A Julia-Wolff-Carathéodory theorem for infinitesimal generators in the unit ball

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Abstract. We prove a Julia-Wolff-Carathédory theorem on angular derivatives of infinitesimal generators of one-parameter semigroups of holomorphic self-maps of the unit ball $B^n \subset \mathbb{C}^n$, starting from results recently obtained by Bracci and Shoikhet.

0. Introduction

The classical Fatou theorem says that a bounded holomorphic function f defined on the unit disk $\Delta \subset \mathbb{C}$ admits non-tangential limit at almost every point of $\partial\Delta$, but it does not say anything about the behavior of $f(\zeta)$ as ζ approaches a specific point σ of the boundary. Of course, to be able to say something in this case one needs some hypotheses on f. For instance, one can assume that, in a very weak sense, $f(\zeta)$ approaches the boundary of Δ at least as fast as ζ. It turns out that under this condition, not only f, but even its derivative admits non-tangential limit. This is the content of the classical *Julia-Wolff-Carathéodory theorem:*

Theorem 0.1: (Julia-Wolff-Carathéodory) *Let $f\colon\Delta \to \Delta$ be a bounded holomorphic function such that*

$$\liminf_{\zeta\to\sigma} \frac{1-|f(\zeta)|}{1-|\zeta|} = \alpha < +\infty \tag{0.1}$$

for some $\sigma \in \partial\Delta$. Then f has non-tangential limit $\tau \in \partial\Delta$ at σ, for all $\zeta \in \Delta$ one has

$$\frac{|\tau - f(\zeta)|^2}{1-|f(\zeta)|^2} \le \alpha \, \frac{|\sigma-\zeta|^2}{1-|\zeta|^2} \,, \tag{0.2}$$

and furthermore both the incremental ratio $\big(\tau - f(\zeta)\big)/(\sigma-\zeta)$ and the derivative $f'(\zeta)$ have non-tangential limit $\alpha\bar\sigma\tau$ at σ.

This results from the work of several authors: Julia [Ju1, Ju2], Wolff [Wo], Carathéodory [C], Landau and Valiron [L-V], R. Nevanlinna [N] and others (see, e.g., [B] and [A1] for proofs, history and applications).

As already noticed by Korányi and Stein ([Ko], [K-S], [St]) when they extended Fatou's theorem to several complex variables, for domains in $\mathbb{C}^n$ the notion of non-tangential limit is not the right one to consider. Actually, it turns out that for generalizing the Julia-Wolff-Carathéodory theorem

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from the unit disk to the unit ball $B^n \subset \mathbb{C}^n$ one needs two different notions of limit at the boundary, both stronger than non-tangential limit.

A function $f: B^n \to \mathbb{C}$ has non-tangential limit $L \in \mathbb{C}$ at a boundary point $p \in \partial B^n$ if $f(z) \to L$ as $z \to p$ staying inside cones with vertex at p; a stronger notion of limit can be obtained by using approach regions larger than cones.

In the unit disk, as approach regions for the non-tangential limit one can use Stolz regions, since they are angle-shaped nearby the vertex. In the unit ball $B^n \subset \mathbb{C}^n$ the natural generalization of a Stolz region is the *Korányi region* $K(p, M)$ of *vertex* $p \in \partial B^n$ and *amplitude* $M > 1$ given by

$$K(p, M) = \left\{ z \in B^n \;\middle|\; \frac{|1 - \langle z, p \rangle|}{1 - \|z\|} < M \right\} ,$$

where $\|\cdot\|$ denotes the euclidean norm and $\langle \cdot, \cdot \rangle$ the canonical hermitian product. We shall say that a function $f: B^n \to \mathbb{C}$ has *K-limit* (or *admissible limit*) $L \in \mathbb{C}$ at $p \in \partial B^n$, and we shall write $K\text{-}\lim_{z \to p} f(z) = L$, if $f(z) \to L$ as $z \to p$ staying inside any Korányi region $K(p, M)$. Since a Korányi region $K(p, M)$ approaches the boundary non-tangentially along the normal direction at p but tangentially along the complex tangential directions at p, it turns out that having K-limit is stronger than having non-tangential limit. However, the best generalization of Julia's lemma to B^n is the following result (proved by Hervé [H] in terms of non-tangential limits and by Rudin [R] in general):

Theorem 0.2: *Let $f: B^n \to B^m$ be a holomorphic map such that*

$$\liminf_{z \to p} \frac{1 - \|f(z)\|}{1 - \|z\|} = \alpha < +\infty ,$$

for some $p \in \partial B^n$. Then f admits K-limit $q \in \partial B^m$ at p, and furthermore for all $z \in B^n$ one has

$$\frac{|1 - \langle f(z), q \rangle|^2}{1 - \|f(z)\|^2} \le \alpha \frac{|1 - \langle z, p \rangle|^2}{1 - \|z\|^2} .$$

To obtain a complete generalization of the Julia-Wolff-Carathéodory theorem for B^n one needs a different notion of limit, still stronger than non-tangential limit, but weaker than K-limit.

A crucial one-variable result relating limits along curves and non-tangential limits is *Lindelöf's theorem.* Given $\sigma \in \partial \Delta$, a *$\sigma$-curve* is a continuous curve $\gamma: [0, 1) \to \Delta$ such that $\gamma(t) \to \sigma$ as $t \to 1^-$. Then Lindelöf [Li] proved that if a bounded holomorphic function $f: \Delta \to \mathbb{C}$ admits limit $L \in \mathbb{C}$ along a given σ-curve then it admits limit L along all non-tangential σ-curves — and thus it has non-tangential limit L at σ.

Trying to generalize this theorem to several complex variables, Čirka [Č] realized that for a bounded holomorphic function the existence of the limit along a (suitable) p-curve (where $p \in \partial B^n$) implies not only the existence of the non-tangential limit, but also the existence of the limit along any curve belonging to a larger class of curves, including some tangential ones — but it does not in general imply the existence of the K-limit. To describe the version (due to Rudin [R]) of Čirka's result we shall need in this paper, let us introduce a bit of terminology.

Let $p \in \partial B^n$. As before, a *p-curve* is a continuous curve $\gamma: [0, 1) \to B^n$ such that $\gamma(t) \to p$ as $t \to 1^-$. A p-curve is *special* if

$$\lim_{t \to 1^-} \frac{\|\gamma(t) - \langle \gamma(t), p \rangle p\|^2}{1 - |\langle \gamma(t), p \rangle|^2} = 0 ; \tag{0.3}$$

and, given $M>1$, it is *M-restricted* if

$$\frac{|1-\langle\gamma(t),p\rangle|}{1-|\langle\gamma(t),p\rangle|}<M$$

for all $t\in[0,1)$. We also say that γ is *restricted* if it is M-restricted for some $M>1$. In other words, γ is restricted if and only if $t\mapsto\langle\gamma(t),p\rangle$ goes to 1 non-tangentially in Δ.

It is not difficult to see that non-tangential curves are special and restricted; on the other hand, a special restricted curve approaches the boundary non-tangentially along the normal direction, but it can approach the boundary tangentially along complex tangential directions. Furthermore, a special M-restricted p-curve is eventually contained in any $K(p,M')$ with $M'>M$, and conversely a special p-curve eventually contained in $K(p,M)$ is M-restricted. However, $K(p,M)$ can contain p-curves that are restricted but not special: for these curves the limit in (0.3) might be a strictly positive number.

