcal{U}.$$

The definition is well-posed because for every k the disjoint union

$$\Gamma_1(k) \cup \Gamma_2(k) \cup \Gamma_3(k) = \{n \in \mathbb{N} \mid n \geq k\} \in \mathcal{U},$$

and so exactly one set $\Gamma_i(k)$ belongs to $\mathcal{U}$. The function χ is the desired 3-coloring. In fact, if by contradiction $\chi(k) = \chi(f(k)) = i$ for some k, then we could pick $n \in \Gamma_i(k) \cap \Gamma_i(f(k)) \in \mathcal{U}$ and have $\chi_n(k) = \chi_n(f(k))$, against the hypothesis on χ_n. (The same argument could be used to extend this lemma to functions $f : I \to I$ over arbitrary infinite sets I.) □

Remark 2.6. The second part of the above proof could also be easily carried out by using nonstandard methods. Indeed, by *saturation* one can pick a hyperfinite set $H \subset {}^*\mathbb{N}$ containing all (finite) natural numbers. By *transfer* from the "finite approximation" result proved above, there exists an internal 3-coloring $\Phi : H \to \{1,2,3\}$ such that $\Phi(\xi) \neq \Phi({}^*f(\xi))$ whenever both $\xi, {}^*f(\xi) \in H$. Then the restriction $\chi = \Phi\restriction_{\mathbb{N}} : \mathbb{N} \to \{1,2,3\}$ gives the desired 3-coloring.

As a corollary, we obtain the

Theorem 2.7. *Let $f : \mathbb{N} \to \mathbb{N}$ and $\alpha \in {}^*\mathbb{N}$. If ${}^*f(\alpha) \underset{u}{\sim} \alpha$ then ${}^*f(\alpha) = \alpha$.*

Proof. If ${}^*f(\alpha) \neq \alpha$, then $\alpha \in {}^*A$ where $A = \{n \mid f(n) \neq n\}$. Pick any function $g : \mathbb{N} \to \mathbb{N}$ that agrees with f on A and such that $g(n) \neq n$ for all $n \in \mathbb{N}$. Since $\alpha \in {}^*A \subseteq {}^*\{n \mid g(n) = f(n)\}$, we have that ${}^*g(\alpha) = {}^*f(\alpha)$. Apply the previous theorem to g and pick a 3-coloring $\chi : \mathbb{N} \to \{1,2,3\}$ such that $\chi(n) \neq \chi(g(n))$ for all n. Then ${}^*\chi({}^*f(\alpha)) = {}^*\chi({}^*g(\alpha)) \neq {}^*\chi(\alpha)$. Now let $X = \{n \in \mathbb{N} \mid \chi(n) = i\}$ where $i = {}^*\chi(\alpha)$. Clearly, $\alpha \in {}^*X$ but ${}^*f(\alpha) \notin {}^*X$, and hence ${}^*f(\alpha) \underset{u}{\not\sim} \alpha$. □

Two important properties of u-equivalence are the following.

Proposition 2.8. *Let $\alpha \in {}^*A$, and let f be 1-1 when restricted to A. Then*

(1) *There exists a bijection φ such that ${}^*f(\alpha) = {}^*\varphi(\alpha)$;*
(2) *For every $g : \mathbb{N} \to \mathbb{N}$, ${}^*f(\alpha) \underset{u}{\sim} {}^*g(\alpha) \Rightarrow {}^*f(\alpha) = {}^*g(\alpha)$.*

Proof. (1). We can assume that $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ infinite, as otherwise the thesis is trivial. Then $\alpha \in {}^*A$ implies that A is infinite, and so we can partition $A = B \cup C$ into two disjoint infinite sets B and C where, say, $\alpha \in {}^*B$. Since f is 1-1, we can pick a bijection φ that agrees with f on B, so that ${}^*\varphi(\alpha) = {}^*f(\alpha)$ as desired.

(2). By the previous point, ${}^*f(\alpha) = {}^*\varphi(\alpha)$ for some bijection φ. Then

$${}^*g(\alpha) \underset{u}{\sim} {}^*\varphi(\alpha) \;\Rightarrow\; {}^*\varphi^{-1}({}^*g(\alpha)) \underset{u}{\sim} {}^*\varphi^{-1}({}^*\varphi(\alpha)) = \alpha \;\Rightarrow\; {}^*\varphi^{-1}({}^*g(\alpha)) = \alpha,$$

and hence ${}^*g(\alpha) = {}^*\varphi(\alpha) = {}^*f(\alpha)$. □

We remark that property (2) of the above proposition does not hold if we drop the hypothesis that f is 1-1. (In Section 3 we shall address the question of the existence of infinite points $\alpha \in {}^*\mathbb{N}$ with the property that ${}^*f(\alpha) \underset{u}{\sim} {}^*g(\alpha) \Rightarrow {}^*f(\alpha) = {}^*g(\alpha)$ for all $f, g : \mathbb{N} \to \mathbb{N}$.)

Proposition 2.9. *If ${}^*f(\alpha) \underset{u}{\sim} \beta$ and ${}^*g(\beta) \underset{u}{\sim} \alpha$ for suitable f and g, then ${}^*\varphi(\alpha) \underset{u}{\sim} \beta$ for some bijection φ.*

Proof. By the hypotheses, ${}^*g({}^*f(\alpha)) \underset{u}{\sim} {}^*g(\beta) \underset{u}{\sim} \alpha$ and so ${}^*g({}^*f(\alpha)) = \alpha$. If $A = \{n \mid g(f(n)) = n\}$, then $\alpha \in {}^*A$ and f is 1-1 on A. By the previous proposition, there exists a bijection φ such that ${}^*f(\alpha) = {}^*\varphi(\alpha)$, and hence ${}^*\varphi(\alpha) \underset{u}{\sim} \beta$. □

We recall that the *image* of an ultrafilter $\mathcal{U}$ under a function $f:\mathbb{N}\to\mathbb{N}$ is the ultrafilter

$$f(\mathcal{U}) \;=\; \{A\subseteq\mathbb{N}\mid f^{-1}(A)\in\mathcal{U}\}.$$

Notice that if $f\equiv_{\mathcal{U}} g$, *i.e.* if $\{n\mid f(n)=g(n)\}\in\mathcal{U}$, then $f(\mathcal{U})=g(\mathcal{U})$.

Proposition 2.10. *For every $f:\mathbb{N}\to\mathbb{N}$ and $\alpha\in{}^*\mathbb{N}$, the image ultrafilter $f(\mathfrak{U}_\alpha)=\mathfrak{U}_{^*f(\alpha)}$.*

Proof. For every $A\subseteq\mathbb{N}$, one has the chain of equivalences:

$$A\in\mathfrak{U}_{^*f(\alpha)} \;\Leftrightarrow\; {}^*f(\alpha)\in{}^*A \;\Leftrightarrow\; \alpha\in{}^*(f^{-1}(A)) \;\Leftrightarrow$$
$$\Leftrightarrow\; f^{-1}(A)\in\mathfrak{U}_\alpha \;\Leftrightarrow\; A\in f(\mathfrak{U}_\alpha).$$

□

Let us now show how the above results about u-equivalence are just reformulation in a nonstandard context of fundamental properties of ultrafilter theory.

Theorem 2.11. *Let $f:\mathbb{N}\to\mathbb{N}$ and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. If $f(\mathcal{U})=\mathcal{U}$ then $\{n\mid f(n)=n\}\in\mathcal{U}$.*

Proof. Let $\alpha\in{}^*\mathbb{N}$ be such that $\mathcal{U}=\mathfrak{U}_\alpha$. By the hypothesis, $\mathfrak{U}_\alpha=f(\mathfrak{U}_\alpha)=\mathfrak{U}_{^*f(\alpha)}$, *i.e.* $\alpha \underset{u}{\sim} {}^*f(\alpha)$ and so, by the previous theorem, ${}^*f(\alpha)=\alpha$. But then $\{n\mid f(n)=n\}\in\mathcal{U}$ because $\alpha\in{}^*\{n\mid f(n)=n\}$. □

Recall the *Rudin-Keisler pre-ordering* $\leq_{RK}$ on ultrafilters:

$$\mathcal{V}\leq_{RK}\mathcal{U} \iff f(\mathcal{U})=\mathcal{V} \text{ for some function } f.$$

In this case, we say that $\mathcal{V}$ is *Rudin-Keisler below* $\mathcal{U}$ (or $\mathcal{U}$ is *Rudin-Keisler above* $\mathcal{V}$). It is readily verified that $g(f(\mathcal{U}))=(g\circ f)(\mathcal{U})$, so $\leq_{RK}$ satisfies the transitivity property, and $\leq_{RK}$ is actually a pre-ordering. Notice that $\mathfrak{U}_\alpha\leq_{RK}\mathfrak{U}_\beta$ means that ${}^*f(\beta)\underset{u}{\sim}\alpha$ for some function f.

Proposition 2.12. *$\mathcal{U}\leq_{RK}\mathcal{V}$ and $\mathcal{V}\leq_{RK}\mathcal{U}$ if and only if $\mathcal{U}\cong\mathcal{V}$ are isomorphic,* i.e. *there exists a bijection $\varphi:\mathbb{N}\to\mathbb{N}$ such that $\varphi(\mathcal{U})=\mathcal{V}$.*

Proof. Let $\mathcal{U}=\mathfrak{U}_\alpha$ and $\mathcal{V}=\mathfrak{U}_\beta$. If $\mathcal{U}\leq_{RK}\mathcal{V}$ and $\mathcal{V}\leq_{RK}\mathcal{U}$, then there exist functions $f,g:\mathbb{N}\to\mathbb{N}$ such that ${}^*f(\alpha)\underset{u}{\sim}\beta$ and ${}^*g(\beta)\underset{u}{\sim}\alpha$. But then, by Proposition 2.9, there exists a bijection $\varphi:\mathbb{N}\to\mathbb{N}$ such that ${}^*\varphi(\alpha)\underset{u}{\sim}\beta$, and hence $\varphi(\mathcal{U})=\mathfrak{U}_{^*\varphi(\alpha)}=\mathfrak{U}_\beta=\mathcal{V}$, as desired. The other implication is trivial. □

We close this section by showing that all infinite numbers α have "large" and "spaced" u-monads, in the sense that $u(\alpha)$ is both a left and a right unbounded subset of the infinite numbers ${}^*\mathbb{N}\setminus\mathbb{N}$, and that different elements of $u(\alpha)$ are placed at infinite distance. (The property of $\mathfrak{c}^+$-*saturation* is essential here.)

Theorem 2.13. [43, 44] *Let $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ be infinite. Then:*

(1) *For every $\xi \in {}^*\mathbb{N}$, there exists an internal 1-1 map $\varphi : {}^*\mathbb{N} \to u(\alpha) \cap (\xi, +\infty)$. In consequence, the set $u(\alpha) \cap (\xi, +\infty)$ contains $|{}^*\mathbb{N}|$-many elements and it is unbounded in ${}^*\mathbb{N}$.*
(2) *For every infinite $\xi \in {}^*\mathbb{N} \setminus \mathbb{N}$, the set $u(\alpha) \cap [0, \xi)$ contains at least $\mathfrak{c}^+$-many elements. In consequence, $u(\alpha)$ is unbounded leftward in ${}^*\mathbb{N} \setminus \mathbb{N}$.*
(3) *If $\alpha \underset{u}{\sim} \beta$ and $\alpha \neq \beta$, then the distance $|\alpha - \beta| \in {}^*\mathbb{N} \setminus \mathbb{N}$ is infinite.*

Proof. (1). Since α is infinite, every $X \in \mathfrak{U}_\alpha$ is an infinite set and so for each $k \in \mathbb{N}$ there exists a 1-1 function $f : \mathbb{N} \to X \cap (k, +\infty)$. By *transfer*, for every $\xi \in {}^*\mathbb{N}$ the following internal set is non-empty:

$$\Gamma(X) = \{\, \varphi : {}^*\mathbb{N} \to {}^*X \cap (\xi, +\infty) \mid \varphi \text{ internal 1-1} \,\}.$$

Notice that $\Gamma(X_1) \cap \cdots \cap \Gamma(X_n) = \Gamma(X_1 \cap \cdots \cap X_n)$, and hence the family $\{\, \Gamma(X) \mid X \in \mathfrak{U}_\alpha \}$ has the finite intersection property. By $\mathfrak{c}^+$-*saturation*, we can pick $\varphi \in \bigcap_{X \in \mathfrak{U}_\alpha} \Gamma(X)$. Clearly, range($\varphi$) is an internal subset of $u(\alpha) \cap (\xi, +\infty)$ with the same cardinality as ${}^*\mathbb{N}$. Since range(φ) is internal and hyperinfinite, it is necessarily unbounded in ${}^*\mathbb{N}$.

(2). For any given $\xi \in {}^*\mathbb{N} \setminus \mathbb{N}$, the family $\{{}^*X \cap [0, \xi) \mid X \in \mathfrak{U}_\alpha\}$ is closed under finite intersections, and all its elements are non-empty. So, by $\mathfrak{c}^+$-*saturation*, there exists

$$\zeta \in \bigcap_{X \in \mathfrak{U}_\alpha} {}^*X \cap [0, \xi).$$

Clearly $\zeta \in u(\alpha) \cap [0, \xi)$, and this shows that $u(\alpha)$ is unbounded leftward in ${}^*\mathbb{N} \setminus \mathbb{N}$. Now fix ξ infinite. By what we have just proved, the family of open intervals

$$\mathcal{G} = \{(k, \zeta) \mid k \in \mathbb{N} \text{ and } \zeta \in u(\alpha) \cap [0, \xi)\}$$

has empty intersection. Since $\mathcal{G}$ satisfies the finite intersection property, and $\mathfrak{c}^+$-*saturation* holds, it must be $|\mathcal{G}| \geq \mathfrak{c}^+$, and hence also $|u(\alpha) \cap [0, \xi)| \geq \mathfrak{c}^+$.

