Differential 1-forms on diffeological spaces and diffeological gluing

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Abstract

This paper aims to describe the behavior of diffeological differential 1-forms under the operation of gluing of diffeological spaces along a smooth map (the results obtained actually apply to all k-forms with $k > 0$). In the diffeological context, two constructions regarding diffeological forms are available, that of the vector space $\Omega^1(X)$ of all 1-forms, and that of the (pseudo-)bundle $\Lambda^1(X)$ of values of 1-forms. We describe the behavior of the former under an arbitrary gluing of two diffeological spaces, while for the latter, we limit ourselves to the case of gluing along a diffeomorphism.
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Introduction

The aim of this work is rather modest; it is to examine the behavior of diffeological differential forms (1-forms, usually, but a lot of it naturally holds for forms of higher order) under the operation of diffeological gluing. In fact, assuming that the notation is known, the main question can be stated very simply: *if a diffeological space X_1 is glued to a diffeological space X_2 along a map f, how can we obtain the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$ out of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$?*

Of course, we make no assumption, as to any of these symbols or terms being known (although the explanation of them can be found in the excellent book [4]), so here we give a rough description of their meaning, and give precise definitions of the most important ones in the first two sections. A *diffeological space* is a set equipped with a *diffeology*, a set of maps into it that are declared to be smooth. There are ensuing notions of smooth maps between such spaces, the induced diffeologies of all kinds, among which we mention in particular the *subset diffeology* and the *quotient diffeology*, for the simple reason that they provide for any subset, and any quotient, of a diffeological space, being in turn a diffeological space, in striking contrast with the category of smooth manifolds.

This latter property makes for the operation of diffeological gluing to be well-defined in the diffeological context. In essence, we are talking about the notion of topological gluing: given two sets (say, they are topological spaces) X_1 and X_2 and a map $f : X_1 \supset Y \to X_2$, the usual gluing procedure yields the space $(X_1 \sqcup X_2)/_{x_2=f(x_1)}$, which for a continuous f has a natural topology. Now, the just-mentioned property of diffeology ensures the same thing, if we assume that f is smooth as a map on Y, which inherits its diffeology from X_1.

There is a certain correlation between this operation being well-defined in the diffeological setting, and the fact that the diffeological counterpart of a vector bundle is a *diffeological vector pseudo-bundle*, and in general it is not a bundle at all. The reason for it not being a bundle, not in the usual sense, is simply that it is allowed to have fibres of different dimensions (which are still required to be vector spaces with smooth operations), and the necessity of such objects for diffeology is not just aprioristic; they arise naturally in various aspects of the theory (see Example 4.13 of [2] for an instance of this). The aforementioned correlation with the operation of gluing is that such pseudo-bundles, when they are not too intricate, can frequently be obtained by applying diffeological gluing to a collection of usual smooth vector bundles.

Diffeological vector spaces and particularly diffeological pseudo-bundles give the appropriate framework for differential forms on diffeological spaces. By itself, a differential k-form on a diffeological space X is just a collection of usual k-forms, one for each plot and defined on the domain of the definition of the plot; a very natural smooth compatibility is imposed on this collection to ensure consistency with (usual) smooth substitutions on the domains of plots. The collection of all possible k-forms defined in this fashion is naturally a diffeological vector space and is denoted by $\Omega^1(X)$, but it does not fiber naturally over X; a further construction, a certain space $\Lambda^1(X)$, has a pseudo-bundle structure, and this is our main object of study.

The main results These regard three main points: the diffeological vector space $\Omega^1(X_1 \cup_f X_2)$, the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$, and construction of so-called pseudo-metrics on the latter. As for the former, our main result is as follows.

Theorem 1. *Let X_1 and X_2 be two diffeological spaces, let $f : X_1 \supset Y \to X_2$ be a smooth map, and let $\alpha_1 : Y \hookrightarrow X_1$ and $\alpha_2 : f(Y) \hookrightarrow X_2$ be the natural inclusions. Then $\Omega^1(X_1 \cup_f X_2)$ is diffeomorphic to the subset of $\Omega^1(X_1) \times \Omega^1(X_2)$ of all pairs (ω_1, ω_2) such that $\alpha_1^*\omega_1 = f^*\alpha_2^*\omega_2$.*

The description is much less straightforward when it comes to the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$, and we only give it in the case where f is a diffeological diffeomorphism. Even in that case, it is easier to say what it is not rather than what it is. We indicate here (the precise statements appear in the paper itself) that each fibre of $\Lambda^1(X_1 \cup_f X_2)$ coincides either with a fibre of $\Lambda^1(X_1)$, or one of $\Lambda^1(X_2)$, or with a subset of the direct product of the two. Accordingly, $\Lambda^1(X_1 \cup_f X_2)$ is equipped with two standard projections, each of them defined on a proper subset of it, to $\Lambda^1(X_1)$ one, to $\Lambda^1(X_2)$ the other. The diffeology of $\Lambda^1(X_1 \cup_f X_2)$ can be characterized as the coarsest one for which these two projections are smooth:

Theorem 2. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supset Y \to X_2$ be a diffeomorphism with its image. The space $\Lambda^1(X_1 \cup_f X_2)$ is obtained as*

$$\left(\cup_{x\in X_1\setminus Y}\Lambda^1_x(X_1)\right) \cup \left(\cup_{y\in Y}\Lambda^1_y(X_1) \times_{comp} \Lambda^1_{f(y)}(X_2)\right) \cup \left(\cup_{x\in X_2\setminus f(Y)}\Lambda^1_x(X_2)\right)$$

and is endowed with the coarsest diffeology such that the natural projections $\tilde{\rho}_1^\Lambda$, of the first two factors to $\Lambda^1(X_1)$, and $\tilde{\rho}_2^\Lambda$, of the last two factors to $\Lambda^1(X_2)$, are smooth.

As an application of this, we consider (under appropriate assumptions) a construction of a *pseudo-metric* on $\Lambda^1(X_1 \cup_f X_2)$, which is a counterpart of a Riemannian metric for finite-dimensional diffeological vector pseudo-bundles.

Acknowledgments The scope of this work is very much limited, but nonetheless carrying it out required a degree of patience and good humor. I may or may not have a natural propensity to these, but in any case it certainly helped to have a good example, for which I must most convincedly thank Prof. Riccardo Zucchi.

1 Main definitions

We recall here as briefly as possible the basic definitions (and some facts) regarding diffeological spaces, diffeological pseudo-bundles, and diffeological gluing; the definitions regarding differential forms are collected in the section that follows.

1.1 Diffeological spaces

The notion of a **diffeological space** is due to J.M. Souriau [11], [12]; it is defined as a(n arbitrary) set X equipped with a **diffeology**. A diffeology, or a diffeological structure, on X is a set $\mathcal{D}$ of maps $U \to X$, where U is any domain in $\mathbb{R}^n$ (and, for a fixed X, this n might vary); the set $\mathcal{D}$ must possess the following properties. First, it must include all constant maps into X; second, for any $p \in \mathcal{D}$ its pre-composition $p \circ g$ with any usual smooth map g must again belong to $\mathcal{D}$; and third, if $p : U \to X$ is a set map and U admits an open cover by some sub-domains U_i such that $p|_{U_i} \in \mathcal{D}$ then necessarily $p \in \mathcal{D}$. The maps that compose a given diffeology $\mathcal{D}$ on X are called **plots** of $\mathcal{D}$ (or of X).

Finer and coarser diffeologies on a given set On a fixed set X, there can be many diffeologies; and these being essentially sets of maps, it makes sense (in some cases) to speak of one being included in another;[1] the former is then said to be **finer** and the latter, to be **coarser**. It is particularly useful, on various occasions, to consider the finest (or the coarsest) diffeology possessing a given property P; many definitions are stated in such terms (although the diffeology thus defined can, and usually is, also be given an explicit description).

Smooth maps, pushforwards, and pullbacks Given two diffeological spaces X and Y, a set map $f : X \to Y$ is said to be **smooth** if for any plot p of X the composition $f \circ p$ is a plot of Y. The *vice versa* (that is, that every plot of Y admits, at least locally, such a form for some p) does not have to be true, but if it is, one says that the diffeology of Y is the **pushforward** of the one of X by the map f; or, accordingly, that the diffeology of X is the **pullback** of that of Y.

Topological constructions and diffeologies Given one or more (as appropriate) diffeological spaces, there are standard diffeological counterparts of all the basic set-theoretic and topological constructions, such as taking subspaces, quotients, direct products, and disjoint unions (with more complicated constructions following automatically). What we mean by a standard diffeological counterpart is of course the choice of diffeology, since the underlying set is known. Thus, any subset X' of a diffeological space X has the standard diffeology that is called the **subset diffeology** and that consists of precisely those of plots of X whose range is contained in X'; the quotient of X by any equivalence relation $\sim$ has the **quotient diffeology** that is the pushforward of the diffeology of X by the quotient projection $X \to X/\sim$. The direct product of a collection of diffeological spaces carries the **direct product diffeology** that is the coarsest diffeology such that all projections on individual factors are smooth; and the disjoint union, the **disjoint union diffeology**, defined as the finest diffeology such that the inclusion of each component is a smooth map.

Diffeologies on spaces of functions For any pair X and Y of diffeological spaces, we can consider the space $C^\infty(X, Y)$ of all smooth (in the diffeological sense) maps $X \to Y$. This space is also endowed with its standard diffeology that is called the **functional diffeology** and that can be defined as follows. A map $q : U \to C^\infty(X, Y)$ is a plot for this functional diffeology if and only if for every plot $p : U' \to X$ of X the natural evaluation map $U \times U' \ni (u, u') \to q(u)(p(u')) \in Y$ is a plot of Y.

1.2 Diffeological vector pseudo-bundles

We briefly mention this concept, since we will need it in order to consider $\Lambda^1(X)$ (see Introduction and the following Section). A smooth surjective map $\pi : V \to X$ between two diffeological spaces V and X is a **diffeological vector pseudo-bundle** if for all $x \in X$ the pre-image $\pi^{-1}(x)$ carries a vector space structure, and the corresponding addition $V \times_X V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ maps are smooth (for the natural diffeologies on their domains). This is a diffeological counterpart of the usual smooth vector bundle; we stress however that it does not include the requirement of local triviality. Indeed, various examples that motivated the concept do not enjoy this property, although there are contexts in which it is necessary to add the assumption of it.[2]

Diffeological vector spaces and operations on them Each fibre of a diffeological vector pseudo-bundle is a vector space and a diffeological space at the same time; and the operations are actually smooth maps for the subset diffeology. Thus, the fibres are **diffeological vector spaces** (that are defined as vector spaces endowed with a diffeology for which the addition and scalar multiplication maps are smooth). We briefly mention that all the basic operations on vector spaces (subspaces, quotients, direct sums, tensor products, and duals) have their diffeological counterparts (see [14], [16], also [6]), in the sense of there being a standard choice of diffeology on the resulting vector space. Thus, a subspace

[1] Formally speaking, the diffeologies on any given set are partially ordered with respect to inclusion and form a complete lattice; see Chapter 1 of [4] for more details.

[2] These contexts mostly have to do with with attempts to endow these pseudo-bundles with a kind of pseudo-Riemannian structure; we will not deal with these in the present paper.

is endowed with the subset diffeology, the quotient space, with a quotient one, the direct sum carries the product diffeology, and the tensor product, the quotient diffeology relative to the product diffeology on the (free) product of its factors.

The case of the dual spaces is worth mentioning in a bit more detail, mainly because there usually is not the standard isomorphism by duality between V and V^*, not even for finite-dimensional V. Indeed, the diffeological dual V^* is defined as $C^\infty(V,\mathbb{R})$, where $\mathbb{R}$ has standard diffeology, and, unless V is also standard, V^* has smaller dimension than V. The diffeology on V^* is the functional diffeology (see above). Notice also that if V is finite-dimensional, V^* is always a standard space (see [8]).

Operations of diffeological vector pseudo-bundles The usual operations on vector bundles (direct sum, tensor product, dual bundle) are defined for diffeological vector pseudo-bundles as well (see [14]), although in the absence of local trivializations defining them does not follow the standard strategy. Indeed, they are defined by carrying out these same operations fibrewise (which is still standard), but then are endowed with a diffeology that either described explicitly, or characterized as the finest diffeology inducing the already-existing diffeology on each fibre. For instance, if $\pi_1 : V_1 \to X$ and $\pi_2 : V_2 \to X$ are two finite-dimensional diffeological vector pseudo-bundles over the same base space X, their direct sum $\pi_1 \oplus \pi_2 : V_1 \oplus V_2 \to X$ is defined by setting $V_1 \oplus V_2 := \cup_{x\in X}\left(\pi_1^{-1}(x)\oplus\pi_2^{-1}(x)\right)$ and endowing it with the finest diffeology such that the corresponding subset diffeology on each fibre $\pi_1^{-1}(x)\oplus\pi_2^{-1}(x)$ is its usual direct sum diffeology (see [14]). We will not make much use of most of these operations and so do not go into more detail about them (see [14]; also [9] and [10] for more details), mentioning the only property that we will need in the sequel and that regards sub-bundles.

Let $\pi : V \to X$ be a diffeological vector pseudo-bundle. For each $x \in X$ let $W_x \leqslant \pi^{-1}(x)$ be a vector subspace, and let $W = \cup_{x\in X} W_x$. It is endowed with the obvious projection onto X, and as a subset of V, it carries the subset diffeology (which on each fibre W_x induces the same diffeology as that relative to the inclusion $W_x \leqslant \pi^{-1}(x)$). This diffeology makes W into a diffeological vector pseudo-bundle and is said to be a **sub-bundle** of V; we stress that there are no further conditions on the choice of W_x, as long as each of them is a vector subspace in the corresponding fibre.

Pseudo-metrics It is known (see [4]) that a finite-dimensional diffeological vector space admits a smooth scalar product if and only if it is a standard space; in general, the closest that comes to a scalar product on such a space is a smooth symmetric semi-definite positive bilinear form of rank equal to the dimension of the diffeological dual (see [8]). Such a form is called a **pseudo-metric** on the space in question.

It is then obvious that neither a diffeological vector pseudo-bundle would usually admit a diffeologically smooth Riemannian metric (it would have to have all standard fibres, and this condition is still not sufficient). However, it mat admit the extension of the notion of pseudo-metric (called pseudo-metric as well), which is just a section of the tensor square of the dual pseudo-bundle such that its value at each point is a pseudo-metric, in the sense of diffeological vector spaces, on the corresponding fibre. The precise definition is as follows.

Let $\pi : V \to X$ be a finite-dimensional diffeological vector pseudo-bundle. A **pseudo-metric** on it is a smooth section $g : X \to V^* \otimes V^*$ such that for all $x \in X$ the value $g(x)$ is a smooth symmetric semidefinite-positive bilinear form on $\pi^{-1}(x)$ of rank equal to $\dim((\pi^{-1}(x))^*)$ (see [9] or [10] for more details).

1.3 Diffeological gluing

This concept, which is central to the present paper, is just a natural carry-over of the usual topological gluing to the diffeological context.

1.3.1 Gluing of diffeological spaces and maps between them

Gluing together two diffeological spaces along a map between subsets of them is the main buidling block of this construction. It then naturally extends to define a gluing of smooth maps, and in particular (also a central concept for us) of diffeological pseudo-bundles.

Diffeological spaces Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supset Y \to X_2$ be a smooth (for the subset diffeology on Y) map. The result of (diffeological) gluing of X_1 to X_2 along f is the space $X_1 \cup_f X_2$ defined by

$$X_1 \cup_f X_2 = (X_1 \sqcup X_2)/\sim,$$

where $\sim$ is the equivalence relation determined by f, that is, $Y \ni y \sim f(y)$. The diffeology on $X_1 \cup_f X_2$, called the **gluing diffeology**, is the pushforward of the disjoint union diffeology on $X_1 \sqcup X_2$ by the quotient projection $\pi : X_1 \sqcup X_2 \to X_1 \cup_f X_2$. Since a pushforward diffeology (equivalently, quotient diffeology) is the finest one making the defining projection smooth, it is quite obvious that the gluing diffeology is the finest one induced[3] by the diffeologies on its factors. Indeed, it frequently turns out to be weaker than other natural diffeologies that the resulting space might carry, as it occurs for the union of the coordinate axes in $\mathbb{R}^2$, whose gluing diffeology (relative to gluing of the two standard axes at the origin) is finer than the subset diffeology relative to its inclusion in $\mathbb{R}^2$, see Example 2.67 in [15].

The standard disjoint cover of $X_1 \cup_f X_2$ There is a technical convention, which comes in handy when working with glued spaces, for instance, when defining maps on them (see below for an instance of this). It is based on the trivial observation that the following two maps are inductions,[4] and their ranges form a disjoint cover of $X_1 \cup_f X_2$:

$$i_1^{X_1} : X_1 \setminus Y \hookrightarrow (X_1 \sqcup X_2) \to X_1 \cup_f X_2 \text{ and } i_2^{X_2} : X_2 \hookrightarrow (X_1 \sqcup X_2) \to X_1 \cup_f X_2,$$

where in both cases the second arrow is the quotient projection π. We will omit the upper index when it is clear which glued space we are referring to.

Smooth maps Let us now have two pairs of diffeological spaces, X_1, X_2 and Z_1, Z_2, with a gluing within each pair given respectively by $f : X_1 \supseteq Y \to X_2$ and $g : Z_1 \supseteq Y' \to Z_2$. Then, under a specific condition called **(f, g)-compatibility**, two maps $\varphi_i \in C^\infty(X_i, Z_i)$ for $i = 1, 2$ induce a well-defined map in $C^\infty(X_1 \cup_f X_2, Z_1 \cup_g Z_2)$.

The (f, g)-compatibility means that $\varphi_1(Y) = Y'$ and $g \circ \varphi_1|_Y = \varphi \circ f$. The induced map, denoted by $\varphi_1 \cup_{(f,g)} \varphi_2$, is given by

$$(\varphi_1 \cup_{(f,g)} \varphi_2)(x) = \begin{cases} i_1^{Z_1}(\varphi_1((i_1^{X_1})^{-1}(x))) & \text{if } x \in \text{Range}(i_1^{X_1}) \\ i_2^{Z_2}(\varphi_2((i_2^{X_2})^{-1}(x))) & \text{if } x \in \text{Range}(i_2^{X_2}). \end{cases}$$

Furthermore, the assignment $(\varphi_1, \varphi_2) \mapsto \varphi_1 \cup_{(f,g)} \varphi_2$ defines a map $C^\infty(X_1, Z_1) \times_{comp} C^\infty(X_2, Z_2) \to C^\infty(X_1 \cup_f X_2, Z_1 \cup_g Z_2)$ from the set of all (f, g)-compatible pairs (φ_1, φ_2) to $C^\infty(X_1 \cup_f X_2, Z_1 \cup_g Z_2)$. This map is smooth for the functional diffeology on the latter space and for the subset diffeology (relative to the product diffeology on the ambient space $C^\infty(X_1, Z_1) \times C^\infty(X_2, Z_2)$) on its domain of definition $C^\infty(X_1, Z_1) \times_{comp} C^\infty(X_2, Z_2)$ (see [10]).

1.3.2 Gluing of pseudo-bundles

Gluing of two diffeological vector pseudo-bundles is an operation which is essentially a special case of gluing of two smooth maps (see immediately above). We mention it separately since it is prominent to our subject (in fact, we would like to describe differential 1-forms on a glued space $X_1 \cup_f X_2$ in reference to the pseudo-bundles of differential 1-forms of its factors).