With these definitions in place, we shall say that a function $f\colon B^n\to\mathbb{C}$ has *restricted K-limit* (or *hypoadmissible limit*) $L\in\mathbb{C}$ at $p\in\partial B^n$, and we shall write $K'\text{-}\lim\limits_{z\to p}f(z)=L$, if $f\bigl(\gamma(t)\bigr)\to L$ as $t\to 1^-$ for any special restricted p-curve $\gamma\colon[0,1)\to B^n$. It is clear that the existence of the K-limit implies the existence of the restricted K-limit, that in turns implies the existence of the non-tangential limit; but none of these implications can be reversed (see, e.g., [R] for examples in the ball).

Finally, we say that a function $f\colon B^n\to\mathbb{C}$ is *K-bounded* at $p\in\partial B^n$ if it is bounded in any Korányi region $K(p,M)$, where the bound can depend on $M>1$. Then the version of Čirka's generalization of Lindelöf's theorem we shall need is the following:

Theorem 0.3: (Rudin [R]) *Let $f\colon B^n\to\mathbb{C}$ be a holomorphic function K-bounded at $p\in\partial B^n$. Assume there is a special restricted p-curve $\gamma^o\colon[0,1)\to B^n$ such that $f\bigl(\gamma^o(t)\bigr)\to L\in\mathbb{C}$ as $t\to 1^-$. Then f has restricted K-limit L at p.*

We can now deal with the generalization of the Julia-Wolff-Carathéodory theorem to several complex variables. With respect to the one-dimensional case there is an obvious difference: instead of only one derivative we have to consider a whole (Jacobian) matrix of them, and there is no reason they should all behave in the same way. And indeed they do not, as shown in Rudin's version of the Julia-Wolff-Carathéodory theorem for the unit ball:

Theorem 0.4: (Rudin [R]) *Let $f\colon B^n\to B^m$ be a holomorphic map such that*

$$\liminf_{z\to p}\frac{1-\|f(z)\|}{1-\|z\|}=\alpha<+\infty\,,$$

for some $p\in\partial B^n$. Then f admits K-limit $q\in\partial B^m$ at p. Furthermore, if we set $f_q(z)=\bigl\langle f(z),p\bigr\rangle q$ and denote by df_z the differential of f at z, we have:
(i) *the function $\bigl(1-\bigl\langle f(z),q\bigr\rangle\bigr)/(1-\langle z,p\rangle)$ is K-bounded and has restricted K-limit α at p;*
(ii) *the map $(f(z)-f_q(z))/(1-\langle z,p\rangle)^{1/2}$ is K-bounded and has restricted K-limit O at p;*
(iii) *the function $\bigl\langle df_z(p),q\bigr\rangle$ is K-bounded and has restricted K-limit α at p;*
(iv) *the map $(1-\langle z,p\rangle)^{1/2}d(f-f_q)_z(p)$ is K-bounded and has restricted K-limit O at p;*
(v) *if v is any vector orthogonal to p, the function $\bigl\langle df_z(v),q\bigr\rangle/(1-\langle z,p\rangle)^{1/2}$ is K-bounded and has restricted K-limit 0 at p;*
(vi) *if v is any vector orthogonal to p, the map $d(f-f_q)_z(v)$ is K-bounded at p.*

In the last twenty years this theorem (as well as Theorems 0.2 and 0.3) has been extended to domains much more general than the unit ball: for instance, strongly pseudoconvex domains, convex domains of finite type, and polydisks (see, e.g., [A1], [A2], [A3], [A5], [AT], [A6], [AMY] and references therein). But in this paper we are interested in a different kind of generalization, that we are now going to describe.

Let $\mathrm{Hol}(B^n, B^n)$ denote the space of holomorphic self-maps of B^n, endowed with the usual compact-open topology. A *one-parameter semigroup* of holomorphic self-maps of B^n is a continuous semigroup homomorphism $\Phi\colon \mathbb{R}^+ \to \mathrm{Hol}(B^n, B^n)$. In other words, writing φ_t instead of $\Phi(t)$, we have $\varphi_0 = \mathrm{id}_{B^n}$, the map $t \mapsto \varphi_t$ is continuous, and the semigroup property $\varphi_t \circ \varphi_s = \varphi_{t+s}$ holds (see, e.g., [A1], [RS2] or [S] for an introduction to the theory of one-parameter semigroups of holomorphic maps).

One-parameter semigroups can be seen as the flow of a vector field (see, e.g., [A4]). Indeed, given a one-parameter semigroup Φ, it is possible to prove that there exists a holomorphic map $G\colon B^n \to \mathbb{C}^n$, the *infinitesimal generator* of the semigroup, such that

$$\frac{\partial \Phi}{\partial t} = G \circ \Phi \;. \tag{0.4}$$

The infinitesimal generator can be obtained by the following formula:

$$G(z) = \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t} \;. \tag{0.5}$$

Remark 0.5: In some papers (e.g., in [ERS] and [RS1]), the infinitesimal generator is defined as the solution of the equation

$$\frac{\partial \Phi}{\partial t} + G \circ \Phi = O \;,$$

that is with a change of sign with respect to our definition. This should be kept in mind when reading the literature on this subject.

Somewhat surprisingly, in 2008 Elin, Reich and Shoikhet [ERS] discovered a Julia's lemma for infinitesimal generators, just assuming that the radial limit of the generator at a point $p \in \partial B^n$ vanishes (roughly speaking, this means that p is a boundary fixed point for the associated semigroup):

Theorem 0.6: ([ERS, Theorem p. 403]) *Let $G\colon B^n \to \mathbb{C}^n$ be the infinitesimal generator on B^n of a one-parameter semigroup $\Phi = \{\varphi_t\}$, and let $p \in \partial B^n$ be such that*

$$\lim_{t \to 1^-} G(tp) = O \;. \tag{0.6}$$

Then the following assertions are equivalent:

(I) *we have*

$$\alpha = \liminf_{t \to 1^-} \mathrm{Re}\, \frac{\langle G(tp), p \rangle}{t - 1} < +\infty \;;$$

(II) *we have*

$$\beta = 2 \sup_{z \in B^n} \mathrm{Re} \left[\frac{\langle G(z), z \rangle}{1 - \|z\|^2} - \frac{\langle G(z), p \rangle}{1 - \langle z, p \rangle} \right] < +\infty \;;$$

(III) *there exists $\gamma \in \mathbb{R}$ such that for all $z \in B^n$ we have*

$$\frac{|1 - \langle \varphi_t(z), p \rangle|^2}{1 - \|\varphi_t(z)\|^2} \le e^{\gamma t} \frac{|1 - \langle z, p \rangle|^2}{1 - \|z\|^2} \;.$$

Furthermore, if any of these assertions holds then $\alpha = \beta = \inf \gamma$ *and we also have*

$$\lim_{t\to 1^-} \frac{\langle G(tp), p\rangle}{t-1} = \beta \,. \tag{0.7}$$

If (0.6) and any (and hence all) of the equivalent conditions (I)–(III) holds we say that $p \in \partial B^n$ is a *boundary regular null point* of G with *dilation* $\beta \in \mathbb{R}$.