(3). For every $n \geq 2$, let k_n be the remainder of the Euclidean division of α by n, and consider the set $X_n = \{x \cdot n + k_n \mid x \in \mathbb{N}\}$. Then $\alpha \in {}^*X_n$ and $\alpha \underset{u}{\sim} \beta$ implies that also $\beta \in {}^*X_n$, so $\alpha - \beta$ is a multiple of n. Since $\beta \neq \alpha$, it must be $|\alpha - \beta| \geq n$. As this is true for all $n \geq 2$, we conclude that α and β have infinite distance. □

3. Hausdorff S-topologies and Hausdorff ultrafilters

It is natural to ask about properties of the *ultrafilter map*:

$$\mathfrak{U} : {}^*\mathbb{N} \to \beta\mathbb{N} \quad \text{where} \quad \mathfrak{U} : \alpha \mapsto \mathfrak{U}_\alpha .$$

We already noticed that if one assumes $\mathfrak{c}^+$-*saturation* then $\mathfrak{U}$ is onto $\beta\mathbb{N}$, *i.e.* every ultrafilter on $\mathbb{N}$ is of the form $\mathfrak{U}_\alpha$ for a suitable $\alpha \in {}^*\mathbb{N}$. However, in this section no saturation property will be assumed.

As a first (negative) result, let us show that the ultrafilter map is never a bijection.

Proposition 3.1. *In any model of nonstandard analysis, if the ultrafilter map $\mathfrak{U} : {}^*\mathbb{N} \twoheadrightarrow \beta\mathbb{N}$ is onto then, for every non-principal $\mathcal{U} \in \beta\mathbb{N}$, the set $\{\alpha \in {}^*\mathbb{N} \mid \mathfrak{U}_\alpha = \mathcal{U}\}$ contains at least $\mathfrak{c}$-many elements.*

Proof. Given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, for $X \in \mathcal{U}$ and $k \in \mathbb{N}$ let

$$\Lambda(X,k) \;=\; \{F \in \mathrm{Fin}(\mathbb{N}) \mid F \subset X \;\&\; |F| \geq k\}\,,$$

where we denoted by $\mathrm{Fin}(\mathbb{N}) = \{F \subset \mathbb{N} \mid F \text{ is finite}\}$. Notice that the family of sets $\mathcal{F} = \{\Lambda(X,k) \mid X \in \mathcal{U}\,,\, k \in \mathbb{N}\}$ has the finite intersection property. Indeed, $\Lambda(X_1,k_1) \cap \ldots \cap \Lambda(X_m,k_m) = \Lambda(X,k)$ where $X = X_1 \cap \ldots \cap X_m \in \mathcal{U}$ and $k = \max\{k_1,\ldots,k_m\}$; and every set $\Lambda(X,k) \neq \emptyset$ since all $X \in \mathcal{U}$ are infinite. Now fix a bijection $\Phi : \mathrm{Fin}(\mathbb{N}) \to \mathbb{N}$, and let

$$\Gamma(X,k) \;=\; \{\Psi(F) \mid F \in \Lambda(X,k)\}.$$

Then also the family $\{\Gamma(X,k) \mid X \in \mathcal{U}\,,\, k \in \mathbb{N}\} \subseteq \mathcal{P}(\mathbb{N})$ has the FIP, and so we can extend it to an ultrafilter $\mathcal{V}$ on $\mathbb{N}$. By the hypothesis on the ultrafilter map there exists $\beta \in {}^*\mathbb{N}$ such that $\mathfrak{U}_\beta = \mathcal{V}$; in particular, $\beta \in \bigcap_{X,k} {}^*\Gamma(X,k)$, and so $\beta = {}^*(\Psi(G))$ for a suitable $G \in \bigcap_{X,k} {}^*\Lambda(X,k)$. Then $G \subseteq {}^*X$ for all $X \in \mathcal{U}$, and hence $\mathfrak{U}_\gamma = \mathcal{U}$ for all $\gamma \in G$. Moreover, $|G| \geq k$ for all $k \in \mathbb{N}$, and so G is an infinite internal set. Finally, we use the following general fact: "Every infinite internal set has at least the cardinality of the continuum". To prove this last property, notice that if A is infinite and internal then there exists a (internal) 1-1 map $f : \{1,\ldots,\nu\} \to A$ for some infinite $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. Now, consider the unit real interval $[0,1]$ and define $\Psi : [0,1] \to \{1,\ldots,\nu\}$ by putting $\Psi(r) = \min\{1 \leq i \leq \nu \mid r \leq i/\nu\}$. The map Ψ is 1-1 because $\Psi(r) = \Psi(r') \Rightarrow |r - r'| \leq 1/\nu \approx 0 \Rightarrow r = r'$, and so we conclude that $\mathfrak{c} = |[0,1]| \leq |\{1,\ldots,\nu\}| \leq |A|$, as desired. (When $\mathfrak{c}^+$-*saturation holds*, then $|\{\alpha \in {}^*\mathbb{N} \mid \mathfrak{U}_\alpha = \mathcal{U}\}| \geq \mathfrak{c}^+$ by Theorem 2.13.) □

We now show that the ultrafilter map is tied up with a topology that is naturally considered in a nonstandard setting. (The notion of S-topology was introduced by A. Robinson himself, the “inventor” of nonstandard analysis.)

Definition 3.2. For every set X, the *S-topology* on ${}^*X$ is the topology having the family $\{{}^*A \mid A \subseteq X\}$ as a basis of open sets.

The capital letter “S” stands for “standard”, and in fact hyper-extensions ${}^*A$ are often called *standard sets* in the literature of nonstandard analysis. The adjective “standard” originated from the distinction between a *standard* universe and a *nonstandard* universe, according to the most used approaches to nonstandard analysis. However, such a distinction is not needed, and indeed one can adopt a foundational framework where there is a single mathematical universe, and take hyper-extensions of *any* object under study (see, *e.g.*, [2]).

Every basic open set ${}^*A$ is also closed because ${}^*X \setminus {}^*A = {}^*(X \setminus A)$, and so the S-topologies are totally disconnected. A first relationship between S-topology and ultrafilter map is the following.

Proposition 3.3. *The S-topology on ${}^*\mathbb{N}$ is compact if and only if the ultrafilter map $\mathfrak{U} : {}^*\mathbb{N} \to \beta\mathbb{N}$ is onto.*

Proof. According to one of the equivalent definitions of compactness, the S-topology is compact if and only if every non-empty family $\mathcal{C}$ of closed sets with the FIP has non-empty intersection $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$. Without loss of generality, one can assume that $\mathcal{C}$ is a family of hyper-extensions. Notice that $\mathcal{C} = \{{}^*A_i \mid i \in I\}$ has the FIP if and only if $\mathcal{C}' = \{A_i \mid i \in I\} \subset \mathcal{P}(\mathbb{N})$ has the FIP, and so we can extend $\mathcal{C}'$ to an ultrafilter $\mathcal{V}$ on $\mathbb{N}$. If the ultrafilter map is onto $\beta\mathbb{N}$, then $\mathcal{V} = \mathfrak{U}_\alpha$ for a suitable α, and therefore $\alpha \in \bigcap_{i\in I} {}^*A_i \neq \emptyset$.

Conversely, if $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$, then $\mathcal{C} = \{{}^*X \mid X \in \mathcal{U}\}$ is a family of closed sets with the FIP. If α is any element in the intersection of $\mathcal{C}$, then $\mathfrak{U}_\alpha = \mathcal{U}$. □

In consequence of the above proposition, the S-topology on ${}^*\mathbb{N}$ is compact when the $\mathfrak{c}^+$-*saturation* property holds. More generally, κ-*saturation* implies that the S-topology is compact on every hyper-extension ${}^*X$ where $2^{|X|} < \kappa$. (Actually, the κ-*enlarging* property suffices: see Definition 4.1.)

A natural question that one may ask is whether the S-topologies are Hausdorff or not. This depends on the considered model, and giving a complete answer turns out to be a difficult issue involving deep set-theoretic matters, which will be briefly discussed below. So, it is not surprising that this simple question was not addressed explicitly in the early literature in nonstandard analysis, despite the fact that the S-topology was a common object of study.

As a first remark, notice that having a Hausdorff S-topology on ${}^*X$ is preserved when passing to lower cardinalities.

Proposition 3.4. *If the S-topology is Hausdorff on* ${}^*X$ *and* $|Y| \leq |X|$*, then the S-topology is Hausdorff on* ${}^*Y$ *as well.*

Proof. Fix a 1-1 map $f : Y \to X$. Given $\xi \neq \eta$ in ${}^*Y$, consider ${}^*f(\xi) \neq {}^*f(\eta)$ in ${}^*X$. By the hypothesis, we can pick disjoint sets $A, B \subseteq X$ with ${}^*f(\xi) \in {}^*A$ and ${}^*f(\eta) \in {}^*B$. Then $C = f^{-1}(A)$ and $D = f^{-1}(B)$ are disjoint subsets of Y such that $\xi \in {}^*C$ and $\eta \in {}^*D$. □

Recall a notion that was introduced in [29]: A model of nonstandard analysis is *κ-constrained* if the following property holds:

$$\forall X\ \forall \xi \in {}^*X\ \exists A \subseteq X \text{ such that } |A| \leq \kappa \text{ and } \xi \in {}^*A.$$

We remark that any ultrapower model of nonstandard analysis constructed by means of an ultrafilter over a set of cardinality κ is κ-constrained.

In the countable case, the notion of *constrained* already appeared at the beginnings of nonstandard analysis, under the name of *σ-quasi standardness* (see W.A.J. Luxemburg's lecture notes [39]). The existence of nonstandard universes which are not κ-constrained for any κ is problematic and appears to be closely related to the existence large cardinals. The reader interested in this foundational issue is referred to [29].

Proposition 3.5. *Assume that our model of nonstandard analysis is* κ*-constrained. Then the S-topology is Hausdorff on* ${}^*\kappa$ *if and only if the S-topology is Hausdorff on every hyper-extension* ${}^*X$*.*

Proof. Let $\xi, \eta \in {}^*X$ be given. By the property of κ-constrained, we can pick sets A, B with $\xi \in {}^*A$, $\eta \in {}^*B$ and $|A|, |B| \leq \kappa$. Since $|A \cup B| \leq \kappa$, by the previous Proposition 3.4, the S-topology on ${}^*(A \cup B)$ is Hausdorff. Pick disjoint subsets $C, D \subseteq A \cup B$ with $\xi \in {}^*C$ and $\eta \in {}^*D$. Then $\xi \in {}^*(C \cap X)$ and $\eta \in {}^*(D \cap X)$ where $C \cap X$ and $D \cap X$ are disjoint subsets of X. □

So, in any ultrapower model of nonstandard analysis determined by an ultrafilter on $\mathbb{N}$, if the S-topology is Hausdorff on ${}^*\mathbb{N}$ then it is Hausdorff on *all* hyper-extensions ${}^*X$.

Proposition 3.6. *The S-topology on* ${}^*X$ *is Hausdorff if and only if the ultrafilter map* $\mathfrak{U} : {}^*X \to \beta X$ *is 1-1.*

Proof. By definition, the S-topology on ${}^*X$ is Hausdorff if and only for every pair of elements $\xi \neq \eta$ in ${}^*X$ there exist basic open sets ${}^*A, {}^*B \subseteq {}^*X$ such that $\xi \in {}^*A$, $\eta \in {}^*B$ and ${}^*A \cap {}^*B = \emptyset$. But this means that $A \in \mathfrak{U}_\xi$ and $B \in \mathfrak{U}_\eta$ for suitable disjoint sets $A \cap B = \emptyset$. We reach the thesis by noticing that this last property holds if and only if the ultrafilters $\mathfrak{U}_\xi$ and $\mathfrak{U}_\eta$ are different. $\square$

Hausdorff S-topologies are tied up with special ultrafilters.

Proposition 3.7. *Let $\mathcal{U}$ be a non-principal ultrafilter on the set I. Then the following are equivalent:*

(1) *In the ultrapower model of nonstandard analysis determined by $\mathcal{U}$, the S-topology on ${}^*I$ is Hausdorff.*
(2) *In any model of nonstandard analysis, if $\mathcal{U} = \mathfrak{U}_\alpha$ is generated by a point $\alpha \in {}^*I$, then:*

$$ {}^*f(\alpha) \underset{u}{\sim} {}^*g(\alpha) \implies {}^*f(\alpha) = {}^*g(\alpha). $$

(3) *For every $f, g : I \to I$,*

$$ f(\mathcal{U}) = g(\mathcal{U}) \implies f \equiv_{\mathcal{U}} g, \text{ i.e. } \{i \in I \mid f(i) = g(i)\} \in \mathcal{U}. $$

Proof. (1) $\Leftrightarrow$ (3). Notice first that if $\xi = [f]_{\mathcal{U}}$ is the element of ${}^*I$ given by the $\mathcal{U}$-equivalence class of the function $f : I \to I$, then $f(\mathcal{U}) = \mathfrak{U}_\xi$. Indeed, for every $A \subseteq I$, one has $A \in f(\mathcal{U}) \Leftrightarrow f^{-1}(A) \in \mathcal{U} \Leftrightarrow \{i \in I \mid f(i) \in A\} \in \mathcal{U} \Leftrightarrow \xi = [f]_{\mathcal{U}} \in {}^*A \Leftrightarrow A \in \mathfrak{U}_\xi$. Now let $\xi = [f]_{\mathcal{U}}$ and $\eta = [g]_{\mathcal{U}}$ be arbitrary elements of ${}^*I = I^I/\mathcal{U}$. By definition, $\xi = \eta \Leftrightarrow f \equiv_{\mathcal{U}} g$; besides, by what just seen above, $f(\mathcal{U}) = g(\mathcal{U}) \Leftrightarrow \mathfrak{U}_\xi = \mathfrak{U}_\eta$. Now, by the previous proposition the S-topology on I is Hausdorff if and only if the ultrafilter map on I is 1-1, and hence the thesis follows.

