SOME APPLICATIONS OF NUMEROSITIES IN MEASURE THEORY

VIERI BENCI, EMANUELE BOTTAZZI, AND MAURO DI NASSO

Abstract. We present some applications of the notion of numerosity to measure theory, including the construction of a non-Archimedean model for the probability of infinite sequences of coin tosses.

Introduction

The idea of numerosity as a notion of measure for the size of infinite sets was introduced by the first named author in [1], and then given sound logical foundations in [3]. A theory of numerosities have been then developed in a sequel of papers (see, *e.g.*, [5, 8]). The main feature of numerosities is that they satisfy the same basic formal properties as finite cardinalities, including the fact that proper subsets must have strictly smaller sizes. This has to be contrasted with Cantorian cardinalities, where every infinite set have proper subsets of the same cardinality.

In this paper we will present three applications of numerosity in topics of measure theory. The first one is about the existence of "inner measures" associated to any given non-atomic pre-measure. The second application is focused on sets of real numbers. We show that elementary numerosities provide a useful tool with really strong compatibility properties with respect to the Lebesgue measure. For instance, intervals of equal length can be given the same numerosity, and any interval of rational length p/q has a numerosity which is exactly p/q. We derive consequences about the existence of totally defined finitely additive measures that extend the Lebesgue measure. Finally, the third application is about non-Archimedean probability. Following ideas from [6], we consider a model for infinite sequences of coin tosses which is coherent with the original view of Laplace. Indeed, probability of an event is

2000 *Mathematics Subject Classification.* 26E30; 28E15; 26E35; 60A05.
Key words and phrases. Non-Archimedean mathematics, measure theory, nonstandard analysis, numerosities.

defined as the numerosity of positive outcomes divided by the numerosity of all possible outcomes; moreover, the probability of cylindrical sets exactly coincides with the usual Kolmogorov probability.

1. Terminology and preliminary notions

We fix here our terminology, and recall a few basic facts from measure theory and numerosity theory that will be used in the sequel.

Let us first agree on notation. We write $A \subseteq B$ to mean that A is a subset of B, and we write $A \subset B$ to mean that A is a *proper* subset of B. The *complement* of a set A is denoted by A^c, and its *powerset* is denoted by $\mathcal{P}(A)$. We use the symbol $\sqcup$ to denote *disjoint unions*. By $\mathbb{N}$ we denote the set of *positive integers*. For an ordered field $\mathbb{F}$, we denote by $[0,\infty)_{\mathbb{F}} = \{x \in \mathbb{F} \mid x \geq 0\}$ the set of its non-negative elements. We will write $[0,+\infty]_{\mathbb{R}}$ to denote the set of non-negative real numbers plus the symbol $+\infty$, where we agree that $x + \infty = +\infty + x = +\infty + \infty = +\infty$ for all $x \in \mathbb{R}$.

Definition 1.1. A *finitely additive measure* is a triple $(\Omega, \mathfrak{A}, \mu)$ where:

- The *space* Ω is a nonempty set;
- $\mathfrak{A}$ is an *algebra of sets* over Ω, *i.e.* a nonempty family of subsets of Ω which is closed under finite unions and intersections, and under relative complements, *i.e.* $A, B \in \mathfrak{A} \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathfrak{A}$;[1]
- $\mu : \mathfrak{A} \to [0,+\infty]_{\mathbb{R}}$ is an *additive function*, *i.e.* $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathfrak{A}$ are disjoint.[2] We also assume that $\mu(\emptyset) = 0$.

The measure $(\Omega, \mathfrak{A}, \mu)$ is called *non-atomic* when all finite sets in $\mathfrak{A}$ have measure zero. We say that $(\Omega, \mathfrak{A}, \mu)$ is a *probability measure* when $\mu : \mathfrak{A} \to [0,1]_{\mathbb{R}}$ takes values in the unit interval, and $\mu(\Omega) = 1$.

For simplicity, in the following we will often identify the triple $(\Omega, \mathfrak{A}, \mu)$ with the function μ.