This result strongly suggests that one should try and prove a Julia-Wolff-Carathéodory theorem for infinitesimal generators along the line of Rudin's Theorem 0.4. This has been partially achieved by Bracci and Shoikhet [BS], who proved the following

Theorem 0.7: ([BS]) *Let* $G\colon B^n \to \mathbb{C}^n$ *be an infinitesimal generator on* B^n *of a one-parameter semigroup, and let* $p \in \partial B^n$*. Assume that*

$$\frac{\langle G(z), p\rangle}{\langle z, p\rangle - 1} \qquad \text{is } K\text{-bounded at } p \tag{0.8}$$

and

$$\frac{G(z) - \langle G(z), p\rangle p}{(\langle z, p\rangle - 1)^{1/2}} \qquad \text{is } K\text{-bounded at } p. \tag{0.9}$$

Then p *is a boundary regular null point for* G*. Furthermore, if* β *is the dilation of* G *at* p *then*

(i) *the function* $\langle G(z), p\rangle / (\langle z, p\rangle - 1)$ *(is* K*-bounded and) has restricted* K*-limit* β *at* p*;*
(ii) *if* v *is a vector orthogonal to* p*, the function* $\langle G(z), v\rangle / (\langle z, p\rangle - 1)^{1/2}$ *is* K*-bounded at* p*;*
(iii) *the function* $\langle dG_z(p), p\rangle$ *is* K*-bounded and has restricted* K*-limit* β *at* p*;*
(iv) *if* v *is a vector orthogonal to* p*, the function* $(\langle z, p\rangle - 1)^{1/2} \langle dG_z(p), v\rangle$ *is* K*-bounded at* p*;*
(v) *if* v *is a vector orthogonal to* p*, the function* $\langle dG_z(v), p\rangle \,/\, (\langle z, p\rangle - 1)^{1/2}$ *is* K*-bounded at* p*.*
(vi) *if* v_1 *and* v_2 *are vectors orthogonal to* p *the function* $\langle dG_z(v_1), v_2\rangle$ *is* K*-bounded at* p*.*

Remark 0.8: In the context of holomorphic maps, conditions (0.8) and (0.9) are a consequence of (the equivalent of) condition (I) in Theorem 0.6, and indeed they appear as part of Theorem 0.4.(i) and (ii); however, the proof in that setting uses in an essential way the fact that there we are dealing with holomorphic *self-maps* of the ball. On the other hand, in our context, (0.9) is *not* a consequence of Theorem 0.6.(I), as Example 1.2 shows, and (0.8) too seems to be stronger than Theorem 0.6.(I); see also similar comments in [BS, Section 4.1]. Thus we have to assume (0.8) and (0.9) as separate hypotheses. Furthermore, Example 1.2 also shows that the exponent 1/2 might not necessarily be the right one to consider in the setting of infinitesimal generators.

Remark 0.9: The assertions in Theorem 0.7.(i), (iii) and (v) follow just assuming (0.8) and that $G(tp) \to O$ as $t \to 1^-$ (see [BS, Proposition 4.1]).

Remark 0.10: The assertions in Theorem 0.7 (and in Theorem 0.12 below) have been numbered so as to reflect the similarities with the assertions in Theorem 0.4. To see this, first of all notice that a boundary regular null point of G is a boundary fixed point of the associated semigroup $\{\varphi_t\}$. So in any comparison we must take ($m = n$ and) $q = p$ in Theorem 0.4; in particular, the analogies between assertions (iii) and (v) in the two statements are obvious. Furthermore we can write

$$\frac{1 - \langle \varphi_t(z), p\rangle}{1 - \langle z, p\rangle} = \frac{\bigl\langle \varphi_t(z) - p, p\bigr\rangle}{\langle z, p\rangle - 1} = \frac{\bigl\langle \varphi_t(z) - z, p\bigr\rangle}{\langle z, p\rangle - 1} + 1 \,,$$

and thus recalling (0.5) it is clear that Theorem 0.7.(i) is the analogue of Theorem 0.4.(i). Moreover, if $\{v_2, \ldots, v_n\}$ is an orthornormal basis of the vector space orthogonal to p we can write

$$G(z) - \bigl\langle G(z), p\bigr\rangle p = \sum_{j=2}^{n} \bigl\langle G(z), v_j\bigr\rangle v_j \;;$$

therefore

$$d(G - \langle G, p\rangle p)_z(\cdot) = \sum_{j=2}^{n} \bigl\langle dG_z(\cdot), v_j \bigr\rangle v_j$$

and the analogies between Theorem 0.4.(ii), (iv) and (vi) and the corresponding statements in Theorem 0.7 become evident.

What is missing in Theorem 0.7 to obtain a complete analogue of Theorem 0.4 is statements about restricted K-limits in cases (ii), (iv) and (v); the aim of this paper is exactly to provide those statements. It turns out that there is an obstruction, parallel to the one telling apart K-limits and restricted K-limits: as better described in Section 1, the curves one would like to use for obtaining the exponent $1/2$ in the statements are restricted but not special, in the sense that the limit in (0.3) is a strictly positive (though finite) number. We are thus led to consider exponents $\gamma < 1/2$: this is not just a technical problem, but an inevitable feature of the theory, and in this way we actually widen the applicability of our results, as Example 1.2 shows.

Our first main theorem then is:

Theorem 0.11: *Let $G\colon B^n \to \mathbb{C}^n$ be an infinitesimal generator on B^n of a one-parameter semigroup, and let $p \in \partial B^n$. Assume that*

$$\frac{\langle G(z), p\rangle}{\langle z, p\rangle - 1} \quad \text{and} \quad \frac{G(z) - \langle G(z), p\rangle p}{(\langle z, p\rangle - 1)^\gamma}$$

are K-bounded at p for some $0 < \gamma < 1/2$. Then $p \in \partial B^n$ is a boundary regular null point for G. Furthermore, if β is the dilation of G at p then:

(i) *the function $\langle G(z), p\rangle / (\langle z, p\rangle - 1)$ (is K-bounded and) has restricted K-limit β at p;*

(ii) *if v is a vector orthogonal to p, the function $\langle G(z), v\rangle / (\langle z, p\rangle - 1)^\gamma$ is K-bounded and has restricted K-limit 0 at p;*

(iii) *the function $\langle dG_z(p), p\rangle$ is K-bounded and has restricted K-limit β at p;*

(iv) *if v is a vector orthogonal to p, the function $(\langle z, p\rangle - 1)^{1-\gamma} \langle dG_z(p), v\rangle$ is K-bounded and has restricted K-limit 0 at p;*

(v) *if v is a vector orthogonal to p, the function $\langle dG_z(v), p\rangle \bigm/ (\langle z, p\rangle - 1)^\gamma$ is K-bounded and has restricted K-limit 0 at p.*

(vi) *if v_1 and v_2 are vectors orthogonal to p the function $(\langle z, p\rangle - 1)^{1/2-\gamma} \langle dG_z(v_1), v_2\rangle$ is K-bounded at p.*

An exact analogue of Theorem 0.4 would be with $\gamma = 1/2$; we can obtain such a statement by assuming a slightly stronger hypothesis on the infinitesimal generator. Under the assumptions of Theorem 0.7 we know that

$$\frac{\bigl\langle G\bigl(\sigma(t)\bigr), p\bigr\rangle}{\langle \sigma(t), p\rangle - 1} = \beta + o(1) \tag{0.10}$$

as $t \to 1^-$ for any special restricted p-curve $\sigma\colon [0,1) \to B^n$. Following ideas introduced in [ESY], [EKRS] and [EJ] in the context of the unit disk, we shall say that p is a *Hölder boundary null point* if there is $\alpha > 0$ such that

$$\frac{\bigl\langle G\bigl(\sigma(t)\bigr), p\bigr\rangle}{\langle \sigma(t), p\rangle - 1} = \beta + o\bigl((1-t)^\alpha\bigr) \tag{0.11}$$

for any special restricted p-curve $\sigma\colon [0,1) \to B^n$ such that $\langle \sigma(t), p\rangle \equiv t$. Then our second main theorem is:

Theorem 0.12: *Let $G: B^n \to \mathbb{C}^n$ be the infinitesimal generator on B^n of a one-parameter semigroup, and let $p \in \partial B^n$. Assume that*

$$\frac{\langle G(z), p\rangle}{\langle z, p\rangle - 1} \quad \textit{and} \quad \frac{G(z) - \langle G(z), p\rangle p}{(\langle z, p\rangle - 1)^{1/2}}$$

are K-bounded at p, and that p is a Hölder boundary null point. Then the statement of Theorem 0.11 holds with $\gamma = 1/2$.