(2) $\Leftrightarrow$ (3). Notice that ${}^*f(\alpha) = {}^*g(\alpha) \Leftrightarrow \alpha \in {}^*\{i \in I \mid f(i) = g(i)\} \Leftrightarrow f \equiv_{\mathfrak{U}_\alpha} g$. The thesis follows by recalling that $\mathfrak{U}_{{}^*f(\alpha)} = f(\mathfrak{U}_\alpha)$ and $\mathfrak{U}_{{}^*g(\alpha)} = g(\mathfrak{U}_\alpha)$. $\square$

Because of the above equivalences, non-principal ultrafilters that satisfy property (3), were named *Hausdorff* in [20]. To the author's knowledge, the problem of existence of such ultrafilters was first explicitly considered by A. Connes in his paper [15] of 1970, where he needed special ultrafilters $\mathcal{U}$ with the property that the maps $[\varphi]_{\mathcal{U}} \mapsto \varphi(\mathcal{U})$ defined on ultrapowers $K^{\mathbb{N}}/\mathcal{U}$ (K a field) be injective into $\beta\mathbb{N}$. He noticed that such a property was satisfied by selective ultrafilters, introduced three years before by G. Choquet [13] under the name of *ultrafiltres absolus*. In consequence, Hausdorff ultrafilters are consistent. Indeed, selective ultrafilters exist under the *continuum hypothesis* (this was already proved by G. Choquet [13] in 1968). However, we remark that the existence of selective ultrafilters cannot be proved in ZFC, as first shown by K. Kunen [34]. Independently, in their 1972 paper [12], G. Cherlin and J. Hirschfeld proved that non-principal ultrafilters exist which are *not* Hausdorff, and asked whether Hausdorff ultrafilters exist at all in ZFC. It is worth remarking that this problem is still open to this day (see [20, 3]).

We close this section by mentioning another result, proved in [19], that connects Hausdorff ultrafilters and nonstandard analysis

Theorem 3.8. *Assume that $\mathfrak{N} \subset \beta\mathbb{N}$ is a set of ultrafilters on $\mathbb{N}$ such that*

- $\mathbb{N} \subsetneq \mathfrak{N}$, i.e. $\mathfrak{N}$ *properly contains all principal ultrafilters;*
- *Every non-principal $\mathcal{U} \in \mathfrak{N}$ is Hausdorff;*
- $\mathfrak{N}$ *is RK-downward closed,* i.e. $\mathcal{U} \in \mathfrak{N}$ *implies that* $f(\mathcal{U}) \in \mathfrak{N}$ *for every* $f : \mathbb{N} \to \mathbb{N}$*;*
- $\mathfrak{N}$ *is "strongly" RK-filtered in the following sense: For every $\mathcal{U}, \mathcal{V} \in \mathfrak{N}$ there exist $\mathcal{W} \in \mathfrak{N}$ and $f, g : \mathbb{N} \to \mathbb{N}$ such that $f(\mathcal{W}) = \mathcal{U}$ and $g(\mathcal{W}) = \mathcal{V}$.*

Then $\mathfrak{N}$ is a set of hypernatural numbers of nonstandard analysis where:

- *For $A \subseteq \mathbb{N}^k$, the hyper-extension ${}^*A \subseteq \mathfrak{N}^k$ is defined by letting for every $\mathcal{U} \in \mathfrak{N}$ and for every $f_1, \dots, f_k : \mathbb{N} \to \mathbb{N}$:*

$$(f_1(\mathcal{U}), \dots, f_k(\mathcal{U})) \in {}^*A \iff \{n \in \mathbb{N} \mid (f_1(n), \dots, f_k(n)) \in A\} \in \mathcal{U}$$

- *For $F : \mathbb{N}^k \to \mathbb{N}$, the hyper-extension ${}^*F : \mathfrak{N}^k \to \mathfrak{N}$ is defined by letting for every $\mathcal{U} \in \mathfrak{N}$ and for every $f_1, \dots, f_k : \mathbb{N} \to \mathbb{N}$:*

$$ {}^*F(f_1(\mathcal{U}), \dots, f_k(\mathcal{U})) \;=\; (F \circ (f_1, \dots, f_k))(\mathcal{U}).$$

($F \circ (f_1, \dots, f_k) : \mathbb{N} \to \mathbb{N}$ is the function $n \mapsto F(f_1(n), \dots, f_k(n))$.)

4. Regular and good ultrafilters

A fundamental notion used in the theory of ultrafilters is that of regularity. We recall that an ultrafilter $\mathcal{U}$ on an infinite set I is called *regular* if there exists a family $\{A_i \mid i \in I\} \subseteq \mathcal{U}$ such that $\bigcap_{i \in I_0} A_i = \emptyset$ for every infinite $I_0 \subseteq I$. When I is countable, it is easily seen that $\mathcal{U}$ is regular if and only if it is non-principal, but in general regularity is a stronger condition. A simple nonstandard characterization holds.

Recall the following weakened version of saturation, where only families of hyper-extensions are considered (compare with Definition 1.1.)

Definition 4.1. Let κ be an infinite cardinal. A model of nonstandard analysis is a *κ-enlargement* if it satisfies the property:

- Let $\mathcal{G}$ be a family of sets with cardinality $|\mathcal{G}| < \kappa$. If $\mathcal{G}$ has the FIP, then $\bigcap_{B\in\mathcal{G}} {}^*B \neq \emptyset$

Proposition 4.2. *Let $\mathcal{U}$ be an ultrafilter over the infinite set I. Then the following are equivalent:*

(1) *$\mathcal{U}$ is regular.*
(2) *In the ultrapower model of nonstandard analysis determined by $\mathcal{U}$, the $|I|^+$-enlarging property holds.*

Proof. (1) $\Rightarrow$ (2). Pick a family $\{C_x \mid x \in I\} \subseteq \mathcal{U}$ such that $\bigcap_{x\in\Lambda} C_x = \emptyset$ whenever $\Lambda \subseteq I$ is infinite. Given a family $\{A_x \mid x \in I\}$ with the FIP, for every $y \in I$ pick an element $\varphi(y) \in \bigcap_{y\in C_x} A_x$ (this is possible because $\{x \mid y \in C_x\}$ is finite). If $\xi = [\varphi]_{\mathcal{U}}$ is the element given by the $\mathcal{U}$-equivalence class of the function φ, then $\xi \in {}^*A_x$ for every $x \in I$, since $\{y \in I \mid \varphi(y) \in A_x\} \supseteq \{y \in I \mid y \in C_x\} = C_x \in \mathcal{U}$.

(2) $\Rightarrow$ (1). Let $\mathrm{Fin}(I)$ be the set of finite parts of I, and for every $x \in I$, let $\widehat{x} = \{a \in \mathrm{Fin}(I) \mid x \in a\}$. The family $\{\widehat{x} \mid x \in I\}$ satisfies the FIP and so, by the κ^+-enlarging property, there exists $A \in \bigcap_{x\in I} {}^*\widehat{x}$. Now pick a function $\varphi : I \to \mathrm{Fin}(I)$ such that $A = [\varphi]_{\mathcal{U}}$ is the $\mathcal{U}$-equivalence class of φ. Then for every $x \in I$, one has that $A \in {}^*\widehat{x}$ if and only if

$$C_x \;=\; \{y \in I \mid \varphi(y) \in \widehat{x}\} \;=\; \{y \in I \mid x \in \varphi(y)\} \,\in \mathcal{U}.$$

The family $\{C_x \mid x \in I\} \subseteq \mathcal{U}$ witnesses the regularity of $\mathcal{U}$ because for every infinite $\Lambda \subseteq I$, the intersection $\bigcap_{x\in\Lambda} C_x = \{y \in I \mid \Lambda \subseteq \varphi(y)\} = \emptyset$. □

One can also easily characterize points that generate regular ultrafilters.

Theorem 4.3. *Given any model of nonstandard analysis, let $\alpha \in {}^*I$. Then the following are equivalent:*

(1) *The ultrafilter $\mathfrak{U}_\alpha$ on I generated by α is regular ;*
(2) *There exists a function $\varphi : I \to Fin(I)$ such that ${}^*x \in {}^*\varphi(\alpha)$ for all $x \in I$.*

Proof. (1) $\Rightarrow$ (2). Given a family $\{C_x \mid x \in I\}$ that witnesses the regularity of $\mathfrak{U}_\alpha$, let $\varphi : I \to \mathrm{Fin}(I)$ be the function defined by setting $\varphi(x) = \{y \in I \mid x \in C_y\}$. If ϑ is the function $y \mapsto C_y$, then we obtain the thesis by the following equivalences, that hold for any $x \in I$:

$$C_x \in \mathfrak{U}_\alpha \;\Leftrightarrow\; \alpha \in {}^*(C_x) = {}^*\vartheta({}^*x) \;\Leftrightarrow\; {}^*x \in \{\xi \in {}^*I \mid \alpha \in {}^*\vartheta(\xi)\} = {}^*\varphi(\alpha).$$

(2) $\Rightarrow$ (1). Pick φ as in the hypothesis, and let

$$C_x \;=\; \{y \in I \mid x \in \varphi(y)\}.$$

Notice that ${}^*x \in {}^*\varphi(\alpha) \Leftrightarrow \alpha \in {}^*(C_x) \Leftrightarrow C_x \in \mathfrak{U}_\alpha$. Moreover, for $X \subseteq I$, whenever it is possible to pick an element $y \in \bigcap_{x\in X} C_x$, one has that $X \subseteq \varphi(y)$, and so X is finite. □

As a corollary of the above characterizations of regularity, we can now give a nonstandard proof of a limiting result about the existence of Hausdorff ultrafilters.

We recall that the *ultrafilter number* $\mathfrak{u}$ denotes the minimum cardinality of any (non-principal) ultrafilter base on $\mathbb{N}$. (A family $\mathcal{B}$ is an *ultrafilter base* on $\mathbb{N}$ if $\{A \subseteq \mathbb{N} \mid \exists B \in \mathcal{B},\ B \subseteq A\}$ is an ultrafilter.) It is not hard to show that $\aleph_0 < \mathfrak{u} \leq \mathfrak{c}$ (see [8, §9]).

Theorem 4.4. [20] *If $\mathcal{U}$ is a regular ultrafilter on a cardinal $\kappa \geq \mathfrak{u}$, then $\mathcal{U}$ is not Hausdorff.*

Proof. By contradiction, pick $\mathcal{U}$ a regular Hausdorff ultrafilter on κ, and consider the corresponding ultrapower model of nonstandard analysis. Then the S-topology on ${}^*\kappa$, and hence on ${}^*\mathbb{N}$, is Hausdorff. Moreover, by the characterization of Proposition 4.2, the κ^+-enlarging property holds. Now pick $\mathcal{G} \subset \mathcal{P}(\mathbb{N})$ a non-principal ultrafilter base of cardinality $\mathfrak{u}$, and for every $A \in \mathcal{G}$, let $\Gamma(A) = \{X \subseteq A \mid X \text{ is infinite}\}$. Clearly the family $\{\Gamma(A) \mid A \in \mathcal{G}\}$ has the FIP, and so by the enlarging property there exists $H \in \bigcap_{A\in\mathcal{G}} {}^*\Gamma(A)$. Then $H \subset {}^*\mathbb{N}$ is hyperinfinite and $\mathfrak{U}_\xi = \mathcal{V}$ for every $\xi \in H$, where $\mathcal{V}$ is the ultrafilter on $\mathbb{N}$ generated by $\mathcal{G}$. This shows that the ultrafilter map $\mathfrak{U} : {}^*\mathbb{N} \to \beta\mathbb{N}$ is not 1-1, and hence the S-topology on ${}^*\mathbb{N}$ is not Hausdorff, a contradiction. □

We recall that an ultrafilter $\mathcal{U}$ is *countably incomplete* if it is not closed under countable intersections, *i.e.* if there exists a countable family $\{A_n\}$ of elements of $\mathcal{U}$ such that $\bigcap_n A_n \notin \mathcal{U}$ (equivalently, one may ask for that intersection to be empty). We remark that an ultrapower modulo $\mathcal{U}$ determines a model of nonstandard analysis if and only if $\mathcal{U}$ is countably incomplete, as otherwise one would have ${}^*\mathbb{N} = \mathbb{N}$. Indeed, if $\xi = [f]_\mathcal{U}$ is an infinite element in the ultrapower ${}^*\mathbb{N} = \mathbb{N}^I/\mathcal{U}$, then for every $n \in \mathbb{N}$, the set $\Gamma_n = \{i \in \mathbb{N} \mid f(i) \neq n\} \in \mathcal{U}$ but the countable intersection $\bigcap_n \Gamma_n = \emptyset$. An ultrafilter $\mathcal{U}$ on an infinite set I is *good* if for every antimonotonic function $f : \text{Fin}(I) \to \mathcal{U}$ there exists an antiadditive $g : \text{Fin}(I) \to \mathcal{U}$ such that $g(a) \subseteq f(a)$ for all $a \in \text{Fin}(I)$. f is called *antimonotonic* if $f(a) \supseteq f(b)$ whenever $a \subseteq b$; and g is called *antiadditive* if $g(a \cup b) = g(a) \cap g(b)$. (See, *e.g.*, [11, §6.1].)

Theorem 4.5. *Let $\mathcal{U}$ be a countably incomplete ultrafilter over the infinite set I. Then the following are equivalent:*

(1) *$\mathcal{U}$ is good.*
(2) *In the ultrapower model of nonstandard analysis determined by $\mathcal{U}$, the κ^+-*saturation *property holds, where $\kappa = |I|$.*

Proof. (1) $\Rightarrow$ (2). Let $\{A_x \mid x \in I\}$ be a family of internal subsets of an hyper-extension ${}^*Y$ with the FIP. For every $x \in I$, pick a function $\varphi_x : I \to \mathcal{P}(Y)$ such that $A_x = [\varphi_x]_{\mathcal{U}}$. By countable incompleteness, we can fix a sequence of sets

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$$

such that $I_n \in \mathcal{U}$ for all n and $\bigcap_n I_n = \emptyset$. Given $a \in \mathrm{Fin}(I)$ of cardinality n, define $f(a) = \{i \in I_n \mid \bigcap_{x \in a} \varphi_x(i) \neq \emptyset\}$. By the hypothesis, the finite intersection $\bigcap_{x \in a} A_x \neq \emptyset$, and so $f(a) \in \mathcal{U}$. Since f is antimonotonic, by the goodness property of $\mathcal{U}$, we can pick an antiadditive $g : \mathrm{Fin}(I) \to \mathcal{U}$ such that $g(a) \subseteq f(a)$ for all $a \in \mathrm{Fin}(I)$. For $j \in I$, let $\Lambda_j = \{x \mid j \in g(\{x\})\}$. If there exist distinct elements $y_1, \ldots, y_k \in \Lambda_j$, then

$$j \in g(\{y_1\}) \cap \ldots \cap g(\{y_k\}) = g(\{y_1, \ldots, y_k\}) \subseteq f(\{y_1, \ldots, y_k\}) \subseteq I_k.$$

By recalling that $\bigcap_k I_k = \emptyset$, we can conclude that every Λ_j must be finite. Notice that $j \in \bigcap_{x \in \Lambda_j} g(\{x\}) = g(\Lambda_j) \subseteq f(\Lambda_j)$ and so we can pick an element $h(j) \in \bigcap_{x \in \Lambda_j} \varphi_x(j) \neq \emptyset$. In this way, for every $x \in I$, we have $j \in g(\{x\}) \Leftrightarrow x \in \Lambda_j \Rightarrow h(j) \in \varphi_x(j)$. If $\xi = [h]_{\mathcal{U}} \in {}^*Y$ is the element in the ultrapower model that corresponds to the function $h : I \to Y$, then $\xi \in \bigcap_{x \in I} A_x$, because for every x one has the inclusion $\{j \in I \mid h(j) \in \varphi_x(j)\} \supseteq g(\{x\}) \in \mathcal{U}$.

