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Generalised relativistic Ohm's laws, extended gauge transformations, and magnetic linking

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Generalisations of the relativistic ideal Ohm's law are presented that include specific dynamical features of the current carrying particles in a plasma. Cases of interest for space and laboratory plasmas are identified where these generalisations allow for the definition of generalised electromagnetic fields that transform under a Lorentz boost in the same way as the real electromagnetic fields and that obey the same set of homogeneous Maxwell's equations. © *2015 AIP Publishing LLC*.
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I. INTRODUCTION

In the fluid description of a plasma, the momentum equation of the lighter particle species, generally the electrons, plays a fundamental role in determining the properties of the large spatial scale, low frequency dynamics. This is particularly evident in the case of the single fluid MHD description where the so called ideal Ohm's law is essentially the momentum equation of massless, cold electrons. More precisely, it expresses the vanishing of the electromagnetic force on the electron fluid under the additional assumption that the electron and the ion fluid velocities can be assumed, within this equation, to coincide. In addition, it is assumed that there is no electron fluid momentum transmitted through collisions to different species (resistivity) or to the electromagnetic fields through high frequency incoherent radiation (radiation friction).

It is a fundamental feature of MHD that, if the ideal Ohm's law applies, the plasma dynamics is constrained by an infinite set of topological invariants, such as the one arising from the conservation of magnetic helicity. These constraints express the invariance in time of *magnetic connections*, i.e., the property that if the ideal Ohm's law holds and the plasma velocity field remains smooth, two fluid elements that at $t=0$ are linked by a magnetic field line remain linked by a magnetic field line at any successive time.[1] This property goes under the abbreviated but suggestive statement that the magnetic field is frozen in the plasma.

It is also well known that a number of physical effects leads to violations of the ideal Ohm's and that these effects are generally related to the appearance of small spatial and/or temporal scales due, e.g., to the nonlinearity of the plasma dynamics. When these violations occur only locally, the magnetic field lines in the plasma undergo the well known process of magnetic reconnection. In this process, the identity of fields lines is lost only inside the reconnection region but the linking between different fluid elements is changed globally, causing a rearrangement of the global magnetic field topology.

There are different ways in which the ideal Ohm's law can be violated: they can be roughly grouped into three different classes. In a first class, the violation amounts to a change of the fluid with respect to which the magnetic field is frozen, as is the case of a two-fluid plasma description where the restriction that the ion fluid and the electron fluid move with the same velocity is relaxed, or in the so called Electron Magnetohydrodynamics[2,3] where ions are taken to be immobile. In this class, which includes the so called Hall-MHD,[4] the magnetic field remains frozen with respect to the electron fluid. As a consequence, the topology of the magnetic field $\vec{B}$ is preserved, although the ion fluid is allowed to slip with respect to the magnetic field.

A second class involves a change in the fields that define the linking. This is the case when the assumption of massless electrons is relaxed, and thus work must be performed in order to accelerate them. In this case is was shown (see Ref. 5) that, if the electron fluid is assumed to be cold, a generalised magnetic field $\vec{B}_e \equiv \nabla \times (\vec{A} - e\vec{u}_e/m_e)$ is frozen in the electron fluid and a generalised ideal Ohm's law can be written in the form

$$\vec{E}_e + \vec{u}_e \times \vec{B}_e/c = 0, \tag{1}$$

where $\vec{E}_e = -\nabla(\phi - e|\vec{u}_e|^2/(2m_e) - \partial_t(\vec{A} - e\vec{u}_e/m_e)/c$. In addition, the fields $\vec{E}_e$ and $\vec{B}_e$ satisfy the homogeneous Maxwell's equation $\nabla \cdot \vec{B}_e = 0$ and $\nabla \times \vec{E}_e = -\partial_t \vec{B}_e/c$. In this case, the topology of the magnetic field $\vec{B}$ is not preserved but the topology of the generalised magnetic field $\vec{B}_e$ is, i.e., B_e-connections are preserved by the electron dynamics. In this case, see Ref. 6 and references therein, magnetic reconnection can only proceed if large gradients of the electron fluid velocity $\vec{u}_e$, or somewhat equivalently of the plasma current density, are produced. A similar result applies if we relax the condition of cold electrons and introduce in Ohm's law the gradient of an isotropic pressure that is a function of the plasma density only. In the non relativistic case, this can be performed by adding the contribution arising from the gradient of the pressure to the gradient of the electrostatic potential ϕ. In this case, if, for example, the electron inertia is neglected, the magnetic field $\vec{B}$ remains frozen in the plasma MHD flow.