We end this paper giving examples of infinitesimal generators with a Hölder boundary null point and satisfying the hypotheses of Theorem 0.12.

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1. Proofs

This section is devoted to the proofs of Theorem 0.11 and Theorem 0.12.

Proof of Theorem 0.11. Our hypotheses ensure that $\lim_{t\to 1^-} G(tp) = O$ and therefore, thanks to Theorem 0.6 we have that p is a boundary regular null point for G. Let $\beta \in \mathbb{R}$ be the dilation of G at p.

(i) This follows immediately from our hypotheses, thanks to Theorems 0.3 and 0.6.

(ii) Given a vector v orthogonal to p, the K-boundedness of the function $\langle G(z), v\rangle/(\langle z, p\rangle - 1)^\gamma$ follows immediately from that of $\big(G(z) - \langle G(z), p\rangle p\big)/(\langle z, p\rangle - 1)^\gamma$. Analogously, to prove that the restricted K-limit at p is zero, it suffices to prove

$$K'\text{-}\lim_{z\to p} \frac{G(z) - \langle G(z), p\rangle p}{(\langle z, p\rangle - 1)^\gamma} = 0\ . \tag{1.1}$$

Without loss of generality, we can assume $p = e_1$, and we write $z = (z_1, z')$ with $z' = (z_2, \ldots, z_n)$ for points in $\mathbb{C}^n$. In particular, we can replace $G(z) - \langle G(z), p\rangle p$ by $G(z)' = (G_2(z), \ldots, G_n(z))$ in the statement we would like to prove, and by Theorem 0.3 to get the assertion it suffices to show that

$$\lim_{t\to 1^-} \frac{G_j(te_1)}{(t-1)^\gamma} = 0 \tag{1.2}$$

for all $j = 2, \ldots, n$.

Since G is an infinitesimal generator with boundary regular null point e_1 having dilation $\beta \in \mathbb{R}$, Theorem 0.6 implies that

$$\mathrm{Re}\left[\frac{\langle G(z), z\rangle}{1 - \|z\|^2} - \frac{G_1(z)}{1 - z_1}\right] \le \frac{\beta}{2} \tag{1.3}$$

for any $z \in B^n$.

Given $j \in \{2, \ldots, n\}$, fix $0 < \varepsilon < 1$ and $\theta \in \mathbb{R}$; for $t \in (0, 1)$, set

$$z_t = te_1 + e^{-i\theta}\varepsilon(1-t)^{1-\gamma}e_j \in B^n\ .$$

In particular, $t \mapsto z_t$ is a special restricted e_1-curve, and we have

$$1 - \|z_t\|^2 = (1-t)(1 + t - \varepsilon^2(1-t)^{1-2\gamma})\ .$$

Now, (1.3) evaluated in z_t becomes

$$\mathrm{Re}\left[\frac{tG_1(z_t)+e^{i\theta}\varepsilon(1-t)^{1-\gamma}G_j(z_t)}{1-\|z_t\|^2}-\frac{G_1(z_t)}{1-\langle z_t,e_1\rangle}\right]\le\frac{\beta}{2}\;.$$

Therefore

$$\begin{aligned}\mathrm{Re}\left[\frac{e^{i\theta}\varepsilon(1-t)^{1-\gamma}G_j(z_t)}{1-\|z_t\|^2}\right]&\le\frac{\beta}{2}+\mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t,e_1\rangle}\right]-t\,\mathrm{Re}\left[\frac{G_1(z_t)}{1-\|z_t\|^2}\right]\\&=\frac{\beta}{2}+\mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t,e_1\rangle}\right]\left(1-\frac{t(1-\langle z_t,e_1\rangle)}{1-\|z_t\|^2}\right)\\&=\frac{\beta}{2}+\mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t,e_1\rangle}\right]\left(1-\frac{t}{1+t-\varepsilon^2(1-t)^{1-2\gamma}}\right)\;.\end{aligned}$$

Furthermore

$$\begin{aligned}\mathrm{Re}\left[\frac{e^{i\theta}\varepsilon(1-t)^{1-\gamma}G_j(z_t)}{1-\|z_t\|^2}\right]&=\frac{\varepsilon(1-t)^{1-\gamma}(1-\langle z_t,e_1\rangle)^\gamma}{1-\|z_t\|^2}\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t,e_1\rangle)^\gamma}\\&=\frac{\varepsilon}{1+t-\varepsilon^2(1-t)^{1-2\gamma}}\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t,e_1\rangle)^\gamma}\;.\end{aligned}$$

Recalling Theorem 0.7, and in particular (0.10), we get

$$\begin{aligned}\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t,e_1\rangle)^\gamma}&\le\left(\frac{\beta}{2}+\mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t,e_1\rangle}\right]\left(1-\frac{t}{1+t-\varepsilon^2(1-t)^{1-2\gamma}}\right)\right)\frac{1+t-\varepsilon^2(1-t)^{1-2\gamma}}{\varepsilon}\\&=\frac{\beta}{2}\cdot\frac{1+t-\varepsilon^2(1-t)^{1-2\gamma}}{\varepsilon}+\mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t,e_1\rangle}\right]\left(\frac{1+t-\varepsilon^2(1-t)^{1-2\gamma}}{\varepsilon}-\frac{t}{\varepsilon}\right)\\&=\frac{\beta}{2}\cdot\frac{1+t-\varepsilon^2(1-t)^{1-2\gamma}}{\varepsilon}+\big(-\beta+o(1)\big)\left(\frac{1-\varepsilon^2(1-t)^{1-2\gamma}}{\varepsilon}\right)\\&=\frac{\beta}{2}\frac{\varepsilon^2(1-t)^{1-2\gamma}+t-1}{\varepsilon}+o(1)\;.\end{aligned}$$

Letting $t\to1^-$ we obtain

$$\limsup_{t\to1^-}\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t,e_1\rangle)^\gamma}\le0$$

for all $\varepsilon>0$ and $\theta\in\mathbb{R}$. Now letting $\varepsilon\to0^+$ we find

$$\limsup_{t\to1^-}\frac{\mathrm{Re}[e^{i\theta}G_j(te_1)]}{(1-t)^\gamma}\le0$$

for all $\theta\in\mathbb{R}$, and this is possible if and only if

$$\lim_{t\to1^-}\frac{G_j(te_1)}{(1-t)^\gamma}=0\;,$$

and (1.2) follows.

(iii) The proof is analogous to the one given in [BS]; we recall it here for the sake of completeness.

Without loss of generality, we can assume $p = e_1$. Let $M' > M > 1$ and set $\delta := \frac{1}{3}(\frac{1}{M} - \frac{1}{M'})$. Thanks to [R, Lemma 8.5.5], for any $z \in K(e_1, M)$ and $(\lambda, u') \in \mathbb{C} \times \mathbb{C}^{n-1}$ with $|\lambda| \leq \delta|z_1 - 1|$ and $\|u'\| \leq \delta|z_1 - 1|^{1/2}$, we have $(z_1 + \lambda, z' + u') \in K(e_1, M')$.