(2) $\Rightarrow$ (1). Let $f : \mathrm{Fin}(I) \to \mathcal{U}$ be antimonotonic. For $a \in \mathrm{Fin}(I)$, let $G_a : I \to \mathcal{P}(\mathrm{Fin}(I))$ be the function where $G_a(i) = \{b \supseteq a \mid i \in f(b)\}$. Notice that $G_a(i) \cap G_b(i) = G_{a \cup b}(i)$. For every $x \in I$, $A_x = [G_{\{x\}}]_{\mathcal{U}}$ is an internal subset of ${}^*\mathrm{Fin}(I)$, and the family $\{A_x \mid x \in I\}$ has the FIP because ${}^*a \in \bigcap_{x \in a} A_x$ for every $a \in \mathrm{Fin}(I)$. Indeed, $a \in \bigcap_{x \in a} G_{\{x\}}(i) = G_a(i) \Leftrightarrow i \in f(a)$, and $f(a) \in \mathcal{U}$. By κ^+-*saturation*, we can pick a function $\vartheta : I \to \mathrm{Fin}(I)$ such that the corresponding element in the ultrapower model $\xi = [\vartheta]_{\mathcal{U}} \in \bigcap_{x \in I} A_x$. Finally, define $g(a) = \{i \in I \mid \vartheta(i) \in G_a(i)\}$. Clearly, $g(a) \in \mathcal{U}$ because $\xi \in \bigcap_{x \in a} A_x$. Moreover, $i \in g(a) \Leftrightarrow \vartheta(i) \in G_a(i) \Leftrightarrow \vartheta(i) \supseteq a$ and $i \in f(\vartheta(i))$, and so $i \in f(\vartheta(i)) \subseteq f(a)$. This shows that $g(a) \subseteq f(a)$. Besides, g is antiadditive because $i \in g(a) \cap g(b) \Leftrightarrow \vartheta(i) \in G_a(i)$ and $\vartheta(i) \in G_b(i) \Leftrightarrow \vartheta(i) \in G_{a \cup b}(i) \Leftrightarrow i \in g(a \cup b)$. □

Points that generate good ultrafilters are characterized as follows.

Theorem 4.6. *Given a model of nonstandard analysis, let $\alpha \in {}^*I$ be such that $\mathfrak{U}_\alpha$ is countably incomplete. Then the following are equivalent:*

(1) $\mathfrak{U}_\alpha$ *is a good ultrafilter;*
(2) *If* $F : I \to \mathcal{P}(Fin(I))$ *has the property that for every finite* $a \subset I$ *there exists* $A \in {}^*F(\alpha)$ *with* ${}^*a \subseteq A$*, then there exists a function* $\vartheta : I \to Fin(I)$ *such that* ${}^*x \in {}^*\vartheta(\alpha) \in {}^*F(\alpha)$ *for all* $x \in I$.

Proof. The proof is similar to the one of the previous proposition.

(1) $\Rightarrow$ (2). Fix a sequence $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$ of sets in $\mathfrak{U}_\alpha$ such that $\bigcap_n I_n = \emptyset$. Now let F satisfy the conditions in (2). For every finite $a = \{x_1, \ldots, x_n\} \subset I$ with n-many elements, put:

$$f(a) \;=\; \{i \in I_n \mid \exists b \in F(i) \text{ such that } a \subseteq b\}.$$

By the hypothesis, there exists $A \in {}^*F(\alpha)$ with ${}^*x_1, \ldots, {}^*x_n \in A$. But then α is an element of ${}^*I_n$ such that ${}^*a \subseteq A$ for a suitable $A \in {}^*F(\alpha)$, and this means that $\alpha \in {}^*(f(a))$, *i.e.* $f(a) \in \mathfrak{U}_\alpha$. Moreover, it directly follows from the definition that $f : \mathrm{Fin}(I) \to \mathfrak{U}_\alpha$ is antimonotonic. So, we can apply the hypothesis and pick an antiadditive function $g : \mathrm{Fin}(I) \to \mathfrak{U}_\alpha$ such that $g(a) \subseteq f(a)$ for all a. For $i \in I$, put $\Lambda(i) = \{x \mid i \in g(\{x\})\}$. Notice that if $a \subseteq \Lambda(i)$ has cardinality n, then $i \in \bigcap_{x \in a} g(\{x\}) = g(a) \subseteq f(a) \subseteq I_n$; and since $\bigcap I_n = \emptyset$, this shows that $\Lambda(i)$ must be finite. Finally, pick any function $\vartheta : I \to \mathrm{Fin}(I)$ such that $\Lambda(i) \subseteq \vartheta(i) \in F(i)$ for all $i \in f(\Lambda(i))$. Since $\alpha \in {}^*(f(\Lambda(i))$, it is ${}^*\Lambda(\alpha) \subseteq {}^*\vartheta(\alpha) \in {}^*F(\alpha)$. Besides, for every $x \in I$ we have that $\alpha \in {}^*(g(\{x\})) = {}^*g(\{{}^*x\})$, so ${}^*x \in {}^*\Lambda(\alpha)$, and hence ${}^*x \in {}^*\vartheta(\alpha)$.

(2) $\Rightarrow$ (1). Given an antimonotonic function $f : \mathrm{Fin}(I) \to \mathfrak{U}_\alpha$, define $F : I \to \mathcal{P}(\mathrm{Fin}(I))$ by putting $F(i) = \{a \in \mathrm{Fin}(I) \mid i \in f(a)\}$. For every $a = \{x_1, \ldots, x_n\} \in \mathrm{Fin}(I)$, $\alpha \in {}^*(f(a)) = {}^*\{i \in I \mid a \in F(i)\}$, and so ${}^*a = \{{}^*x_1, \ldots, {}^*x_n\} \in {}^*F(\alpha)$. Then, by the hypothesis there exists a function $\vartheta : I \to \mathrm{Fin}(I)$ such that ${}^*\vartheta(\alpha) \in {}^*F(\alpha)$ and ${}^*x \in {}^*\vartheta(\alpha)$ for every $x \in I$. Finally, put $g(a) = \{i \in I \mid a \subseteq \vartheta(i) \in F(i)\}$. Notice that $\alpha \in {}^*(g(a))$, because ${}^*a \subseteq {}^*\vartheta(\alpha) \in {}^*F(\alpha)$. It is readily verified that g is antiadditive. Moreover, notice that $\vartheta(i) \in F(i) \Leftrightarrow i \inf(\vartheta(i))$, so $i \in g(a) \Rightarrow a \subseteq \vartheta(i)$ and $i \in f(\vartheta(i)) \Rightarrow i \in f(a)$, and also the desired inclusion $g(a) \subseteq f(a)$ is verified. □

Remark 4.7. A much simplied proof of the above theorem could also be obtained by employing a known property from nonstandard set theory. Precisely, let $\mathfrak{M}$ be a given model of nonstandard analysis, and let $\alpha \in {}^*I$ be such that the generated ultrafilter $\mathfrak{U}_\alpha$ is countably incomplete. Then the model of nonstandard analysis determined by $\mathfrak{U}_\alpha$ is isomorphic to the elementary submodel $\mathfrak{M}[\alpha] \prec \mathfrak{M}$ whose universe is given by the elements that are *standard relative to* α, *i.e.* by the elements of the form ${}^*f(\alpha)$ where f is any function defined on I (see [29,

§6] and references therein). By working inside $\mathfrak{M}[\alpha]$, one can directly use the equivalence of Theorem 4.5 to derive Theorem 4.6.

5. Ultrafilters generated by pairs

As in Section 2, we now work in a fixed $\mathfrak{c}^+$-*saturated* model of nonstandard analysis, and extend the u-equivalence to pairs.

Definition 5.1. The *ultrafilter generated* by an ordered pair $(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ is the family

$$\mathfrak{U}_{(\alpha,\beta)} \;=\; \{X \subseteq \mathbb{N} \times \mathbb{N} \mid (\alpha, \beta) \in {}^*X\}.$$

The $\mathfrak{c}^+$-*saturation* property guarantees that every ultrafilter on $\mathbb{N}\times\mathbb{N}$ is generated by some pair $(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$. The u-equivalence $\underset{u}{\sim}$ relation on ${}^*\mathbb{N} \times {}^*\mathbb{N}$ is defined in exactly in the same way as it was defined on ${}^*\mathbb{N}$, that is we set $(\alpha, \beta) \underset{u}{\sim} (\alpha', \beta')$ when $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_{(\alpha',\beta')}$. We recall that the *Cartesian product of filters*

$$\mathcal{U} \times \mathcal{V} \;=\; \{A \times B \mid A \in \mathcal{U}, B \in \mathcal{V}\}$$

is a filter; however, if $\mathcal{U}$ and $\mathcal{V}$ are non-principal ultrafilters, then $\mathcal{U} \times \mathcal{V}$ is *not* an ultrafilter, and indeed there may be plenty of ultrafilters $\mathcal{W} \supset \mathcal{U} \times \mathcal{V}$ (see Remark 5.5 below).

A canonical class of ultrafilters on the Cartesian product is given by the so-called tensor products: If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on $\mathbb{N}$, the *tensor product* $\mathcal{U} \otimes \mathcal{V}$ is the ultrafilter on $\mathbb{N} \times \mathbb{N}$ defined by setting:

$$X \in \mathcal{U} \otimes \mathcal{V} \iff \{n \mid \{m \mid (n, m) \in X\} \in \mathcal{V}\} \in \mathcal{U}.$$

It is easily seen that $\mathcal{U} \otimes \mathcal{V} \supseteq \mathcal{U} \times \mathcal{V}$. We recall that tensor products are not commutative in all non-trivial cases; indeed, if we denote by $\Delta^+ = \{(n, m) \mid n < m\}$ then $\Delta^+ \in \mathcal{U} \otimes \mathcal{V}$ and $\Delta^+ \notin \mathcal{V} \otimes \mathcal{U}$ for all non-principal $\mathcal{U}$ and $\mathcal{V}$.

A first easy observation about pairs is the following.

Proposition 5.2. $\mathfrak{U}_{(\alpha,\beta)} \supseteq \mathfrak{U}_\alpha \times \mathfrak{U}_\beta$.

Proof. If $A \in \mathfrak{U}_\alpha$ and $B \in \mathfrak{U}_\beta$, then trivially $(\alpha, \beta) \in {}^*A\times{}^*B = {}^*(A\times B)$, *i.e.*, $A \times B \in \mathfrak{U}_{(\alpha,\beta)}$. □

It is also easy to improve on the above property, and characterize those products of ultrafilters that are contained in a given ultrafilter on $\mathbb{N} \times \mathbb{N}$.

Proposition 5.3. $\mathfrak{U}_{(\alpha,\beta)} \supseteq \mathfrak{U}_\gamma \times \mathfrak{U}_\delta$ *if and only if* $\alpha \underset{u}{\sim} \gamma$ *and* $\beta \underset{u}{\sim} \delta$. *In consequence,* $(\alpha, \beta) \underset{u}{\sim} (\gamma, \delta) \Rightarrow \alpha \underset{u}{\sim} \gamma$ *and* $\beta \underset{u}{\sim} \delta$.

Proof. If $\alpha \underset{u}{\sim} \gamma$ and $\beta \underset{u}{\sim} \delta$, then $\mathfrak{U}_{(\alpha,\beta)} \supseteq \mathfrak{U}_\alpha \times \mathfrak{U}_\beta = \mathfrak{U}_\gamma \times \mathfrak{U}_\delta$. Conversely, assume that $\alpha \underset{u}{\not\sim} \gamma$ or $\beta \underset{u}{\not\sim} \delta$, say $\alpha \underset{u}{\not\sim} \gamma$, and pick $A \subseteq \mathbb{N}$ such that $\alpha \in {}^*A$ and $\gamma \notin {}^*A$. Then $A^c \times \mathbb{N} \in \mathfrak{U}_\gamma \times \mathfrak{U}_\delta$ because $\gamma \in {}^*A^c$ and trivially $\delta \in {}^*\mathbb{N}$, but $A^c \times \mathbb{N} \notin \mathfrak{U}_{(\alpha,\beta)}$ because $(\alpha,\beta) \notin {}^*(A^c \times \mathbb{N})$. Finally, if $(\alpha,\beta) \underset{u}{\sim} (\gamma,\delta)$ then $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_{(\gamma,\delta)} \supseteq \mathfrak{U}_\gamma \times \mathfrak{U}_\delta$, and so by what just proved above, it must be $\alpha \underset{u}{\sim} \gamma$ and $\beta \underset{u}{\sim} \delta$. □

The above proposition can be reformulated in "standard" terms as follows:

Proposition 5.4. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on $\mathbb{N}$ and let $\mathcal{W}$ be an ultrafilter on $\mathbb{N} \times \mathbb{N}$. Then $\mathcal{W} \supseteq \mathcal{U} \times \mathcal{V}$ if and only if $\mathcal{U} = \pi_1(\mathcal{W})$ and $\mathcal{V} = \pi_2(\mathcal{W})$ where π_1 and π_2 are the canonical projections on the first and second coordinate, respectively.*

Proof. Pick $\alpha, \beta, \gamma, \delta \in {}^*\mathbb{N}$ such that $\mathcal{W} = \mathfrak{U}_{(\alpha,\beta)}$, $\mathcal{U} = \mathfrak{U}_\gamma$ and $\mathcal{V} = \mathfrak{U}_\delta$. By Proposition 2.4, $\pi_1(\mathcal{W}) = \mathfrak{U}_{{}^*\pi_1(\alpha,\beta)} = \mathfrak{U}_\alpha$ and similarly $\pi_2(\mathcal{W}) = \mathfrak{U}_\beta$. Then apply the previous proposition. □

Remark 5.5. The implication $(\alpha,\beta) \underset{u}{\sim} (\gamma,\delta) \Rightarrow \alpha \underset{u}{\sim} \gamma$ and $\beta \underset{u}{\sim} \delta$ cannot be reversed. Indeed, it is well known that for every non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ there are at least two different ultrafilters $\mathcal{W} \supset \mathcal{U} \times \mathcal{V}$, namely $\mathcal{U} \otimes \mathcal{V}$ and $\sigma(\mathcal{V} \otimes \mathcal{U})$ where $\sigma(n,m) = (m,n)$ is the map that permutes the coordinates. Actually, provided there are no P-points RK-below $\mathcal{U}$ or $\mathcal{V}$, there exist infinitely many $\mathcal{W} \supset \mathcal{U} \times \mathcal{V}$ (see [10] and references therein).