The third class involves phenomena that are the consequence of a momentum transfer to the other particle species either through collisions or higher frequency collective phenomena, i.e., through the effect of collisional or anomalous resistivity. Electron momentum can also be lost through high frequency incoherent radiation (the so called radiation reaction force or radiation friction), or spatially redistributed between different electron fluid elements by electron viscosity. Additional violations in this class arise from electron kinetic effects that are not accounted for within a standard fluid description, such as Landau damping or an anisotropic and in particular, non-gyrotropic pressure tensor. Contrary to the two previous cases, for this class, it is not normally possible to define a generalised magnetic field that remains frozen in a fluid plasma component, however selected.

An important feature of the ideal Ohm's law is that it is in no sense restricted to a non relativistic plasma regime, as it can be written (unmodified) in the fully covariant form

$$F_{\mu\nu}u_\nu = 0, \tag{2}$$

where $F_{\mu\nu}$ is the electromagnetic (e.m.) field tensor, u_μ is a normalised timelike 4-vector ($u_\mu u_\mu = -1$) which we interpret as the fluid velocity 4-vector field of the plasma species with respect to which the magnetic field is frozen.[7] While the ideal Ohm's law is fully covariant, its interpretation in terms of the conservation of magnetic connections is not. In fact, the meaning of magnetic connection and magnetic topology is not clear in a relativistic context because of two related reasons: first, the distinction between electric and magnetic fields is frame dependent, and second, the very concept of field lines, which are defined in coordinate space at a given time, is frame dependent due to the violation of simultaneity in different reference frames of events at different spatial locations. This feature was addressed in Ref. 8 where it was shown that the covariant formulation of magnetic connections can be restored by means of a time resetting projection along the trajectories of the plasma elements. This projection is consistent with the ideal Ohm's law and compensates for the loss of simultaneity in different reference frames between spatially separated events. It was then shown (see Ref. 9) that a frame independent definition of magnetic topology can be recovered by referring to 2D-hypersurfaces in 4D Minkowski space instead of 1D curves in 3D space at fixed time. These hypersurfaces are defined by the two linearly independent 4-vector fields[10] whose contraction with the e.m. field tensor $F_{\mu\nu}$ is identically zero, while the corresponding homogeneous Maxwell equations $\partial_\mu \mathcal{F}_{\mu\nu} = 0$, with $\mathcal{F}_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma}/2$ the dual of the e.m. tensor $F_{\mu\nu}$, play the role of a Frobenius involution condition for the existence of the foliation of Minkowski space defined by these hypersurfaces. The covariant definition of these hypersurfaces makes it possible to define magnetic connection-lines covariantly by taking cuts at the same coordinate time in each reference frame.

In the present article, we address the relativistic covariant formulation of a non-ideal Ohm's law (Sec. II) and look for the conditions that are required in order to define a covariant form of generalised connections. In this context, we note the analysis recently presented in Ref. 11 where generalized magnetic connections are derived for a set of relativistic non-ideal MHD equations that include thermal-inertial, thermal-electromotive, Hall, and current-inertia effects.

We show that the conditions required in order to define a covariant form of generalised connections can be satisfied automatically by introducing a *generalised gauge transformation* of the 4-vector potential A_μ defined by a *gauge field* s_μ that must satisfy a compatibility condition involving the 4-velocity u_μ. We refer in particular, to the case where inertial and thermal electron effects are considered (Sec. III). The results obtained in this section agree with the analysis in Ref. 11 when the difference between the adopted plasma descriptions is taken into account: generalized relativistic MHD equations in Ref. 11, relativistic electron fluid equations coupled to the homogeneous Maxwell's equations in the present article.

An interesting extension to a fluid of relativistic spherical tops is given in Sec. IV, while two dissipative cases, radiation friction and collisional resistivity, are discussed in Secs. V and VI, respectively. The definition of generalised helicity is given in Sec. VII, while the relevance of the present analysis to the development of magnetic reconnection is briefly discussed in the Conclusions.