Remark that a finitely additive measure μ is necessarily *monotone*, *i.e.*

- $\mu(A) \leq \mu(B)$ for all $A, B \in \mathfrak{A}$ with $A \subseteq B$.

[1] Actually, the closure under intersections follow from the other two properties, since $A \cap B = A \setminus (A \setminus B)$.

[2] Such functions μ are sometimes called *contents* in the literature.

Definition 1.2. A finitely additive measure μ defined on a ring of sets $\mathfrak{A}$ is called a *pre-measure* if it is σ-*additive*, *i.e.* if for every countable family $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{A}$ of pairwise disjoint sets whose union lies in $\mathfrak{A}$, it holds:

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) \;=\; \sum_{n=1}^{\infty} \mu(A_n).$$

A *measure* is a pre-measure which is defined on a σ-*algebra*, *i.e.* on an algebra of sets which is closed under countable unions and intersections.

Definition 1.3. An *outer measure* on a set Ω is a function

$$M : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$$

defined on all subsets of Ω which is *monotone* and σ-*subadditive*, *i.e.*

$$M\left(\bigcup_{n\in\mathbb{N}} A_n\right) \;\leq\; \sum_{n\in\mathbb{N}} M(A_n).$$

It is also assumed that $M(\emptyset) = 0$.

Definition 1.4. Given an outer measure M on Ω, the following family is called the *Caratheodory* σ-*algebra* associated to M:

$$\mathfrak{C}_M \;=\; \{X \subseteq \Omega \mid M(Y) = M(X \cap Y) + M(X \setminus Y) \text{ for all } Y \subseteq \Omega\}.$$

A well known theorem of Caratheodory states that the above family is actually a σ-algebra, and that the restriction of M to $\mathfrak{C}_M$ is a *complete* measure, *i.e.* a measure where $M(X) = 0$ implies $Y \in \mathfrak{C}_M$ for all $Y \subseteq X$. This result is usually combined with the property that every pre-measure μ over a ring $\mathfrak{A}$ of subsets of Ω is canonically extended to the outer measure $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]_{\mathbb{R}}$ defined by putting:

$$\overline{\mu}(X) \;=\; \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) \,\middle|\, \{A_n\}_n \subseteq \mathfrak{A} \;\&\; X \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}.$$

Indeed, a fundamental result in measure theory is that the above function $\overline{\mu}$ is actually an outer measure that extends μ, and that the associated Caratheodory σ-algebra $\mathfrak{C}_{\overline{\mu}}$ includes $\mathfrak{A}$. Moreover, such an outer measure $\overline{\mu}$ is *regular*, *i.e.* for all $X \in \mathcal{P}(\Omega)$ there exists $C \in \mathfrak{C}_{\overline{\mu}}$ such that $X \subseteq C$ and $\overline{\mu}(X) = \overline{\mu}(C)$. (See *e.g.* [9] Prop. 20.9.)

Next, we will recall the notion of *elementary numerosity*, a variant of the notion of numerosity that was introduced in [2]. The underlying idea is that of refining the notion of finitely additive measure in such a way that also single points count. To this end, one needs to consider ordered fields that extend the real line.

Recall that every ordered field $\mathbb{F}$ that properly extend $\mathbb{R}$ is necessarily *non-Archimedean*, in that it contains *infinitesimal numbers* $\epsilon \neq 0$ such that $-1/n < \epsilon < 1/n$ for all $n \in \mathbb{N}$. Two elements $\xi, \zeta \in \mathbb{F}$ are called *infinitely close* if $\xi - \zeta$ is infinitesimal; in this case, we write $\xi \approx \zeta$. A number $\xi \in \mathbb{F}$ is called *finite* if $-n < \xi < n$ for some $n \in \mathbb{N}$, and it is called *infinite* otherwise. Clearly, a number ξ is infinite if and only if its reciprocal $1/\xi$ is infinitesimal. We remark that every finite $\xi \in \mathbb{F}$ is infinitely close to a unique real number r, namely $r = \inf\{x \in \mathbb{R} \mid x > \xi\}$. Such a number r is called the *standard part* of ξ, and is denoted by $r = \mathrm{st}(\xi)$. Notice that $\mathrm{st}(\xi+\zeta) = \mathrm{st}(\xi)+\mathrm{st}(\zeta)$ and $\mathrm{st}(\xi \cdot \zeta) = \mathrm{st}(\xi) \cdot \mathrm{st}(\zeta)$ for all finite ξ, ζ. The notion of standard part can be extended to the infinite elements $\xi \in \mathbb{F}$ by setting $\mathrm{st}(\xi) = +\infty$ when ξ is positive, and $\mathrm{st}(\xi) = -\infty$ when ξ is negative.