About the existence of pairs of hypernatural numbers that generate ultrafilters $\mathcal{W} \supseteq \mathcal{U} \times \mathcal{V}$, the following result holds.

Proposition 5.6. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on $\mathbb{N}$, and let $X \in {}^*\mathcal{U}$ and $Y \in {}^*\mathcal{V}$. Then for every ultrafilter $\mathcal{W} \supset \mathcal{U} \times \mathcal{V}$, there exist $\alpha \in X$ and $\beta \in Y$ such that $\mathfrak{U}_{(\alpha,\beta)} = \mathcal{W}$.*

Proof. Since $\mathcal{W} \supset \mathcal{U} \times \mathcal{V}$, the intersection $(U \times V) \cap W \neq \emptyset$ is non-empty for every $U \in \mathcal{U}$, $V \in \mathcal{V}$, and $W \in \mathcal{W}$. Then, by *transfer*, $(X \times Y) \cap Z \neq \emptyset$ for all $Z \in {}^*\mathcal{W}$. In particular, the family $\{(X \times Y) \cap {}^*W \mid W \in \mathcal{W}\}$ has the finite intersection property. By $\mathfrak{c}^+$-*saturation*, we can pick a pair $(\alpha,\beta) \in X \times Y$ such that $(\alpha,\beta) \in {}^*W$ for all $W \in \mathcal{W}$, so that $\mathfrak{U}_{(\alpha,\beta)} = \mathcal{W}$. (This proof was communicated to the author by Karel Hrbáček, and it is reproduced here under his permission.) □

Now let us fix some useful notation. For $X \subseteq \mathbb{N} \times \mathbb{N}$, $n \in \mathbb{N}$, and $\xi \in {}^*\mathbb{N}$:

- $X_n = \{\, m \in \mathbb{N} \mid (n,m) \in X\}$ is the vertical n-fiber of X;
- ${}^*X_\xi = \{\zeta \in {}^*\mathbb{N} \mid (\xi,\zeta) \in {}^*X\}$ is the vertical ξ-fiber of ${}^*X$.

Notice that ${}^*X_\xi = {}^*\chi(\xi)$ is the value taken at ξ by the hyper-extension of the sequence $\chi(n) = X_n$. Notice also that for finite $k \in \mathbb{N}$, one has ${}^*X_k = {}^*(X_k)$.

It directly follows from the definitions that

$$X \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \iff \alpha \in {}^*\{n \mid X_n \in \mathfrak{U}_\beta\} \iff {}^*X_\alpha \in {}^*\mathfrak{U}_\beta.$$

Theorem 5.7. *Let $\alpha, \beta \in {}^*\mathbb{N}$ be infinite numbers. Then the following properties are equivalent:*

1. $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$*;*
2. (α, β) *generates a tensor product;*
3. *For every $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, if $\beta \in {}^*F(\alpha)$, then ${}^*F(\alpha) \in {}^*\mathfrak{U}_\beta$;*
4. *For every $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, if ${}^*F(\alpha) \in {}^*\mathfrak{U}_\beta$, then $\beta \in {}^*F(\alpha)$;*
5. *For every $X \subseteq \mathbb{N} \times \mathbb{N}$, if $(n, \beta) \in {}^*X$ for all $n \in \mathbb{N}$, then $(\alpha, \beta) \in {}^*X$;*
6. *For every $X \subseteq \mathbb{N} \times \mathbb{N}$, if $(\alpha, \beta) \in {}^*X$, then $(n, \beta) \in {}^*X$ for some $n \in \mathbb{N}$;*
7. *For every $f : \mathbb{N} \to \mathbb{N}$, if ${}^*f(\beta) \notin \mathbb{N}$, then ${}^*f(\beta) > \alpha$.*

Proof. (1) $\Leftrightarrow$ (2). One implication is trivial. Conversely, let us assume that $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\gamma \otimes \mathfrak{U}_\delta$ for some γ, δ. We have seen in Proposition 5.2 that $\mathfrak{U}_{(\alpha,\beta)}$ extends $\mathfrak{U}_\alpha \times \mathfrak{U}_\beta$; moreover, the tensor product $\mathfrak{U}_\gamma \otimes \mathfrak{U}_\delta$ extends $\mathfrak{U}_\gamma \times \mathfrak{U}_\delta$. But then it must be $\mathfrak{U}_\alpha = \mathfrak{U}_\gamma$ and $\mathfrak{U}_\beta = \mathfrak{U}_\delta$, as otherwise there would be disjoint sets in $\mathfrak{U}_{(\alpha,\beta)}$.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). Given a function $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$, let

$$\Theta(F) = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid m \in F(n)\}$$

be the set of pairs whose vertical n-fibers $\Theta(F)_n = F(n)$. Then, we have $\Theta(F) \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \Leftrightarrow {}^*\Theta(F)_\alpha = {}^*F(\alpha) \in {}^*\mathfrak{U}_\beta$. Moreover, $\beta \in {}^*F(\alpha) \Leftrightarrow (\alpha, \beta) \in {}^*\Theta(F) \Leftrightarrow \Theta(F) \in \mathfrak{U}_{(\alpha,\beta)}$. Now notice that for every $X \subseteq \mathbb{N} \times \mathbb{N}$, one has $X = \Theta(F)$ where $F(n) = X_n$ is the sequence of the n-fibers of X. In consequence, properties (3) and (4) are equivalent to the conditions $\mathfrak{U}_{(\alpha,\beta)} \subseteq \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$ and $\mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \subseteq \mathfrak{U}_{(\alpha,\beta)}$, respectively. The proof is complete because inclusions between ultrafilters imply equalities.

(5) $\Leftrightarrow$ (6). This is straightforward, because property (5) for a set X is the contrapositive of property (6) for the complement X^c.

(1) $\Rightarrow$ (5). Notice that $(n, \beta) \in {}^*X \Leftrightarrow \beta \in {}^*(X_n) \Leftrightarrow X_n \in \mathfrak{U}_\beta$. So, by the hypothesis, the set $\{n \mid X_n \in \mathfrak{U}_\beta\} = \mathbb{N}$. As trivially $\mathbb{N} \in \mathfrak{U}_\alpha$, we conclude that $X \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta = \mathfrak{U}_{(\alpha,\beta)}$, and hence $(\alpha, \beta) \in {}^*X$.

(5) $\Rightarrow$ (7). Let $X = \{(n, m) \mid n < f(m)\}$. Since ${}^*f(\beta)$ is infinite, we have that $(n, \beta) \in {}^*X$ for all finite $n \in \mathbb{N}$. But then $(\alpha, \beta) \in {}^*X$, *i.e.* $\alpha < {}^*f(\beta)$.

(7) ⇒ (1) ([44], Th. 3.4). The set $\Delta^+ = \{(n,m) \mid n < m\}$ belongs to $\mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$; moreover, since $\alpha < \beta$, it is also $\Delta^+ \in \mathfrak{U}_{(\alpha,\beta)}$. Thus, it suffices to show that the implication $X \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \Rightarrow X \in \mathfrak{U}_{(\alpha,\beta)}$ holds for all $X \subseteq \Delta^+$. By definition, $X \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \Leftrightarrow \alpha \in {}^*\{n \mid X_n \in \mathfrak{U}_\beta\}$, and so $X_n \in \mathfrak{U}_\beta$ for infinitely many n. In consequence, for every m one can always find $n > m$, and hence $(n,m) \notin X$, such that $X_n \in \mathfrak{U}_\beta$. This means that

$$F(m) = \{n \mid X_n \in \mathfrak{U}_\beta \;\&\; (n,m) \notin X\} \neq \emptyset.$$

If $f : \mathbb{N} \to \mathbb{N}$ is the function $f(m) = \min F(m)$, then the number ${}^*f(\beta)$ is infinite. Indeed, if by contradiction ${}^*f(\beta) = k \in \mathbb{N}$, then we would have $k \in {}^*F(\beta)$, that is $({}^*X)_k = {}^*(X_k) \in {}^*\mathfrak{U}_\beta$ and $(k,\beta) \notin {}^*X$, and hence $X_k \in \mathfrak{U}_\beta$ and $X_k \notin \mathfrak{U}_\beta$, a contradiction. Now, by (7) it follows that $\alpha < {}^*f(\beta) = \min {}^*F(\beta)$, and so $\alpha \notin {}^*F(\beta)$. This means that it is not the case that both ${}^*X_\alpha \in {}^*\mathfrak{U}_\beta$ and $(\alpha,\beta) \notin {}^*X$. We reach the thesis $(\alpha,\beta) \in {}^*X$ by recalling that ${}^*X_\alpha \in {}^*\mathfrak{U}_\beta \Leftrightarrow X \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$. □

Definition 5.8. We say that $(\alpha,\beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ is a *tensor pair* if it satisfies all the equivalent conditions in the previous theorem.

Notice that if $n \in \mathbb{N}$ is finite, then all pairs (n,β) are trivially tensor pairs. The property of generating tensor products is preserved under a special class of maps.

Proposition 5.9. *If (α,β) is a tensor pair then for every $f, g : \mathbb{N} \to \mathbb{N}$ also $({}^*f(\alpha), {}^*g(\beta))$ is a tensor pair. In "standard" terms, this means that if (f,g) is the map $(n,m) \mapsto (f(n), g(m))$, then for all ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the image $(f,g)(\mathcal{U} \otimes \mathcal{V}) = f(\mathcal{U}) \otimes g(\mathcal{V})$.*

Proof. We use the characterization of tensor pairs as given by (6) of Proposition 5.7. Notice that ${}^*(f,g) = ({}^*f, {}^*g)$, and so

$$({}^*f(\alpha), {}^*g(\beta)) \in {}^*X \;\Leftrightarrow\; (\alpha,\beta) \in ({}^*f, {}^*g)^{-1}({}^*X) \;=\; {}^*[(f,g)^{-1}(X)].$$

By the hypothesis, we can pick $n \in \mathbb{N}$ such that $(n,\beta) \in {}^*[(f,g)^{-1}(X)]$, that is $({}^*f(n), {}^*g(\beta)) \in {}^*X$, where ${}^*f(n) = f(n) \in \mathbb{N}$. □

As we already noticed in Remark 5.5, both the tensor product $\mathcal{U} \otimes \mathcal{U}$ and its image $\sigma(\mathcal{U} \otimes \mathcal{U})$ under the map $\sigma(n,m) = (m,n)$ extend the Cartesian product $\mathcal{U} \times \mathcal{U}$. However, there is another canonical ultrafilter that extend the product of an ultrafilter $\mathcal{U}$ by itself, namely the *diagonal ultrafilter* determined by $\mathcal{U}$:

$$\Delta_{\mathcal{U}} \;=\; \{X \subseteq \mathbb{N} \times \mathbb{N} \mid \{n \mid (n,n) \in X\} \in \mathcal{U}\}.$$

Clearly $\Delta_{\mathcal{U}} \cong \mathcal{U}$. Notice that the diagonal $\Delta = \{(n,n) \mid n \in \mathbb{N}\} \in \Delta_{\mathcal{U}}$, but $\Delta \notin \mathcal{U} \otimes \mathcal{U}$ and $\Delta \notin \sigma(\mathcal{U} \otimes \mathcal{U})$ whenever $\mathcal{U}$ is non-principal. Notice also that if $\mathcal{U} = \mathfrak{U}_\alpha$, then $\Delta_{\mathcal{U}} = \mathfrak{U}_{(\alpha,\alpha)}$.

Proposition 5.10. *Let $\mathcal{U}$ be non-principal, and let $\mathcal{W}$ be an ultrafilter that extends the product filter $\mathcal{U}\times\mathcal{U}$. Then $\mathcal{W}\leq_{RK}\mathcal{U}$ if and only if $\mathcal{W}=\Delta_{\mathcal{U}}$.*

Proof. Let $\mathcal{U}=\mathfrak{U}_\alpha$. If h is the function such that $h(n)=(n,n)$ for all n, then $h(\mathcal{U})=\mathfrak{U}_{{}^*h(\alpha)}=\mathfrak{U}_{(\alpha,\alpha)}=\Delta_{\mathcal{U}}$, and so $\Delta_{\mathcal{U}}\leq_{RK}\mathcal{U}$. Conversely, let $\mathcal{W}\supset\mathcal{U}\times\mathcal{U}$ and assume that $F(\mathfrak{U}_\alpha)=\mathcal{W}$ for a suitable function $F:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$, say $F(n)=(f(n),g(n))$. By Proposition 5.3, we can pick $\beta\underset{u}{\sim}\gamma\underset{u}{\sim}\alpha$ such that $\mathcal{W}=\mathfrak{U}_{(\beta,\gamma)}$. Then

$$\mathfrak{U}_{(\beta,\gamma)}=F(\mathfrak{U}_\alpha)=\mathfrak{U}_{{}^*F(\alpha)}=\mathfrak{U}_{({}^*f(\alpha),{}^*g(\alpha))}.$$

Thus, ${}^*f(\alpha)\underset{u}{\sim}\beta\underset{u}{\sim}\alpha$ and ${}^*g(\alpha)\underset{u}{\sim}\gamma\underset{u}{\sim}\alpha$ and so, by Theorem 2.7, ${}^*f(\alpha)={}^*g(\alpha)=\alpha$. We conclude that $\mathcal{W}=\mathfrak{U}_{(\alpha,\alpha)}=\Delta_{\mathcal{U}}$. □

Corollary 5.11. *If $\mathcal{U}$ is a non-principal ultrafilter, then $\mathcal{U}\otimes\mathcal{U}\not\leq_{RK}\mathcal{U}$ (and hence $\mathcal{U}\otimes\mathcal{U}\not\cong\mathcal{U}$).*

We close this section by showing that tensor pairs are found in abundance.