Before proceeding, we recall that the tensor contractions $F_{\mu\nu}F_{\mu\nu}$ and $F_{\mu\nu}\mathcal{F}_{\mu\nu}$ are Lorentz invariants proportional to $|\vec{E}|^2 - |\vec{B}|^2$ and $\vec{E}\cdot\vec{B}$, respectively, and that $F_{\mu\nu}\mathcal{F}_{\mu\nu}$ vanishes if an equation of the form of Eq. (2) holds where, in general, the 4-vector field that is annihilated by the e.m. field tensor $F_{\mu\nu}$ need not be timelike.

II. RELATIVISTIC OHM'S LAW

We write the relativistic Ohm's law in formal terms as

$$F_{\mu\nu}u_\nu = R_\mu, \tag{3}$$

with R_μ a 4-vector field such that

$$u_\mu R_\mu = 0. \tag{4}$$

The 4-vector R_μ is taken to include any non ideal effect not included in Eq. (2). Note however that if R_μ can be put in the form $R_\mu = -F_{\mu\nu}v_\nu$ such that if $u_\mu + v_\mu \neq 0$ is still a timelike 4-vector field, this violation of the ideal Ohm's law can in principle be removed by a different choice of the "reference" 4-velocity.[12] For this to occur the Lorentz invariant $F_{\mu\nu}\mathcal{F}_{\mu\nu}$ must vanish. This case will not be considered in the rest of this article as we will require that the violation of the ideal Ohm's law makes $F_{\mu\nu}\mathcal{F}_{\mu\nu} \neq 0$ at least locally.

Using the standard decomposition[13–15] of the field tensor

$$F_{\mu\nu} = \varepsilon_{\mu\nu\lambda\sigma}b_\lambda u_\sigma + [u_\mu e_\nu - u_\nu e_\mu], \tag{5}$$

where b_μ is the *4-vector magnetic field* and e_μ is the *4-vector electric field*, with $u_\mu e_\mu = 0$ and $u_\mu b_\mu = 0$, we find

$$R_\mu = e_\mu. \tag{6}$$

The 4-vectors e_μ and b_μ in Eq. (5) are related to the standard electric and magnetic fields $\vec{E}$ and $\vec{B}$ in 3D space by

$$b_\mu = \gamma(\vec{B}\cdot\vec{v}\,,\,\vec{B}+\vec{E}\times\vec{v}), \tag{7}$$

and

$$e_\mu = \gamma(\vec{E}\cdot\vec{v}\,,\,\vec{E}+\vec{v}\times\vec{B}), \tag{8}$$

with $e_\mu b_\mu = \vec{E}\cdot\vec{B}$, γ is the relativistic Lorentz factor, and we have used $u_\mu = \gamma(1,\vec{v})$.

The representation given in Eq. (5) is physically convenient as it allows us to separate covariantly the magnetic and the electric parts of the e.m. field tensor relative to a given plasma component moving with 4-velocity u_μ. In the local rest frame of this plasma component, the time components of e_μ and b_μ vanish, while their space components reduce to the standard 3-D electric and magnetic fields. In addition, as shown by Eqs. (3) and (6), the electric part vanishes if the ideal Ohm's law holds. In this case, we can use $e_\mu = 0$ in order to express b_μ in terms of $\vec{B}$ and $\vec{v}$ only, and magnetic connections, defined by the cuts at constant time of the 2D-hypersurfaces generated by the 4-vectors b_μ and u_μ, are preserved.

In order to search for generalized connections when $e_\mu \neq 0$, we consider the magnetic part of $F_{\mu\nu}$, introduce a generalised magnetic 4-vector field $\hat{b}_\mu \equiv b_\mu + d_\mu$, and define the generalised field tensor

$$F^b_{\mu\nu} = \varepsilon_{\mu\nu\lambda\sigma}\hat{b}_\lambda u_\sigma. \tag{9}$$

Then, we look for the conditions such that $F^b_{\mu\nu}$ satisfies the homogeneous Maxwell's equations

$$\partial_\mu \mathcal{F}^b_{\mu\nu} = 0, \tag{10}$$

where the dual tensor $\mathcal{F}^b_{\mu\nu}$ is defined by

$$\mathcal{F}^b_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda\sigma}F^b_{\lambda\sigma}/2 = u_\mu\hat{b}_\nu - \hat{b}_\mu u_\nu. \tag{11}$$