Let $\pi_1 : V_1 \to X_1$ and $\pi_2 : V_2 \to X_2$ be two diffeological vector pseudo-bundles, let $f : X_1 \supseteq Y \to X_2$ be a smooth map defined on some subset of X_1, and let $\tilde{f} : \pi_1^{-1}(Y) \to \pi_2^{-1}(f(Y))$ be a smooth lift of f whose restriction to each fibre in $\pi_1^{-1}(Y)$ is linear. The definitions given so far allow us to consider (without any further comment) the spaces $V_1 \cup_{\tilde{f}} V_2$ and $X_1 \cup_f X_2$, and the map $\pi_1 \cup_{(\tilde{f},f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \to X_1 \cup_f X_2$ between them. It then follows from the assumptions on $\tilde{f}$ that this latter map is, in turn, a

[3]We use the term informally at the moment; it stands for whatever diffeology can be obtained in not-too-artificial a way from those on the factors.

[4]An injective map $f : X \to Y$ between two diffeological spaces is called an induction if the diffeology of X is the pullback of the subset diffeology on $f(X) \subset Y$.

diffeological vector pseudo-bundle, with operations on fibres inherited from either V_1 or V_2, as appropriate (see [9]).

This gluing operation is relatively well-behaved with respect to the usual operations on smooth vector bundles, which, as we mentioned above, extend to the diffeological pseudo-bundles. More precisely, it commutes with the direct sum and tensor product, but in general not with taking dual pseudo-bundles. We do not give more details about these, since we will not need them.

2 Diffeological differential 1-forms

For diffeological spaces, there exists a rather well-developed theory of differential k-forms on them (see [4], Chapter 6, for a detailed exposition). We now recall the case $k = 1$ (some definitions are given also for generic k).

2.1 Differential 1-forms and differentials of functions

We briefly go over those definitions that will be needed in what follows.

2.1.1 The definition of a 1-form

A differential 1-form on a diffeological space X is defined by assigning to each plot $p : \mathbb{R}^k \supset U \to X$ a (usual) differential 1-form $\omega(p)(u) = f_1(u)du_1 + \ldots + f_k(u)du_k \in \Lambda^1(U)$ such that this assignment satisfies the following compatibility condition: if $q : U' \to X$ is another plot of X such that there exists a usual smooth map $F : U' \to U$ with $q = p \circ F$ then $\omega(q)(u') = F^*(\omega(p)(u))$.

The definition of a diffeological k-forms is the same, except that the differential forms assigned to the domains of plots are k-forms.

2.1.2 Locality of differential forms

Let X be a diffeological space, and let ω_1 and ω_2 be two differential k-forms on X. Let $x \in X$; the forms ω_1 and ω_2 *have the same germ at* x if for every plot $p : U \to X$ such that $U \ni 0$ and $p(0) = x$ there exists an open neighborhood U' of 0 in V such that the restrictions of $\omega_1(p) \in \Lambda^k(U)$ and of $\omega_2(p) \in \Lambda^k(U)$ to U' coincide, $\omega_1(p)(u') = \omega_2(p)(u')$ for all $u' \in U'$. The **locality property** states that two differential forms ω_1 and ω_2 coincide if and only if they have the same germ at every point of X; alternatively, if and only if there exists a D-open covering $\{X^{(i)}\}$ of X such that the restrictions of ω_1 and ω_2 to each $X^{(i)}$ coincide.

2.1.3 The differential of a function

Let X be a diffeological space, and let $f : X \to \mathbb{R}$ be a diffeologically smooth function on it; recall that this means that for every plot $p : U \to X$ the composition $f \circ p : U \to \mathbb{R}$ is smooth in the usual sense, therefore $d(f \circ p)$ is a differential form on U. It is quite easy to see that the assignment $p \mapsto d(f \circ p) =: \omega_p$ is a differential 1-form on X; indeed, let $g : V \to U$ be a smooth function. The smooth compatibility condition $\omega_{p \circ g} = g^*(\omega_p)$ is then equivalent to $d((f \circ p) \circ g) = g^*(d(f \circ p))$, which is a standard property of usual differential forms.

2.2 The space $\Omega^1(X)$ of 1-forms

The set of all differential 1-forms on X is denoted by $\Omega^1(X)$; it carries a natural functional diffeology with respect to which it becomes a diffeological vector space. There is also a (pointwise) quotient of it over the forms degenerating at the given point; the collection of such quotients forms a (pseudo-)bundle $\Lambda^1(X)$.

The functional diffeology on $\Omega^1(X)$ The addition and the scalar multiplication operations, that make $\Omega^1(X)$ into a vector space, are given pointwise (meaning the points in the domains of plots). The already-mentioned functional diffeology on $\Omega^1(X)$ is characterized by the following condition:

- a map $q : U' \to \Omega^1(X)$ is a plot of $\Omega^1(X)$ if and only if for every plot $p : U \to X$ the map $U' \times U \to \Lambda^1(\mathbb{R}^n)$ given by $(u', u) \mapsto q(u')(p)(u)$ is smooth, where $U \subset \mathbb{R}^n$.

The expression $q(u')(p)$ stands for the 1-form on the domain of definition of p, *i.e.*, the domain U, that the differential 1-form $q(u')$ on X assigns to the plot p of X.

The spaces $\Omega^k(X)$ for any k These are defined in the same way as $\Omega^1(X)$, *i.e.* they are the sets of all differential k-forms on X. The functional diffeology is also similarly defined, with the evaluation map taking values in $\Lambda^k(\mathbb{R}^n)$.

2.3 The bundle of k-forms $\Lambda^k(X)$

Once again, our main interest here is the case of $k = 1$; we treat the general case simply because it does not change much.

The fibre $\Lambda^k_x(X)$ There is a natural quotienting of $\Omega^k(X)$, which gives, at every point $x \in X$, the set of all distinct values, at x, of the differential k-forms on X. This set is called $\Lambda^k_x(X)$; its precise definition is as follows.

Let X be a diffeological space, and let x be a point of it. A plot $p : U \to X$ is *centered at* x if $U \ni 0$ and $p(0) = x$. Let $\sim_x$ be the following equivalence relation: two k-forms $\alpha, \beta \in \Omega^k(X)$ are equivalent, $\alpha \sim_x \beta$, if and only if, for every plot p centered at x, we have $\alpha(p)(0) = \beta(p)(0)$. The class of α for the equivalence relation $\sim_x$ is called **the value of α at the point x** and is denoted by α_x. The set of all the values at the point x, for all k-forms on X, is denoted by $\Lambda^k_x(X)$:

$$\Lambda^k_x(X) = \Omega^k(X)/\sim_x = \{\alpha_x \mid \alpha \in \Omega^k(X)\}.$$

An element $\alpha \in \Lambda^k_x(X)$ is said to be a **k-form of X at the point x** (and x is said to be the *basepoint of* α). The space $\Lambda^k_x(X)$ is then called the **space of k-forms of X at the point x**.

The space $\Lambda^k_x(X)$ as a quotient of $\Omega^k(X)$ Two k-forms α and β have the same value at the point x if and only if their difference vanishes at this point: $(\alpha - \beta)_x = 0$. The set $\{\alpha \in \Omega^k(X) \mid \alpha_x = 0_x\}$ of the k-forms of X vanishing at the point x is a vector subspace of $\Omega^k(X)$; furthermore,

$$\Lambda^k_x(X) = \Omega^k(X)/\{\alpha \in \Omega^k(X) \mid \alpha_x = 0_x\}.$$

In particular, as a quotient of a diffeological vector space by a vector subspace, the space $\Lambda^k_x(X)$ is naturally a diffeological vector space; the addition and the scalar multiplication on $\Lambda^k_x(X)$ are well-defined for any choice of representatives.

The k-forms bundle $\Lambda^k(X)$ The **bundle of k-forms over** X, denoted by $\Lambda^k(X)$, is the union of all spaces $\Lambda^k_x(X)$:

$$\Lambda^k(X) = \coprod_{x \in X} \Lambda^k_x(X) = \{(x, \alpha) \mid \alpha \in \Lambda^k_x(X)\}.$$

It has the obvious structure of a pseudo-bundle over X. The bundle $\Lambda^k(X)$ is endowed with the diffeology that is the pushforward of the product diffeology on $X \times \Omega^k(X)$ by the projection $\Pi : X \times \Omega^k(X) \to \Lambda^k(X)$ acting by $\Pi(x, \alpha) = (x, \alpha_x)$. Note that for this diffeology the natural projection $\pi : \Lambda^k(X) \to X$ is a local subduction;[5] furthermore, each subspace $\pi^{-1}(x)$ is smoothly isomorphic to $\Lambda^k_x(X)$.

[5] A surjective map $f : X \to Y$ between two diffeological spaces is called a subduction if the diffeology of Y coincides with the pushforward of the diffeology of X by f.

The plots of the bundle $\Lambda^k(X)$ A map $p : U \ni u \mapsto (p_1(u), p_2(u)) \in \Lambda^k(X)$ defined on some domain U in some $\mathbb{R}^m$ is a plot of $\Lambda^k(X)$ if and only if the following two conditions are fulfilled:

1. The map p_1 is a plot of X;
2. For all $u \in U$ there exists an open neighborhood U' of u and a plot $q : U' \to \Omega^k(X)$ (recall that $\Omega^k(X)$ is considered with its functional diffeology described above) such that for all $u' \in U'$ we have $p_2(u') = q(u')(p_1)(u')$.

In other words, a plot of $\Lambda^k(X)$ is locally represented by a pair, consisting of a plot of X and a plot of $\Omega^k(X)$ (with the same domain of definition).

3 The spaces $\Omega^1(X_1 \cup_f X_2)$, $\Omega^1(X_1 \sqcup X_2)$, and $\Omega^1(X_1) \times \Omega^1(X_2)$

In this section X_1 and X_2 are two diffeological spaces, and $f : X_1 \supset Y \to X_2$ is a smooth map that defines a gluing between them. Our aim here is to describe how the space $\Omega^1(X_1 \cup_f X_2)$ is related to the spaces $\Omega^1(X_1)$ and $\Omega^1(X_2)$.

Since the space $X_1 \cup_f X_2$ is a quotient of the disjoint union $X_1 \sqcup X_2$, the natural projection $\pi : (X_1 \sqcup X_2) \to X_1 \cup_f X_2$ yields the corresponding pullback map $\pi^* : \Omega^1(X_1 \cup_f X_2) \to \Omega^1(X_1 \sqcup X_2)$ (see [4], Section 6.38); as we show immediately below, the latter space is diffeomorphic to $\Omega^1(X_1) \times \Omega^1(X_2)$. We then consider the image of π^* (this space is sometimes called the space of *basic* forms); we show that, although in general π^* is not surjective, it is a diffeomorphism with its image. Finally, we describe, in as much detail as possible, the structure of this image.

3.1 The diffeomorphism $\Omega^1(X_1 \sqcup X_2) = \Omega^1(X_1) \times \Omega^1(X_2)$

This is a rather easy and, in any case, expected fact, but for completeness we provide a proof.

Theorem 3.1. *The spaces $\Omega^1(X_1 \sqcup X_2)$ and $\Omega^1(X_1) \times \Omega^1(X_2)$ are diffeomorphic, for the usual functional diffeology on $\Omega^1(X_1 \sqcup X_2)$ and the product diffeology on $\Omega^1(X_1) \times \Omega^1(X_2)$.*

Proof. Let us first describe a bijection $\varphi : \Omega^1(X_1 \sqcup X_2) \to \Omega^1(X_1) \times \Omega^1(X_2)$. Let $\omega \in \Omega^1(X_1 \sqcup X_2)$, so that for every plot p of $X_1 \sqcup X_2$ there is a usual differential 1-form $\omega(p)$. Furthermore, every plot of X_1 is naturally a plot of $X_1 \sqcup X_2$ (and the same is true for every plot of X_2), therefore

$$\{\omega(p) \,|\, p \in \mathrm{Plots}(X_1 \sqcup X_2)\} \supset \{\omega_1(p_1) \,|\, p_1 \in \mathrm{Plots}(X_1)\},$$

where $\omega_1(p_1)$ is the differential 1-form (on the domain of definition of p_1) assigned by ω to the plot[6] of $X_1 \sqcup X_2$ obtained by composing p_1 with the natural inclusion $X_1 \hookrightarrow X_1 \sqcup X_2$. Furthermore, there is an analogous inclusion for X_2, that is,

$$\{\omega(p) \,|\, p \in \mathrm{Plots}(X_1 \sqcup X_2)\} \supset \{\omega_2(p_2) \,|\, p_2 \in \mathrm{Plots}(X_2)\}.$$

Notice, finally, that as sets,

$$\{\omega(p) \,|\, p \in \mathrm{Plots}(X_1 \sqcup X_2)\} = \{\omega_1(p_1) \,|\, p_1 \in \mathrm{Plots}(X_1)\} \cup \{\omega_2(p_2) \,|\, p_2 \in \mathrm{Plots}(X_2)\};$$

indeed, it is a general property of the disjoint union diffeology (see [4], Ex. 22 on p.23) that for any plot $p : U \to X_1 \sqcup X_2$ we have $U = U_1 \cup U_2$, where $U_1 \cap U_2 = \emptyset$, and if U_i is non-empty then $p|_{U_i}$ is a plot of X_i. We indicate this fact by writing $\omega = \omega_1 \cup \omega_2$.

Observe now that each ω_i is a well-defined differential 1-form on X_i; indeed, it is defined for all plots of X_i (these being plots of $X_1 \sqcup X_2$), and it satisfies the smooth compatibility condition simply because ω does. On the other hand, for any two forms ω_i on X_i their formal union $\omega_1 \cup \omega_2$ yields a differential form on $X_1 \sqcup X_2$, by the already-cited property of the disjoint union diffeology (since X_1 and X_2 are disjoint,

[6] We did not formally introduce the notation $\mathrm{Plots}(X)$; its meaning as the set of all plots of X should be completely obvious.

the smooth compatibility condition is empty in this case). Thus, setting $\varphi(\omega_1 \cup \omega_2) = (\omega_1, \omega_2)$ yields a well-defined bijection $\Omega^1(X_1 \sqcup X_2) \leftrightarrow \Omega^1(X_1) \times \Omega^1(X_2)$; let us show that it is both ways smooth.

Let $q : U' \to \Omega^1(X_1 \sqcup X_2)$ be a plot; we need to show that $\varphi \circ q$ is a plot of $\Omega^1(X_1) \times \Omega^1(X_2)$. Notice that each $q(u')$ writes in the form $q(u') = q_1(u') \cup q_2(u')$, and $(\varphi \circ q)(u') = (q_1(u'), q_2(u'))$. It suffices to show that each q_i, defined by $q_i(u')(p) = q(u)(p)$ whenever p coincides with a plot of X_i, is a plot of $\Omega^1(X_i)$. For it to be so, for any arbitrary plot $p_i : U_i \to X_i$ the evaluation $U' \times U_i' \ni (u', u_i) \mapsto (q_i(u')(p_i))(u_i) \in \Lambda^1(U_i)$ should be smooth (in the usual sense). Now, the pair of plots p_1, p_2 defines a plot $p_1 \sqcup p_2 : U_1 \sqcup U_2 \to X_1 \sqcup X_2$[7] of $X_1 \sqcup X_2$. The evaluation of $q(u')$ on this plot, smooth by hypothesis, is $(u', u_1) \mapsto (q(u')(p_1))(u_1) = (q_1(u')(p_1))(u_1)$ for $u_1 \in U_1$ and $(u', u_2) \mapsto (q(u')(p_2))(u_2) = (q_2(u')(p_2))(u_2)$ for $u_2 \in U_2$, by definitions of q_1 and q_2, so we are finished.

The proof works in a very similar manner for the inverse map φ^{-1}. Indeed, let $q_i : U' \to \Omega^1(X_i)$ for $i = 1, 2$ be a pair of plots of $\Omega^1(X_1)$, $\Omega^1(X_2)$ respectively; such pair represents a plot of the direct product $\Omega^1(X_1) \times \Omega^1(X_2)$. We need to show that $\varphi^{-1} \circ (q_1, q_2) : U' \to \Omega^1(X_1 \sqcup X_2)$ is a plot of $\Omega^1(X_1 \sqcup X_2)$. Notice first of all that $(\varphi^{-1} \circ (q_1, q_2))(u') = q_1(u') \cup q_2(u')$. To show that the assignment $u' \mapsto (q_1(u') \cup q_2(u'))$ defines a plot of $\Omega^1(X_1 \sqcup X_2)$, consider a plot $p = p_1 \sqcup p_2 : U_1 \sqcup U_2 \to X_1 \sqcup X_2$ of $X_1 \sqcup X_2$ and the evaluation of $q_1(u') \cup q_2(u')$ on it. The same formulas as above show that we actually a disjoint union of the evaluations of q_1 and q_2, smooth by assumption, so we are finished. □

3.2 The image of the pullback map $\Omega^1(X_1 \cup_f X_2) \to \Omega^1(X_1 \sqcup X_2)$

We now begin to consider the pullback map $\pi^* : \Omega^1(X_1 \cup_f X_2) \to \Omega^1(X_1 \sqcup X_2)$. Recall that, given a differential 1-form ω on $X_1 \cup_f X_2$, the form $\pi^*(\omega)$ is defined by the following rule: if p is a plot of $X_1 \sqcup X_2$ then the (usual) differential form $(\pi^*(\omega))(p)$ is the form $\omega(\pi \circ p)$ (see [4], Chapter 6, for this and other standard facts regarding the behavior of diffeological forms under smooth maps).

Following from this definition and from the already-established diffeomorphism $\Omega^1(X_1 \sqcup X_2) \cong \Omega^1(X_1) \times \Omega^1(X_2)$, each form in $\Omega^1(X_1 \cup_f X_2)$ splits as a pair of forms, one in $\Omega^1(X_1)$, the other in $\Omega^2(X_2)$. This point of view can be used to show that the pullback map is *not* in general surjective, which we do in the section that follows, using the notion of an f-invariant 1-form and that of a pair of compatible 1-forms (these notions serve also to describe the image of the pullback map).

3.2.1 The map π^* composed with $\Omega^1(X_1 \sqcup X_2) \to \Omega^1(X_2)$ is surjective

Let us make our first observation regarding the properties of the pullback map. This property, stated in the title of the section, follows easily from the existence of the induction $i_2^{X_2} : X_2 \to X_1 \cup_f X_2$ (see Section 1.3.2); the composition $[\Omega^1(X_1 \sqcup X_2) \to \Omega^1(X_2)] \circ \pi^*$ is the map $(i_2^{X_2})^*$, whose surjectivity follows from it being an induction.

3.2.2 Determining the projection to $\Omega^1(X_1)$: f-equivalent plots and f-invariant forms

As follows from the gluing construction, there is in general not an induction of X_1 into $X_1 \cup_f X_2$; the map $i_1^{X_1}$ is an induction, but it is defined on the proper subset $X_1 \setminus Y$ of X_1. Obviously, there is a natural map $i_1' : X_1 \to X_1 \cup_f X_2$ obtained by taking the composition of the inclusion $X_1 \hookrightarrow (X_1 \sqcup X_2)$ and the projection $(X_1 \sqcup X_2) \to X_1 \cup_f X_2$; in general, it is not an induction, so the corresponding pullback map $(i_1')^*$ *a priori* is not surjective. It is rather clear that this is correlated to f being, or not, injective, so in general the forms in $\Omega^1(X_1)$ contained in the image of $(i_1')^*$ should possess the property described in the second of the following definitions (we need an auxiliary term first).

Definition 3.2. *Two plots p_1 and p_1' of X_1 are said to be f-**equivalent** if they have the same domain of definition U and for all $u \in U$ such that $p_1(u) \neq p_1'(u)$ we have $p_1(u), p_1'(u) \in Y$ and $f(p_1(u)) = f(p_1(u'))$.*

Thus, two plots on the same domain are f-equivalent if they differ only at points of the domain of gluing, and among such, only at those that are identified by f.