Now, fix $z \in K(e_1, M)$ and let $r = r(z) := \delta|z_1 - 1|$. By Cauchy's formula, we have

$$\begin{aligned}\langle dG_z(e_1), e_1\rangle &= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\langle G(z_1 + \zeta, z'), e_1\rangle}{\zeta^2} d\zeta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\langle G(z_1 + re^{i\theta}, z'), e_1\rangle}{z_1 + re^{i\theta} - 1} \frac{z_1 + re^{i\theta} - 1}{re^{i\theta}} d\theta \,.\end{aligned}$$

The first factor in the integral is bounded because $(z_1 + re^{i\theta}, z') \in K(e_1, M')$; furthermore, we also have $|(z_1 + re^{i\theta} - 1)/re^{i\theta}| \leq 1 + 1/\delta$, and thus we are done.

To prove that the restricted K-limit at p is β, by Theorem 0.3 it suffices to prove that

$$\lim_{t\to 1^-} \langle dG_{te_1}(e_1), e_1\rangle = \beta.$$

Thanks to [BCD, Theorem 0.4], we have that $\lim_{t\to 1^-} \frac{d}{dt}(G_1(te_1)) = \beta$, and then we are done, because $\frac{d}{dt}(G_1(te_1)) = \langle dG_{te_1}(e_1), e_1\rangle$.

(iv) Without loss of generality we can assume $p = e_1$ and $v = e_2$, so that the quotient we would like to study is

$$(z_1 - 1)^{1-\gamma} \frac{\partial G_2}{\partial z_1}(z) \,.$$

The proof of the K-boundedness is again an application of the Cauchy formula. As before, let $M' > M > 1$ and set $\delta := \frac{1}{3}(\frac{1}{M} - \frac{1}{M'})$. Thanks to [R, Lemma 8.5.5], for any $z \in K(e_1, M)$ and $(\lambda, u') \in \mathbb{C} \times \mathbb{C}^{n-1}$ with $|\lambda| \leq \delta|z_1 - 1|$ and $\|u'\| \leq \delta|z_1 - 1|^{1/2}$, we have $(z_1 + \lambda, z' + u') \in K(e_1, M')$.

Now, fix $z \in K(e_1, M)$ and let $r = r(z) := \delta|z_1 - 1|$. By Cauchy's formula, we have

$$\begin{aligned}|z_1 - 1|^{1-\gamma} \frac{\partial G_2}{\partial z_1}(z) &= \frac{|z_1 - 1|^{1-\gamma}}{2\pi i} \int_{|\zeta|=r} \frac{G_2(z_1 + \zeta, z')}{\zeta^2} d\zeta \\ &= \frac{1}{2\pi\delta}\int_{-\pi}^{\pi} \frac{G_2(z_1 + re^{i\theta}, z')}{|z_1 + re^{i\theta} - 1|^{\gamma}} \left|\frac{z_1 + re^{i\theta} - 1}{z_1 - 1}\right|^{\gamma} \frac{|z_1 - 1|}{|z_1 - 1|e^{i\theta}} d\theta \,.\end{aligned}$$

The choice of r ensures that $(z_1 + \zeta, z') \in K(e_1, M')$; thus the first factor in the integral is bounded, and, since an easy computation shows that $\frac{|z_1 + re^{i\theta} - 1|}{|z_1 - 1|} \leq 1 + \delta$, we are done.

To prove that the restricted K-limit at p vanishes, thanks to Theorem 0.3, it suffices to show that

$$\lim_{t\to 1^-} (t - 1)^{1-\gamma} \frac{\partial G_2}{\partial z_1}(te_1) = 0 \,. \tag{1.4}$$

Indeed, choose $\varepsilon \in (0, 1)$, and for any $t \in (0, 1)$, let $\sigma_t \colon \varepsilon\Delta \to B^n$ be defined by

$$\sigma_t(\zeta) = (t + \zeta(1 - t))e_1 \,.$$

Then $\sigma_t(0) = te_1$ and $\sigma_t'(0) = (1 - t)e_1$. Moreover, for any $\zeta \in \varepsilon\Delta$ we have

$$\frac{|1 - t - \zeta(1 - t)|}{1 - |t + \zeta(1 - t)|} = \frac{(1 - t)|1 - \zeta|}{1 - |1 - (1 - t)(1 - \zeta)|} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \,.$$

Therefore $\sigma_t(\overline{\varepsilon\Delta}) \subset K(e_1, M)$ for all $M > \frac{1+\varepsilon}{1-\varepsilon}$. In particular, for all $\theta \in \mathbb{R}$, the e_1-curve $t \mapsto \sigma_t(\varepsilon e^{i\theta})$ is special and M-restricted. Now,

$$\begin{aligned}(t-1)^{1-\gamma}\frac{\partial G_2}{\partial z_1}(te_1) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{G_2(t+\varepsilon(1-t)e^{i\theta}, O')}{(t+\varepsilon(1-t)e^{i\theta}-1)^\gamma}\,\frac{(t+\varepsilon(1-t)e^{i\theta}-1)^\gamma}{\varepsilon(1-t)e^{i\theta}}(t-1)^{1-\gamma}d\theta \\ &= \frac{-1}{2\pi}\int_{-\pi}^{\pi}\frac{G_2(t+\varepsilon(1-t)e^{i\theta}, O')}{(t+\varepsilon(1-t)e^{i\theta}-1)^\gamma}\,\frac{(1-\varepsilon e^{i\theta})^\gamma}{\varepsilon e^{i\theta}}d\theta\ .\end{aligned}$$

The second factor of the integrand is bounded, and the first factor converges punctually and boundedly to 0 as $t \to 1$, thanks to (ii); therefore (1.4) follows from the dominated convergence theorem.

(v) Without loss of generality we can assume $p = e_1$ and $v = e_2$, so that the quotient we would like to study is

$$\frac{1}{(z_1-1)^\gamma}\frac{\partial G_1}{\partial z_2}(z)\ .$$

The proof of the K-boundedness is yet another application of the Cauchy formula. Let $M' > M > 1$; set $\delta := \frac{1}{3}(\frac{1}{M} - \frac{1}{M'})$, and $r = r(z) := \delta|z_1 - 1|^{1-\gamma}$; [R, Lemma 8.5.5] ensures that if $z \in K(e_1, M)$ then $z + re^{i\theta}e_2 \in K(e_1, M')$ for all $\theta \in \mathbb{R}$. Then Cauchy's formula yields

$$\begin{aligned}\frac{1}{|z_1-1|^\gamma}\frac{\partial G_1}{\partial z_2}(z) &= \frac{1}{2\pi i|z_1-1|^\gamma}\int_{|\zeta|=r}\frac{G_1(z+\zeta e_2)}{\zeta^2}d\zeta \\ &= \frac{1}{2\pi\delta}\int_{-\pi}^{\pi}\frac{G_1(z+re^{i\theta}e_2)}{|z_1-1|e^{i\theta}}d\theta\ ,\end{aligned}$$

and the K-boundness follows.