Definition 1.5. An *elementary numerosity* on the set Ω is a function

$$\mathfrak{n} : \mathcal{P}(\Omega) \longrightarrow [0, +\infty)_{\mathbb{F}}$$

defined on all subsets of Ω, taking values in an ordered field $\mathbb{F} \supseteq \mathbb{R}$ that extends the real line, and that satisfies the following two properties:

(1) *Additivity*: $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B)$ whenever $A \cap B = \emptyset$;
(2) *Unit size*: $\mathfrak{n}(\{x\}) = 1$ for every point $x \in \Omega$.

Notice that if Ω is a finite set, then the only elementary numerosity is the finite cardinality. On the other hand, when Ω is infinite, then the numerosity function must also take "infinite" values, and so the field $\mathbb{F}$ must be non-Archimedean. It is worth remarking that also Cantorian cardinality satisfies the above properties (1), (2), but the sum operation between cardinals is really far from being a ring operation.[3]

As straight consequences of the definition, we obtain that elementary numerosities can be seen as generalizations of finite cardinalities. Indeed, one can easily show that

- $\mathfrak{n}(A) = 0$ if and only if $A = \emptyset$;
- If $A \subset B$ is a proper subset, then $\mathfrak{n}(A) < \mathfrak{n}(B)$;
- If F is a finite set of cardinality n, then $\mathfrak{n}(F) = n$.

Given an elementary numerosity and a "measure unit" $\beta \in \mathbb{F}$, there is a canonical way to construct a (real-valued) finitely additive measure.

[3] Recall that for infinite cardinals κ, ν it holds $\kappa + \nu = \max\{\kappa, \nu\}$.

Definition 1.6. If $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{\mathbb{F}}$ is an elementary numerosity, and $\beta \in \mathbb{F}$ is a positive number, the map $\mathfrak{n}_\beta : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$ is defined by setting

$$\mathfrak{n}_\beta(A) \; = \; \mathrm{sh}\left(\frac{\mathfrak{n}(A)}{\beta}\right).$$

Proposition 1.7. *$\mathfrak{n}_\beta$ is a finitely additive measure. Moreover, $\mathfrak{n}_\beta$ is non-atomic if and only if β is an infinite number.*

Proof. For all disjoint $A, B \subseteq \Omega$, one has:

$$\begin{aligned}\mathfrak{n}_\beta(A \cup B) \; &= \; \mathrm{st}\left(\frac{\mathfrak{n}(A \cup B)}{\beta}\right) \; = \; \mathrm{st}\left(\frac{\mathfrak{n}(A)}{\beta} + \frac{\mathfrak{n}(B)}{\beta}\right) \\ &= \; \mathrm{st}\left(\frac{\mathfrak{n}(A)}{\beta}\right) + \mathrm{st}\left(\frac{\mathfrak{n}(B)}{\beta}\right) \; = \; \mathfrak{n}_\beta(A) + \mathfrak{n}_\beta(B).\end{aligned}$$

Notice that the measure $\mathfrak{n}_\beta$ is non-atomic if and only if $\mathfrak{n}_\beta(\{x\}) = \mathrm{sh}(1/\beta) = 0$, and this holds if and only if β is infinite. $\square$

The relevant result about elementary numerosities that we will use in the sequel, is the following representation theorem, that was proved in [2]:

Theorem 1.8. *Let $(\Omega, \mathfrak{A}, \mu)$ be a non-atomic finitely additive measure on the infinite set Ω, and let $\mathfrak{B} \subseteq \mathfrak{A}$ be a subalgebra that does not contain nonempty null sets. Then there exist*

- *a non-Archimedean field $\mathbb{F} \supset \mathbb{R}$;*
- *an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{\mathbb{F}}$;*

such that:

(1) *$\mu(B) = \mu(B') \Leftrightarrow \mathfrak{n}(B) = \mathfrak{n}(B')$ for all $B, B' \in \mathfrak{B}$ of finite measure;*
(2) *For every set $Z \in \mathfrak{A}$ of positive finite measure, if $\beta = \mathfrak{n}(Z)/\mu(Z)$ then $\mu(A) = \mathfrak{n}_\beta(A)$ for all $A \in \mathfrak{A}$.*

2. Numerosities and inner measures

In this section we will use elementary numerosities to prove a general existence result about "inner" measures.

Theorem 2.1. *Let $\mathfrak{A}$ be an algebra of subsets of Ω and let $\mu : \mathfrak{A} \to [0,+\infty]_{\mathbb{R}}$ be a non-atomic pre-measure. Assume that μ is non-trivial, in the sense that there are sets $Z \in A$ with $0 < \mu(Z) < +\infty$. Then, along with the associated outer measure $\overline{\mu}$, there exists an "inner" finitely additive measure*

$$\underline{\mu} : \mathcal{P}(\Omega) \to [0,+\infty]_{\mathbb{R}}$$

such that:

(1) *$\underline{\mu}(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$, the Caratheodory σ-algebra associated to μ. In particular, $\underline{\mu}(A) = \mu(A) = \overline{\mu}(A)$ for all $A \in \mathfrak{A}$.*

(2) *$\underline{\mu}(X) \leq \overline{\mu}(X)$ for all $X \subseteq \Omega$.*

Proof. By *Caratheodory extension theorem*, the restriction of $\overline{\mu}$ to $\mathfrak{C}_\mu$ is a measure that agrees with μ on $\mathfrak{A}$. Now we apply Theorem 1.8 to the measure $(\mathfrak{C}_\mu, \mathfrak{A}, \overline{\mu})$, and obtain the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{\mathbb{F}}$. By property (2) in the Theorem, if we pick any number $\beta = \frac{\mathfrak{n}(Z)}{\mu(Z)}$ where $0\mu(Z) < +\infty$, then $\mathfrak{n}_\beta(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$. We claim that $\underline{\mu} = \mathfrak{n}_\beta : \mathcal{P}(\Omega) \to [0,+\infty]_{\mathbb{R}}$ is the desired "inner" finitely additive measure.

Property (1) is trivially satisfied by our definition of $\underline{\mu}$, so let us turn to (2). For every $X \subseteq \Omega$, by definition of outer measure we have that for every $\epsilon > 0$ there exists a countable union $A = \bigcup_{n=1}^{\infty} A_n$ of sets $A_n \in \mathfrak{A}$ such that $A \supseteq X$ and $\sum_{n=1}^{\infty} \mu(A_n) \leq \overline{\mu}(X) + \epsilon$. Notice that A belongs to the σ-algebra generated by $\mathfrak{A}$, and hence $A \in \mathfrak{C}_\mu$. In consequence, $\underline{\mu}(A) = \mathfrak{n}_\beta(A) = \overline{\mu}(A)$. Finally, by monotonicity of the finitely additive measure $\underline{\mu}$, and by σ-subadditivity of the outer measure $\overline{\mu}$, we obtain:

$$\underline{\mu}(X) \ \leq \ \underline{\mu}(A) \ = \ \overline{\mu}(A) \ \leq \ \sum_{n=1}^{\infty} \overline{\mu}(A_n) \ = \ \sum_{n=1}^{\infty} \mu(A_n) \ \leq \ \overline{\mu}(X) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, the desired inequality $\underline{\mu}(X) \leq \overline{\mu}(X)$ follows. $\square$

It seems of some interest to investigate the properties of the extension of the Caratheodory algebra given by family of all sets for which the outer measure coincides with the above "inner measure":

$$\mathfrak{C}(\mathfrak{n}_\beta) \ = \ \left\{ X \subseteq \Omega \mid \underline{\mu}(X) = \overline{\mu}(X) \right\}.$$

Clearly, the properties of $\mathfrak{C}(\mathfrak{n}_\beta)$ may depend on the choice of the elementary numerosity $\mathfrak{n}$.