[7]The meaning of this notation is that $p(u) = p_1(u)$ for $u \in U_1$ and $p(u) = p_2(u)$ for $u \in U_2$ (we could also say that $p_i = p|_{U_i}$); the disjoint union $U_1 \sqcup U_2$ is considered as a disconnected domain in a Euclidean space large enough to contain both, and possibly applying a shift if both contain zero.

Definition 3.3. *A form $\omega_1 \in \Omega^1(X_1)$ is said to be f-**invariant** if for any two plots f-equivalent $p_1, p_1' : U \to X_1$ we have $\omega_1(p_1) = \omega_1(p_1')$.*

As we will see with more precision below, this notion is designed to ensure that an f-invariant form descends to a (portion of a) well-defined form on the result of gluing of X_1 to another diffeological space.

3.2.3 $\Omega_f^1(X_1)$ is a vector subspace of $\Omega^1(X_1)$

This is a consequence of the following statement.

Lemma 3.4. *Let $\omega_1', \omega_1'' \in \Omega^1(X_1)$ be two f-invariant forms, and let $\alpha \in \mathbb{R}$. Then the forms $\omega_1' + \omega_1''$ and $\alpha\omega_1'$ are f-invariant forms.*

Proof. Let $p', p'' : U \to X_1$ be two plots of X_1 with the following property: if $u \in U$ is such that $p'(u) \neq p''(u)$ then $p'(u), p''(u) \in Y$ and $f(p'(u)) = f(p''(u))$. The assumption that ω_1', ω_1'' are f-invariant means that we have $\omega_1'(p') = \omega_1'(p'')$ and $\omega_1''(p') = \omega_1''(p'')$. The same equalities should now be checked for $\omega_1' + \omega_1''$ and $\alpha\omega_1'$, and these follow immediately from the definition of the addition and scalar multiplication in $\Omega^1(X_1)$. Specifically,

$$(\omega_1' + \omega_1'')(p') = \omega_1'(p') + \omega_1''(p') = \omega_1'(p'') + \omega_1''(p'') = (\omega_1' + \omega_1'')(p'') \text{ and}$$

$$(\alpha\omega_1')(p') = \alpha(\omega_1'(p')) = \alpha(\omega_1'(p'')) = (\alpha\omega_1')(p'').$$

□

We thus obtain that $\Omega_f^1(X_1)$ is a vector subspace of $\Omega^1(X_1)$. In particular, its intersection with any other vector subspace of $\Omega^1(X_1)$ is a vector subspace itself, so we always have a well-defined quotient (in the sense of vector spaces).

3.2.4 Characterizing the basic forms relative to π^*

We now establish the following statement.

Theorem 3.5. *Let ω_i be a differential 1-form on X_i, for $i = 1, 2$. The pair (ω_1, ω_2) belongs to the image of the pullback map π^* if and only if ω_1 is f-compatible, and for every plot p_1 of the subset diffeology on Y we have*

$$\omega_1(p_1) = \omega_2(f \circ p_1).$$

Proof. Suppose that $(\omega_1, \omega_2) = \pi^*(\omega)$ for some $\omega \in \Omega^1(X_1 \cup_f X_2)$. That ω_1 has to be f-compatible, has already been seen. Recall also that by definition $\omega_i(p_i) = \omega(\pi \circ p_i)$ for $i = 1, 2$ and any plot p_i of X_i.

Let us check that the second condition indicated in the statement holds. Let p_1 be a plot for the subset diffeology of Y; then $f \circ p_1$ is a plot of X_2. Furthermore, $\pi \circ \pi_1 = \pi \circ f \circ p_1$ by the very construction of $X_1 \cup_f X_2$. Therefore we have:

$$\omega_1(p_1) = \omega(\pi \circ p_1) = \omega(\pi \circ f \circ p_1) = \omega_2(f \circ p_1),$$

as wanted.

Let us now prove the reverse. Suppose that we are given two forms ω_1 and ω_2, satisfying the condition indicated; let us define ω. Recall that, as we have already mentioned, it suffices to define ω on plots with connected domains. Let $p : U \to X_1 \cup_f X_2$ be such a plot; then it lifts either to a plot p_1 of X_1 or to a plot p_2 of X_2. In the former case we define $\omega(p) = \omega_1(p_1)$, in the latter case we define $\omega(p) = \omega_2(p_2)$. Finally, if p is defined on a disconnected domain, $\omega(p)$ is defined by the collection of the values of its restriction to the corresponding connected components.

Let us show that this definition is well-posed (which it may not be *a priori* if p happens to lift to two distinct plots). Now, if p lifts to a plot of X_2 then this lift is necessarily unique, since $i_2 : X_2 \to X_1 \cup_f X_2$ is an induction. Suppose now that p lifts to two distinct plots $p_1 : U \to X_1$ and $p_1' : U \to X_1$ of X_1. It is then clear that p_1 and p_1' differ only at points of Y, and among such, only at those that have the same image under f. More precisely, for any $u \in U$ such that $p_1(u) \neq p_1'(u)$, we have $p_1(u), p_1'(u) \in Y$ and

$f(p_1(u)) = f(p_1'(u))$. Thus, for ω to be well-defined we must have that $\omega_1(p_1)(u) = \omega_1(p_1')(u)$ for all such u.

What we now need to check is whether ω thus defined satisfies the smooth compatibility condition. Let $q : U' \to X_1 \cup_f X_2$ be another plot of $X_1 \cup_f X_2$ for which there exists a smooth map $g : U' \to U$ such that $q = p \circ g$. We need to check that $\omega(q) = g^*(\omega(p))$.

Suppose first that p lifts to a plot p_2 of X_2. Then we have $p = \pi \circ p_2$, so $q = \pi \circ p_2 \circ g$. Notice that $p_2 \circ g$ is also a plot of X_2 and is a lift of q; thus, according to our definition $\omega(q) = \omega_2(p_2 \circ g) = g^*(\omega_2(p_2)) = g^*(\omega(p))$. If now p lifts to a plot p_1 of X_1 the same argument is sufficient, whence the conclusion. □

The theorem just proved motivates the following definition, which will serve to characterize the basic forms in $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$.

Definition 3.6. *Let $\omega_i \in \Omega^1(X_i)$ for $i = 1, 2$. We say that ω_1 and ω_2 are **compatible** with respect to the gluing along f if for every plot p_1 of the subset diffeology on the domain Y of f we have*

$$\omega_1(p_1) = \omega_2(f \circ p_1).$$

3.3 The pullback map is a diffeomorphism $\Omega^1(X_1 \cup_f X_2) \cong \Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$

We now obtain our first definite conclusion regarding the space $\Omega^1(X_1 \cup_f X_2)$; namely, in this section we construct a smooth inverse of the map π^*, which obviously ensures the claim in the title of the section.

3.3.1 Constructing the inverse of π^*

Let us first define this map; in the next section we will prove that it is smooth.

The induced 1-form $\omega_1 \cup_f \omega_2$ on $X_1 \cup_f X_2$ Let ω_1 be an f-invariant 1-form on X_1, and let ω_2 be a 1-form on X_2 such that ω_1 and ω_2 are compatible. Let $p : U \to X_1 \cup_f X_2$ be an arbitrary plot of $X_1 \cup_f X_2$; the form $(\omega_1 \cup_f \omega_2)(p) \in C^\infty(U, \Lambda^1(U))$ is defined as follows.

Let $u \in U$; in any connected neighborhood of $x = p(u)$ the plot p lifts to either a plot p_1 of X_1 or a plot p_2 of X_2. We define, accordingly,

$$(\omega_1 \cup_f \omega_2)(p)(u) := \omega_i(p_i)(u).$$

Lemma 3.7. *If ω_1 is f-invariant, and ω_1 and ω_2 are compatible with each other, the differential 1-form $\omega_1 \cup_f \omega_2$ is well-defined.*

Proof. We need to show that for each plot $p : U \to X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ the form $(\omega_1 \cup_f \omega_2)(p) \in C^\infty(U, \Lambda^1(U))$ is well-defined, *i.e.*, that it does not depend on the choice of the lift of the plot p. Obviously, it suffices to assume that U is connected, which then implies that p lifts either to a plot of X_1 or to a plot of X_2. If p has a unique lift, then there is nothing to prove. Suppose that p has two distinct lifts, p' and p''. Notice that X_2 smoothly injects into $X_1 \cup_f X_2$, therefore p' and p'' cannot be both plots of X_2.

Assume first that one of them, say p', is a plot of X_1, while the other, p'', is a plot of X_2. Since they project to the same map to $X_1 \cup_f X_2$, we can conclude that $p'' = f \circ p'$, so

$$(\omega_1 \cup_f \omega_2)(p) = \omega_1(p') = \omega_2(f \circ p') = \omega_2(p''),$$

by the compatibility of the forms ω_1 and ω_2 with each other.

Assume now that p' and p'' are both plots of X_1. Once again, since they project to the same plot of $X_1 \cup_f X_2$, for every $u \in U$ such that $p'(u) \neq p''(u)$ we have $p'(u), p''(u) \in Y$ and $f(p'(u)) = f(p''(u))$, that is, that they are f-equivalent; since ω_1 is assumed to be f-invariant, we obtain that

$$\omega_1(p') = \omega_1(p'') = (\omega_1 \cup_f \omega_2)(p).$$

We can thus conclude that for each plot $p : U \to X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ the form $(\omega_1 \cup_f \omega_2)(p) \in C^\infty(U, \Lambda^1(U))$ is well-defined. It remains to observe that the resulting collection $\{(\omega_1 \cup_f \omega_2)(p)\}$ of usual differential 1-forms satisfies the smooth compatibility condition for diffeological differential forms for all the same reasons as those given at the end of the proof of Theorem 3.5. □

The map $\mathcal{L}:\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_1\cup_f X_2)$ As we have seen above, the assignment

$$(\omega_1,\omega_2)\mapsto\omega_1\cup_f\omega_2$$

to any two compatible differential 1-forms $\omega_1\in\Omega^1_f(X_1)$ and $\omega_2\in\Omega^1(X_2)$, of the differential 1-form $\omega_1\cup_f\omega_2$ is well-defined. This yields a map $\mathcal{L}$ defined on the set $\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)$ of all pairs of compatible 1-forms (such that the first component of the pair is f-invariant), with range the space of 1-forms on $X_1\cup_f X_2$.

Lemma 3.8. *The map $\mathcal{L}$ is the inverse of the pullback map $\pi^*:\Omega^1(X_1\cup_f X_2)\to\Omega^1(X_1)\times\Omega^1(X_2)$.*

Proof. This follows from construction. Indeed, let $\omega\in\Omega^1(X_1\cup_f X_2)$; recall that $\pi^*(\omega)=(\omega_1,\omega_2)$, where for every plot p_i of X_i we have $\omega_i(p_i)=\omega(\pi\circ p_i)$. Furthermore, by Theorem 3.5, the form ω_1 is f-invariant and the two forms ω_1 and ω_2 are compatible with each other. Therefore the pair (ω_1,ω_2) is in the domain of $\mathcal{L}$, and by construction $\omega_1\cup_f\omega_2$ is precisely ω.

Vice versa, let (ω_1,ω_2) be in the domain of definition of $\mathcal{L}$. It suffices to observe that $\pi^*(\omega_1\cup_f\omega_2)=(\omega_1',\omega_2')$ with $\omega_i'(p_i)=(\omega_1\cup_f\omega_2)(\pi\circ p_i)=\omega_i(p_i)$ for any plot p_i of X_i, which means that $\omega_i'=\omega_i$ for $i=1,2$. □

3.3.2 The inverse of the pullback map is smooth

To prove that the map π^* is a diffeomorphism, it remains to show that its inverse $\mathcal{L}$ is a smooth map, for the standard diffeologies on its domain and its range. Specifically, the range carries the standard diffeology of the space of 1-forms on a diffeological space (see Section 2), while the domain is endowed with the subset diffeology relative to the inclusion $\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)\subset\Omega^1(X_1)\times\Omega^1(X_2)$ (this direct product has, as usual, the product diffeology relative to the standard diffeologies on $\Omega^1(X_1)$ and $\Omega^1(X_2)$).

Theorem 3.9. *The map $\mathcal{L}:\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_1\cup_f X_2)$ is smooth.*

Proof. Consider a plot $p:U\to\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)$. First of all, by definition of a subset diffeology and a product diffeology, we can assume that U is small enough so that for every $u\in U$ we have $p(u)=(p_1(u),p_2(u))$, where $p_1:U\to\Omega^1_f(X_1)$ is a plot of $\Omega^1_f(X_1)$ (considered with the subset diffeology relative to the inclusion $\Omega^1_f(X_1)$), $p_2:U\to\Omega^1(X_2)$ is a plot of $\Omega^1(X_2)$, and $p_1(u)$ and $p_2(u)$ are compatible with respect to f, for all $u\in U$.

That p_i is a plot of $\Omega^1(X_i)$, by definition of the standard diffeology on the latter, means that for every plot $q_i:U_i'\to X_i$ the map $U\times U_i'\to\Lambda^1(\mathbb{R})$, acting by $(u,u_i')\mapsto(p_i(u))(q_i)(u_i')$, is smooth (in the usual sense). The compatibility of the forms $p_1(u)$ and $p_2(u)$ means that $p_2(u)(f\circ q_1)=p_1(u)(q_1)$, for all u and for all plots q_1 of the subset diffeology of Y.

Suppose we are given p_1 and p_2 satisfying all of the above. We need to show that $(\mathcal{L}\circ(p_1,p_2)):U\to\Omega^1(X_1\cup_f X_2)$ is a plot of $\Omega^1(X_1\cup_f X_2)$. This, again, amounts to showing that for any plot $q:U'\to X_1\cup_f X_2$ the evaluation map

$$(u,u')\mapsto(\mathcal{L}(p_1(u),p_2(u)))(q)(u')$$

defined a usual smooth map $U\times U'\to\Lambda^1(\mathbb{R}^n)$ (for $U'\subset\mathbb{R}^n$).

Assume that U' is connected so that q lifts either to a plot q_1 of X_1, or a plot q_2 of X_2. It may furthermore lift to more than one plot of X_1, or it may lift to both a plot of X_1 and a plot of X_2. Suppose first that q lifts to a precisely one plot, say a plot q_i of X_i. Then

$$(u,u')\mapsto(\mathcal{L}(p_1(u),p_2(u)))(q)(u')=p_i(u)(q_i)(u')\in\Lambda^1(\mathbb{R}^n);$$

this is a smooth map, since each p_i is a plot of $\Omega^1(X_i)$.

Suppose now that q lifts to two distinct plots q_1 and q_1' of X_1. In this case, however, $p_1(u)(q_1)=p_1(u)(q_1')$ because $p_1(u)$ is f-compatible for any $u\in U$ by assumption, so we get the desired conclusion as in the previous case. Finally, if q lifts to both q_1 and q_2 (each q_i being a plot of X_i) then $q_2=f\circ q_1$, and we obtain the claim by using the compatibility of the pair of forms $p_1(u),p_2(u)$ for each u. □

Corollary 3.10. *The pullback map π^* is a diffeomorphism with its image.*

Proof. We have just seen that π^* has a smooth inverse $\mathcal{L}$. It remains to observe that π^* itself is smooth, because the pullback of a smooth map is always smooth itself (see [4], Section 6.38). □

4 The space $\Omega^1_f(X_1) \times_{comp} \Omega^1(X_2)$ as a sub-direct product $\Omega^1_f(X_1)$ and $\Omega^1(X_2)$

What we mean by a sub-direct product[8] of any direct product $X \times Y$ of two sets is any subset such that both projections on the two factors X and Y are surjective. Thus, the question of whether $\Omega^1_f(X_1) \times_{comp} \Omega^1(X_2)$ is a sub-direct product of $\Omega^1_f(X_1)$ and $\Omega^1(X_2)$ takes the form of the following two: first, if $\omega_1 \in \Omega^1_f(X_2)$ is an arbitrary f-invariant form, does there always exist $\omega_2 \in \Omega^1(X_2)$ compatible with it?, and *vice versa*, if $\omega_2 \in \Omega^1(X_2)$ is any form, does there exist an f-invariant form ω_1 on X_1, compatible with ω_2 as well? An additional assumption on f, namely, that it be a subduction, ensures that the answer is positive in both cases. Proving this requires an intermediate construction.

4.1 The space of f-equivalence classes X_1^f

The intermediate construction just mentioned is a certain auxiliary space X_1^f, which is a quotient of X_1. The aim of introducing it is to identify the space $X_1 \cup_f X_2$ with a result of a specific gluing of X_1^f to X_2; and under the assumption that f is a subduction, this other gluing turns out to be a diffeomorphism.

The space X_1^f and the map $f_\sim$ In order to consider the projection $\Omega^1_f(X_1) \times_{comp} \Omega^1(X_2) \to \Omega^1(X_2)$, we introduce a slightly different form of our glued space $X_1 \cup_f X_2$. Let us define X_1^f to be the diffeological[9] quotient of X_1 by the equivalence relation $y_1 \sim y_1 \Leftrightarrow f(y_1) = f(y_2)$, that is:

$$X_1^f := X_1/_{(f(y_1)=f(y_2))}.$$

Let $\pi_1^f : X_1 \to X_1^f$ be the quotient projection, and let us define the map $f_\sim : X_1^f \supseteq \pi_1^f(Y) \to X_2$ induced by f. This map is given by the condition $f_\sim \circ \pi_1^f = f$.

Lemma 4.1. *The map $f_\sim$ is injective and smooth. It is a diffeomorphism with its image if and only if f is a subduction.*

Proof. The injectivity of $f_\sim$ is by construction (we actually defined the space X_1^f so that the pushforward of f to it be injective), and its smoothness follows from the definition of the quotient diffeology. Recall now that a subduction is a smooth map such that the diffeology on its target space is the pushforward of the diffeology of its domain by the map. Thus, the assumption that f, considered as a map $Y \to f(Y)$, is a subduction means that for every plot q of the subset diffeology on $f(Y)$, defined on a sufficiently small neighborhood, there is a plot p of the subset diffeology on Y such that $f \circ p = q$. Therefore $f_\sim \circ (\pi_1^f \circ p) = q$, and so $(f_\sim)^{-1} \circ q = \pi_1^f \circ p$ for any plot q of $f(Y)$ and for an appropriate plot p of Y. Since $\pi_1^f \circ p$ is a plot of $\pi_1^f(Y)$, we conclude that $(f_\sim)^{-1}$ is smooth, and so $f_\sim$ is a diffeomorphism with its image. We obtain the *vice versa* by applying the same reasoning in the reverse order. □

Lifts of plots of X_1^f By definition of the quotient diffeology, every plot p of X_1^f lifts (locally) to a plot of X_1. Two lifts p' and p'' are lifts of the same p if and only if they are f-equivalent.

The diffeomorphism $X_1 \cup_f X_2 \cong X_1^f \cup_{f_\sim} X_2$ The existence of this diffeomorphism is a direct consequence of the definition of gluing. Formally, it is defined as the pushforward of the map $\pi_1^f \sqcup \mathrm{Id}_{X_2} : X_1 \sqcup X_2 \to X_1 \sqcup X_2$ by the two quotient projections, π and π^f respectively. Here by $\pi_1^f \sqcup \mathrm{Id}_{X_2}$ we mean the map on $X_1 \sqcup X_2$, whose value at an arbitrary point $x \in X_1 \sqcup X_2$ is $\pi_1^f(x)$ if $x \in X_1$ and x if $x \in X_2$; the map $\pi^f : X_1^f \sqcup X_2 \to X_1^f \cup_{f_\sim} X_2$ is, as we said, the quotient projection that defines the space $X_1^f \cup_{f_\sim} X_2$.

[8]Which is more or less a standard notion, I guess.