Following the usual procedure where the homogeneous Maxwell's equations allow us to define the 4-vector potential, we set

$$F^b_{\mu\nu} \equiv \partial_\mu A^b_\nu - \partial_\nu A^b_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu s_\nu - \partial_\nu s_\mu, \tag{12}$$

where A_μ is the 4-vector potential that defines $F_{\mu\nu}$, while A^b_μ is a generalized 4-vector potential, and $s_\mu \equiv A^b_\mu - A_\mu$ is a "gauge" field. Combining Eqs. (5), (9), and (12) gives

$$\varepsilon_{\mu\nu\lambda\sigma}\, d_\lambda u_\sigma + u_\mu e_\nu - u_\nu e_\mu = -\partial_\mu s_\nu + \partial_\nu s_\mu. \tag{13}$$

Contracting Eq. (13) with u_μ, we obtain the compatibility condition

$$e_\mu = u_\nu\partial_\nu s_\mu - u_\nu\partial_\mu s_\nu = \partial_\tau s_\mu - u_\nu\partial_\mu s_\nu, \tag{14}$$

with $\partial_\tau \equiv u_\mu\partial_\mu$ the convective derivative with respect to the proper time τ, while the remaining components of Eq. (13) determine the 4-vector d_μ which can be obtained from Eq. (13) by contracting it with $u_\alpha\varepsilon_{\alpha\beta\mu\nu}$ and using $d_\mu u_\mu = 0$. Any choice of the 4-vector field s_μ compatible with a specified e_μ in Eq. (14) defines a generalised ideal Ohm's law in terms of modified e.m. fields given by the field tensor

$$F^b_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda\sigma}\hat{b}_\lambda u_\sigma \equiv F_{\mu\nu} + \partial_\mu s_\nu - \partial_\nu s_\mu, \tag{15}$$

and generalised conserved connections defined by the cuts at constant time of the 2D-hypersurfaces generated by the 4-vectors $\hat{b}_\mu$ and u_μ.

Note that the choice where s_ν is a 4-gradient corresponds to $e_\mu = 0$ and is simply a standard gauge transformation of the vector potential A_μ that does not affect the e.m. fields.

III. RELATIVISTIC INERTIAL OHM'S LAW

An interesting choice of the gauge 4-vector s_μ is

$$s_\mu = \Pi u_\mu, \tag{16}$$

with Π a scalar field. From Eq. (14), we obtain

$$\begin{aligned} e_\mu &= \partial_\tau(\Pi u_\mu) + \partial_\mu\Pi = (u_\mu u_\nu + \delta_{\mu\nu})\partial_\nu\Pi + \Pi\partial_\tau u_\mu \\ &= \partial_\nu[(u_\mu u_\nu + \delta_{\mu\nu})\Pi] + u_\mu u_\nu(\Pi/n)\partial_\nu n, \end{aligned} \tag{17}$$

where $u_\mu u_\nu + \delta_{\mu\nu}$ is the projector perpendicular to u_μ, and the scalar function n is defined by the continuity equation $\partial_\nu(nu_\nu) = 0$. Thus, we can write

$$n\, e_\mu = \partial_\nu[nu_\mu u_\nu\Pi] + n\partial_\mu\Pi. \tag{18}$$

If we set $\Pi \equiv (P+\epsilon)/(nq)$, where we interpret P and ϵ as the invariant pressure and mass-energy density, n as the invariant numerical density with m and q as the mass and charge, respectively, we obtain

$$nq\, e_\mu = nq\, F_{\mu\nu}u_\nu = \partial_\nu[\, u_\mu u_\nu(P+\epsilon)] + n\,\partial_\mu[(P+\epsilon)/n], \tag{19}$$

that gives

$$nq\, e_\mu = \partial_\nu T_{\mu\nu} - (P/n)\partial_\mu n + n\,\partial_\mu(\epsilon/n), \tag{20}$$

where $T_{\mu\nu} \equiv (P+\epsilon)u_\mu u_\nu + P\delta_{\mu\nu}$ can be interpreted as the fluid energy momentum tensor.