Theorem 5.12. *Let $\alpha,\beta\in{}^*\mathbb{N}$ be infinite. Then:*

(1) *The set $R_{\alpha,\beta}=\{\beta'\mid\beta'\underset{u}{\sim}\beta\ \&\ (\alpha,\beta')$ tensor pair$\}$ contains $|{}^*\mathbb{N}|$-many elements and it is unbounded in ${}^*\mathbb{N}$;*
(2) *The set $L_{\alpha,\beta}=\{\alpha'\mid\alpha'\underset{u}{\sim}\alpha\ \&\ (\alpha',\beta)$ tensor pair$\}$ is unbounded leftward in ${}^*\mathbb{N}\setminus\mathbb{N}$, and hence it contains at least $\mathfrak{c}^+$-many elements.*

Proof. (1). We prove the thesis by showing that there exists an internal hyperinfinite set of elements $\beta'\underset{u}{\sim}\beta$ such that (α,β') is a tensor pair. Let $B\subseteq\mathbb{N}$ and $f:\mathbb{N}\to\mathbb{N}$ be such that $\beta\in{}^*B$ and ${}^*f(\beta)\notin\mathbb{N}$. Then ${}^*f(\beta)\in{}^*(f(B))$ implies that the image set $f(B)$ is infinite, and so the following property holds:

$$\forall y\in\mathbb{N}\ \exists\,\sigma:\mathbb{N}\to B\text{ 1-1 s.t. }\forall x\in\mathbb{N}\ f(\sigma(x))>y.$$

By *transfer*, it follows that the following internal set is non-empty:

$$\Gamma_{B,f}\ =\ \{\sigma:{}^*\mathbb{N}\to{}^*B\text{ internal}\mid\sigma\text{ 1-1 }\&\ {}^*f({}^*\sigma(\nu))>\alpha\text{ for all }\nu\in{}^*\mathbb{N}\}.$$

Notice that $\Gamma_{B_1,f_1}\cap\ldots\cap\Gamma_{B_k,f_k}=\Gamma_{B,f}$ where $B=B_1\cap\ldots\cap B_k$ and $f=\min\{f_1,\ldots,f_k\}$. Then, the family $\{\Gamma_{B,f}\mid\beta\in{}^*B\,\&\,{}^*f(\beta)\notin\mathbb{N}\}$ has the finite intersection property and so, by $\mathfrak{c}^+$-enlarging, we can pick an element

$$\tau\ \in\ \bigcap\{{}^*\Gamma_{B,f}\mid\beta\in{}^*B\ \&\ {}^*f(\beta)\notin\mathbb{N}\}\,.$$

Notice that $\tau(\nu)\underset{u}{\sim}\beta$ for all $\nu\in{}^*\mathbb{N}$, and therefore range(τ) is an hyperinfinite set of elements $\beta'\underset{u}{\sim}\beta$ such that ${}^*f(\beta')>\alpha$ whenever ${}^*f(\beta)\notin\mathbb{N}$.

Finally, since $\beta' \underset{u}{\sim} \beta$, one has that ${}^*f(\beta) \notin \mathbb{N} \Leftrightarrow {}^*f(\beta') \notin \mathbb{N}$; so, condition (7) of Theorem 5.7 applies, and all pairs (α, β') are tensor pairs.

(2). We recall that by $\mathfrak{c}^+$-*saturation*, the coinitiality of ${}^*\mathbb{N} \setminus \mathbb{N}$ is greater than $\mathfrak{c}$, and so we can pick an infinite ξ such that $\xi < {}^*f(\beta)$ for all f with ${}^*f(\beta) \notin \mathbb{N}$. Then, by (7) of Theorem 5.7, (α', β) is a tensor pair for all $\alpha' < \xi$, and the thesis follows because the set $u(\alpha) \cap [0, \xi)$ is unbounded leftward in ${}^*\mathbb{N} \setminus \mathbb{N}$, by Theorem 2.13. □

6. Hyper-shifts

The following notion was introduced by M. Beiglböck in [4]:

Definition 6.1. Let $A \subseteq \mathbb{N}$ and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. The *ultrafilter-shift* of A by $\mathcal{U}$ is defined as the set

$$A - \mathcal{U} \ = \ \{n \in \mathbb{N} \mid A - n \in \mathcal{U}\}.$$

We now introduce a class of subsets of $\mathbb{N}$, which are found as segments of hyper-extensions, and that precisely correspond to ultrafilter-shifts.

Definition 6.2. The *hyper-shift* of a set $A \subseteq \mathbb{N}$ by a number $\gamma \in {}^*\mathbb{N}$ is the following set:

$$A_\gamma \ = \ ({}^*A - \gamma) \cap \mathbb{N} \ = \ \{n \in \mathbb{N} \mid \gamma + n \in {}^*A\}.$$

It is readily seen that hyper-shifts are coherent with respect to finite shifts, intersections, unions, and complements.

Proposition 6.3. *For every $A, B \subseteq \mathbb{N}$, for every $n \in \mathbb{N}$, and for every $\gamma \in {}^*\mathbb{N}$, the following equalities hold:*

1. $(A - n)_\gamma = A_\gamma - n\,;$
2. $(A \cap B)_\gamma = A_\gamma \cap B_\gamma\,;$
3. $(A \cup B)_\gamma = A_\gamma \cup B_\gamma\,;$
4. $(A^c)_\gamma = (A_\gamma)^c.$

Proposition 6.4. *For every $A \subseteq \mathbb{N}$ and for every $\gamma \in {}^*\mathbb{N}$,*

$$A_\gamma \ = \ A - \mathfrak{U}_\gamma.$$

Proof. The following chain of equivalences is directly obtained from the definitions: $n \in A_\gamma \Leftrightarrow \gamma + n \in {}^*A \Leftrightarrow \gamma \in {}^*A - n = {}^*(A - n) \Leftrightarrow A - n \in \mathfrak{U}_\gamma \Leftrightarrow n \in A - \mathfrak{U}_\gamma$. □

Ultrafilter-shifts (and their nonstandard counterparts, the hyper-shifts) have a precise combinatorial meaning corresponding to a notion of embeddability.

Definition 6.5. Let $A, B \subseteq \mathbb{N}$. We say that A is *exactly embedded* in B, and write $A \leq_e B$, if for every finite interval I there exists x such that $x + (A \cap I) = B \cap (x + I)$.

A similar relation between sets of natural numbers, named "finite embeddability", has been considered in [17, §4] (see also [9], where that notion was extended to ultrafilters). The difference is that "A finitely embedded in B" only requires the inclusion $x + (A \cap I) \subseteq B \cap (x + I)$.

With respect to finite configurations, $A \leq_e B$ tells that B is at least as combinatorially rich as A. For example, if A contains arbitrarily long arithmetic progressions and $A \leq_e B$, then also B contains arbitrarily long arithmetic progressions.

Proposition 6.6. *For $A, B \subseteq \mathbb{N}$, the following are equivalent:*

(1) $A \leq_e B$;
(2) *A is an ultrafilter-shift of B ;*
(3) *$A = B_\gamma$ for some $\gamma \in {}^*\mathbb{N}$.*

Proof. (2) $\Leftrightarrow$ (3) is given by Proposition 6.4.

(1) $\Rightarrow$ (3). Given $n \in \mathbb{N}$, let us consider the sets

$$\Lambda_n \;=\; \{x \in \mathbb{N} \mid x + (A \cap [1, n]) = B \cap [x + 1, x + n]\}\,.$$

By the hypothesis, $\Lambda_n \neq \emptyset$ for all $n \in \mathbb{N}$ and so, by *overspill*, we can pick $\gamma \in {}^*\Lambda_\nu$ for some infinite $\nu \in {}^*\mathbb{N}$. (We denoted by ${}^*\Lambda_\nu = {}^*F(\nu)$ the value taken at ν by the hyper-extension of the function $F(n) = \Lambda_n$.) Then $\gamma + ({}^*A \cap [1, \nu]) = {}^*B \cap [\gamma + 1, \gamma + \nu]$, which implies $A = ({}^*B - \gamma) \cap \mathbb{N} = B_\gamma$.

(3) $\Rightarrow$ (1). If $A = B_\gamma$, then for every interval $I \subset \mathbb{N}$ we have ${}^*A \cap I = A \cap I = B_\gamma \cap I = ({}^*B - \gamma) \cap I$. So, γ is a witness of the following property: "$\exists \xi \in {}^*\mathbb{N},\ \xi + ({}^*A \cap I) = {}^*B \cap (\xi + I)$." By *transfer* we obtain the thesis: "$\exists x \in \mathbb{N},\ x + (A \cap I) = B \cap (x + I)$." □

Let us now see how ultrafilter-shifts and hyper-shifts can be used to characterize density. Recall the following

Definition 6.7. The *Schnirelmann density* of $A \subseteq \mathbb{N}$ is defined as

$$\sigma(A) \;=\; \inf_{n \in \mathbb{N}} \frac{|A \cap [1, n]|}{n}.$$

The *asymptotic density* of $A \subseteq \mathbb{N}$ is defined as

$$d(A) \;=\; \lim_{n \to \infty} \frac{|A \cap [1, n]|}{n},$$

when the limit exists. Otherwise, one defines the *upper density* $\overline{d}(A)$ and the *lower density* $\underline{d}(A)$ by taking the limit superior or the limit inferior of the above sequence, respectively.

Another notion that is used in combinatorial number theory is the following uniform version of the asymptotic density.

Definition 6.8. The *Banach density* of $A \subseteq \mathbb{N}$ is defined as

$$\mathrm{BD}(A) \;=\; \lim_{n\to\infty} \frac{\max_{k\in\mathbb{N}} |A\cap[k+1,k+n]|}{n}.$$

Notice that the sequence $a_n = \max_k |A\cap[k+1,k+n]|$ is *subadditive*, *i.e.* it satisfies $a_{n+m} \leq a_n + a_m$. In consequence, one can show that $\lim_n a_n/n$ actually exists, and in fact $\lim_n a_n/n = \inf_n a_n/n$. It is readily checked that for every $A \subseteq \mathbb{N}$:

$$\sigma(A) \;\leq\; \underline{d}(A) \;\leq\; \overline{d}(A) \;\leq\; \mathrm{BD}(A).$$

On the other hand, there exist sets where all the above inequalities are strict.

Positive Banach densities are preserved under exact embeddings (but the same property does not hold neither for Schnirelmann nor for asymptotic densities).

Proposition 6.9. *If $B \leq_e A$ then $BD(B) \leq BD(A)$. In consequence:*

(1) *$BD(A-\mathcal{U}) \leq BD(A)$ for all ultrafilters $\mathcal{U}$ on $\mathbb{N}$;*
(2) *$BD(A_\gamma) \leq BD(A)$ for all $\gamma \in {}^*\mathbb{N}$.*

Proof. Let $\langle I_n \mid n\in\mathbb{N}\rangle$ be a sequence of intervals with length $|I_n| = n$ and such that $\lim_n |B\cap I_n|/n = \mathrm{BD}(B)$. By the hypothesis, for every n there exists x_n such that $x_n+(B\cap I_n) = A\cap J_n$ where $J_n = x_n+I_n$ is the interval of length n obtained by shifting I_n by x_n. Then $\mathrm{BD}(A) \geq \lim_n |A\cap J_n|/n = \lim_n |B\cap I_n|/n = \mathrm{BD}(B)$. □

The Banach density of a set equals the maximum density of its ultrafilter-shifts.

Theorem 6.10. *For every $A \subseteq \mathbb{N}$ there exists a hyper-shift A_γ such that*

$$\sigma(A_\gamma) \;=\; d(A_\gamma) \;=\; BD(A_\gamma) \;=\; BD(A).$$

Equivalently, there exists an ultrafilter $\mathcal{U} = \mathfrak{U}_\gamma$ on $\mathbb{N}$ such that

$$\sigma(A-\mathcal{U}) \;=\; d(A-\mathcal{U}) \;=\; BD(A-\mathcal{U}) \;=\; BD(A).$$

Proof. By the nonstandard characterization of limit, given any infinite N, we can pick an interval $I = [\Omega+1,\Omega+N] \subset {}^*\mathbb{N}$ of length N such that $\|{}^*A\cap I\|/N = a \approx \mathrm{BD}(A)$. (We use delimiters $\|X\|$ to denote the internal cardinality of an internal set X, and $\xi\approx\eta$ to mean that ξ and η are infinitely close, that is, $\xi-\eta$ is infinitesimal.) Now fix an infinite ν such that $\nu/N \approx 0$.

- *Claim.* There exists γ such that for every $1 \leq i \leq \nu$:

$$\frac{\|{}^*A\cap[\gamma+1,\gamma+i]\|}{i} \;\geq\; a - \frac{\nu}{N}.$$

Notice that the above claim yields the thesis. Indeed, for every finite $n \in \mathbb{N}$, trivially $n \leq \nu$, and so

$$\frac{|A_\gamma \cap [1,n]|}{n} \;=\; \frac{\|{}^*A \cap [\gamma+1, \gamma+n]\|}{n} \;\geq\; a - \frac{\nu}{N} \;\approx\; \mathrm{BD}(A).$$

This implies that $\sigma(A_\gamma) \geq \mathrm{BD}(A)$, and the thesis follows because $\sigma(A_\gamma) \leq \underline{d}(A_\gamma) \leq \overline{d}(A_\gamma) \leq \mathrm{BD}(A_\gamma) \leq \mathrm{BD}(A)$. To prove the claim, let us proceed by contradiction, and for every γ, let

$$\psi(\gamma) \;=\; \min\left\{1 \leq i \leq \nu \;\middle|\; \frac{\|{}^*A \cap [\gamma+1, \gamma+i]\|}{i} < a - \frac{\nu}{N}\right\}.$$

By internal induction, define $\xi_1 = \Omega$, $\xi_{j+1} = \xi_j + \psi(\xi_j)$, and stop at step μ when $\Omega + N - \nu \leq \xi_\mu < \Omega + N$. Then we would have

$$a - \frac{\nu}{N} \;\leq\; \frac{\|{}^*A \cap [\Omega+1, \xi_\mu]\|}{N} \;=\; \frac{1}{N}\sum_{j=1}^{\mu-1} \|{}^*A \cap [\xi_j + 1, \xi_{j+1}]\| \;<$$

$$<\; \frac{1}{N}\sum_{j=1}^{\mu-1} \psi(\xi_j)\left(a - \frac{\nu}{N}\right) \;=\; \frac{\xi_\mu - \xi_1}{N}\left(a - \frac{\nu}{N}\right) \;<\; a - \frac{\nu}{N},$$

that gives the desired contradiction. □

A series of relevant results in additive combinatorics have been recently obtained by R. Jin by nonstandard analysis. In some of them, nonstandard properties of Banach density like the one stated in above theorem, play an important role (see, *e.g.*, the survey [30] and references therein).