Theorem 2.1 ensures that the inclusion $\mathfrak{C}_\mu \subseteq \mathfrak{C}(\mathfrak{n}_\beta)$ always holds. Moreover, this inclusion is an equality if and only if all $X \notin \mathfrak{C}_\mu$ satisfy the inequality $\underline{\mu}(X) < \overline{\mu}(X)$. It turns out that, when $\mu(\Omega) < +\infty$, this property is equivalent to a number of other statements.

Proposition 2.2. *If $\mu(\Omega) < +\infty$, then the following are equivalent:*

1. $\mathfrak{C}_\mu = \mathfrak{C}(\mathfrak{n}_\beta)$.
2. $X \notin \mathfrak{C}_\mu \Rightarrow \underline{\mu}(X) < \overline{\mu}(X)$ *and* $\underline{\mu}(X^c) < \overline{\mu}(X^c)$.
3. $\underline{\mu}(X) = \overline{\mu}(X) \Longleftrightarrow \underline{\mu}(X^c) = \overline{\mu}(X^c)$.
4. $\underline{\mu}(X) = 0 \Longleftrightarrow \overline{\mu}(X) = 0$.

If $\mu(\Omega) = +\infty$, then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. We have already seen that (1) and (2) are equivalent.

(2) $\Rightarrow$ (3). Suppose towards a contradiction that (2) holds but (3) is false. The latter hypothesis ensures the existence of a set X such that $\underline{\mu}(X) = \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$. Thanks to Theorem 2.1, we deduce that $X \notin \mathfrak{C}_\mu$. By (2) we get the contradiction $\underline{\mu}(X) < \overline{\mu}(X)$.

(3) $\Rightarrow$ (4). The implication $\overline{\mu}(X) = 0 \Rightarrow \underline{\mu}(X) = 0$ is always true. On the other hand, if $\underline{\mu}(X) = 0$, then $\underline{\mu}(X^c) = \underline{\mu}(\Omega) = \overline{\mu}(\Omega)$. By the inequality $\underline{\mu}(X^c) \leq \overline{\mu}(X^c)$, we deduce $\overline{\mu}(X^c) = \overline{\mu}(\Omega) = \underline{\mu}(X^c)$ and, thanks to (3), also $\overline{\mu}(X) = 0$ follows.

(4) $\Rightarrow$ (2), under the hypothesis that $\mu(\Omega) < +\infty$. Suppose towards a contradiction that (4) holds but (2) is false. The latter hypothesis ensures the existence of a set $X \notin \mathfrak{C}_\mu$ satisfying $\underline{\mu}(X) = \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$. Thanks to Propositions 20.9 and 20.11 of [9], we can find a set $A \in \mathfrak{C}_\mu$ satisfying $A \supset X$, $\overline{\mu}(A) = \overline{\mu}(X)$ and $\overline{\mu}(A \setminus X) > 0$. From the hypothesis $\underline{\mu}(X) = \overline{\mu}(X)$ we obtain the following equalities:

$$\underline{\mu}(X) \;=\; \overline{\mu}(X) \;=\; \overline{\mu}(A) \;=\; \underline{\mu}(A).$$

The above equalities and the hypothesis $\mu(\Omega) < +\infty$ imply $\underline{\mu}(A \setminus X) = 0$. By (4), we obtain the contradiction $\overline{\mu}(A \setminus X) = 0$. $\square$