If we further assume that ϵ and P are functions of n only, consistently with the assumption made in the nonrelativistic case in the Introduction, and use the thermodynamic relationship (see, e.g., Eq. (8) of Ref. 15 with all dependences on the entropy density dropped having in effect assumed that the entropy density of the fluid is uniform and constant)

$$P = n\partial\epsilon/\partial n - \epsilon, \tag{21}$$

we find that the last two terms in Eq. (20) cancel. Writing the dissipationless relativistic single fluid momentum-energy equation in the form

$$\partial_\nu T_{\mu\nu} = nq\, F_{\mu\nu}u_\nu, \tag{22}$$

we see that, if ϵ and P are functions of n only, the choice of the gauge field $s_\mu = [(P+\epsilon)/(nq)]u_\mu$ defines conserved generalized connections in agreement with Ref. 11. In the cold $P = 0$ limit, $\Pi = m/q$ and the combination $A_\mu + s_\mu$ reduces, aside for a multiplication factor, to the standard (cold) fluid canonical momentum $mu_\mu + qA_\mu$. In the non relativistic limit for the electron fluid, the generalised e.m. fields that are obtained from the generalised 4-vector potential $A_\mu + mu_\mu/q$ reduce to the electric and magnetic fields $\vec{E}_e$ and $\vec{B}_e$ defined in the Introduction.

IV. RELATIVISTIC FLUID OF SPHERICAL TOPS

An interesting choice of s_μ that can be related to the motion of relativistic spherical tops,[16] in view of the description of a classical fluid of electrons with an internal degree of freedom (such as spin, see, e.g., the recent Ref. 17, Chap. 7, and references therein), is

$$s_\mu = C\,\sigma_{\mu\nu}\sigma_{\nu\lambda}u_\lambda, \tag{23}$$

with $\sigma_{\mu\nu}$ an antisymmetric matrix function and C a scalar constant. We obtain

$$e_\mu = C\,[\partial_\tau(\sigma_{\mu\nu}\sigma_{\nu\lambda}u_\lambda) - u_\nu\partial_\mu(\sigma_{\nu\beta}\sigma_{\beta\lambda}u_\lambda)]. \tag{24}$$

If $\sigma_{\mu\nu}$ is taken to be constant, Eq. (24) can be written as

$$e_\mu = C[(\sigma_{\mu\nu}\sigma_{\nu\lambda}\partial_\tau u_\lambda) - \partial_\mu(\Sigma_\nu\Sigma_\nu)/2], \tag{25}$$

with $\Sigma_\mu = \sigma_{\mu\nu}u_\nu$. The gauge field in Eq. (23) is compatible with an equation of motion with a modified inertia term $(\sigma_{\mu\nu}\sigma_{\nu\lambda}\partial_\tau u_\lambda)$ and a gradient force term $-\partial_\mu(\Sigma_\nu\Sigma_\nu)/2$, which ensures the consistency of the constraint $u_\mu u_\mu = -1$.

V. RELATIVISTIC RADIATION REACTION ON A COLD FLUID PLASMA

A different result can be expected in a "dissipative" case setting

$$s_\mu = C\,\partial_\tau u_\mu. \tag{26}$$

This choice requires the introduction in the fluid momentum equation of the second (proper) time convective derivative of the fluid 4-velocity and could be used to make a comparison with the radiation reaction force[18] on a cold relativistic plasma due to emission of (classical) incoherent high frequency radiation (see also Ref. 19 for a thermal relativistic plasma). We obtain

$$\begin{aligned} e_\mu &= C\,[\partial_\tau(\partial_\tau u_\mu) - u_\nu\partial_\mu(\partial_\tau u_\nu)] \\ &= C\,[\partial_\tau\partial_\tau u_\mu + (\partial_\mu u_\nu)(\partial_\tau u_\nu)], \end{aligned} \tag{27}$$

which can be rewritten more transparently as

$$e_\mu = C\,(\delta_{\mu\nu} + u_\mu u_\nu)[\partial_\tau\partial_\tau u_\nu - u_\alpha\partial_\nu\partial_\tau u_\alpha], \tag{28}$$

where $(\delta_{\mu\nu} + u_\mu u_\nu)$ is the projector perpendicular to u_μ and $(\partial_\mu u_\nu)(\partial_\tau u_\nu)$.