7. Nonstandard characterizations in the space $(\beta\mathbb{N}, \oplus)$

The space of ultrafilters $\beta\mathbb{N}$ can be equipped with a "pseudo-sum" operation $\oplus$ that extends the usual sum between natural numbers. The resulting space $(\beta\mathbb{N}, \oplus)$ and its generalizations have been extensively studied during the last decades, producing plenty of interesting applications in Ramsey theory and combinatorics of numbers (see the monography [27]).

Definition 7.1. The *pseudo-sum* $\mathcal{U} \oplus \mathcal{V}$ of two ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ is the image $S(\mathcal{U} \otimes \mathcal{V})$ of the tensor product $\mathcal{U} \otimes \mathcal{V}$ under the sum function $S(n,m) = n + m$. Equivalently, for every $A \subseteq \mathbb{N}$:

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U},$$

where $A - n = \{m \in \mathbb{N} \mid m + n \in A\}$ is the *leftward n-shift* of A.

Notice that $\oplus$ actually extends the sum on $\mathbb{N}$; indeed, if $\mathcal{U}_n$ and $\mathcal{U}_m$ are the principal ultrafilters determined by n and m respectively, then $\mathcal{U}_n \oplus \mathcal{U}_m = \mathcal{U}_{m+n}$ is the principal ultrafilter determined by $n+m$.

The following property is a straight consequence of the definitions.

Proposition 7.2. *$\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$ if and only if there exists a tensor pair (α, β) such that $\mathfrak{U}_\alpha = \mathcal{U}$, $\mathfrak{U}_\beta = \mathcal{V}$ and $\mathfrak{U}_{\alpha+\beta} = \mathcal{W}$. In consequence:*

$$\{\mathcal{U} \oplus \mathcal{V} \mid \mathcal{U}, \mathcal{V} \in \beta\mathbb{N}\} \;=\; \{\mathfrak{U}_{\alpha+\beta} \mid (\alpha, \beta) \text{ tensor pair}\}.$$

Proof. Given $\mathcal{W} = \mathfrak{U}_\xi \oplus \mathfrak{U}_\eta$, pick a pair (α, β) with $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\xi \otimes \mathfrak{U}_\eta$. Then (α, β) is a tensor pair such that $\mathfrak{U}_\alpha = \mathfrak{U}_\xi$, $\mathfrak{U}_\beta = \mathfrak{U}_\eta$, and $\mathcal{W} = S(\mathfrak{U}_\xi \otimes \mathfrak{U}_\eta) = S(\mathfrak{U}_{(\alpha,\beta)}) = \mathfrak{U}_{\alpha+\beta}$. Conversely, if (α, β) is a tensor pair, then $\mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta = S(\mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta) = S(\mathfrak{U}_{(\alpha,\beta)}) = \mathfrak{U}_{\alpha+\beta}$. $\square$

We recall that the space $\beta\mathbb{N}$ of ultrafilters on $\mathbb{N}$ is endowed with the topology as determined by the following family of basic (cl)open sets, for $A \subseteq \mathbb{N}$:

$$\mathcal{O}_A \;=\; \{\mathcal{U} \mid A \in \mathcal{U}\}.$$

In this way, $\beta\mathbb{N}$ is the *Stone-Čech compactification* of the discrete space $\mathbb{N}$. Indeed, $\beta\mathbb{N}$ is Hausdorff and compact, and if one identifies each $n \in \mathbb{N}$ with the corresponding principal ultrafilter, then $\mathbb{N}$ is dense in $\beta\mathbb{N}$. Moreover, one can prove that every $f : \mathbb{N} \to K$ where K is Hausdorff compact space K has a (unique) continuous extension $\beta f : \beta\mathbb{N} \to K$. When equipped with the pseudo-sum operation, $\beta\mathbb{N}$ has the structure of a *compact topological left semigroup*, because for any fixed $\mathcal{V}$, the "product on the left" by $\mathcal{U}$:

$$\psi_\mathcal{V} : \mathcal{U} \mapsto \mathcal{U} \oplus \mathcal{V}$$

is a continuous function. We remark that the pseudo-sum operation is associative, but it fails badly to be commutative (see Proposition 7.4).

Connections between pseudo-sums and hyper-shifts are established in the next proposition.

Proposition 7.3. *Let $\alpha, \beta, \gamma \in {}^*\mathbb{N}$. Then:*

(1) $A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \;\Leftrightarrow\; A_\beta \in \mathfrak{U}_\alpha \;\Leftrightarrow\; \alpha \in {}^*(A_\beta)$.
(2) *For every $n \in \mathbb{N}$, $(A-n)_\beta = A_\beta - n$.*
(3) *For every $n \in \mathbb{N}$, $A - n \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \;\Leftrightarrow\; n \in (A_\beta)_\alpha$.*
(4) *$\mathfrak{U}_\gamma = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \;\Leftrightarrow\; A_\gamma = (A_\beta)_\alpha$ for every $A \subseteq \mathbb{N}$.*
(5) *If (α, β) is a tensor pair, then $A_{\alpha+\beta} = (A_\beta)_\alpha$ for every $A \subseteq \mathbb{N}$.*

Proof. Equivalences in (1) directly follow from the definitions. (2) is proved by the chain of equivalences: $k \in (A-n)_\beta \Leftrightarrow k+\beta \in {}^*(A-n) = {}^*A - n \Leftrightarrow k+\beta+n \in {}^*A \Leftrightarrow k+n \in A_\beta \Leftrightarrow k \in A_\beta - n$. (3) By using the previous properties, we obtain $A - n \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow \alpha \in {}^*[(A-n)_\beta] =$

${}^*(A_\beta - n) = {}^*(A_\beta) - n \Leftrightarrow \alpha + n \in {}^*A_\beta \Leftrightarrow n \in (A_\beta)_\alpha$. (4) Assume first $\mathfrak{U}_\gamma = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta$. For every $n \in \mathbb{N}$, by (3) one has that $n \in (A_\beta)_\alpha \Leftrightarrow A - n \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow A - n \in \mathfrak{U}_\gamma \Leftrightarrow \gamma \in {}^*(A-n) = {}^*A - n \Leftrightarrow n \in A_\gamma$, and so $(A_\beta)_\alpha = A_\gamma$. Conversely, assume by contradiction that we can pick $A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta$ with $A \notin \mathfrak{U}_\gamma$. Then $\alpha \in {}^*(A_\beta)$ and $\gamma \notin {}^*A$, and hence $(A_\beta)_\alpha \neq A_\gamma$ because $0 \in (A_\beta)_\alpha$ but $0 \notin A_\gamma$. (5) By Proposition 7.2, $\mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta = \mathfrak{U}_{\alpha+\beta}$, and so the thesis directly follows from (4). □

As a first example of use of hyper-shifts in $(\beta\mathbb{N}, \oplus)$, let us prove the continuity of the "product on the left" functions. This is easily done by showing that $\psi^{-1}_{\mathfrak{U}_\beta}(\mathcal{O}_A) = \mathcal{O}_{A_\beta}$ for every $\beta \in {}^*\mathbb{N}$ and for every $A \subseteq \mathbb{N}$; indeed, for every $\alpha \in {}^*\mathbb{N}$ one has:

$$\mathfrak{U}_\alpha \in \psi^{-1}_{\mathfrak{U}_\beta}(\mathcal{O}_A) \;\Leftrightarrow\; \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \in \mathcal{O}_A \;\Leftrightarrow$$

$$A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \;\Leftrightarrow\; A_\beta \in \mathfrak{U}_\alpha \;\Leftrightarrow\; \mathfrak{U}_\alpha \in \mathcal{O}_{A_\beta}.$$

As one can readily verify, $\mathcal{U}_n \oplus \mathcal{V} = \mathcal{V} \oplus \mathcal{U}_n$ for every principal ultrafilter $\mathcal{U}_n$. We now use hyper-shifts to show that the center of $(\beta\mathbb{N}, \oplus)$ actually contains only principal ultrafilters.

Theorem 7.4. *For every non-principal ultrafilter $\mathcal{U}$ there exists a non-principal ultrafilter $\mathcal{V}$ such that $\mathcal{U} \oplus \mathcal{V} \neq \mathcal{V} \oplus \mathcal{U}$.*

Proof. Pick an infinite γ such that $\mathcal{U} = \mathfrak{U}_\gamma$, and let ν be such that $\nu^2 \leq \gamma < (\nu+1)^2$. Let us assume that ν is even (the case ν odd is entirely similar), and let

$$A \;=\; \bigcup_{n \text{ even}} [n^2, (n+1)^2).$$

We distinguish two cases. If $(\nu+1)^2 - \gamma$ is infinite, let $\beta = (\nu+1)^2$. Notice that $A_\gamma = \mathbb{N}$ because $\gamma + n \in {}^*A$ for all n, and $A_\beta = \emptyset$ because $\beta + n \notin {}^*A$ for all n. Then $A \in \mathfrak{U}_\beta \oplus \mathfrak{U}_\gamma$ because trivially $A_\gamma \in \mathfrak{U}_\beta$, and $A \notin \mathfrak{U}_\gamma \oplus \mathfrak{U}_\beta$ because trivially $A_\beta \notin \mathfrak{U}_\gamma$. If $(\nu+1)^2 - \gamma$ is finite, let $\beta = \nu^2$. In this case A_γ is finite, and $A_\beta = \mathbb{N}$. Then $A \notin \mathfrak{U}_\beta \oplus \mathfrak{U}_\gamma$ because $A_\gamma \notin \mathfrak{U}_\beta$, and $A \in \mathfrak{U}_\gamma \oplus \mathfrak{U}_\beta$ because $A_\beta \in \mathfrak{U}_\gamma$. □

8. Idempotent ultrafilters

Ultrafilters that are idempotent with respect to pseudo-sums play an instrumental role in applications.

Definition 8.1. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is called *idempotent* if $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$, *i.e.* if

$$A \in \mathcal{U} \iff \{n \mid A - n \in \mathcal{U}\} \in \mathcal{U}.$$

Theorem 8.2. *Let $\alpha \in {}^*\mathbb{N}$. The following properties are equivalent:*

(1) $\mathfrak{U}_\alpha$ *is idempotent;*
(2) *There exists a tensor pair* (α, β) *such that* $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \alpha + \beta$ *;*
(3) *For every* $A \subseteq \mathbb{N}$, $A_\alpha = (A_\alpha)_\alpha$ *;*
(4) *For every* $A \subseteq \mathbb{N}$, $(A \cap A_\alpha)_\alpha = A_\alpha$ *;*
(5) *If* $\alpha \in {}^*A$ *then* $\alpha \in {}^*(A_\alpha)$ *;*
(6) *If* $\alpha \in {}^*A$ *then* $A \cap A_\alpha \neq \emptyset$ *;*
(7) *If* $\alpha \in {}^*A$ *then there exists* $B \subseteq A \cap B_\alpha$ *such that* $\alpha \in {}^*B$*.*
(8) *For every* $A \in \mathfrak{U}_\alpha$ *there exists* $a \in A$ *such that* $A - a \in \mathfrak{U}_\alpha$ *;*
(9) *For every* $A \in \mathfrak{U}_\alpha$ *there exists* $B \subseteq A$ *such that* $B \in \mathfrak{U}_\alpha$ *and* $B - b \in \mathfrak{U}_\alpha$ *for all* $b \in B$*.*

Proof. (1) ⇔ (2). By Theorem 5.12, we can pick β such that (α, β) is a tensor pair and $(\alpha, \beta) \underset{u}{\sim} (\alpha, \alpha)$. If $\mathfrak{U}_\alpha$ is idempotent, then

$$\mathfrak{U}_\beta \;=\; \mathfrak{U}_\alpha \;=\; \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha \;=\; S(\mathfrak{U}_\alpha \otimes \mathfrak{U}_\alpha) \;=\; S(\mathfrak{U}_{(\alpha,\beta)}) \;=\; \mathfrak{U}_{\alpha+\beta},$$

where S denotes the sum function. Conversely, by Proposition 7.2,

$$\mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha \;=\; \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \;=\; \mathfrak{U}_{\alpha+\beta} \;=\; \mathfrak{U}_\alpha.$$

(1) ⇔ (3) is given by property (4) of Proposition 7.3.

(3) ⇔ (4). Notice that $(A \cap A_\alpha)_\alpha = A_\alpha \cap (A_\alpha)_\alpha$, so property (4) is equivalent to $A_\alpha \subseteq (A_\alpha)_\alpha$ for every $A \subseteq \mathbb{N}$, and one implication is trivial. The converse implication $(A_\alpha)_\alpha \subseteq A_\alpha$ is proved by considering complements:

$$(A_\alpha)^c \;=\; (A^c)_\alpha \;\subseteq\; [(A^c)_\alpha]_\alpha \;=\; [(A_\alpha)^c]_\alpha \;=\; [(A_\alpha)_\alpha]^c.$$

(1) ⇔ (5). We recall that $\alpha \in {}^*(A_\alpha) \Leftrightarrow A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$. So (5) states the inclusion between ultrafilters $\mathfrak{U}_\alpha \subseteq \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$, which is equivalent to equality $\mathfrak{U}_\alpha = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$.