[9]That is, endowed with the quotient diffeology.

4.2 The linear diffeomorphism $\Omega^1_f(X_1) \cong \Omega^1(X_1^f)$

The reason that explains the introduction of the space X_1^f is that it allows to consider, instead of a subset of 1-forms on X_1, the space of all 1-forms on X_1^f; and to obtain $X_1 \cup_f X_2$ by gluing X_1^f to X_2 along a bijective map (a diffeomorphism if we assume f to be a subduction, see above).

Proposition 4.2. *The pullback map $(\pi_1^f)^* : \Omega^1(X_1^f) \to \Omega^1_f(X_1)$ is a diffeomorphism.*

Proof. Let us first show that $(\pi_1^f)^*$ takes values in $\Omega^1_f(X_1)$. Let $\omega \in \Omega^1(X_1^f)$; its image $(\pi_1^f)^*(\omega)$ is defined by setting, for every plot p_1 of X_1, that $(\pi_1^f)^*(\omega)(p_1) = \omega(\pi_1^f \circ p_1)$. We need to show that $(\pi_1^f)^*(\omega)$ is f-invariant, so let p_1 and p_1' be two f-equivalent plots; then we have $\pi_1^f \circ p_1 = \pi_1^f \circ p_1'$, and so $(\pi_1^f)^*(\omega)(p_1) = (\pi_1^f)^*(\omega)(p_1')$. Thus, the range of $(\pi_1^f)^*$ is contained in $\Omega^1_f(X_1)$.

Let us show $(\pi_1^f)^*$ is a bijection by constructing its inverse. Let ω_1 be an f-invariant 1-form on X_1, and let us assign to it a form ω_1^f on X_1^f by setting $\omega_1^f(p_1^f) = \omega_1(p_1)$, where p_1 is any lift to X_1 of the plot p_1^f. We need to show that this is well-defined, *i.e.*, $\omega_1(p_1)$ does not depend on the choice of a specific lift.[10] Indeed, let p_1 and p_1' be two lifts of some p_1^f; this means, first, that they have the same domain of definition U and, second, that for any $u \in U$ such that $p_1(u) \neq p_1'(u)$, we have $p_1(u), p_1'(u) \in Y$ and $f(p_1(u)) = f(p_1'(u))$. In other words, they are f-equivalent, so by f-invariance of ω_1 we have $\omega_1(p_1) = \omega_1(p_1')$. The form ω_1^f is therefore well-defined, and the fact that the assignment $\Omega^1_f \ni \omega_1 \mapsto \omega_1^f \in \Omega^1(X_1^f)$ is obvious from the construction.

Thus, $(\pi_1^f)^*$ is a bijective map and, as any pullback map, it is smooth. It thus remains to show that its inverse, that we have just constructed, is smooth (with respect to the usual functional diffeology of a space of forms; obviously, the diffeology of $\Omega^1_f(X_1)$ is the subset diffeology relative to its inclusion in $\Omega^1(X_1)$).

Let $q : U' \to \Omega^1_f(X_1)$ be a plot of $\Omega^1_f(X_1)$; thus, for every plot $p_1 : \mathbb{R}^n \supset U \to X_1$ the evaluation map $(u', u) \mapsto (q(u')(p))(u)$ is a smooth map to $\Lambda^1(\mathbb{R}^n)$, and furthermore for any $u' \in U'$ and for any two f-equivalent plots p_1, p_1' of X_1, *i.e.*, such that $\pi_1^f \circ p_1 = \pi_1 \circ p_1'$, we have $q(u')(p_1) = q(u')(p_1')$. Let us now consider the composition $\left((\pi_1^f)^*\right)^{-1} \circ q$; as always, we need to show that this is a plot of $\Omega^1(X_1^f)$. Since the plots of X_1^f are defined by classes of f-equivalent plots of X_1, and the forms $\left(((\pi_1^f)^*)^{-1} \circ q\right)(u')$ are given by values of $q(u')$ on class representatives, the evaluation map for this composition is simply the same as the one for q, so we get the desired conclusion. □

Thus, the f-invariant forms on X_1 are precisely the pullbacks by the natural projection of the forms on X_1^f. Furthermore, by construction of X_1^f we can, instead of gluing between X_1 and X_2, consider the corresponding gluing between X_1^f and X_2, which has an advantage of being a gluing along a bijective map.

4.3 The space $\Omega^1(X_1^f \cup_{f_\sim} X_2)$

We have already given a description of the space $\Omega^1(X_1 \cup_f X_2)$ in terms of $\Omega^1_f(X_1)$ and $\Omega^1(X_2)$. We now use the presentation of $X_1 \cup_f X_2$ as $X_1^f \cup_{f_\sim} X_2$, to write the same space $\Omega^1(X_1 \cup_f X_2) = \Omega^1(X_1^f \cup_{f_\sim} X_2)$ in terms of $\Omega^1(X_1^f)$ and $\Omega^1(X_2)$.

Compatibility of $\omega_1 \in \Omega^1(X_1^f)$ and $\omega_2 \in \Omega^1(X_2)$ The notion of compatibility admits an obvious extension to the case of a 1-form $\omega_1^f \in \Omega^1(X_1^f)$ and a 1-form $\omega_2 \in \Omega^1(X_2)$. This notion is the same as the f-compatibility, but considered with respect to $f_\sim$. Specifically, ω_1^f and ω_2 are said to be **$f_\sim$-compatible** if for every plot p_1^f of $Y^f = \pi_1^f(Y)$, considered with the subset diffeology relative to the inclusion $Y^f \subseteq X_1^f$ we have

$$\omega_1^f(p_1^f) = \omega_2(f_\sim \circ p_1^f).$$

We then easily obtain the following.

[10] At least one lift always exists, by the properties of the quotient diffeology.

Lemma 4.3. *The forms* ω_1^f *and* ω_2 *are* $f_\sim$*-compatible if and only if* $\omega_1 = (\pi_1^f)^*(\omega_1^f)$ *and* ω_2 *are* f*-compatible.*

Proof. Let $p_1^f : U \to X_1^f$ be a plot of X_1^f, and let $\{p_1^{(i)}\}$ be the collection of all its lifts to X_1; this collection is then an equivalence class by f-equivalence, and moreover, we always have

$$f_\sim \circ p_1^f = f \circ p_1^{(i)}.$$

This ensures that the equalities $\omega_1^f(p_1^f) = \omega_2(f \circ p_1^{(i)})$ and $\omega_1(p_1^{(i)}) = \omega_2(f \circ p_1^{(i)})$ hold simultaneously, whence the claim. □

The diffeomorphism $\Omega^1(X_1 \cup_f X_2) \cong \Omega^1(X_1^f) \times_{comp} \Omega^1(X_2)$ The lemma just proven, together with Proposition 4.2, trivially imply that

$$\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2) \cong \Omega^1(X_1^f) \times_{comp} \Omega^1(X_2).$$

This, together with Corollary 3.10 (and Theorem 3.9), yields immediately the following.

Proposition 4.4. *There is a natural diffeomorphism*

$$\Omega^1(X_1 \cup_f X_2) \cong \Omega^1(X_1^f) \times_{comp} \Omega^1(X_2),$$

that filters through the pullback map π^*.

Notice that this diffeomorphism is given by

$$((\pi_1^f)^* \times \mathrm{Id}_{X_2}^*) \circ [\Omega^1(X_1 \sqcup X_2) \to \Omega^1(X_1) \times \Omega^1(X_2)] \circ \pi^*.$$

4.4 The surjectivity of the projections to $\Omega_f^1(X_1)$ and $\Omega^1(X_2)$

We are now ready to prove the final statement of this section, that is, that the natural projections of $\Omega^1(X_1 \cup_f X_2)$ to the factors of $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$ are surjective, if f is assumed to be a subduction onto its image. More precisely, the projection onto $\Omega^1(X_2)$ is surjective even without this assumption (see Section 3.2.1). It remains to establish the claim for $\Omega_f^1(X_1)$.

4.4.1 The space $\Omega^1(X_1^f) \times_{comp} \Omega^1(X_2)$ is a sub-direct product

The statement in the title of the section is the main argument in the proof that $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$ is a sub-direct product of its factors; and the statement itself easily follows from what we have already established. Indeed, $f_\sim$ being a diffeomorphism with its image immediately implies the following fact.

Lemma 4.5. *The map* $i_1^{X_1^f} : X_1^f \to X_1^f \cup_{f_\sim} X_2$ *given by the composition of the natural inclusion* $X_1^f \hookrightarrow X_1^f \sqcup X_2$ *with the quotient projection* $\pi^f : X_1^f \sqcup X_2 \to X_1^f \cup_{f_\sim} X_2$ *is an induction.*

Also immediate from this lemma is the following.

Corollary 4.6. *The pullback map* $(i_1^{X_1^f})^* : \Omega^1(X_1^f \cup_{f_\sim} X_2) \to \Omega^1(X_1^f)$ *is surjective.*

It thus remains to notice that $(i_1^{X_1^f})^*$ is precisely the composition of the diffeomorphism $\Omega^1(X_1^f \cup_{f_\sim} X_2) \cong \Omega^1(X_1^f) \times_{comp} \Omega^1(X_2)$ with the projection of the latter onto its first factor.

4.4.2 The projection $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2) \to \Omega_f^1(X_1)$ is surjective

We now collect everything together to obtain the final statement of this section. (We give the statement for both projections at the same time).

Theorem 4.7. *Suppose that* f *is a subduction onto its image; let* $i_j : X_j \hookrightarrow (X_1 \sqcup X_2)$ *for* $j = 1, 2$ *be the natural inclusions. Then the maps* $i_1^* \circ \pi^* : \Omega^1(X_1 \cup_f X_2) \to \Omega_f^1(X_1)$ *and* $i_2^* \circ \pi^* : \Omega^1(X_1 \cup_f X_2) \to \Omega^1(X_2)$ *are surjective.*

Proof. It suffices to consider $\Omega^1(X_1 \cup_f X_2)$ as $\Omega^1(X_1^f \cup_{f_\sim} X_2) \cong \Omega^1(X_1^f) \times_{comp} \Omega^1(X_2)$, to identify the latter space with $\Omega_f^1(X_1) \times_{comp} \Omega^1(X_2)$ via the diffeomorphism $(\pi_1^f)^* \times \mathrm{Id}_{X_2}^*$, and deduce the claim from Corollary 4.6. □

5 Vanishing 1-forms and the pullback map π^*

Recall that each fibre of the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$ is the quotient over the subspace of forms vanishing at the given point. In this section we consider how the vanishing of forms interacts with the pullback map π^*.

Let $x \in X_1 \cup_f X_2$, and let $\omega \in \Omega^1(X_1 \cup_f X_2)$. Recall that ω vanishes at x if for every plot $p : U \to X_1 \cup_f X_2$ such that $U \ni 0$ and $p(0) = x$ we have $\omega(p)(0) = 0$. Let us consider the pullback form $\pi^*(\omega)$, written as $\pi^*(\omega) = (\omega_1, \omega_2)$; the fact that ω vanishes at some point x might then imply that either ω_1 or ω_2, or both, vanish at one of, or all, lifts of x; and going still furthermore, some kind of a reverse of this statement might hold. Below we discuss precisely this kind of question, concentrating on the structure of the pullback of a form on $X_1 \cup_f X_2$ vanishing at some x. Three cases arise there, that depend on the nature of x.

5.1 The pullbacks of forms on $X_1 \cup_f X_2$ vanishing at a point

Let $x \in X_1 \cup_f X_2$, and let $\omega \in \Omega^1(X_1 \cup_f X_2)$ be a form vanishing at x. It is quite obvious then (but worth stating anyhow) that the pullback of ω vanishes at any lift of this point.

Lemma 5.1. *Let $x \in X_1 \cup_f X_2$, let $\omega \in \Omega^1(X_1 \cup_f X_2)$ be a form vanishing at x, and let $\pi^*(\omega) = (\omega_1, \omega_2)$. Let $\tilde{x} \in X_i$ be such that $\pi(\tilde{x}) = x$. Then the corresponding ω_i vanishes at $\tilde{x}$.*

Proof. Let $p_i : U \to X_i$ be a plot centered at $\tilde{x}$; then obviously, $\pi \circ p_i$ is a plot of $X_1 \cup_f X_2$ centered at x. Furthermore, $\omega_i(p)(0) = \omega(\pi \circ p)(0) = 0$, since ω vanishes at x. □

Let us consider the implications of this lemma. Note first of all that at this point it is convenient to consider X_1^f instead of X_1, identifying $\Omega_f^1(X_1)$ with $\Omega^1(X_1)$ and, whenever it is convenient, the space $X_1 \cup_f X_2$ with $X_1^f \cup_{f_\sim} X_2$. Recall that π^f stands for the obvious projection $X_1 \sqcup X_2 \to X_1^f \cup_{f_\sim} X_2$, and let $x \in X_1^f \cup_{f_\sim} X_2$; there are three cases (two of which are quite similar).

If $x \in i_1(X_1 \setminus Y)$ then it has a unique lift, both with respect to π^f and with respect to π; this lift furthermore is contained in X_1^f and X_1 respectively. Therefore

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) \cong (\pi^f)^*(\Omega_x^1(X_1^f \cup_{f_\sim} X_2)) \subseteq$$
$$\subseteq \Omega^1_{(\pi^f)^{-1}(x)}(X_1^f) \times_{comp} \Omega^1(X_2) \cong (\Omega_f^1)_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2).$$

The case when $x \in i_2(X_2 \setminus f(Y))$ is similar; the lift of x is also unique then, and belongs to $X_2 \setminus f(Y)$. We thus have a similar sequence of inclusions:

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) \cong (\pi^f)^*(\Omega_x^1(X_1^f \cup_{f_\sim} X_2)) \subseteq$$
$$\subseteq \Omega^1(X_1^f) \times_{comp} \Omega^1_{(\pi^f)^{-1}(x)}(X_2) \cong \Omega_f^1(X_1) \times_{comp} \Omega^1_{\pi^{-1}(x)}(X_2).$$

Now, in the third case, which is the one of $x \in \pi^f(Y)$, it admits precisely two lifts via π^f, one to a point $\tilde{x}_1 \in X_1^f$, the other to a point $\tilde{x}_2 \in X_2$. By Lemma 5.1,

$$\pi^*(\Omega_x^1(X_1 \cup_f X_2)) \cong (\pi^f)^*(\Omega_x^1(X_1^f \cup_{f_\sim} X_2)) \subseteq \Omega^1_{\tilde{x}_1}(X_1^f) \times_{comp} \Omega^1_{\tilde{x}_2}(X_2);$$

note that in this case, $\tilde{x_1}$, which is a point of X_1^f, may have multiple (possibly infinitely many) lifts to X_1.

5.2 Classification of pullback spaces according to the point of vanishing

The discussion carried out in the section immediately above allows us to obtain the following statement.

Proposition 5.2. *Let $x \in X_1 \cup_f X_2$. Then:*

1. *If* $x \in i_1(X_1 \setminus Y)$ *then*

$$\pi^*(\Omega^1_x(X_1 \cup_f X_2)) \subseteq (\Omega^1_f)_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2).$$

2. *If* $x \in i_2(X_2 \setminus f(Y))$ *then*

$$\pi^*(\Omega^1_x(X_1 \cup_f X_2)) \subseteq \Omega^1_f(X_1) \times_{comp} \Omega^1_{\pi^{-1}(x)}(X_2).$$

3. *If* $x \in \pi(Y) = i_2(f(Y))$*, and* $\tilde{x}_1 \in X_1^f$ *and* $\tilde{x}_2 \in X_2$ *are the two points in* $(\pi^f)^{-1}(x)$*, then*

$$(\pi^f)^*(\Omega^1_x(X_1 \cup_f X_2)) \subseteq \Omega^1_{\tilde{x}_1}(X_1^f) \times_{comp} \Omega^1_{\tilde{x}_2}(X_2).$$

The questions that arise now are, whether any, or all, of the three inclusions are actually identities, and, for the third item, how the space $\Omega^1_{\tilde{x}_1}(X_1^f)$ is related to one or more spaces of vanishing f-compatible forms on X_1.

5.3 The reverse inclusion for points in $i_2(X_2 \setminus f(Y))$ and $i_1(X_1 \setminus Y)$

This follows from a rather simple observation. If $\omega_2 \in \Omega^1(X_2)$ is a form that also belongs to the image of the projection $\Omega^1_f(X_1) \times_{comp} \Omega^1(X_2) \to \Omega^1(X_2)$ (that is, there exists an f-compatible form ω_1 on X_1 such that ω_1 and ω_2 are compatible between them), and ω_2 vanishes at some point x_2, then any form on $X_1 \cup_f X_2$ to which ω_2 projects, also vanishes, at the point of $X_1 \cup_f X_2$ that corresponds to x_2.

Lemma 5.3. *Let* $(\omega_1, \omega_2) \in \Omega^1_f(X_1) \times_{comp} \Omega^1(X_2)$*, and let* $\tilde{x} \in X_2$ *be such that* ω_2 *vanishes at* $\tilde{x}$*. Then* $\omega_1 \cup_f \omega_2$ *vanishes at* $x := \pi(\tilde{x})$.

Proof. Let $p : U \to X_1 \cup_f X_2$ be a plot centered at x. As we have noted above, $\tilde{x}$ is the only lift of x to X_2 (although it may have lifts to X_1 as well), and any lift of p to a plot of X_2 is centered at $\tilde{x}$. Note also that at least one such lift exists, by definition of a pushforward diffeology and the disjoint union diffeology on $X_1 \sqcup X_2$. It remains to observe that if p_2 is such a lift then by construction $(\omega_1 \cup_f \omega_2)(p)(0) = \omega_2(p_2)(0) = 0$, whence the claim. ◻

This lemma, together with the second point of Proposition 5.2, immediately implies the following.

Corollary 5.4. *If* $x \in i_2(X_2 \setminus f(Y))$ *then*

$$\pi^*(\Omega^1_x(X_1 \cup_f X_2)) = \Omega^1_f(X_1) \times_{comp} \Omega^1_{\pi^{-1}(x)}(X_2).$$

The case of a point in $i_1(X_1 \setminus Y)$ is completely analogous to that of a point in $i_2(X_2 \setminus f(Y))$, since the main argument is based on the same property, that of there being a unique lift of the point of vanishing. We therefore immediately state the final conclusion.

Corollary 5.5. *If* $x \in i_1(X_1 \setminus Y)$ *then*

$$\pi^*(\Omega^1_x(X_1 \cup_f X_2)) = (\Omega^1_f)_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2).$$

5.4 The case of points in $\pi(Y) = i_2(f(Y))$

For points such as these, we already have the inclusion $(\pi^f)^*(\Omega^1_x(X_1 \cup_f X_2)) \subseteq \Omega^1_{\tilde{x}_1}(X_1^f) \times_{comp} \Omega^1_{\tilde{x}_2}(X_2)$. The questions to consider now are, first, whether it is actually an identity, and then, how the right-hand side is related to one or more subspaces of f-invariant forms on X_1 vanishing at points in the lift of x.