Now we would like to prove that the restricted K-limit at p vanishes. Let $\Phi: B^2 \to B^n$ be given by $\Phi(\zeta, \eta) = \zeta e_1 + \eta e_2$, and put $H = \Xi \circ \Phi$, where

$$\Xi(z) = \frac{\langle G(z), z\rangle}{1-\|z\|^2} - \frac{G_1(z)}{1-z_1}\ .$$

Hence

$$H(\zeta,\eta) = \frac{G_1(\zeta,\eta,0,\ldots,0)\overline{\zeta} + G_2(\zeta,\eta,0,\ldots,0)\overline{\eta}}{1-|\zeta|^2-|\eta|^2} - \frac{G_1(\zeta,\eta,0,\ldots,0)}{1-\zeta}\ .$$

Now we expand H in power series with respect to η:

$$H(\zeta,\eta) = H(\zeta,0) + \frac{\partial H}{\partial\eta}(\zeta,0)\eta + \frac{\partial H}{\partial\overline{\eta}}(\zeta,0)\overline{\eta} + O(|\eta|^2)\ . \qquad (1.5)$$

We have

$$H(\zeta,0) = G_1(\zeta,O')\left[\frac{\overline{\zeta}}{1-|\zeta|^2} - \frac{1}{1-\zeta}\right] = -G_1(\zeta,O')\frac{1}{1-|\zeta|^2}\frac{1-\overline{\zeta}}{1-\zeta}\ ;$$

$$\frac{\partial H}{\partial\eta}(\zeta,0) = \frac{\partial G_1}{\partial z_2}(\zeta,O')\left[\frac{\overline{\zeta}}{1-|\zeta|^2} - \frac{1}{1-\zeta}\right] = -\frac{\partial G_1}{\partial z_2}(\zeta,O')\frac{1}{1-|\zeta|^2}\frac{1-\overline{\zeta}}{1-\zeta}\ ;$$

and

$$\frac{\partial H}{\partial\overline{\eta}}(\zeta,0) = \frac{G_2(\zeta,O')}{1-|\zeta|^2}\ .$$

Recalling (1.3) we get

$$\frac{\beta}{2} \geq \operatorname{Re} H(\zeta,\eta) = \operatorname{Re}\left[H(\zeta,0) + \frac{\partial H}{\partial \eta}(\zeta,0)\eta + \frac{\partial H}{\partial \overline{\eta}}(\zeta,0)\overline{\eta} + O(|\eta|^2)\right]$$

$$= \frac{1}{1-|\zeta|^2}\operatorname{Re}\left[-\left(G_1(\zeta,O') + \eta\frac{\partial G_1}{\partial z_2}(\zeta,O')\right)\frac{|1-\zeta|^2}{(1-\zeta)^2} + G_2(\zeta,O')\overline{\eta} + O\big((1-|\zeta|^2)|\eta|^2\big)\right] ,$$

and thus

$$-\frac{\beta}{2}\frac{1-|\zeta|^2}{|1-\zeta|^2} \leq \operatorname{Re}\left[\frac{G_1(\zeta,O')}{(1-\zeta)^2} + \frac{\eta}{(1-\zeta)^2}\frac{\partial G_1}{\partial z_2}(\zeta,O') - \frac{\overline{\eta}G_2(\zeta,O')}{|1-\zeta|^2} + O\left(\frac{1-|\zeta|^2}{|1-\zeta|^2}|\eta|^2\right)\right] . \tag{1.6}$$

Fix $c>0$ and for $t\in[0,1)$ put

$$\zeta_t = t + ic(1-t) .$$

In particular,

$$1-\zeta_t = (1-t)(1-ic) , \quad |1-\zeta_t| = (1-t)(1+c^2)^{1/2} \quad \text{and} \quad \frac{1}{1-\zeta_t} = \frac{1}{1-t}\frac{1+ic}{1+c^2} .$$

It is easy to check that $\zeta_t\in\Delta$ if $1-t<2/(1+c^2)$, and in this case

$$1-|\zeta_t|^2 = 1-t^2-c^2(1-t)^2 = (1-t)\big(1+t-(1-t)c^2\big) < 2(1-t) .$$

Moreover, if $1-t<1/(1+c^2)$ we have $1-|\zeta_t|^2>1-t$, and thus we can find $\eta_t\in\mathbb{C}$ such that

$$2(1-t) > 1-|\zeta_t|^2 > |\eta_t|^2 > 1-t ;$$

in particular, $(\zeta_t,\eta_t)\in B^2$, and we choose the argument of η_t so that

$$\frac{\eta_t}{(1-\zeta_t)^2}\frac{\partial G_1}{\partial z_2}(\zeta_t,O') = -\left|\frac{\eta_t}{(1-\zeta_t)^2}\frac{\partial G_1}{\partial z_2}(\zeta_t,O')\right| \in \mathbb{R}^- .$$

Now we compute (1.6) in (ζ_t,η_t). Multiplying by $|1-\zeta_t|^{2-\gamma}$ and dividing by $|\eta_t|$ we get

$$\left|\frac{1}{(1-\zeta_t)^\gamma}\frac{\partial G_1}{\partial z_2}(\zeta_t,O')\right| \leq \operatorname{Re}\left[\frac{G_1(\zeta_t,O')}{1-\zeta_t}\frac{|1-\zeta_t|^{2-\gamma}}{(1-\zeta_t)|\eta_t|}\right] + \frac{|G_2(\zeta_t,O')|}{|1-\zeta_t|^\gamma} + O\left(\frac{1-|\zeta_t|^2}{|1-\zeta_t|^\gamma}|\eta_t|\right)$$
$$+ \frac{\beta}{2}\frac{1-|\zeta_t|^2}{|1-\zeta_t|^\gamma|\eta_t|} .$$

Applying (0.10) we obtain

$$\begin{aligned}
&\left|\frac{1}{(1-\zeta_t)^\gamma}\frac{\partial G_1}{\partial z_2}(\zeta_t,O')\right| \\
&\quad\leq \frac{|1-\zeta_t|^{2-\gamma}}{|\eta_t|}\operatorname{Re}\left[\frac{-\beta+o(1)}{1-t}\frac{1+ic}{1+c^2}\right] + \frac{|G_2(\zeta_t,O')|}{|1-\zeta_t|^\gamma} + O\left(\frac{1-|\zeta_t|^2}{|1-\zeta_t|^\gamma}|\eta_t|\right) \\
&\qquad + \frac{\beta}{2}\frac{1-|\zeta_t|^2}{|1-\zeta_t|^\gamma|\eta_t|} \\
&\quad\leq \frac{(1-t)^{2-\gamma}(1+c^2)^{1-\gamma/2}}{(1-t)^{1/2}}\frac{-\beta+o(1)}{(1-t)(1+c^2)} + \frac{|G_2(\zeta_t,O')|}{|1-\zeta_t|^\gamma} + O\left(\frac{1-|\zeta_t|^2}{|1-\zeta_t|^\gamma}|\eta_t|\right) \\
&\qquad + \frac{\beta}{2}\frac{1-|\zeta_t|^2}{(1-t)^\gamma(1+c^2)^{\gamma/2}|\eta_t|} \\
&\quad\leq \frac{(-\beta+o(1))(1-t)^{1/2-\gamma}}{(1+c^2)^{\gamma/2}} + \frac{|G_2(\zeta_t,O')|}{|1-\zeta_t|^\gamma} + O\left(\frac{2(1-t)}{(1-t)^\gamma(1+c^2)^{\gamma/2}}\sqrt{2}(1-t)^{1/2}\right) \\
&\qquad + \frac{\beta}{2}\frac{2(1-t)}{(1-t)^\gamma(1+c^2)^{\gamma/2}(1-t)^{1/2}} \\
&\quad\leq o\big((1-t)^{1/2-\gamma}\big) + \frac{|G_2(\zeta_t,O')|}{|1-\zeta_t|^\gamma} + O\big((1-t)^{3/2-\gamma}\big) .
\end{aligned}$$

Since $t\mapsto \zeta_t e_1$ is a special restricted curve we can apply (ii) obtaining

$$\limsup_{t\to 1^-}\left|\frac{1}{(1-\zeta_t)^\gamma}\frac{\partial G_1}{\partial z_2}(\zeta_t,O')\right|\le 0\;.$$

So we get

$$\lim_{t\to 1^-}\frac{1}{(\zeta_t-1)^\gamma}\frac{\partial G_1}{\partial z_2}(\zeta_t,O')=0$$

and the assertion follows from Theorem 0.3.