(5) ⇒ (6). Since $\alpha \in {}^*A \cap {}^*(A_\alpha) = {}^*(A \cap A_\alpha)$, by *transfer* we obtain the thesis.

(6) ⇒ (5). Assume by contradiction that $\alpha \in {}^*A$ but $\alpha \notin {}^*(A_\alpha)$. Then $\alpha \in {}^*B$ where $B = A \cap (A_\alpha)^c$. This is against the hypothesis because $B \cap B_\alpha = (A \cap (A_\alpha)^c) \cap (A_\alpha \cap [(A_\alpha)_\alpha]^c) = \emptyset$.

(3) & (5) ⇒ (7). Let $B = A \cap A_\alpha$. Then, $\alpha \in {}^*A \cap {}^*(A_\alpha) = {}^*B$. Moreover, trivially $B \subseteq A$, and also

$$B \;\subseteq\; A_\alpha \;=\; A_\alpha \cap (A_\alpha)_\alpha \;=\; (A \cap A_\alpha)_\alpha \;=\; B_\alpha.$$

(7) ⇒ (6). Given $\alpha \in {}^*A$, pick B as in the hypothesis. Notice that $B \subseteq A$; moreover $B \subseteq B_\alpha \subseteq (A \cap B_\alpha)_\alpha = A_\alpha \cap (B_\alpha)_\alpha \subseteq A_\alpha$, and hence $B \subseteq A \cap A_\alpha$. We conclude by noticing that $\alpha \in {}^*B$, and so, by *transfer*, $B \neq \emptyset$.

Finally, it is easily verified that properties (8) and (9) are the "standard" counterparts of properties (6) and (7), respectively. □

We remark that the existence of idempotent ultrafilters is not a trivial matter: it is proved as an application of a general result by R. Ellis about the existence of idempotents in every compact Hausdorff topological left semigroup (see, *e.g.*, [27]). Indeed, $(\beta\mathbb{N}, \oplus)$ is a compact Hausdorff topological left semigroup.

Historically, the first application of idempotent ultrafilters in combinatorics of numbers was a short and elegant proof of Hindman's theorem found by F. Galvin and S. Glazer. The original argument used by N. Hindman in his proof [25] was really intricated. Actually, Hindman himself wrote in [26]: *"If the reader has a graduate student that she wants to punish, she should make him read and understand that original proof"*. A detailed report about the discovery of the ultrafilter proof can be found in [26].

Theorem 8.3 (Hindman)**.** *For every finite partition $\mathbb{N} = C_1 \cup \ldots \cup C_r$ of the natural numbers, there exists an infinite set X such that every (finite) sum of distinct elements from X belongs to the same piece C_i.*

Proof. Pick α such that $\mathfrak{U}_\alpha$ is idempotent, and let C_i be the piece of the partition such that $\alpha \in {}^*C_i$. By (7) of Theorem 8.2, we can fix a set $B \subseteq C_i \cap B_\alpha$ with $\alpha \in {}^*B$. Notice that $x \in B \Rightarrow x \in B_\alpha \Leftrightarrow \alpha + x \in {}^*B \Leftrightarrow \alpha \in {}^*(B - x)$.

Now pick any $x_1 \in B$. Then α witnesses the existence in ${}^*B$ of elements larger than x_1 that belong to ${}^*(B - x_1)$. By *transfer*, we obtain the existence in B of an element $x_2 > x_1$ such that $x_2 \in B - x_1$. Notice that $x_1, x_2, x_1 + x_2 \in B$, and hence $\alpha \in {}^*(B - x_2)$ and $\alpha \in {}^*(B - x_1 - x_2)$. Similarly as above, α witnesses the existence in ${}^*B$ of elements larger than x_2 that belong to ${}^*[(B - x_1) \cap (B - x_2) \cap (B - x_1 - x_2)]$. Again by using *transfer*, we get the existence in B of an element $x_3 > x_2$ such that $x_3 \in (B - x_1) \cap (B - x_2) \cap (B - x_1 - x_2)$, and so we have that $x_1, x_2, x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 \in B$. By iterating the process, we eventually obtain a set $X = \{x_1 < x_2 < \ldots < x_n < \ldots\}$ such that every sum of distinct elements from X belongs to B, and hence to the same piece C_i of the partition, as desired. □

We recall that in the definition of the pseudo-sum $\mathcal{U} \oplus \mathcal{V}$, one considers *leftward* shifts $A - n = \{m \mid m + n \in A\}$. By considering instead *rightward* shifts $A + n = \{a + n \mid a \in A\}$, one obtains the following operation.

Definition 8.4. Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on $\mathbb{N}$, where $\mathcal{V}$ is non-principal. The ultrafilter $\mathcal{U} \star \mathcal{V}$ is defined by setting for every $A \subseteq \mathbb{N}$:

$$A \in \mathcal{U} \star \mathcal{V} \;\Leftrightarrow\; \{n \mid A + n \in \mathcal{V}\} \in \mathcal{U}.$$

Notice that one can identify $\mathcal{U} \star \mathcal{V}$ with the image $D(\mathcal{U} \otimes \mathcal{V})$ of the tensor product $\mathcal{U} \otimes \mathcal{V}$ under the difference function $D(n,m) = m - n$. Indeed, although D takes values in $\mathbb{Z}$, one has that $\mathbb{N} \in \mathcal{U} \star \mathcal{V}$ whenever $\mathcal{V}$ is non-principal and so, in this case, one can restrict to subsets of $\mathbb{N}$.

In a similar way as done in Theorem 8.2, one can prove several nonstandard characterizations of ultrafilters that are idempotent with respect to $\star$. In particular, corresponding to item (2) in Theorem 8.2, it is shown that $\mathfrak{U}_\alpha \star \mathfrak{U}_\alpha = \mathfrak{U}_\alpha$ if and only if there exists a tensor pair (α, β) such that $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \beta - \alpha$. The problem is that there can be no such pair!

Theorem 8.5. *There exist pairs (α, β) such that $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \beta - \alpha$, but not one of them is a tensor pair. In consequence, there exist no ultrafilters $\mathcal{U}$ such that $\mathcal{U} \star \mathcal{U} = \mathcal{U}$.*

Proof. For every $A \subseteq \mathbb{N}$, let us consider the set

$$\Gamma(A) \;=\; \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid \text{either } a, b, b-a \in A \text{ or } a, b, b-a \in A^c\}.$$

We want to show that the family $\{\Gamma(A) \mid A \subseteq \mathbb{N}\}$ has the finite intersection property. Once this is proved, by $\mathfrak{c}^+$-*saturation* one can pick a pair $(\alpha, \beta) \in \bigcap_{A \subseteq \mathbb{N}} {}^*\Gamma(A)$. It is then easily verified that $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \beta - \alpha$.

Given $A_1, \ldots, A_n$, pick a finite partition $\mathbb{N} = C_1 \cup \ldots \cup C_r$ such that, for every piece C_i and every set A_j, one has that either $C_i \subseteq A_j$ or $C_i \subseteq A_j^c$. Now, by Rado's theorem, the equation $X - Y = Z$ is *partition regular* on $\mathbb{N}$, and so we can pick elements $x, y, z \in C_i$ in one piece of the partition that satisfy $x - y = z$; in consequence, $(x,y) \in \bigcap_{i=1}^r \Gamma(C_j) \subseteq \bigcap_{j=1}^n \Gamma(A_i) \neq \emptyset$. We recall that an equation $f(X_1, \ldots, X_n) = 0$ is called *partition regular* on $\mathbb{N}$ when for every finite partition of $\mathbb{N}$ there exist $x_1, \ldots, x_n$ in the same piece of the partition that solve the equation, *i.e.* $f(x_1, \ldots, x_n) = 0$. Rado's theorem states that a linear equation $c_1 X_1 + \ldots + c_n X_n = 0$ (where the $c_i \neq 0$) is partition regular on $\mathbb{N}$ if and only if there exists a sum of distinct coefficients that equals zero (see [23, Ch.3]).

Let us now turn to the negative result, and assume $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \beta - \alpha$. Notice that both α and β must be multiples of every (finite) natural number. Indeed, given $n \geq 2$, the u-equivalence of α and β implies that $\alpha \equiv \beta \mod n$, and hence $\beta - \alpha \equiv 0 \mod n$. But $\beta - \alpha$ is u-equivalent to both α and β, and so $\alpha \equiv \beta \equiv \beta - \alpha \equiv 0 \mod n$. Now, let us consider the functions $f, g : \mathbb{N} \to \mathbb{N} \cup \{0\}$ where $f(n)$ is the greatest exponent such that $3^{f(n)}$ divides n, and $g(n) = n/3^{f(n)}$. By what proved above, both ${}^*f(\alpha)$ and ${}^*f(\beta)$ are infinite. Since $\alpha \underset{u}{\sim} \beta$, we have ${}^*g(\alpha) \underset{u}{\sim} {}^*g(\beta)$, and so ${}^*g(\alpha) \equiv {}^*g(\beta) = j \mod 3$, where either $j = 1$ or $j = 2$. Now assume by contradiction that (α, β) is a tensor pair.

By Proposition 5.9, also $({}^*f(\alpha), {}^*f(\beta))$ is a tensor pair, and since both components are infinite, it is ${}^*f(\alpha) < {}^*f(\beta)$. Then we have:

$$\beta - \alpha \;=\; 3^{{}^*f(\beta)} \cdot {}^*g(\beta) \;-\; 3^{{}^*f(\alpha)} \cdot {}^*g(\alpha) \;=\; 3^{{}^*f(\alpha)} \cdot (3^{\nu} \cdot {}^*g(\beta) - {}^*g(\alpha))$$

where $\nu = {}^*f(\beta) - {}^*f(\alpha) > 0$. In consequence, ${}^*f(\beta - \alpha) = {}^*f(\alpha)$ and ${}^*g(\beta-\alpha) = 3^{\nu}\cdot{}^*g(\beta) - {}^*g(\alpha) \equiv -{}^*g(\alpha) \equiv -j \mod 3$, while $\beta - \alpha \underset{u}{\sim} \alpha \Rightarrow {}^*g(\beta-\alpha) \underset{u}{\sim} {}^*g(\alpha) \Rightarrow {}^*g(\beta-\alpha) \equiv {}^*g(\alpha) \equiv j \mod (3)$. We must conclude that $-j \equiv j \mod 3$, and hence $j = 0$, a contradiction. (The last part of this argument is essentially the same as the one used by Hindman in [24, §4] to prove the non-existence of ultrafilters $\mathcal{U} = \mathcal{U} \star \mathcal{U}$.) $\square$

9. Final remarks and open questions

Since a first draft of this paper was written in 2009, several applications of the presented nonstandard approach to the use of ultrafilters appeared in the literature. In [18], iterated nonstandard extensions were used to characterize idempotent ultrafilters along the lines of Theorem 8.2; and by using suitable linear combinations of idempotent ultrafilters, a new proof of a version of Radó's Theorem was given. Partition regularity of (nonlinear) polynomial equations by nonstandard methods is the subject-matter of the paper [36]. In [9], a notion of finite embeddability between sets and between ultrafilters is investigated, also with the use of the hyper-shifts of §5. The papers [37, 38] continue that line of research: the nonstandard approach is exploited to further investigating the relationships between finite embeddability relations, algebraic properties in $(\beta\mathbb{N}, \oplus)$, and combinatorial structure of sets of natural numbers.

We like to close this paper with some remarks about idempotent ultrafilters. To this day, basically the only known proof of their existence is grounded on an old result by R. Ellis, namely the fact that every compact Hausdorff topological left semigroup has idempotents (see [21]). Since idempotent ultrafilters are widely used in applications, it seems desirable to better understand them; to this end, a solution to the following problem would be valuable.

- Open problem #1: *Find an alternative, nonstandard proof of the existence of idempotent ultrafilters.*

Our notions of u-equivalence and of tensor pair, and hence of idempotent ultrafilter, can be generalized to models M of any first-order theory $\mathsf{T} \supseteq \mathsf{PA}$ that extends *Peano Arithmetic.* (By this we mean that T is a collection of sentences in a first-order language $\mathcal{L}$ that extends the language of PA.)

We recall that the *type* of an element $a \in M$ is the set of all formulas with one free variable that are satisfied by a in M:

$$tp(a) \;=\; \{\varphi(x) \mid M \models \varphi(a)\}.$$

Another notion that makes sense for models M of theories $\mathsf{T} \supseteq \mathsf{PA}$ is the following. Call a pair $(a,b) \in M \times M$ *independent* when for every formula $\varphi(x,y)$, if $M \models \varphi(a,b)$ then $M \models \varphi(k,b)$ for some $k \in \mathbb{N}$. (This definition corresponds to the notion of *heir of a type*, as used in stable theories.)

If $\mathrm{Th}(\mathbb{N})$ is the first-order theory of the natural numbers in the full language that contains a symbol for every relation, function and constant of $\mathbb{N}$, then $M \models \mathrm{Th}(\mathbb{N})$ means that $M = {}^*\mathbb{N}$ is the set of hypernatural numbers of a model of nonstandard analysis. In this case, trivially every subset $A \subseteq \mathbb{N}$ is definable, and hence $tp(a) = tp(b)$ if and only if $a \underset{u}{\sim} b$. Moreover, (a,b) is independent means that (a,b) is a tensor pair (see (6) of Theorem 5.7). Thus, by using property (2) of Theorem 8.2, one could propose the following generalization of idempotent ultrafilter.

Definition 9.1. Let $M \models \mathsf{T} \supseteq \mathsf{PA}$. We say that an element $\alpha \in M$ is *idempotent* if there exists an independent pair (α,β) such that $tp(\alpha) = tp(\beta) = tp(\alpha+\beta)$.

- Open problem # 2: *Given a first-order theory* $\mathsf{T} \supseteq \mathsf{PA}$*, find sufficient conditions for models* $M \models \mathsf{T}$ *to contain idempotent elements.*

We recall that in any $\mathfrak{c}^+$-*saturated* model of nonstandard analysis, every ultrafilter on $\mathbb{N}$ is generated by some element $\alpha \in {}^*\mathbb{N}$. In consequence, all $\mathfrak{c}^+$-saturated models of $\mathrm{Th}(\mathbb{N})$ contain idempotent elements. Isolating model-theoretic properties that guarantee the existence of idempotent elements would probably be useful also to attack the previous open problem.

10. References

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