3. Numerosities and Lebesgue measure

In this section, we show that elementary numerosities exist which are consistent with Lebesgue measure in a strong sense. Precisely, the following result holds:

Theorem 3.1. *Let* $(\mathbb{R}, \mathfrak{L}, \mu_L)$ *be the* Lebesgue measure *over* $\mathbb{R}$. *Then there exists an elementary numerosity* $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \to [0, +\infty)_{\mathbb{F}}$ *such that:*

1. $\mathfrak{n}([x, x+a)) = \mathfrak{n}([y, y+a))$ *for all* $x, y \in \mathbb{R}$ *and for all* $a > 0$.
2. $\mathfrak{n}([x, x+a)) = a \cdot \mathfrak{n}([0, 1))$ *for all rational numbers* $a > 0$.

(3) *$st\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right) = \mu_L(X)$ for all $X \in \mathfrak{L}$.*

(4) *$st\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right) \leq \overline{\mu}_L(X)$ for all $X \subseteq \mathbb{R}$.*

Proof. Notice that the family of half-open intervals

$$\mathfrak{I} \;=\; \{[x, x+a) \mid x \in \mathbb{R} \;\&\; a > 0\}$$

generates a subalgebra $\mathfrak{B} \subset \mathfrak{L}$ whose nonempty sets have all finite positive measure. Then, by combining Theorems 1.8 and 2.1, we obtain the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \to [0, +\infty)_{\mathbb{F}}$ such that, for $\beta = \mathfrak{n}([0,1)) = \frac{\mathfrak{n}([0,1))}{\mu_L([0,1))}$, one has:

(i) $\mathfrak{n}(X) = \mathfrak{n}(Y)$ for all $X, Y \in \mathfrak{B}$ with $\mu_L(X) = \mu_L(Y)$;

(ii) $\mathfrak{n}_\beta(X) = \mu_L(X)$ for all $X \in \mathfrak{L}$;

(iii) $\mathfrak{n}_\beta(X) \leq \overline{\mu}_L(X)$ for all $X \subseteq \mathbb{R}$.

Since $[x, x+a) \in \mathfrak{B}$ for all $x \in \mathbb{R}$ and for all $a > 0$, property (1) directly follows from (i). In order to prove (2), it is enough to show that $\mathfrak{n}([0,a)) = a \cdot \mathfrak{n}([0,1))$ for all positive $a \in \mathbb{Q}$. Given $p, q \in \mathbb{N}$, by (1) and additivity we have that

$$\mathfrak{n}\left(\left[0, \frac{p}{q}\right)\right) \;=\; \mathfrak{n}\left(\bigsqcup_{i=0}^{p-1}\left[\frac{i}{q}, \frac{i+1}{q}\right)\right) \;=\; \sum_{i=0}^{p-1}\mathfrak{n}\left(\left[\frac{i}{q}, \frac{i+1}{q}\right)\right) \;=\; p \cdot \mathfrak{n}\left(\left[0, \frac{1}{q}\right)\right).$$

In particular, for $p = q$ we get that $\mathfrak{n}([0,1)) = q \cdot \mathfrak{n}([0, 1/q))$, and hence property (2) follows:

$$\mathfrak{n}\left(\left[0, \frac{p}{q}\right)\right) \;=\; \frac{p}{q} \cdot \mathfrak{n}\left([0,1)\right).$$

Finally, (ii) and (iii) directly correspond to properties (3) and (4), respectively. $\square$

Remark 3.2. Let $\{X_n \mid n \in \mathbb{N}\}$ be a countable family of isometric, pairwise disjoint, non-Lebesgue measurable sets such that the union $A = \bigcup_{n \in \mathbb{N}} X_n$ is measurable with positive finite measure. (*E.g.*, one can consider a *Vitali set* on $[0,1)$ and take the countable family of its rational translations modulo 1.) Let $\mathfrak{n}$ be an elementary numerosity as given by the above theorem, and consider the finitely additive measure $\mathfrak{n}_\beta$ with $\beta = \mathfrak{n}(A)/\mu(A)$. Then, one and only one of the following holds:

- $\mathfrak{n}_\beta(X_n) = 0$ for all $n \in \mathbb{N}$. In this case, the measure $\mathfrak{n}_\beta$ is not σ-additive because $\mathfrak{n}_\beta(A) = \mu_L(A) > 0$.