(vi) Without loss of generality we can assume $p=e_1$, $v_1=e_2$, and $v_2=e_3$, so that the function we would like to study is

$$(z_1-1)^{\frac{1}{2}-\gamma}\frac{\partial G_3}{\partial z_2}(z)\;.$$

We argue as usual.

Let $M'>M>1$ and set $\delta:=\frac{1}{3}(\frac{1}{M}-\frac{1}{M'})$. Thanks to [R, Lemma 8.5.5], for any $z\in K(e_1,M)$ and $u'\in\mathbb{C}^{n-1}$ with $\|u'\|\le\delta|z_1-1|^{1/2}$ we have $(z_1,z'+u')\in K(e_1,M')$.

Now, fix $z\in K(e_1,M)$ and let $r=r(z):=\delta|z_1-1|^{1/2}$. By Cauchy's formula, we have

$$\begin{aligned}|z_1-1|^{\frac{1}{2}-\gamma}\frac{\partial G_3}{\partial z_2}(z)&=\frac{|z_1-1|^{\frac{1}{2}-\gamma}}{2\pi i}\int_{|\zeta|=r}\frac{G_3(z+\zeta e_2)}{\zeta^2}d\zeta\\&=\frac{1}{2\pi\delta}\int_{-\pi}^{\pi}\frac{G_3(z+re^{i\theta}e_2)}{|z_1-1|^\gamma e^{i\theta}}d\theta\;.\end{aligned}$$

The choice of r ensures that $z+re^{i\theta}e_2\in K(e_1,M')$, and the assertion follows from (ii). □

An accurate examination of the proof of the previous theorem reveals that the main point is the proof of part (ii). As soon as the statement of Theorem 0.11.(ii) holds for some $0<\gamma\le 1/2$ (with $\gamma=1/2$ included) then the rest of the Theorem follows with the same γ (again, $\gamma=1/2$ included). The proof of Theorem 0.11.(ii) we presented however breaks down for $\gamma=1/2$ because the curve

$$(0,1)\ni t\mapsto z_t=te_1+e^{-i\theta}\varepsilon(1-t)^{1-\gamma}e_j\in B^n$$

is *not* special if $\gamma=1/2$; the limit (0.3) is a strictly positive (though finite) number.

Remark 1.1: Even assuming that the hypotheses of Theorem 0.11 are satisfied with $\tilde\gamma\ge 1/2$, as explained above with this proof we can only obtain the thesis for all exponents $\gamma<1/2$.

Furthermore the exponent 1/2, which is the natural one to consider in the setting of self-maps, it is not necessarily the right one for infinitesimal generators, as next example shows.

Example 1.2: Let $G\colon B^2\to\mathbb{C}^2$ be defined as

$$G(z,w)=(-z(1-z),-w(1-z)^{-\alpha})\;,$$

with $0<\alpha<1/2$. It is easy to check that G is an infinitesimal generator, since it vanishes at the origin and $\mathrm{Re}\langle G(z,w),(z,w)\rangle\le 0$ for every $(z,w)\in B^2$. Moreover, G satisfies the hypotheses of Theorem 0.11 with $p=e_1$ and $\gamma=1/2-\alpha$, but $G_2(z,w)/(z-1)^\beta$ is not K-bounded for any $\beta>1/2-\alpha$. Indeed, given $c\in(0,1)$, all points of the form $(t,c\sqrt{1-t^2})$, with $t\in[0,1)$, belong to a Korányi region of vertex e_1, whereas $G_2(t,c\sqrt{1-t^2})/(t-1)^\beta$ is not bounded as t tends to 1, for $1/2-\alpha-\beta<0$. Furthermore $G_2(z,w)/(z-1)^\beta$ does not even have a restricted K-limit at e_1. In fact, choosing $\rho > 1$ such that $\beta > \rho/2 - \alpha$, the curve $\sigma_\rho \colon [0,1) \to B^2$ defined by $\sigma_\rho(t) = (t, c(1-t^2)^{\rho/2})$, with $c \in (0,1)$, is a special restricted e_1-curve such that $G_2(\sigma_\rho(t))/(t-1)^\beta$ diverges as t tends to 1. This example can be easily generalized to any dimension.

On the other hand, we can get the statement with exponent $\gamma = 1/2$ by using the notion of Hölder boundary null point, as follows:

Proof of Theorem 0.12. As explained above, it suffices to prove that

$$\lim_{t\to 1^-} \frac{G_j(te_1)}{(t-1)^{1/2}} = 0 \tag{1.7}$$

for all $j = 2, \ldots, n$.

Let $\alpha > 0$ be given by the definition of Hölder boundary null point; we can clearly assume that $\alpha < 1$. Given $j \in \{2, \ldots, n\}$, fix $0 < \varepsilon < 1$ and $\theta \in \mathbb{R}$; for $t \in (0,1)$, set

$$z_t = te_1 + e^{-i\theta}\varepsilon(1-t)^{1/2+\alpha} e_j \in B^n\ .$$

In particular, $t \mapsto z_t$ is a special restricted e_1-curve such that $\langle z_t, e_1\rangle \equiv t$, and we have

$$1 - \|z_t\|^2 = (1-t)(1 + t - \varepsilon^2(1-t)^{2\alpha})\ .$$

Now, (1.3) evaluated in z_t becomes

$$\mathrm{Re}\left[\frac{tG_1(z_t) + e^{i\theta}\varepsilon(1-t)^{1/2+\alpha}G_j(z_t)}{1-\|z_t\|^2} - \frac{G_1(z_t)}{1-\langle z_t, e_1\rangle}\right] \le \frac{\beta}{2}\ .$$

Therefore

$$\mathrm{Re}\left[\frac{e^{i\theta}\varepsilon(1-t)^{1/2+\alpha}G_j(z_t)}{1-\|z_t\|^2}\right] \le \frac{\beta}{2} + \mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t, e_1\rangle}\right]\left(1 - \frac{t}{1+t-\varepsilon^2(1-t)^{2\alpha}}\right)\ .$$

Furthermore

$$\begin{aligned}
\mathrm{Re}\left[\frac{e^{i\theta}\varepsilon(1-t)^{1/2+\alpha}G_j(z_t)}{1-\|z_t\|^2}\right] &= \frac{\varepsilon(1-t)^{1/2+\alpha}(1-\langle z_t, e_1\rangle)^{1/2}}{1-\|z_t\|^2}\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t, e_1\rangle)^{1/2}} \\
&= \frac{\varepsilon(1-t)^{1+\alpha}}{1-t^2-\varepsilon^2(1-t)^{1+2\alpha}}\frac{|G_j(z_t)|}{(1-\langle z_t, e_1\rangle)^{1/2}} \\
&= \frac{\varepsilon(1-t)^{\alpha}}{1+t-\varepsilon^2(1-t)^{2\alpha}}\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t, e_1\rangle)^{1/2}}\ .
\end{aligned}$$