5.4.1 The reverse inclusion $(\pi^f)^*(\Omega^1_x(X_1\cup_f X_2))\supseteq\Omega^1_{\tilde{x}_1}(X_1^f)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)$

Let $x\in X_1\cup_f X_2$, and let $\tilde{x}_1\in X_1^f$, $\tilde{x}_2\in X_2$ be such that $\pi^f(\tilde{x}_i)=x$, which is equivalent to $f_\sim(\tilde{x}_1)=\tilde{x}_2$. Let $\omega_1^f\in\Omega^1_{\tilde{x}_1}(X_1^f)$ and $\omega_2\in\Omega^1_{\tilde{x}_2}(X_2)$ be 1-forms compatible with $f_\sim$. Consider $((\pi^f)^*)^{-1}(\omega_1^f,\omega_2)=\omega_1^f\cup_{f_\sim}\omega_2$;

Lemma 5.6. *The form $((\pi^f)^*)^{-1}(\omega_1^f,\omega_2)$ vanishes at x.*

Proof. Let $p:U\to X_1\cup X_2$ be a plot centered at x; assume U to be connected. Then p lifts to a plot p_i of X_i. Suppose it lifts to a plot p_2 of X_2; since the lift of x to X_2 is unique, it has to be $\tilde{x}_2$, which implies that p_2 is centered at $\tilde{x}_2$, and therefore

$$(((\pi^f)^*)^{-1}(\omega_1^f,\omega_2))(p)(0)=\omega_2(p_2)(0)=0.$$

Assume now that p lifts to a plot p_1 of X_1. Notice that $\pi^f\circ p_1$ is a plot of X_1^f, and it is centered at $\tilde{x}_1$, since the lift of x to X_1^f. Thus, we have again

$$(((\pi^f)^*)^{-1}(\omega_1^f,\omega_2))(p)(0)=\omega_1^f(p_1)(0)=0,$$

and the lemma is proven. □

Corollary 5.7. *For x, $\tilde{x}_1$, and $\tilde{x}_2$ as above, we have*

$$(\pi^f)^*(\Omega^1_x(X_1\cup_f X_2))=\Omega^1_{\tilde{x}_1}(X_1^f)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2).$$

Let us now turn to the relation of the space $\Omega^1_{\tilde{x}_1}(X_1^f)$ to the subspaces of vanishing forms in $\Omega^1_f(X_1)$. Recall that we have already established the diffeomorphism of $\Omega^1_f(X_1)$ and $\Omega^1(X_1^f)$, so we are essentially asking, what becomes of subspaces of forms vanishing at a given point of X_1^f under this diffeomorphism (the pullback map $(\pi_1^f)^*$).

5.4.2 The pullback space $(\pi_1^f)^*(\Omega^1_y(X_1^f))$ for $y\in\pi_1^f(Y)$

Fix a point $y\in\pi_1^f(Y)$; let first $\omega\in\Omega^1_y(X_1^f)$ be a form vanishing at y. By the argument identical to that in the proof of Lemma 4.1, the pullback form $(\pi_1^f)^*(\omega)\in\Omega^1_f(X_1)$ vanishes at any lift of y. Indeed, if $\tilde{y}\in Y\subset X_1$ is such that $\pi_1^f(\tilde{y})=y$, and $p:U\to X_1$ is a plot centered at $\tilde{y}$, then $\pi_1^f\circ p$ is a plot centered at y, and

$$((\pi_1^f)^*(\omega))(p)(0)=\omega(\pi_1^f\circ p)(0)=0.$$

Thus, we obtain the following.

Proposition 5.8. *For any $y\in\pi_1^f(Y)\subset X_1^f$ we have*

$$(\pi_1^f)^*(\Omega^1_y(X_1^f))=\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega^1_f)_{\tilde{y}}(X_1).$$

Proof. The inclusion $(\pi_1^f)^*(\Omega^1_y(X_1^f))\subseteq\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega^1_f)_y(X_1)$ has been proven immediately prior to the statement of the proposition, so it suffices to establish the reverse inclusion. This follows from the definition of the inverse of $(\pi_1^f)^*$. More precisely, suppose that $\omega_1\in\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega^1_f)_y(X_1)$; this means that ω_1 vanishes at every point $\tilde{y}\in X_1$ such that $\pi_1^f(\tilde{y})=y$, which in turn means that for every plot p_1 centered at any such point we have $\omega_1(p_1)(0)=0$. Let ω_1^f be the pushforward of the form ω_1 to X_1^f, that is, $\omega_1^f=((\pi_1^f)^*)^{-1}(\omega_1)$; let p_1^f be any plot of X_1^f centered at y, and let p_1 be a lift of p_1^f to a plot of X_1 (such a lift exists by the definition of the pushforward diffeology), $p_1^f=\pi_1^f\circ p_1$. Notice that p_1^f is centered at some $\tilde{y}$ such that $\pi_1^f(\tilde{y})=y$, and ω_1 vanishes at all such points, therefore $\omega_1(p_1)(0)=0$. Finally, recall that $\omega_1^f(p_1^f)=\omega_1(p_1)$ for any lift p_1 of the plot p_1^f. This allows us to conclude that $\omega_1^f(p_1^f)(0)=\omega_1(p_1)(0)$, and since p_1^f is arbitrary, we further conclude that ω_1^f vanishes at y, as we wanted. □

5.4.3 The subspace $\left(\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega_f^1)_{\tilde{y}}(X_1)\right)\times_{comp}\Omega^1_{\tilde{y}_2}(X_2)$

Here $\tilde{y}_2$ is *the* point of X_2 such that $f(\tilde{y})=\tilde{y}_2$. The structure of the subspace in question depends on whether f is a subduction; we will assume that it is. Recall that, as has been established in the previous section, this ensures that the natural projections $\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1_f(X_1)$ and $\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_2)$ are both surjective. We now need to see whether this holds for subspaces of vanishing forms; to avail ourselves of the tools used previously, we first consider the interaction between the vanishing of forms and the f-invariance. Here is what we mean.

Projection to $\Omega^1_{\tilde{y}_2}(X_2)$ Let us now use the above construction to show that the surjectivity of the projection is preserved for the subspaces of vanishing forms. We start from the second factor; as before, it is the easier case.

Proposition 5.9. *Let $y\in\pi_1^f(Y)$ be any point, and let $\tilde{y}_2\in X_2\cap(\pi^f)^{-1}(y)$. Then the projection $\left(\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega_f^1)_{\tilde{y}}(X_1)\right)\times_{comp}\Omega^1_{\tilde{y}_2}(X_2)\to\Omega^1_{\tilde{y}_2}(X_2)$ is surjective.*

Proof. By Proposition 5.8 it suffices to show that $(\pi_1^f\times\mathrm{Id}_{X_2})^*(\Omega^1_{\pi_1^f(y)}(X_1^f\cup_{f_\sim}X_2))$ is a sub-direct product of $\Omega^1_y(X_1^f)$ and of $\Omega^1_{\tilde{y}_2}(X_2)$, and this follows, again, from $X_1^f\hookrightarrow X_1^f\cup_{f_\sim}X_2$ and $X_2\hookrightarrow X_1^f\cup_{f_\sim}X_2$ being inductions. □

Projection to $\left(\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega_f^1)_{\tilde{y}}(X_1)\right)$ This has just been proven together with the case of the other factor.

Proposition 5.10. *Let f be a subduction, let $y\in\pi_1^f(Y)$ be any point, and let $\tilde{y}_2\in X_2\cap(\pi^f)^{-1}(y)$. Then the projection $\left(\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega_f^1)_{\tilde{y}}(X_1)\right)\times_{comp}\Omega^1_{\tilde{y}_2}(X_2)\to\left(\cap_{\tilde{y}\in(\pi_1^f)^{-1}(y)}(\Omega_f^1)_{\tilde{y}}(X_1)\right)$ is surjective.*

5.5 The pullbacks of the spaces of vanishing forms: summary

We collect here the final conclusions of this section regarding the image of the space $\Omega^1_x(X_1\cup_f X_2)$ under the pullback map $\pi^*:\Omega^1(X_1\cup_f X_2)\to\Omega^1_f(X_1)\times_{comp}\Omega^1(X_2)$.

Theorem 5.11. *Let f be a subduction, and let $x\in X_1\cup_f X_2$. Then the following is true:*

1. *If $x\in i_1(X_1\setminus Y)$ then*
$$\pi^*(\Omega^1_x(X_1\cup_f X_2))=(\Omega^1_f)_{\pi^{-1}(x)}(X_1)\times_{comp}\Omega^1(X_2).$$
2. *If $x\in i_2(X_2\setminus f(Y))$ then*
$$\pi^*(\Omega^1_x(X_1\cup_f X_2))=\Omega^1_f(X_1)\times_{comp}\Omega^1_{\pi^{-1}(x)}(X_2).$$
3. *If $x\in i_2(f(Y))$ then*
$$\pi^*(\Omega^1_x(X_1\cup_f X_2))=\left(\cap_{\tilde{x}\in(\pi_1^f)^{-1}(x)}(\Omega_f^1)_{\tilde{x}}(X_1)\right)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2),$$
where $\tilde{x}_2$ is such that $i_2(\tilde{x}_2)=x$.

6 The pseudo-bundle $\Lambda^1(X_1\cup_f X_2)$ when f is a diffeomorphism with its image

We now turn to the pseudo-bundle $\Lambda^1(X_1\cup_f X_2)$ of the values of differential 1-forms on $X_1\cup_f X_2$. We actually limit ourselves to the case when the gluing map is a diffeomorphism with its image, one, because it is the easiest case (in particular, we can dispense with the f-invariance), and two, because the relative conclusions do apply to the general case, but they will be stated in terms of spaces X_1^f and X_2, and their gluing along $f_\sim$ (from which one may hope, justifiably or not, to draw some conclusions in terms of the initial X_1, X_2, and f).

6.1 The fibres of $\Lambda^1(X_1 \cup_f X_2)$

Here we describe the fibres of $\Lambda^1(X_1 \cup_f X_2)$, *i.e.*, the spaces $\Lambda^1_x(X_1 \cup_f X_2)$, in terms of the spaces $\Lambda^1_{x_1}(X_1)$ and $\Lambda^1_{x_2}(X_2)$ for $x_i \in X_i$ such that $x_i \in \pi^{-1}(x)$.

6.1.1 The fibre $\Lambda^1_x(X_1 \cup_f X_2)$ for $x \in i_1(X_1 \setminus Y) \cup i_2(X_2 \setminus f(Y))$

We first consider the fibres of $\Lambda^1(X_1 \cup_f X_2)$ in the case of $x \in i_1(X_1 \setminus Y)$ and that of $x \in i_2(X_2 \setminus f(Y))$. These two cases are rather straightforward because these are precisely the cases where the point x has a unique lift to $X_1 \sqcup X_2$, and the answer is as follows.

Theorem 6.1. *Let f be a diffeomorphism with its image, and $x \in X_1 \cup_f X_2$ such that $x \notin i_2(f(Y))$. Then*

1. *If $x \in i_1(X_1 \setminus Y)$ then*
$$\Lambda^1_x(X_1 \cup_f X_2) \cong \Lambda^1_{x_1}(X_1), \text{ where } x_1 = \pi^{-1}(x).$$
2. *If $x \in i_2(X_2 \setminus f(Y))$ then*
$$\Lambda^1_x(X_1 \cup_f X_2) \cong \Lambda^1_{x_2}(X_2), \text{ where } x_2 = \pi^{-1}(x).$$

Both diffeomorphisms are induced by the pullback map π^*.

Recall that the f being a diffeomorphism with its image means in particular that $\Omega^1_f(X_1) = \Omega^1(X_1)$, *i.e.* any form is f-invariant.

Proof. Let first $x \in i_1(X_1 \setminus Y)$. Then, by Theorem 5.11 we have $\pi^*(\Omega^1_x(X_1 \cup_f X_2)) = \Omega^1_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2)$, which is a subset in $\Omega^1(X_1) \times_{comp} \Omega^1(X_2) = \pi^*(\Omega^1(X_1 \cup_f X_2))$. Thus, π^* lifts, first of all, to a well-defined map on $\Lambda_x(X_1 \cup_f X_2)$, whose image is the quotient

$$\left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right) / \left(\Omega^1_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2)\right).$$

We need to verify that $\Lambda^1_x(X_1)$ is diffeomorphic to this quotient. Let $\tilde{\rho}_1^\Lambda$ be the map induced by the projection ρ_1 of $\Omega^1(X_1) \times_{comp} \Omega^1(X_2)$ to the first factor, *i.e.*, to $\Omega^1(X_1)$. This means that, if $\lambda^x_\cup$ is the quotient projection $\Omega^1(X_1) \times_{comp} \Omega^1(X_2) \to \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right) / \left(\Omega^1_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2)\right)$, and λ^1_x is the quotient projection $\Omega^1(X_1) \to \Omega^1(X_1)/\Omega^1_{\pi^{-1}(x)}(X_1)$, then we have

$$\tilde{\rho}_1^\Lambda \circ \lambda^x_\cup = \lambda^1_x \circ \rho_1.$$

It is clear that $\tilde{\rho}_1^\Lambda$ is a smooth linear map; furthermore, the fact that $\left(\Omega^1_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2)\right)$ is a sub-direct product ensures that its image is indeed $\Lambda^1_x(X_1)$. It remains to see that it has trivial kernel. To do so, let (ω_1, ω_2) be a representative of an equivalence class, in the quotient

$$\left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right) / \left(\Omega^1_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2)\right)$$

belonging to the kernel of $\tilde{\rho}_1^\Lambda$. It suffices to observe that ω_1 belongs to the kernel of λ^1_x. Since this kernel is composed precisely of the forms vanishing at $\pi_1^{-1}(x)$, we have $\omega_1 \in \Omega^1_{\pi^{-1}(x)}(X_1)$ and, since ω_2 has to be compatible with it, the pair (ω_1, ω_2) belongs to $\Omega^1_{\pi^{-1}(x)}(X_1) \times_{comp} \Omega^1(X_2)$, which proves the first part of the statement.

The proof of the second part is completely analogous, but let us go over it briefly. Let $x \in i_2(X_2 \setminus f(Y))$. We observe that also in this case (and for all the same reasons) the pullback map π^* lifts a map $\lambda^x_\cup$ on $\Lambda_x(X_1 \cup_f X_2)$, whose image is the quotient $\left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right) / \left(\Omega^1(X_1) \times_{comp} \Omega^1_{\pi^{-1}(x)}(X_2)\right)$; we need to show that this quotient is diffeomorphic to $\Lambda^1_x(X_2)$.

Similar to the case of the first factor, let us denote by ρ_2 the projection onto the second factor of $\Omega^1(X_1)\times_{comp}\Omega^1(X_2)$, that is,

$$\rho_2:\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_2),$$

and by λ_x^2 the defining projection of $\Lambda^1_{\pi^{-1}(x)}(X_2)$, that is,

$$\lambda_x^2:\Omega^1(X_2)\to\Omega^1(X_2)/\Omega^1_{\pi^{-1}(x)}(X_2)=\Lambda^1_{\pi^{-1}(x)}(X_2).$$

Then there is a smooth linear map

$$\tilde{\rho}_2^{\Lambda}:\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1(X_1)\times_{comp}\Omega^1_{\pi^{-1}(x)}(X_2)\right)\to\Lambda^1_{\pi^{-1}(x)}(X_2)$$

such that $\tilde{\rho}_2^{\Lambda}\circ\lambda_{\cup}^x=\lambda_x^2\circ\rho_2$; this map is the pushforward of the map ρ_2 by the other two maps. It is well-defined and surjective, since both spaces $\Omega^1(X_1)\times_{comp}\Omega^1(X_2)$ and $\Omega^1(X_1)\times_{comp}\Omega^1_{\pi^{-1}(x)}(X_2)$ are sub-direct products (of their factors). It then suffices to verify that it has trivial kernel, which follows from the equality $\tilde{\rho}_2^{\Lambda}\circ\lambda_{\cup}^x=\lambda_x^2\circ\rho_2$ (the fact $\tilde{\rho}_2^{\Lambda}$ is a smooth linear bijection implies that it also has a smooth inverse due to the properties of the diffeologies involved; recall that both are pushforward diffeologies). ☐

The essence of Theorem 5.11 be summarized by saying that over the points of $X_1\cup_f X_2$ that admit a unique lift to $X_1\sqcup X_2$, the fibre of the bundle $\Lambda^1(X_1\cup_f X_2)$ coincides with a fibre of either $\Lambda^1(X_1)$ or $\Lambda^1(X_2)$, as appropriate.

6.1.2 The fibre $\Lambda^1_x(X_1\cup_f X_2)$ for $x\in i_2(f(Y))$

Then the quotient that we are now interested in is the following one:

$$\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_{\tilde{x}_1}(X_1)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)\right);$$

This is the space that we denote by

$$\Lambda^1_{x_1}(X_1)\times_{comp}\Lambda^1_{x_2}(X_2)$$

we wish to relate it to the spaces $\Lambda^1_{\tilde{x}_1}(X_1)$ and $\Lambda^1_{\tilde{x}_2}(X_2)$. More precisely, we establish the following.

Lemma 6.2. *Let f be a diffeomorphism with image, let $\tilde{x}_2\in f(Y)\subset X_2$, and let $\tilde{x}_1=f^{-1}(\tilde{x}_2)$. Then*

$$\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_{\tilde{x}_1}(X_1)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)\right)$$

is diffeomorphic to a sub-direct product of $\Lambda^1_{\tilde{x}_1}(X_1)$ and $\Lambda^1_{\tilde{x}_2}(X_2)$.

Proof. The proof consists of two parts. We first show that there is a natural embedding of the quotient $\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_{\tilde{x}_1}(X_1)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)\right)$ into the direct product $\Lambda^1_{\tilde{x}_1}(X_1)\times\Lambda^1_{\tilde{x}_2}(X_2)$, and then, that the restrictions of the two direct product projection (to the embedded quotient) are surjections onto the factors $\Lambda^1_{\tilde{x}_1}(X_1)$ and $\Lambda^1_{\tilde{x}_2}(X_2)$.

Let $(\omega_1,\omega_2),(\omega_1',\omega_2')\in\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)$ be two pairs that represent the same coset of $\Omega^1_{\tilde{x}_1}(X_1)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)$. Since the latter is a subdirect product of its factors, this is equivalent to the following two conditions: ω_1 and ω_1' representing the same coset of $\Omega^1_{\tilde{x}_1}(X_1)$ in $\Omega^1(X_1)$, and ω_2 and ω_2' representing the same coset of $\Omega^1_{\tilde{x}_2}(X_2)$ in $\Omega^1(X_2)$. Thus, the pairs (ω_1,ω_2) and (ω_1',ω_2') represent the same element of $\Lambda^1_{\tilde{x}_1}(X_1)\times\Lambda^1_{\tilde{x}_2}(X_2)$. This implies that the assignment

$$\Omega^1(X_1)\ni\omega_1\mapsto[\omega_1]\in\Lambda^1_{\tilde{x}_1}(X_1),\quad\Omega^1(X_2)\ni\omega_2\mapsto[\omega_2]\in\Lambda^1_{\tilde{x}_2}(X_2)$$

yields a well-defined smooth linear map

$$\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_{\tilde{x}_1}(X_1)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)\right)\to\Lambda^1_{\tilde{x}_1}(X_1)\times\Lambda^1_{\tilde{x}_2}(X_2)$$

acting by $(\omega_1,\omega_2)\mapsto([\omega_1],[\omega_2])$. It now remains to observe that its compositions with the projections $\Lambda^1_{\tilde{x}_1}(X_1)\times\Lambda^1_{\tilde{x}_2}(X_2)\to\Lambda^1_{\tilde{x}_1}(X_1)$ and $\Lambda^1_{\tilde{x}_1}(X_1)\times\Lambda^1_{\tilde{x}_2}(X_2)\to\Lambda^1_{\tilde{x}_2}(X_2)$ are both surjective maps, and this, again, follows from the spaces $\Omega^1(X_1)\times_{comp}\Omega^1(X_2)$ and $\Omega^1_{\tilde{x}_1}(X_1)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2)$ being sub-direct products of their respective factors. ☐

Corollary 6.3. *Under the assumptions of the above lemma, we have a diffeomorphism*

$$\Lambda^1_x(X_1 \cup_f X_2) \cong \Lambda^1_{\tilde{x}_1}(X_1) \times_{comp} \Lambda^1_{\tilde{x}_2}(X_2).$$

Notice that the above statement could be seen as making a reference to an independent notion of compatibility for elements of $\Lambda^1_{x_1}(X_1)$ and $\Lambda^1_{x_2}(X_2)$. The appropriate definition is as follows.