- $\mathfrak{n}_\beta(X_n) = \epsilon > 0$ for some $n \in \mathbb{N}$. In this case, $\mathfrak{n}_\beta$ is not invariant with respect to isometries, as otherwise one would get the contradiction $\mu_L(A) = \mathfrak{n}_\beta(A) \geq \sum_{n\in\mathbb{N}} \mathfrak{n}_\beta(X_n) = \sum_{n\in\mathbb{N}} \epsilon = +\infty$.

4. Numerosities and probability of infinite coin tosses

The last application of elementary numerosities that we present in this paper is about the existence of a non-Archimedean probability for infinite sequences of coin tosses, which we propose as a sound mathematical model for Laplace's original ideas.

Recall the *Kolmogorovian framework*:

- The *sample space*
$$\Omega \;=\; \{H,T\}^{\mathbb{N}} \;=\; \{\omega \mid \omega : \mathbb{N} \to \{H,T\}\}$$
is the set of sequences which take either H ("head") or T ("tail") as values.
- A *cylinder set* of codimension n is a set of the form:[4]
$$C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)} \;=\; \{\omega \in \Omega \mid \omega(i_s) = t_s \;\text{ for } s = 1,\ldots,n\}$$

From the probabilistic point of view, the cylinder set $C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}$ represents the event that for every $s = 1,\ldots,n$, the i_s-th coin toss gives t_s as outcome. Notice that the family $\mathfrak{C}$ of all finite disjoint unions of cylinder sets is an algebra of sets over Ω.

- The function $\mu_C : \mathfrak{C} \to [0,1]$ is defined by setting:
$$\mu_C\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}\right) \;=\; 2^{-n}$$
for all cylindrical sets, and then it is extended to a generic element of $\mathfrak{C}$ by finite additivity:

$$\mu_C\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)} \cup \ldots \cup C^{(j_1,\ldots,i_m)}_{(u_1,\ldots,u_m)}\right) = \mu_C\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}\right) + \ldots + \mu_C\left(C^{(j_1,\ldots,i_m)}_{(u_1,\ldots,u_m)}\right).$$

It is shown that μ_C is a probability pre-measure on the ring $\mathfrak{C}$.

Let $\mathfrak{A}$ be the σ-algebra generated by the ring of cylinder sets $\mathfrak{C}$, and let $\mu : \mathfrak{A} \to [0,1]$ be the unique probability measure that extends μ_C, as guaranteed by *Caratheodory extension theorem.* The triple $(\Omega, \mathfrak{A}, \mu)$ is named the *Kolmogorovian probability* for *infinite sequences of coin tosses.*

In [6] it is proved the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{\mathbb{F}}$ which is coherent with the pre-measure μ_C. Namely, by considering the ratio $P(E) = \mathfrak{n}(E)/\mathfrak{n}(\Omega)$ between the numerosity of

[4] We agree that $i_1 < \ldots < i_n$.

the given event E and the numerosity of the whole space Ω, then one obtains a *non-Archimedean* finitely additive probability

$$P : \mathcal{P}(\Omega) \longrightarrow [0,1]_{\mathbb{F}}$$

that satisfies the following properties:

(1) If $F \subset \Omega$ is finite, then for all $E \subseteq \Omega$, the conditional probability
$$P(E|F) \ = \ \frac{|E \cap F|}{|F|}.$$
(2) P agrees with μ_C over all cylindrical sets:
$$P\left(C_{(t_1,\ldots,t_n)}^{(i_1,\ldots,i_n)}\right) \ = \ \mu_C\left(C_{(t_1,\ldots,t_n)}^{(i_1,\ldots,i_n)}\right) \ = \ 2^{-n}.$$

We are now able to refine this result by showing that, up to infinitesimals, we can take P to agree with μ on the whole σ-algebra $\mathfrak{A}$.