Using (0.11) we then get

$$\begin{aligned}
\frac{\mathrm{Re}[e^{i\theta}G_j(z_t)]}{(1-\langle z_t, e_1\rangle)^{1/2}} &\le \left(\frac{\beta}{2} + \mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t, e_1\rangle}\right]\left(1 - \frac{t}{1+t-\varepsilon^2(1-t)^{2\alpha}}\right)\right)\frac{1+t-\varepsilon^2(1-t)^{2\alpha}}{\varepsilon(1-t)^\alpha} \\
&= \frac{\beta}{2}\cdot\frac{1+t-\varepsilon^2(1-t)^{2\alpha}}{\varepsilon(1-t)^\alpha} + \mathrm{Re}\left[\frac{G_1(z_t)}{1-\langle z_t, e_1\rangle}\right]\left(\frac{1+t-\varepsilon^2(1-t)^{2\alpha}}{\varepsilon(1-t)^\alpha} - \frac{t}{\varepsilon(1-t)^\alpha}\right) \\
&= \frac{\beta}{2}\cdot\frac{1+t-\varepsilon^2(1-t)^{2\alpha}}{\varepsilon(1-t)^\alpha} + \big(-\beta + o\big((1-t)^\alpha\big)\big)\left(\frac{1-\varepsilon^2(1-t)^{2\alpha}}{\varepsilon(1-t)^\alpha}\right) \\
&= \frac{\beta}{2}\frac{\varepsilon^2(1-t)^\alpha - (1-t)^{1-\alpha}}{\varepsilon} + o(1)\ .
\end{aligned}$$

Letting $t \to 1^-$ we obtain

$$\limsup_{t\to 1^-} \frac{\operatorname{Re}[e^{i\theta} G_j(z_t)]}{(1-\langle z_t, e_1\rangle)^{1/2}} \le 0$$

for all $\varepsilon > 0$ and $\theta \in \mathbb{R}$. Now letting $\varepsilon \to 0^+$ we find

$$\limsup_{t\to 1^-} \frac{\operatorname{Re}[e^{i\theta} G_j(te_1)]}{(1-t)^{1/2}} \le 0$$

for all $\theta \in \mathbb{R}$, and this is possible if and only if

$$\lim_{t\to 1^-} \frac{G_j(te_1)}{(1-t)^{1/2}} = 0 ,$$

and we are done. □

We end this paper giving examples of infinitesimal generators having a Hölder boundary null point.

Example 1.3: Let $p = e_1$, and $G\colon B^n \to \mathbb{C}^n$ be an infinitesimal generator with $K'\text{-}\lim\limits_{z\to e_1} G(z) = O$. Setting $G_1 = \langle G, e_1\rangle$, condition (0.11) can be written as

$$G_1\bigl(\sigma(t)\bigr) = \beta(t-1) + o\bigl((1-t)^{1+\alpha}\bigr)$$

for any special e_1-curve $\sigma\colon [0,1) \to B^n$ such that $\langle \sigma(t), e_1\rangle \equiv t$. In particular, if G_1 is of class $C^{1+\alpha'}$ at e_1 for some $\alpha' > \alpha$ then (0.11) is satisfied, and e_1 is a Hölder boundary null point for G.

To give an explicit example, let us recall that if $F\colon B^n \to B^n$ is a holomorphic self-map of B^n then $G = \mathrm{id} - F$ is an infinitesimal generator (see, e.g., [RS2, Theorem 6.16] and [S, Corollary 3.3.1]). Recalling Theorem 0.4, to get an example of infinitesimal generator having e_1 as Hölder boundary null point and satisfying the hypotheses of Theorem 0.11 it thus suffices to find F having K-limit e_1 at e_1, with $\liminf\limits_{z\to e_1} (1 - \|F(z)\|)/(1-\|z\|) < +\infty$ and such that

$$F_1\bigl(\sigma(t)\bigr) = t + \beta(1-t) + o\bigl((1-t)^{1+\alpha}\bigr)$$

for any special e_1-curve $\sigma\colon [0,1) \to B^n$ such that $\langle \sigma(t), e_1\rangle \equiv t$. For example, we can just take maps of the form $F(z) = f(z_1)e_1$ with f given by

$$f(\zeta) = \zeta + \beta(1-\zeta) + c(1-\zeta)^{1+\alpha'} = 1 - a(1-\zeta) + c(1-\zeta)^{1+\alpha'} ; \tag{1.8}$$

thus we just need to choose $a = 1 - \beta > 0$ and $c > 0$ so that $f(\Delta) \subseteq \Delta$. Put $w = 1 - \zeta$; then $|f(\zeta)| < 1$ if and only if $|1 - aw + cw^{1+\alpha'}| < 1$ if and only if

$$a^2|w|^2 + c^2|w|^{2(1+\alpha')} + 2c\operatorname{Re}(w^{1+\alpha'}) < 2a\operatorname{Re} w + 2ac|w|^2 \operatorname{Re}(w^{\alpha'}) . \tag{1.9}$$

First of all, write $w = |w|e^{i\theta}$, with $|\theta| < \pi/2$. Then

$$\operatorname{Re}(w^{\alpha'}) = |w|^{\alpha'} \cos(\alpha'\theta) \ge \varepsilon_{\alpha'} |w|^{\alpha'} ,$$

where $\varepsilon_{\alpha'} = \cos(\alpha'\pi/2) > 0$. Recalling that $|w| < 2$, it follows that taking $c < 2^{1-\alpha'}\varepsilon_{\alpha'} a$ we get

$$c^2|w|^{2(1+\alpha')} < 2^{\alpha'} c^2 |w|^{2+\alpha'} < 2ac\varepsilon_{\alpha'} |w|^{2+\alpha'} \le 2ac|w|^2 \operatorname{Re}(w^{\alpha'}) . \tag{1.10}$$

Now, if $|\theta| \geq \pi/2(1+\alpha')$ then $\mathrm{Re}(w^{1+\alpha'}) \leq 0$. Since $|1-w| < 1$ implies $|w|^2 < 2\,\mathrm{Re}(w)$, in this case we get

$$a^2|w|^2 + 2c\,\mathrm{Re}(w^{1+\alpha'}) < 2a^2\,\mathrm{Re}\,w < 2a\,\mathrm{Re}\,w \tag{1.11}$$

as soon as $a < 1$.

If instead $|\theta| < \pi/2(1+\alpha')$, we have $|\,\mathrm{Im}\,w| < C_{\alpha'}\,\mathrm{Re}\,w$, where $C_{\alpha'} = \tan\bigl(\pi/2(1+\alpha')\bigr)$, and thus $|w| < D_{\alpha'}\,\mathrm{Re}\,w$, where $D_{\alpha'} = \sqrt{1+C_{\alpha'}^2}$. Hence

$$a^2|w|^2 + 2c\,\mathrm{Re}(w^{1+\alpha'}) < \bigl[2a^2 + 2cD_{\alpha'}^{1+\alpha'}(\mathrm{Re}\,w)^{\alpha'}\bigr]\,\mathrm{Re}\,w < 2a\,\mathrm{Re}\,w \tag{1.12}$$

as soon as $a^2 + 2^{\alpha'}D_{\alpha'}^{1+\alpha'}c < a$. Since we already requested that $c < 2^{1-\alpha'}\varepsilon_{\alpha'}a$, it suffices to have $a < (1 + 2\varepsilon_{\alpha'}D_{\alpha'}^{1+\alpha'})^{-1}$.

Putting together (1.9), (1.10), (1.11) and (1.12) it follows that if $a < (1 + 2\varepsilon_{\alpha'}D_{\alpha'}^{1+\alpha'})^{-1}$ and $c < 2^{1-\alpha'}\varepsilon_{\alpha'}a$, then the function f given by (1.8) maps Δ into itself, as we wanted.

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