While, taking the electron distribution function to be a δ function in momentum space, the term in Eq. (28) that involves the second derivative of the 4-velocity with respect to the proper time τ can be related to the single particle Lorentz-Abraham-Dirac (LAD) equation,[18] the second involves the coordinate derivatives of the acceleration 4-vector $\partial_\tau u_\mu$. Thus, the introduction in the 4-momentum equation of a LAD-force term does not lead to a generalised Ohm's law compatible with Eq. (10) unless the contribution of the second term vanishes. Conversely, one can use Eq. (28) to split the LAD force into a term that defines a generalised ideal Ohm's law and a term that cannot be included in such a framework. This is important when looking, as is done in Sec. VII, for conservation laws of the plasma dynamics.

VI. RESISTIVE OHM'S LAW

A similar splitting of the term that violates the ideal Ohm's law can in principle be found in the case of a resistive Ohm's law.

We write the relativistic covariant form[20] of the resistive term as

$$e_\mu = \eta\,(\delta_{\mu\nu} + u_\mu u_\nu)j_\nu = \eta\,(j_\mu - \rho u_\mu), \tag{29}$$

where η is a scalar resistivity, j_μ is the current density four vector, and $\rho \equiv u_\mu j_\mu$ is the invariant charge density. The projector operator, which is required in order to satisfy the constraint $e_\mu u_\mu = 0$, subtracts from j_μ the current density arising from the charge advected by the fluid 4-velocity u_μ which is not affected by resistivity. For the sake of simplicity, we take η to be a constant.

Using the inhomogeneous Maxwell's equation $\partial_\nu F_{\mu\nu} = (4\pi/c)\,j_\mu$, we write Eq. (29) as

$$e_\mu = (c\eta/4\pi)\,(\partial_\nu F_{\mu\nu} + u_\mu u_\nu\partial_\alpha F_{\alpha\nu}), \tag{30}$$

and compare it with Eq. (14) that requires $e_\mu = \partial_\tau s_\mu - u_\nu\partial_\mu s_\nu$. A possible choice for s_μ, in a sense the counterpart of the choice made in Eq. (26) as it involves integration with respect to the proper time τ' along the fluid trajectories instead of differentiation, is to take

$$s_\mu = \int^\tau d\tau' e_\mu(\tau') = (c\eta/4\pi)\int^\tau d\tau'(\partial_\nu F_{\mu\nu} + u_\mu u_\nu\partial_\alpha F_{\alpha\nu})', \tag{31}$$

which leaves the term $u_\nu\partial_\mu s_\nu$ unbalanced. Similarly to the result of Sec. V, the unbalanced term depends on the coordinate derivatives of s_μ.

VII. GENERALISED MAGNETIC HELICITY

As already mentioned, if the generalised Ohm's law can be written in the form $F^b_{\mu\nu}u_\mu = 0$ with $\partial_\mu\mathcal{F}^b_{\mu\nu} = 0$, it is possible to define in a covariant way generalised magnetic connections between plasma elements. In this section, we consider the generalisation on magnetic helicity that in the case of the ideal Ohm's law in Eq. (1) is represented by the 4-vector

$$K_\mu = \mathcal{F}_{\mu\nu}A_\nu. \tag{32}$$

The 4-vector K_μ, which is defined modulo a 4-divergence because of the standard gauge freedom in the choice of the vector potential $(A_\mu \rightarrow A_\mu + \partial_\mu\psi)$, satisfies the continuity equation

$$\partial_\mu K_\mu = \mathcal{F}_{\mu\nu} F_{\mu\nu}/2 = 0, \tag{33}$$

where the last equality holds because of the ideal Ohm's law. In the framework of the above analysis, Eqs. (32) and (33) can be generalised by defining

$$K_\mu^b = \mathcal{F}_{\mu\nu}^b A_\nu^b, \tag{34}$$

where $\mathcal{F}_{\mu\nu}^b$ and A_ν^b are defined by Eqs. (11) and (12), and K_μ^b satisfies the continuity equation

$$\partial_\mu K_\mu^b = \varepsilon_{\mu\nu\alpha\beta}\, \partial_\mu[(A_\nu + s_\nu)\partial_\alpha(A_\beta + s_\beta)] = 0. \tag{35}$$

Referring, for example, to the case of the ideal inertial Ohm's law for cold electrons, we see that this generalised continuity equation involves the conservation of the sum of the magnetic helicity defined by Eq. (32), of a term proportional to the fluid 4-helicity defined by $\Omega_{\mu\nu} u_\nu$ where $\Omega_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta}\partial_\alpha u_\beta$ is the fluid vorticity, and of two mixed terms proportional to $\mathcal{F}_{\mu\nu} u_\nu$ and $\Omega_{\mu\nu} A_\nu$, respectively.