Theorem 4.1. *Let* $(\Omega, \mathfrak{A}, \mu)$ *be the Kolmogorovian probability for infinite coin tosses. Then there exists an elementary numerosity* $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{\mathbb{F}}$ *such that the corresponding non-Archimedean probability* $P(E) = \mathfrak{n}(E)/\mathfrak{n}(\Omega)$ *satisfies the above properties* (1) *and* (2), *along with the additional condition:*

(3) $st(P(E)) = \mu(E)$ *for all* $E \in \mathfrak{A}$.

Proof. Recall that the family $\mathfrak{C} \subset \mathfrak{A}$ of finite disjoint unions of cylinder sets is an algebra whose nonempty sets have all positive measure. So, by applying Theorems 1.8 and 2.1, we obtain an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{\mathbb{F}}$ such that for every positive number of the form $\beta = \frac{\mathfrak{n}(Z)}{\mu(Z)}$ (where $0 < \mu(Z) < +\infty$), one has:

(*i*) $\mathfrak{n}(C) = \mathfrak{n}(C')$ whenever $C, C' \in \mathfrak{C}$ are such that $\mu(C) = \mu(C')$;
(*ii*) $\mathfrak{n}_\beta(E) = \mu(E)$ for all $E \in \mathfrak{A}$.

Property (1) trivially follows by recalling that elementary numerosities of finite sets agree with cardinality:

$$P(E|F) \ = \ \frac{P(E \cap F)}{P(F)} \ = \ \frac{\frac{\mathfrak{n}(E\cap F)}{\mathfrak{n}(\Omega)}}{\frac{\mathfrak{n}(F)}{\mathfrak{n}(\Omega)}} \ = \ \frac{\mathfrak{n}(E \cap F)}{\mathfrak{n}(F)} \ = \ \frac{|E \cap F|}{|F|}.$$

Let us now turn to condition (2). Notice that for any fixed n-tuple of indices $(i_1, \ldots, i_n)$:

- There are exactly 2^n-many different n-tuples $(t_1, \ldots, t_n)$ of heads and tails;
- The associated cylinder sets $C_{(t_1,\ldots,t_n)}^{(i_1,\ldots,i_n)}$ are pairwise disjoint and their union equals the whole sample space Ω.

By (i), all those cylinder sets of codimension n have the same numerosity $\eta = \mathfrak{n}\left(C^{(i_1,\dots,i_n)}_{(t_1,\dots,t_n)}\right)$ and so, by additivity, it must be $\mathfrak{n}(\Omega) = 2^n \cdot \eta$. We conclude that

$$P\left(C^{(i_1,\dots,i_n)}_{(t_1,\dots,t_k)}\right) \;=\; \frac{\mathfrak{n}\left(C^{(i_1,\dots,i_n)}_{(t_1,\dots,t_k)}\right)}{\mathfrak{n}(\Omega)} \;=\; \frac{\eta}{2^n \cdot \eta} \;=\; 2^{-n}.$$

We are left to prove (3). By taking as $\beta = \frac{\mathfrak{n}(\Omega)}{\mu(\Omega)} = \mathfrak{n}(\Omega)$, property (ii) ensures that for every $E \in \mathfrak{A}$:

$$\mu(E) \;=\; \mathfrak{n}_\beta(E) \;=\; \mathrm{st}\left(\frac{\mathfrak{n}(E)}{\beta}\right) \;=\; \mathrm{st}\left(\frac{\mathfrak{n}(E)}{\mathfrak{n}(\Omega)}\right) \;=\; \mathrm{st}(P(E)).$$

$\square$

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Dipartimento di Matematica, Università di Pisa, Italy and Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia
E-mail address: benci@dma.unipi.it

Dipartimento di Matematica, Università di Trento, Italy.
E-mail address: emanuele.bottazzi@unitn.it

Dipartimento di Matematica, Università di Pisa, Italy.
E-mail address: dinasso@dm.unipi.it