Definition 6.4. *Let $[\omega_i] \in \Lambda^1_{x_i}(X_i)$ for $i = 1, 2$. We say that $[\omega_1]$ and $[\omega_2]$ are **compatible** if for any $\omega_i' \in [\omega_i]$ the forms ω_1' and ω_2' are compatible.*

6.2 Re-interpreting the compatibility

We now describe another way of viewing the compatibility, or, more precisely, the space of compatible pairs of 1-forms. The map f has the pullback map $f^* : \Omega^1(f(Y)) \to \Omega^1(Y)$, so the spaces $\Omega^1(f(Y))$ and $\Omega^1(Y)$ serve as the projective images of the spaces $\Omega^1(X_2)$ and $\Omega^1(X_1)$ respectively. Indeed, let $\alpha_1 : Y \hookrightarrow X_1$ and $\alpha_2 : f(Y) \hookrightarrow X_2$ be the natural inclusions; the projections just mentioned are their pullbacks:

$$\alpha_2^* : \Omega^1(X_2) \to \Omega^1(f(Y)) \quad \text{and} \quad \alpha_1^* : \Omega^1(X_1) \to \Omega^1(Y).$$

What we then have is the following.

Proposition 6.5. *Let $\omega_i \in \Omega^1(X_i)$ for $i = 1, 2$. Then ω_1 and ω_2 are compatible if and only we have*

$$f^*(\alpha_2^*(\omega_2)) = \alpha_1^*(\omega_1).$$

Proof. Suppose first that ω_1 and ω_2 are compatible; consider $f^*(\alpha_2^*(\omega_2)), \alpha_1^*(\omega_1) \in \Omega^1(Y)$. Let $p : U \to Y$ be a plot for the subset diffeology of Y; then

$$\alpha_1^*(\omega_1)(p) = \omega_1(\alpha_1 \circ p) = \omega_1(p),$$

where we identify the plot p with $\alpha_1 \circ p$, as is typical for the plots in a subset diffeology. Likewise,

$$f^*(\alpha_2^*(\omega_2))(p) = \alpha_2^*(\omega_2)(f \circ p) = \omega_2(\alpha_2 \circ (f \circ p)) = \omega_2(f \circ p),$$

where again we identify $\alpha_2 \circ (f \circ p)$ and $f \circ p$. By the compatibility of ω_1 and ω_2, we have that $\omega_1(p) = \omega_2(f \circ p)$, which implies that $f^*(\alpha_2^*(\omega_2))(p) = \alpha_1^*(\omega_1)(p)$ for all plots in the subset diffeology of Y; this means precisely that $f^*(\alpha_2^*(\omega_2))$ and $\alpha_1^*(\omega_1)$ are equal as forms in $\Omega^1(Y)$.

The *vice versa* of this statement is obtained from the same two equalities, by assuming first that $f^*(\alpha_2^*(\omega_2))(p) = \alpha_1^*(\omega_1)(p)$ for all p and concluding that then also $\omega_1(p) = \omega_2(f \circ p)$, which is the condition for the compatibility of forms ω_1 and ω_2. □

The proposition just proven allows us to give some alternative viewpoints on the subspace $\Omega^1(X_1) \times_{comp} \Omega^1(X_2) \cong \pi^*(X_1 \cup_f X_2)$ (recall that formally, $\pi^*(\Omega^1(X_1 \cup_f X_2))$ is contained in $\Omega^1(X_1 \sqcup X_2)$) of compatible forms. The main one is contained in the following statement.

Corollary 6.6. *Let $(\alpha_1^*, f^* \circ \alpha_2^*) : \Omega^1(X_1) \times \Omega^1(X_2) \to \Omega^1(Y) \times \Omega^1(Y)$ be the direct product map. Then*

$$\pi^*(\Omega^1(X_1 \cup_f X_2)) \cong \Omega^1(X_1) \times_{comp} \Omega^1(X_2) = (\alpha_1^*, f^* \circ \alpha_2^*)^{-1} \left(diag(\Omega^1(Y) \times \Omega^1(Y)) \right).$$

The space $\mathrm{diag}(\Omega^1(Y) \times \Omega^1(Y))$ is the usual diagonal of the direct product $\Omega^1(Y) \times \Omega^1(Y)$ and is trivially identified with just $\Omega^1(Y)$.

6.3 Canonical decomposition of $\Lambda^1(X_1 \cup_f X_2)$

The fibrewise description of the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$ suggests that it has a natural decomposition into three pieces, two of which are essentially portions of the pseudo-bundles $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$, and the third is constructed from the two as a subset of their fibrewise direct product. This decomposition is in accordance with the decomposition of the base space $X_1 \cup_f X_2$ as the disjoint union

$$i_1(X_1 \setminus Y) \sqcup i_2(X_2 \setminus f(Y)) \sqcup i_2(f(Y)),$$

and is denoted as follows:

$$\begin{cases} \Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)} & = & \bigcup_{x\in i_1(X_1\setminus Y)}\Lambda^1_x(X_1\cup_f X_2), \\ \Lambda^1(X_1\cup_f X_2)|_{i_2(X_2\setminus f(Y))} & = & \bigcup_{x\in i_2(X_2\setminus f(Y))}\Lambda^1_x(X_1\cup_f X_2), \\ \Lambda^1(X_1\cup_f X_2)|_{i_2(f(Y))} & = & \bigcup_{x\in i_2(f(Y))}\Lambda^1_x(X_1\cup_f X_2). \end{cases}$$

6.3.1 The counterparts of the three components of $\Lambda^1(X_1\cup_f X_2)$

The above decomposition of $\Lambda^1(X_1\cup_f X_2)$ is stressed by the following consequence of Theorem 7.1 and Corollary 7.3:

$$\begin{cases} \bigcup_{x\in i_1(X_1\setminus Y)}\Lambda^1_x(X_1\cup_f X_2) & \leftrightarrow & L_1:=\cup_{x\in X_1\setminus Y}\Lambda^1_x(X_1) \\ \bigcup_{x\in i_2(X_2\setminus f(Y))}\Lambda^1_x(X_1\cup_f X_2) & \leftrightarrow & L_2:=\cup_{x\in X_2\setminus f(Y)}\Lambda^1_x(X_2) \\ \bigcup_{x\in i_2(f(Y))}\Lambda^1_x(X_1\cup_f X_2) & \leftrightarrow & L_Y:=\cup_{y\in Y}\{y\}\times\left(\Lambda^1_y(X_1)\times_{comp}\Lambda^1_{f(y)}(X_2)\right). \end{cases}$$

The four spaces in the first two cases come with natural diffeologies, that are subset diffeologies relative to their inclusions into $\Lambda^1(X_1\cup_f X_2)$ (the two left-hand sides), and into $\Lambda^1(X_1)$ or $\Lambda^1(X_2)$, as appropriate. In the third case, the space on the left-hand side is again a subset of $\Lambda^1(X_1\cup_f X_2)$ and so can be endowed with the corresponding subset diffeology. On the other hand, the space on the right-hand side is an independent construction (that is, its explicit definition and its identification with $\bigcup_{x\in i_2(f(Y))}\Lambda^1_x(X_1\cup_f X_2)$ are the only descriptions that we have of it), which does not automatically provide them with a diffeology (and so we must specify one).

6.3.2 The covering map for the decomposition

The starting point now is the presentation of $\Lambda^1(X_1\cup_f X_2)$ as the diffeological quotient

$$\left((X_1\cup_f X_2)\times(\Omega^1(X_1)\times_{comp}\Omega^1(X_2))\right)/\left(\cup_{x\in X_1\cup_f X_2}(\{x\}\times\pi^*(\Omega^1_x(X_1\cup_f X_2)))\right).$$

(Notice that here the space being quotiented can be seen as a trivial (pseudo-)bundle over $X_1\cup_f X_2$, while the space by which we quotient, is its diffeological sub-bundle). In particular, the diffeology on $\Lambda^1(X_1\cup_f X_2)$ is the pushforward of the diffeology of $(X_1\cup_f X_2)\times(\Omega^1(X_1)\times_{comp}\Omega^1(X_2))$; we will use the fact that the diffeology on the latter is a restriction of the product diffeology on a direct product of three factors $((X_1\cup_f X_2),(\Omega^1(X_1),\Omega^1(X_2)))$.

Recall that $\rho_i:\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_i)$ is the projection of $\Omega^1(X_1)\times_{comp}\Omega^1(X_2)$ onto its i-th factor, for $i=1,2$. Since they obviously preserve the subspaces of vanishing forms, they induce well-defined bijections $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}\to L_1$ and $\Lambda^1(X_1\cup_f X_2)|_{i_2(X_2\setminus f(Y))}\to L_2$; these are the two of our desired maps (see below for details). Finally, we observe the following:

$$\alpha_1^*\circ\rho_1=f^*\circ\alpha_2^*\circ\rho_2;$$

this is also obvious from the discussion in Section 7.2.

6.3.3 The bijection $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}\leftrightarrow L_1$

As we already mentioned,

$$\begin{cases} \Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)} & = & \bigcup_{x\in X_1\setminus Y}\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_x(X_1)\times_{comp}\Omega^1(X_2)\right), \\ \Lambda^1(X_1)\supset L_1 & = & \bigcup_{x\in X_1\setminus Y}\Omega^1(X_1)/\Omega^1_x(X_1). \end{cases}$$

We construct the desired bijection between them from the map $\rho_1:\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_1)$. Since we obviously have

$$\rho_1(\Omega^1_x(X_1)\times_{comp}\Omega^1(X_2))\subseteq\Omega^1_x(X_1),$$

ρ_1 induces a well-defined map

$$\rho_1^\Lambda:\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}\to L_1.$$

Proposition 6.7. *The map ρ_1^Λ is bijective, linear on each fibre, and smooth for the subset diffeologies relative to the inclusions*

$$\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}\subset\Lambda^1(X_1\cup_f X_2)\quad\text{and}\quad L_1\subset\Lambda^1(X_1).$$

Proof. Recall that the entire $\Lambda^1(X_1\cup_f X_2)$ is defined as the following diffeological quotient:

$$\left((X_1\cup_f X_2)\times(\Omega^1(X_1)\times_{comp}\Omega^1(X_2))\right)/\left(\cup_{x\in X_1\cup_f X_2}(\{x\}\times\pi^*(\Omega_x^1(X_1\cup_f X_2)))\right);$$

in particular, the diffeology on $\Lambda^1(X_1\cup_f X_2)$ is the pushforward of the product diffeology on $(X_1\cup_f X_2)\times(\Omega^1(X_1)\times_{comp}\Omega^1(X_2))$. This implies that any plot $p:U\to\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}$ of $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}$ locally (that is, assuming that U is small enough) lifts to a map of form $(i_1\circ p_1,p_1^\Omega\times p_2^\Omega)$, where p_1 is a plot of $X_1\setminus Y\subset X_1$, and p_i^Ω is a plot of $\Omega^1(X_i)$ for $i=1,2$, where furthermore $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u\in U$.

Since ρ_1^Λ is defined as the pushforward, by the quotient projections, of the map ρ_1, the composition $\rho_1^\Lambda\circ p$ is a plot of L_1 if the composition $(i_1^{-1},\rho_1)\circ(i_1\circ p_1,p_1^\Omega\times p_2^\Omega)$ is a plot of $X_1\times\Omega^1(X_1)$; and that it is one, is rather obvious, since

$$(i_1^{-1},\rho_1)\circ(i_1\circ p_1,p_1^\Omega\times p_2^\Omega)=(p_1,p_1^\Omega),$$

which is indeed a plot of $X_1\times\Omega^1(X_1)$ by the choice of p_1 and p_1^Ω. □

6.3.4 The bijection $\Lambda^1(X_1\cup_f X_2)|_{i_2(X_2\setminus f(Y))}\leftrightarrow L_2$

This case is entirely analogous to the previous one. Indeed, we have

$$\begin{cases}\Lambda^1(X_1\cup_f X_2)|_{i_2(X_2\setminus f(Y))} &= \bigcup_{x\in X_2\setminus f(Y)}\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1(X_1)\times_{comp}\Omega_x^1(X_2)\right),\\ \Lambda^1(X_2)\supset L_2 &= \bigcup_{x\in X_2\setminus f(Y)}\Omega^1(X_2)/\Omega_x^1(X_2).\end{cases}$$

Once again, the projection

$$\rho_2:\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_2)$$

induces a well-defined bijective and linear on each fibre map

$$\rho_2^\Lambda:\Lambda^1(X_1\cup_f X_2)|_{i_2(X_2\setminus f(Y))}\to L_2.$$

The following is then established exactly as Proposition 6.7.

Proposition 6.8. *The map ρ_2^Λ is smooth for the subset diffeologies relative to the inclusions*

$$\Lambda^1(X_1\cup_f X_2)|_{i_2(X_2\setminus f(Y))}\subset\Lambda^1(X_1\cup_f X_2)\quad\text{and}\quad L_2\subset\Lambda^1(X_2).$$

6.3.5 The case of $\Lambda^1(X_1\cup_f X_2)|_{i_2(f(Y))}$

This is somewhat different from the previous two, in that the corresponding collection of fibres is not contained in either $\Lambda^1(X_1)$ or $\Lambda^1(X_2)$. It can however by seen as a subspace of the following:

$$\left(\bigcup_{y\in Y}\Lambda_y^1(X_1)\right)\times\left(\bigcup_{y\in Y}\Lambda_{f(y)}^1(X_2)\right),$$

that is, we consider the restriction of the pseudo-bundle $\Lambda^1(X_1)$ onto $(\pi_1^\Lambda)^{-1}(Y)$, that is, the pseudo-bundle

$$\lambda_1^Y:\bigcup_{y\in Y}\Lambda_y^1(X_1)=:\Lambda^1(X_1)|_Y\to Y$$

over Y, and the analogously defined restricted pseudo-bundle

$$\lambda_2^Y:\bigcup_{y\in Y}\Lambda_{f(y)}^1(X_2)=:\Lambda^1(X_2)|_{f(Y)}\to f(Y)\to Y,$$

which in this case is post-composed with the diffeomorphism f^{-1}, in order to obtain again a pseudo-bundle over Y. We then take the usual product over Y of λ_1^Y and λ_2^Y, obtaining a pseudo-bundle $\lambda_1^Y\times_Y\lambda_2^Y$ with the total space $\left(\bigcup_{y\in Y}\Lambda_y^1(X_1)\right)\times_Y\left(\bigcup_{y\in Y}\Lambda_{f(y)}^1(X_2)\right)$, that contains L_Y.

The two descriptions of $\Lambda^1(X_1\cup_f X_2)|_{i_2(f(Y))}$ By analogy with the first two components in the decomposition of $\Lambda^1(X_1\cup_f X_2)$ that we are considering at the moment, we can write

$$\left\{\begin{array}{rcl}\Lambda^1(X_1\cup_f X_2)|_{i_2(f(Y))} & = & \bigcup_{y\in Y}\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_y(X_1)\times_{comp}\Omega^1_{f(y)}(X_2)\right)\\ \Lambda^1(X_1)|_Y\times_{comp}\Lambda^1(X_2)|_{f(Y)}\supset L_Y & = & \bigcup_{y\in Y}\left(\Omega^1(X_1)/\Omega^1_y(X_1)\right)\times_{comp}\left(\Omega^1(X_2)/\Omega^1_{f(y)}(X_2)\right).\end{array}\right.$$

The second equality requires us to specify once more what is being meant by the compatibility of the equivalence class in $\Omega^1(X_1)/\Omega^1_y(X_1)$ with an equivalence class in $\Omega^1(X_2)/\Omega^1_{f(y)}(X_2)$. By definition, two such classes are compatible if every form in the first class (an element of $\Omega^1(X_1)/\Omega^1_y(X_1)$) is compatible with every form in the second class (an element of $\Omega^1(X_2)/\Omega^1_{f(y)}(X_2)$).[11] We also note, although this should be quite clear, that the space $\Omega^1_y(X_1)\times_{comp}\Omega^1_{f(y)}(X_2)$ is by definition the space of all pairs (ω_1,ω_2) such that ω_1 and ω_2 are compatible, ω_1 vanishes at y, and ω_2 vanishes at $f(y)$.

The diffeology on L_Y We have just described L_Y as a subset of the direct product (over Y) of $\Lambda^1(X_1)|_Y$ and $\Lambda^1(X_2)|_{f(Y)}$, each of which carries the corresponding subset diffeology. Thus, their direct product carries the product diffeology; we endow L_Y with the subset diffeology relative to that.

The map $\Lambda^1(X_1\cup_f X_2)|_{i_2(f(Y))}\to L_Y$ To construct this map, we use again the two projections ρ_1 and ρ_2; this time, we start by considering the (essentially trivial) map $(\rho_1,\rho_2):\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_1)\times\Omega^1(X_2)$. We claim that it induces a well-defined and bijective map $\Lambda^1(X_1\cup_f X_2)|_{i_2(f(Y))}\to L_Y$.

Proposition 6.9. *The map*

$$\rho^\Lambda:\bigcup_{y\in Y}\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)/\left(\Omega^1_y(X_1)\times_{comp}\Omega^1_{f(y)}(X_2)\right)\to$$

$$\to\bigcup_{y\in Y}\left(\Omega^1(X_1)/\Omega^1_y(X_1)\right)\times_{comp}\left(\Omega^1(X_2)/\Omega^1_{f(y)}(X_2)\right)$$

that acts by assigning, for each $y\in Y$, to any coset

$$(\omega_1,\omega_2)+\left(\Omega^1_y(X_1)\times_{comp}\Omega^1_{f(y)}(X_2)\right)$$

with compatible ω_1 and ω_2, the pair of cosets

$$\left(\omega_1+\Omega^1_y(X_1),\omega_2+\Omega^1_{f(y)}(X_2)\right)$$

is well-defined, bijective, and two ways smooth.

Proof. The map ρ^Λ is of course well-defined as a map into

$$\left(\Omega^1(X_1)/\Omega^1_y(X_1)\right)\times\left(\Omega^1(X_2)/\Omega^1_{f(y)}(X_2)\right);$$

what we need to check is that its range is contained

$$\left(\Omega^1(X_1)/\Omega^1_y(X_1)\right)\times_{comp}\left(\Omega^1(X_2)/\Omega^1_{f(y)}(X_2)\right),$$

that is, if (ω_1,ω_2) is a compatible pair then $\omega_1+\Omega^1_y(X_1)$ and $\omega_2+\Omega^1_{f(y)}(X_2)$ are compatible cosets. □

[11] We note without explaining that this is a far more restrictive condition than that of the existence of just one form in the first class compatible with at least one form in the second class.

6.4 Characterizing the plots of $\Lambda^1(X_1 \cup_f X_2)$

Let us now describe the local shape of plots of $\Lambda^1(X_1 \cup_f X_2)$, based on its definition as the quotient

$$\left((X_1 \cup_f X_2) \times \Omega^1(X_1 \cup_f X_2)\right) / \left(\bigcup_{x \in X_1 \cup_f X_2} \{x\} \times \Omega^1_x(X_1 \cup_f X_2) \right)$$

and the above decomposition. Recall that

$$\lambda_\cup : (X_1 \cup_f X_2) \times \Omega^1(X_1 \cup_f X_2) \to \Lambda^1(X_1 \cup_f X_2)$$

is the corresponding quotient projection; let $\pi^\Lambda : \Lambda^1(X_1 \cup_f X_2) \to X_1 \cup_f X_2$ be the pseudo-bundle projection.