VIII. CONCLUSIONS

The dynamics of relativistic plasmas is a subject of great present interest for both laboratory plasma physics[21,22] and astrophysical plasmas[23,24] and, in particular, for the conversion of electromagnetic field energy into kinetic and thermal energy of the plasma particles, and vice versa. In this context, the equations of magnetohydrodynamics have been extended (see Refs. 13 and 14) and used in numerical simulations (see, e.g., Ref. 25) so as to include fluid and thermal velocities close to the speed of light, and the concept of reconnection of magnetic field lines, a fundamental process in plasmas, has been extended to relativistic regimes.[26] Magnetic reconnection is in fact ubiquitous in magnetised plasmas and can be viewed as a process that converts magnetic energy inside highly inhomogeneous regions into plasma particle energy and as a process that modifies the magnetic topology, more precisely the connections drawn by the magnetic field lines. These processes are made possible by local effects that are outside the large spatial-scale, long time-interval description of (ideal) MHD theory. Thus, an important point in this relativistic extension of MHD is to provide a frame independent definition of magnetic reconnection. Such a definition is not obvious both from a theoretical and an observational point of view, since the distinction between electric and magnetic fields is frame dependent and the tracing of field lines, which are defined in coordinate space at a given time, is also frame dependent due to the violation of simultaneity in different reference frames of events at different spatial locations. This point was addressed in Refs. 8 and 9.

A second important point is to find a covariant relativistic extension of the generalised magnetic connections that are known to occur when the ideal Ohm's law is violated by terms that can be accounted for by defining generalised e.m. fields.[11] Generalised e.m. fields must satisfy:

(1) an ideal Ohm's law (see Eq. (1)),
(2) and a set of equations analogous to the homogeneous Maxwell's equations (see Eq. (10)).

The inclusion of electron inertia terms in the ideal Ohm's equation expressed in terms of the electron fluid velocity is a well studied example[27] in the nonrelativistic case.

The connections between plasma elements defined by the generalised electromagnetic fields that satisfy conditions (1) and (2) are conserved by the plasma dynamics and, for the electron inertia case, it was shown in the literature (see, e.g., Refs. 28–30 and references therein) that they can have important consequences on the development of magnetic reconnection. In fact in this case, the generalized magnetic field $\vec{B}_e$ cannot reconnect, and thus the reconnection of $\vec{B}$ can only proceed by developing increasingly steeper layers of the electron velocity, and thus of the plasma current density, on scalelength related to the so called electron inertial skin depth.

In the present article, we have examined within a formal framework whether such extensions can be performed in a covariant relativistic way in terms of a *gauge* 4-vector field that adds to the 4-vector potential field so as to implement conditions (1) and (2) automatically.

We have shown that the extension that includes the electron inertial term is straightforward in a cold relativistic plasma and leads to a generalised vector potential that is proportional to the well known fluid canonical momentum. It can easily be extended to include electron thermal effects if the pressure and mass-energy density are assumed to be functions of the electron density only.

We have also exploited the formal framework developed in Sec. II in order to examine more exotic situations, such as the equation for relativistic spherical tops with the aim of looking how to include in this formalism internal degrees of freedom of the particles in the plasma.

Dissipative terms ranging from a relativistic formulation of resistivity (frequently used in the study of relativistic reconnection, see, e.g., Ref. 31) and the radiation reaction force (of interest for present laser plasma interactions[32] and suggested as a mechanism for relativistic reconnection in Ref. 33) have been examined. It appears that only part of their contributions can be accounted for by generalised e.m. fields and that the part that cannot be accounted for involves coordinate derivatives and is thus related to inhomogeneities in the dissipation process between neighbouring plasma elements.

Finally, in Sec. VI, we have shown that, when generalised e.m. fields can be defined according to the requirements (1) and (2), a generalised helicity 4-vector field can be constructed that has vanishing 4-divergence, i.e., that obeys a conservation law expressed by a continuity equation as is the case for the helicity 4-vector field in the context of ideal MHD.

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