6.4.1 General considerations

Since the diffeology of $\Lambda^1(X_1 \cup_f X_2)$ is by definition the quotient diffeology, *i.e.*, the pushforward of the diffeology on $(X_1 \cup_f X_2) \times \Omega^1(X_1 \cup_f X_2)$ by π^Λ, any plot of $\Lambda^1(X_1 \cup_f X_2)$ locally lifts to a plot of $(X_1 \cup_f X_2) \times \Omega^1(X_1 \cup_f X_2)$. We will identify this space with

$$(X_1 \cup_f X_2) \times \pi^*(\Omega^1(X_1 \cup_f X_2)) = (X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right),$$

and from now on we will assume that the domain of definition of π^Λ is this latter space, $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$. The properties of the product, the quotient, and the gluing diffeologies then trivially imply the following:

Lemma 6.10. *Let $p : U' \to \Lambda^1(X_1 \cup_f X_2)$ be any plot of $\Lambda^1(X_1 \cup_f X_2)$. Then there exists a sub-domain $U \subseteq U'$ such that $\pi^\Lambda \circ (p|_U)$ lifts, via the projection $X_1 \sqcup X_2 \to X_1 \cup_f X_2$ that defines $X_1 \cup_f X_2$, to either a plot p_1 of X_1 or a plot p_2 of X_2.*

Proof. By definition of a quotient diffeology, there exists a sub-domain U_1 of U' such that $p|_{U_1}$ lifts to a plot $\tilde{p}_1$ of $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$. By definition of the product diffeology, there exists a sub-domain $U_2 \subseteq U_1$ such that the restriction $\tilde{p}_1|_{U_2}$ has form $(p'_\cup, p^{\Omega}{}')$, where $p_\cup$ is a plot of $X_1 \cup_f X_2$ and p^Ω is a plot of $\Omega^1(X_1) \times_{comp} \Omega^1(X_2)$. Since the subset diffeology on the latter is a subset of the product diffeology on $\Omega^1(X_1) \times \Omega^1(X_2)$, there exists a sub-domain $U_3 \subseteq U_2$ such that $\tilde{p}_1|_{U_3} = (p_\cup{}'', p^\Omega{}'')$, where $p^\Omega{}''$ has form $p^\Omega{}'' = (p_1^\Omega{}', p_2^\Omega{}')$ for some plots $p_1^\Omega{}'$ of $\Omega^1(X_1)$ and $p_2^\Omega{}'$ of $\Omega^1(X_2)$. Set U to be any connected component of U_3, and let $\tilde{p} = \tilde{p}_1|_U$, which writes as

$$\tilde{p} = (p_\cup, (p_1^\Omega, p_2^\Omega)),$$

where $p_\cup = p_\cup{}''|_U$ is a plot of $X_1 \cup_f X_2$, $p_1^\Omega = p_1^\Omega{}'|_U$ is a plot of $\Omega^1(X_1)$, and $p_2^\Omega = p_2^\Omega{}'|_U$ is a plot of $\Omega^1(X_2)$. It now suffices to observe that $p_\cup = \pi^\Lambda \circ (p|_U)$ is a plot of $X_1 \cup_f X_2$ defined on a connected domain, and (as follows from the gluing construction) any such plot lifts to a plot of exactly one of the factors. □

Remark 6.11. *Let $p : U \to \Lambda^1(X_1 \cup_f X_2)$ be a plot such that U is as in the above lemma. Then we can summarize by saying that the lift $\tilde{p}$ of p to $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$ has either form*

$$\tilde{p} = \begin{cases} (i_1 \circ p_1, (p_1^\Omega, p_2^\Omega)) & \text{on } (\pi^\Lambda \circ p|_{U'})^{-1}(X_1 \setminus Y) \\ (i_2 \circ f \circ p_1, (p_1^\Omega, p_2^\Omega)) & \text{on } (\pi^\Lambda \circ p|_{U'})^{-1}(Y) \end{cases}$$

or form

$$\tilde{p} = (i_2 \circ p_2, (p_1^\Omega, p_2^\Omega)) \text{ if } Range(\pi^\Lambda \circ p|_{U'}) \subseteq i_2(X_2).$$

6.4.2 The two diffeologies $\mathcal{D}_1^\Omega$ and $\mathcal{D}_2^\Omega$ on $\Omega^1(Y)$

As we have just seen (and as is clear from the definition of $\Lambda^1(X_1 \cup_f X_2)$ itself), we have to consider frequently pairs of form (p_1^Ω, p_2^Ω), where $p_1^\Omega : U \to \Omega^1(X_1)$ and $p_2^\Omega : U \to \Omega^1(X_2)$ are plots of, respectively, $\Omega^1(X_1)$ and $\Omega^1(X_2)$ such that $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u \in U$. Furthermore, we have already seen that this compatibility can be expressed as the requirement

$$\alpha_1^*(p_1^\Omega(u)) = (f^* \circ \alpha_2^*)(p_2^\Omega(u)) \quad \text{for all } u \in U.$$

Observe now that $\alpha_1^* \circ p_1^\Omega : U \to \Omega^1(Y)$, which is a plot for the standard (functional) diffeology on $\Omega^1(Y)$ by the smoothness of α_1^*, can also be viewed as a plot of the diffeology on the same $\Omega^1(Y)$ that is obtained by pushing forward the diffeology of $\Omega^1(X_1)$ by the map α_1^*.

We denote this pushforward diffeology by $\mathcal{D}_1^\Omega$. It follows from what has just been said that $\mathcal{D}_1^\Omega$ is contained in the standard diffeology of $\Omega^1(Y)$. We do not discuss whether it is *properly* contained in it (we do not know), since the condition that will usually be sufficient for us is that the two diffeologies be equal to each other:

$$\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega \subseteq \mathcal{D}^\Omega(Y),$$

where $\mathcal{D}^\Omega(Y)$ is the usual functional diffeology on $\Omega^1(Y)$. The effect that such equality has is summarized in the following statement.

Corollary 6.12. *Let X_1, X_2, and the gluing diffeomorphism f be such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Then, up to restricting a plot on each domain, for every plot $p_1^\Omega : U \to \Omega^1(X_1)$ of $\Omega^1(X_1)$ there exists a plot $p_2^\Omega : U \to \Omega^1(X_2)$ of $\Omega^1(X_2)$ such that $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u \in U$, and* vice versa *for each plot p_2^Ω of $\Omega^1(X_2)$ there exists a plot p_1^Ω of $\Omega^1(X_1)$ such that p_1^Ω and p_2^Ω at each point.*

Proof. This follows directly from the criterion of compatibility of forms and the definition of a pushforward diffeology. Indeed, let $p_1^\Omega : U \to \Omega^1(X_1)$ be a plot of $\Omega^1(X_1)$. By definition of a pushforward diffeology, $\alpha_1^* \circ p_1^\Omega : U \to \Omega^1(Y)$ is a plot of the diffeology $\mathcal{D}_1^\Omega$, and by assumption, it is also a plot of the diffeology $\mathcal{D}_2^\Omega$. Since the latter is also a pushforward diffeology, every plot of it locally has form $f^* \circ \alpha_2^* \circ q$ for some plot q of $\Omega^1(X_2)$. Thus, if we assume that U is small enough, there exists a plot p_2^Ω of $\Omega^1(X_2)$ such that $\alpha_1^* \circ p_1^\Omega = f^* \circ \alpha_2^* \circ p_2^\Omega$; then it follows from the condition of compatibility that $\alpha_1^*(p_1^\Omega(u)) = (f^* \circ \alpha_2^*)(p_2^\Omega(u))$ for all $u \in U$, which is what we wanted. The reverse is obviously obtained by the exact same reasoning. □

6.4.3 Plots of $\Lambda^1(X_1)$ and those of $\Lambda^1(X_1 \cup_f X_2)$

We have said already that it is natural to distinguish between plots of $\Lambda^1(X_1 \cup_f X_2)$ whose compositions with π^Λ lift to plots of X_1 and those whose compositions lift to plots of X_2. It then follows from the shape of the canonical decomposition of $\Lambda^1(X_1 \cup_f X_2)$ (see above) that it might be possible to relate these plots, themselves, to those of $\Lambda^1(X_1)$ and those of $\Lambda^1(X_2)$, respectively, which is what we do in the present section.

From plots of $\Lambda^1(X_1 \cup_f X_2)$ to those of $\Lambda^1(X_1)$ Assume first that p is such that the component $p_\cup$ of its lift $\tilde{p}$ lifts in turn to a plot of X_1. The plot p can then be characterized as follows.

Lemma 6.13. *Let U be a connected domain, and let $p : U \to \Lambda^1(X_1 \cup_f X_2)$ be a map such that the range of the composition of p with the pseudo-bundle projection $\Lambda^1(X_1 \cup_f X_2) \to X_1 \cup_f X_2$ is contained in $\pi(X_1) = i_1(X_1 \setminus Y) \cup i_2(f(Y))$. Then p is a plot of $\Lambda^1(X_1 \cup_f X_2)$ if and only if, up to passing to a smaller sub-domain of U, there exists a lift $\tilde{p} = (p_\cup, (p_1^\Omega, p_2^\Omega))$ of p to a plot of $(X_1 \cup_f X_2) \times \big(\Omega^1(X_1) \times \Omega^1(X_2)\big)$ such that*

$$\alpha_1^* \circ p_1^\Omega = f^* \circ \alpha_2^* \circ p_2^\Omega.$$

Conversely, for any $p_\cup$, p_1^Ω, p_2^Ω that satisfy these conditions, $\tilde{p}$ is a lift of some plot of $\Lambda^1(X_1 \cup_f X_2)$.

Proof. This is a trivial consequence of the definition of the diffeology of $\Lambda^1(X_1 \cup_f X_2)$ as a pushforward of the diffeology on $(X_1 \cup_f X_2) \times \big(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\big)$, that of the gluing diffeology, and the criterion of the compatibility of forms in terms of pullback maps α_1^* and $f^* \circ \alpha_2^*$. □

The *vice versa*: from a plot of $\Lambda^1(X_1)$ to one of $\Lambda^1(X_1\cup_f X_2)$ Let $p:U\to\Lambda^1(X_1)$ be a plot of $\Lambda^1(X_1)$, with U small enough so that p lifts a plot of $X_1\times\Omega^1(X_1)$ of form (p_1,p_1^Ω), where $p_1:U\to X_1$ is a plot of X_1 and $p_1^\Omega:U\to\Omega^1(X_1)$ is a plot of $\Omega^1(X_1)$. We say that p extends to a plot of $\Lambda^1(X_1\cup_f X_2)$ there exists a plot $p_2^\Omega:U\to\Omega^1(X_2)$ of $\Omega^1(X_2)$ such that $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u\in U$; and if such p_2^Ω exists, then for any choice of it the following is a plot of $(X_1\cup_f X_2)\times\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)$:

$$\begin{cases} (i_1\circ p_1,(p_1^\Omega,p_2^\Omega)) & \text{on } p_1^{-1}(X_1\setminus Y) \\ (i_2\circ f\circ p_1,(p_1^\Omega,p_2^\Omega)) & \text{on } p_1^{-1}(Y). \end{cases}$$

As we have essentially said already, the existence of p_2^Ω depends on whether $\alpha_1^*\circ p_1^\Omega$ is a plot of $\mathcal{D}_2^\Omega$. We thus conclude that the following statement is true.

Lemma 6.14. *Let $\tilde{p}_1=(p_1,p_1^\Omega):U\to X_1\times\Omega^1(X_1)$ be a plot of $X_1\times\Omega^1(X_1)$ such that p_1 is a plot of X_1 and p_1^Ω is a plot of $\Omega^1(X_1)$. Then there exists a plot p_2^Ω of $\Omega^1(X_2)$ such that $\tilde{p}:U\to(X_1\cup_f X_2)\times\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)$ defined by*

$$\tilde{p}=\begin{cases} (i_1\circ p_1,(p_1^\Omega,p_2^\Omega)) & \text{on } p_1^{-1}(X_1\setminus Y) \\ (i_2\circ f\circ p_1,(p_1^\Omega,p_2^\Omega)) & \text{on } p_1^{-1}(Y) \end{cases}$$

is a plot of $(X_1\cup_f X_2)\times\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)$ if and only if $\alpha_1^*\circ p_1^\Omega\in\mathcal{D}_2^\Omega$.

The subduction of $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)\cup i_2(f(Y))}$ onto $\Lambda^1(X_1)$ We now consider the portion of the pseudo-bundle $\Lambda^1(X_1\cup_f X_2)$ that lies over $i_1(X_1\setminus Y)\cup i_2(f(Y))$. This is the image of X_1 in the glued space $X_1\cup_f X_2$, and since we assume throughout this section that the gluing map f is a diffeomorphism, it is its diffeomorphic image. The notation $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)\cup i_2(f(Y))}$ is analogous to the previous defined $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)}$, and the space itself is endowed with the subset diffeology relative to its inclusion $\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)\cup i_2(f(Y))}\subset\Lambda^1(X_1\cup_f X_2)$.

Proposition 6.15. *Let X_1, X_2, and the gluing map f be such that $\mathcal{D}_1^\Omega=\mathcal{D}_2^\Omega$. Then the map*

$$\tilde{\rho}_1^\Lambda:\Lambda^1(X_1\cup_f X_2)|_{i_1(X_1\setminus Y)\cup i_2(f(Y))}\to\Lambda^1(X_1)$$

induced by the projection $\rho_1:\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\to\Omega^1(X_1)$ is a subduction.

Proof. It suffices to show that every plot of $\Lambda^1(X_1)$ locally lifts to a plot of form $\tilde{\rho}_1^\Lambda\circ p$, where p is a plot of $\Lambda^1(X_1\cup_f X_2)$ (indeed, since the ranges of all the plots of $\Lambda^1(X_1)$ cover it, the surjectivity will follow from it). Let $p':U\to\Lambda^1(X_1)$ be a plot of $\Lambda^1(X_1)$ (we will assume U small enough as needed), and let $\tilde{p}':U\to X_1\times\Omega^1(X_1)$ be its lift to a plot of $X_1\times\Omega^1(X_1)$, that has form (p_1,p_1^Ω) ($\tilde{p}'$ having this form is one meaning of U being small enough). By the assumption that $\mathcal{D}_1^\Omega=\mathcal{D}_2^\Omega$, there exists then a plot $p_2^\Omega:U\to\Omega^1(X_2)$ such that $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u\in U$. We therefore have that

$$\tilde{p}=\begin{cases} (i_1\circ p_1,(p_1^\Omega,p_2^\Omega)) & \text{on } p_1^{-1}(X_1\setminus Y), \\ (i_2\circ f\circ p_1,(p_1^\Omega,p_2^\Omega)) & \text{on } p_1^{-1}(Y) \end{cases}$$

is a plot of $(X_1\cup_f X_2)\times\left(\Omega^1(X_1)\times_{comp}\Omega^1(X_2)\right)$. Let $p=\lambda_\cup\circ\tilde{p}$ be the corresponding plot of $\Lambda^1(X_1\cup_f X_2)$. Then it follows directly from the construction that $p'=\tilde{\rho}_1^\Lambda\circ p$; since p' corresponds to the local form of any arbitrary plot of $\Lambda^1(X_1)$, we get our claim. □

6.4.4 Plots of $\Lambda^1(X_2)$ and those of $\Lambda^1(X_1\cup_f X_2)$

We now develop the analogous reasoning for the second factor of the gluing, that is, the case of an arbitrary plot p of $\Lambda^1(X_1\cup_f X_2)$ such that $\pi^\Lambda\circ p$ lifts to a plot p_2 of X_2 (this composition being a plot of $X_1\cup_f X_2$, this amounts to saying that it takes values in $i_2(X_2)$). Our treatment of this second case can be made quite brief, as it essentially mimicks the case of the first factor; this is in contrast to what usually happens for our gluing diffeologies, which are usually not symmetric in the two factors. However, although $\pi^\Lambda\circ p$ has then a unique form $\pi^\Lambda\circ p=i_2\circ p_2$, the relevant part of $\Lambda^1(X_1\cup_f X_2)$ (the portion of it that lies over $i_2(X_2)$) has different behavior over $i_2(f(Y))$ and over $i_2(X_2\setminus f(Y))$, just as it occurs for the factor X_1.

From a plot of $\Lambda^1(X_1 \cup_f X_2)$ to one of $\Lambda^1(X_2)$ Let $p: U \to \Lambda^1(X_1 \cup_f X_2)$ be a plot of $\Lambda^1(X_1 \cup_f X_2)$ such that $\pi^\Lambda \circ p$ has form $i_2 \circ p_2$ for some plot p_2 of X_2. Let $\tilde{p}$ be a lift of p to a plot of $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$; we assume, as we can always do, that U is small enough so that $\tilde{p}$ has form $\tilde{p} = (i_2 \circ p_2, p^\Omega)$ for a plot p^Ω of $\Omega^1(X_1) \times_{comp} \Omega^1(X_2)$, which furthermore has form $p^\Omega = (p_1^\Omega, p_2^\Omega)$, with p_i^Ω a plot of $\Omega^1(X_i)$ for $i = 1, 2$. Observe that the pair (p_1^Ω, p_2^Ω) is not arbitrary; it is determined precisely by the following condition: $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u \in U$.

We thus have that p can be written as:

$$p(u) = \begin{cases} (p_1^\Omega(u), p_2^\Omega(u)) + \left(\Omega^1(X_1) \times_{comp} \Omega^1_{p_2(u)}(X_2)\right) & \text{on } p_2^{-1}(X_2 \setminus f(Y)) \\ (p_1^\Omega(u), p_2^\Omega(u)) + \left(\Omega^1_{f^{-1}(p_2(u))}(X_1) \times_{comp} \Omega^1_{p_2(u)}(X_2)\right) & \text{on } p_2^{-1}(f(Y)), \end{cases}$$

to which there naturally corresponds a plot of $\Lambda^1(X_2)$. This plot is defined as the pushforward to $\Lambda^1(X_2)$ of the plot (p_2, p_2^Ω) of $X_2 \times \Omega^1(X_2)$.

The *vice versa*: from a plot of $\Lambda^1(X_2)$ to a plot of $\Lambda^1(X_1 \cup_f X_2)$ Also the treatment of the *vice versa* case is analogous to what we have already done for the first factor. Let $p': U \to \Lambda^1(X_2)$ be a plot; assume that U is small enough that p' lifts to a plot $\tilde{p}'$ of $X_2 \times \Omega^1(X_2)$ that has form $\tilde{p}' = (p_2, p_2^\Omega)$ for some plot p_2 of X_2 and some plot p_2^Ω of $\Omega^1(X_2)$. We wonder whether there is a plot $p_1^\Omega: U \to \Omega^1(X_1)$ of $\Omega^1(X_1)$ such that $\tilde{p} = (i_2 \circ p_2, (p_1^\Omega, p_2^\Omega))$ is a plot of $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$. The answer is entirely analogous the case of the first factor and is as follows.

Lemma 6.16. *Let $p': U \to \Lambda^1(X_2)$ be a plot of $\Lambda^1(X_2)$ that lifts to a plot of $X_2 \times \Omega^1(X_2)$ that has form (p_2, p_2^Ω). Then it extends to a plot of $\Lambda^1(X_1 \cup_f X_2)$, in the sense that $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$ admits a plot of form $(i_2 \circ p_2, (p_1^\Omega, p_2^\Omega))$, if and only if $f^* \circ \alpha_2^* \circ p_2^\Omega \in \mathcal{D}_2^\Omega$.*

Proof. The proof is the same as in the case of the first factor, so we omit it. □

The subduction $\Lambda^1(X_1 \cup_f X_2)|_{i_2(X_2)} \to \Lambda^1(X_2)$ Just as in the case of the first factor, we have the following statement, which also renders that of the preceding lemma more precise. Indeed, it clarifies what we mean by saying that a given plot p' of $\Lambda^1(X_2)$ extends[12] to a plot p of $\Lambda^1(X_1 \cup_f X_2)$; it means that $p' = \tilde{\rho}_2^\Lambda \circ p$.

Proposition 6.17. *Let X_1, X_2, and the gluing map f be such that $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$. Then the map*

$$\tilde{\rho}_2^\Lambda : \Lambda^1(X_1 \cup_f X_2)|_{i_2(X_2)} \to \Lambda^1(X_2)$$

induced by the projection $\rho_2 : \Omega^1(X_1) \times_{comp} \Omega^1(X_2) \to \Omega^1(X_2)$ is a subduction.

Proof. The proof is completely analogous to that of Proposition 6.15, so we omit it. □

6.4.5 The final description of plots of $\Lambda^1(X_1 \cup_f X_2)$

Here we give the final characterization of the diffeology of $\Lambda^1(X_1 \cup_f X_2)$.

Theorem 6.18. *Let X_1 and X_2 be two diffeological spaces, and let $f: X_1 \supset Y \to X_2$ be a diffeomorphism of its domain with its image. Then the diffeology of $\Lambda^1(X_1 \cup_f X_2)$ is the coarsest diffeology such that both maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are smooth.*

Proof. Let $\mathcal{D}'$ be any diffeology on $\Lambda^1(X_1 \cup_f X_2)$ such that $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are subductions, and let $s: U' \to \Lambda^1(X_1 \cup_f X_2)$ be a plot of $\mathcal{D}'$. It suffices to show that for every $u' \in U'$ there is a neighborhood U of u' such that $s|_U$ lifts to a plot of $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$.

Thus, we can assume first of all that U' is connected; this implies that $\pi^\Lambda \circ s$, which is a plot of $X_1 \cup_f X_2$, lifts to either a plot s_1 of X_1 or to a plot s_2 of X_2. Since the two cases are symmetric in the case of gluing along a diffeomorphism, it suffices to consider one of them; so we assume that $\pi^\Lambda \circ s$ lifts to a plot s_1.

[12] Admittedly, one may find our use of the term dubious; indeed, our use of it is not its standard meaning.

Next, $\tilde{\rho}_1^\Lambda \circ s$ is a plot of $\Lambda^1(X_1)$ by assumption. We therefore can assume that U' is small enough so that it lifts to a plot of $X_1 \times \Omega^1(X_1)$ and, furthermore, that this lift has form (s_1', s_1^Ω) for a plot s_1' of X_1 and a plot s_1^Ω of $\Omega^1(X_1)$. It is then trivial to observe that s_1' coincides with s_1 whenever both are defined. Thus, we obtain

$$\tilde{\rho}_1^\Lambda \circ s = \lambda^1 \circ (s_1, s_1^\Omega),$$

where $\lambda^1 : X_1 \times \Omega^1(X_1) \to \Lambda^1(X_1)$ is the defining projection.

Let us now consider $\tilde{\rho}_2^\Lambda \circ s$. This is a plot of $\Lambda^1(X_2)$ which takes values in $\Lambda^1(X_2)|_{f(Y)}$ only. Restricting again, if necessary, the domain U' to its sub-domain U, we can assume that it lifts to a plot of $X_2 \times \Omega^1(X_2)$ and that firthermore this lift has form (s_2, s_2^Ω); it is again trivial to see that $s_2 = f \circ s_1$ whenever the latter expression makes sense.

Let $s_\cup = \begin{cases} i_1 \circ s_1 & \text{on } s_1^{-1}(X_1 \setminus Y) \\ i_2 \circ f \circ s_1 & \text{on } s_1^{-1}(Y); \end{cases}$ it is obvious that $s_\cup \equiv \pi^\Lambda \circ s$. Thus, to obtain our final claim, it suffices to show that $s_1^\Omega(u)$ and $s_2^\Omega(u)$ are compatible for all $u \in U$. This follows from their corresponding to a unique plot s of $\Lambda^1(X_1 \cup_f X_2)$; more precisely, we have, for any fixed $u \in U$, that

$$s(u) = (s_1^\Omega(u), s_2^\Omega(u)) + \pi^*(\Omega^1_{s_1(u)}(X_1 \cup_f X_2)),$$

i.e., the pair $(s_1^\Omega(u), s_2^\Omega(u))$ belongs to to a coset in $\pi^*(\Omega^1(X_1) \times \Omega^1(X_2))$, which, as we know includes compatible pairs only. Since by construction of the maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ we also have $s = \lambda_\cup \circ (s_\cup, (s_1^\Omega, s_2^\Omega))$, we can conclude that s indeed belongs to the standard diffeology on $\Lambda^1(X_1 \cup_f X_2)$, whence the claim. □

7 Existence of a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$

In this section we consider, under the assumption that both $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ admit pseudo-metrics,[13] and that f is a diffeomorphism, the existence of an induced pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$. Since the latter is not the result of a gluing of the former two together, we cannot apply the gluing construction for the pseudo-metrics either. However, we do something similar and obtain one on $\Lambda^1(X_1 \cup_f X_2)$ by combining the given two; this requires additional assumptions on them.

7.1 The compatibility of pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$

It is intuitively clear that it is not possible to get a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$ out of just any two arbitrary pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$. We need a certain compatibility notion; the most natural one is the following.

Definition 7.1. *Let g_1^Λ and g_2^Λ be pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. We say that g_1^Λ and g_2^Λ are **compatible**, if for all $y \in Y$ and for all $\omega, \mu \in (\pi^\Lambda)^{-1}(i_2(f(y)))$ we have*

$$g_1^\Lambda(y)(\tilde{\rho}_1^\Lambda(\omega), \tilde{\rho}_1^\Lambda(\mu)) = g_2^\Lambda(f(y))(\tilde{\rho}_2^\Lambda(\omega), \tilde{\rho}_2^\Lambda(\mu)).$$

There is an equivalent way to define compatibility, stated in the following proposition.

Proposition 7.2. *Two pseudo-metrics g_1^Λ and g_2^Λ are compatible if and only if the following is true: for any $y \in Y$ and for any two compatible pairs (ω', ω'') and (μ', μ''), where $\omega', \mu' \in \Lambda^1(X_1)$ and $\omega'', \mu'' \in \Lambda^1(X_2)$, we have*

$$g_1^\Lambda(y)(\omega', \mu') = g_2^\Lambda(f(y))(\omega'', \mu'').$$

Proof. The compatibility condition given in the statement clearly implies the one given in the definition; indeed, for any $\omega \in \Lambda^1(X_1 \cup_f X_2)$ the forms $\tilde{\rho}_1^\Lambda(\omega)$ and $\tilde{\rho}_2^\Lambda(\omega)$ are compatible by construction. To establish the equivalence, it suffices to show that for any pair of compatible forms (ω_1, ω_2), where $\omega_i \in \Lambda^1(X_i)$, there exists $\omega \in \Lambda^1(X_1 \cup_f X_2)$ such that $\tilde{\rho}_i^\Lambda(\omega) = \omega_i$, and this is a consequence of Lemma 6.2. □

[13] Notice that this assumption includes that of these pseudo-bundles having only finite-dimensional fibres, which they do, for instance, if X_1 and X_2 are subsets of a standard $\mathbb{R}^n$.

7.2 The definition of the induced pseudo-metric g^Λ

We now define the induced pseudo-metric g^Λ on $\Lambda^1(X_1 \cup_f X_2)$. Let g_1^Λ and g_2^Λ be two compatible pseudo-metrics, on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. The map $g^\Lambda : X_1 \cup_f X_2 \to \left(\Lambda^1(X_1 \cup_f X_2)\right)^* \otimes \left(\Lambda^1(X_1 \cup_f X_2)\right)^*$ is defined as follows:

$$g^\Lambda(x)(\cdot,\cdot) = \begin{cases} g_1^\Lambda(i_1^{-1}(x))(\tilde{\rho}_1^\Lambda(\cdot), \tilde{\rho}_1^\Lambda(\cdot)), & \text{if } x \in i_1(X_1 \setminus Y), \\ g_2^\Lambda(i_2^{-1}(x))(\tilde{\rho}_2^\Lambda(\cdot), \tilde{\rho}_2^\Lambda(\cdot)), & \text{if } x \in i_2(X_2 \setminus f(Y)), \\ \frac{1}{2}\left(g_1^\Lambda(f^{-1}(i_2^{-1}(x)))(\tilde{\rho}_1^\Lambda(\cdot), \tilde{\rho}_1^\Lambda(\cdot)) + g_2^\Lambda(i_2^{-1}(x))(\tilde{\rho}_2^\Lambda(\cdot), \tilde{\rho}_2^\Lambda(\cdot))\right), & \text{if } x \in i_2(f(Y)). \end{cases}$$

In the section immediately following, we show that this is indeed a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$, that is, that it has the correct rank on each fibre, and that it is smooth.

7.3 Proving that g^Λ is a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$

It is clear from the construction that there are two items to be checked: one, that g^Λ yields a pseudo-metric on each individual fibre, that is, that it has the maximal rank possible, and two, that it is smooth.

The rank of g^Λ We first check that g^Λ yields pseudo-metrics on individual fibres. By construction, $g^\Lambda(x)$ for $x \in X_1 \cup_f X_2$ is always a smooth symmetric bilinear form on $\Lambda_x^1(X_1 \cup_f X_2)$. Therefore it suffices to show that it has the maximal possible rank on each fibre; this is the content of the following statement.

Lemma 7.3. *For any $x \in X_1 \cup_f X_2$, the rank of g^Λ is equal to* $\dim((\Lambda_x^1(X_1 \cup_f X_2))^*)$.

Proof. Over points in $i_1(X_1 \setminus Y)$ and those in $i_2(X_2 \setminus f(Y))$ this follows directly from the construction. Let $x \in i_2(f(Y))$. Recall that for any such x the fibre $\Lambda_x^1(X_1 \cup_f X_2)$ then has form $\Lambda^1_{f^{-1}(i_2^{-1}(x))}(X_1) \times_{comp} \Lambda^1_{i_2^{-1}(x)}(X_2)$, which in particular is a subspace in $\Lambda^1_{f^{-1}(i_2^{-1}(x))}(X_1) \times \Lambda^1_{i_2^{-1}(x)}(X_2)$. The definition of g^Λ obviously extends to that of a symmetric bilinear form on the latter space, and furthermore, this form is proportional to the usual direct sum form[14] $g_1^\Lambda(f^{-1}(i_2^{-1}(x))) + g_2^\Lambda(i_2^{-1}(x))$, which obviously has the maximal rank possible. Therefore $g^\Lambda(x)$, being a restriction of $g_1^\Lambda(f^{-1}(i_2^{-1}(x))) + g_2^\Lambda(i_2^{-1}(x))$ to the vector subspace $\Lambda^1_{f^{-1}(i_2^{-1}(x))}(X_1) \times_{comp} \Lambda^1_{i_2^{-1}(x)}(X_2)$, has the maximal rank as well. □

The smoothness of g^Λ We now turn to showing that g^Λ is smooth as a map

$$X_1 \cup_f X_2 \to \left(\Lambda^1(X_1 \cup_f X_2)\right)^* \otimes \left(\Lambda^1(X_1 \cup_f X_2)\right)^*;$$

the proof of this is the main ingredient in the statement that follows.

Theorem 7.4. *The map $g^\Lambda : X_1 \cup_f X_2 \to \left(\Lambda^1(X_1 \cup_f X_2)\right)^* \otimes \left(\Lambda^1(X_1 \cup_f X_2)\right)^*$ is a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$.*

Proof. As is the usual procedure with proving the smoothness of a prospective pseudo-metrics, it is sufficient to choose plots $p : U \to X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ and $q, s : U' \to \Lambda^1(X_1 \cup_f X_2)$, and to show that the evaluation map

$$(u, u') \mapsto g^\Lambda(p(u))(q(u'), s(u')),$$

defined on the set of all (u, u') such that $\pi^\Lambda(q(u')) = \pi^\Lambda(s(u')) = p(u)$, is smooth as a map into the standard $\mathbb{R}$. Also as usual, we can assume that U is connected and, subsequently, that U' is such that the plots q, s have one of the standard forms described in the previous section, *i.e.*, they lift to products of plots for the three-factor product space $(X_1 \cup_f X_2) \times \left(\Omega^1(X_1) \times_{comp} \Omega^1(X_2)\right)$. In particular, we may assume that p either lifts to a plot p_1 of X_1 or a plot p_2 of X_2; notice that these two cases are perfectly analogous in the current context.

Suppose that p lifts to a plot p_1 of X_1. The value of the corresponding evaluation map in this case is

[14] Defined by requiring the two factors to be orthogonal, and the restriction to each factor to coincide with the given form g_i^Λ.

$g^\Lambda(p(u))(q(u'),s(u'))=$

$$=\left\{\begin{array}{ll} g_1^\Lambda(p_1(u))(\tilde{\rho}_1^\Lambda(q(u')),\tilde{\rho}_1^\Lambda(s(u'))) & \text{on } p_1^{-1}(X_1\setminus Y) \\ \frac{1}{2}\left(g_1^\Lambda(p_1(u))(\tilde{\rho}_1^\Lambda(q(u')),\tilde{\rho}_1^\Lambda(s(u')))+g_2^\Lambda(f(p_1(u)))(\tilde{\rho}_2^\Lambda(q(u')),\tilde{\rho}_2^\Lambda(s(u')))\right) & \text{on } p_1^{-1}(Y). \end{array}\right.$$

It now suffices to apply the compatibility condition to the second part of the formula to obtain that

$$g^\Lambda(p(u))(q(u'),s(u'))=g_1^\Lambda(p_1(u))(\tilde{\rho}_1^\Lambda(q(u')),\tilde{\rho}_1^\Lambda(s(u')))$$

on the entire range of p, so it is smooth, because g_1^Λ is smooth (by the assumption on it).

Likewise, if p lifts to a plot p_2 of X_2 then the corresponding evaluation has form

$g^\Lambda(p(u))(q(u'),s(u'))=$

$$=\left\{\begin{array}{ll} g_2^\Lambda(p_2(u))(\tilde{\rho}_2^\Lambda(q(u')),\tilde{\rho}_2^\Lambda(s(u'))) & \text{on } p_2^{-1}(X_2\setminus f(Y)) \\ \frac{1}{2}\left(g_1^\Lambda(f^{-1}(p_2(u)))(\tilde{\rho}_1^\Lambda(q(u')),\tilde{\rho}_1^\Lambda(s(u')))+g_2^\Lambda(p_2(u))(\tilde{\rho}_2^\Lambda(q(u')),\tilde{\rho}_2^\Lambda(s(u')))\right) & \text{on } p_2^{-1}(f(Y)). \end{array}\right.$$

In this case the compatibility condition ensures that

$$g^\Lambda(p(u))(q(u'),s(u'))=g_2^\Lambda(p_2(u))(\tilde{\rho}_2^\Lambda(q(u')),\tilde{\rho}_2^\Lambda(s(u'))),$$

and so we obtain a smooth function by the assumption on g^Λ. Both cases having thus been considered, we get the final claim. □

A direct consequence of the theorem just proven is the following statement.

Corollary 7.5. *Let X_1 and X_2 be two diffeological spaces, and let f be a map defined on a subset of X_1 that is a diffeomorphism of its domain with its image. If $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ admit compatible pseudo-metrics, then $\Lambda^1(X_1\cup_f X_2)$ carries a pseudo-metric induced by them.*

8 Concluding remarks: $\Lambda^1(X_1\cup_f X_2)$ for non-injective f, and the case of $k\geqslant 2$

In this concluding section we collect what can be more-or-less easily said about the case of non-injective gluing maps, and make final observations regarding higher-degree forms.

8.1 The fibres of $\Lambda^1(X_1\cup_f X_2)$

Let us assume that f is a subduction, but that at least one point in $X_1\cup_f X_2$ has more than one lift to $X_1\sqcup X_2$. This is the case where certain points of $X_1\cup_f X_2$, specifically those in $x\in i_2(f(Y))$, may have more than one lift to X_1, and to describe the fibre of $\Lambda^1(X_1\cup_f X_2)$ we must refer to the third case of Proposition 5.2 in its full form. Recall that by the Proposition just mentioned, if $x\in i_2(f(Y))$, then

$$\pi^*(\Omega_x^1(X_1\cup_f X_2))=\left(\cap_{\tilde{x}\in Y_x}(\Omega_f^1)_{\tilde{x}}(X_1)\right)\times_{comp}\Omega^1_{\tilde{x}_2}(X_2),$$

and so the relationship with the fibres of $\Lambda^1(X_1)$ is far from evident.

8.1.1 The diffeomorphism $\Lambda^1_{(\pi^f)^{-1}(x)}(X_1^f)\cong(\Lambda_f^1)_{\pi^{-1}(x)}(X_1)$ for $x\in X_1^f\setminus\pi_1^f(Y)$

We have seen in Section 5 that there is a natural diffeomorphism $\Omega^1(X_1^f)\cong\Omega_f^1(X_1)$; we now shall see what becomes of it when we consider pseudo-bundles $\Lambda^1_{(\pi^f)^{-1}(x)}(X_1^f)$ and $(\Lambda_f^1)_{\pi^{-1}(x)}(X_1)$ for x that lies outside of the domain of gluing Y. Since such a point has a unique lift[15] to X_1, it is quite easy to see that the diffeomorphism still holds.

[15] As one might imagine, this is the crux of the matter.

Proposition 8.1. *For all $x \in X_1^f \setminus \pi_1^f(Y)$ there is a natural diffeomorphism*

$$\Lambda^1_{(\pi^f)^{-1}(x)}(X_1^f) \to (\Lambda^1_f)_{\pi^{-1}(x)}(X_1)$$

that lifts to the pullback map $(\pi_1^f)^* : \Omega^1(X_1^f) \to \Omega^1_f(X_1)$.

Proof. This follows essentially from the results already established, that is, from the fact that $(\pi_1^f)^*$ preserves the subspaces of vanishing 1-forms; this follows from it being essentially a component of the whole pullback map π^*, and Theorem 5.11, part 1. □

8.1.2 The points in $i_1(X_1 \setminus Y)$ and $i_2(X_2 \setminus f(Y))$

This is the easier case, and we dispense with it right away, using the results of the previous section.

Theorem 8.2. *Let f be a subduction, and let $x \in X_1 \cup_f X_2$. Then*

1. *If $x \in i_1(X_1 \setminus Y)$ then*
$$\Lambda^1_x(X_1 \cup_f X_2) \cong (\Lambda^1_f)_{\pi^{-1}(x)}(X_1).$$
2. *If $x \in i_2(X_2 \setminus f(Y))$ then*
$$\Lambda^1_x(X_1 \cup_f X_2) \cong \Lambda^1_{\pi^{-1}(x)}(X_2).$$

Proof. Recall first of all the identification $X_1 \cup_f X_2 = X_1^f \cup_{f_\sim} X_2$; since $f_\sim$ is a diffeomorphism with its image, by Theorem 6.1 we have that

$$\Lambda^1_x(X_1 \cup_f X_2) \cong \Lambda^1_{(\pi^f)^{-1}(x)}(X_1^f) \text{ and } \Lambda^1_x(X_1 \cup_f X_2) \cong \Lambda^1_{\pi^{-1}(x)}(X_2);$$

in particular, the second part of the statement is thus proven. The first part of the statement (the case of the space X_1) follows now from Proposition 8.1. □

8.1.3 A point $x \in X_1^f$ such that $(\pi_1^f)^{-1}(x)$ has more than one point

This is the case of a point $x \in \pi_1^f(Y)$. We have already seen that for such a point the space $\Omega^1_x(X_1^f)$ of all 1-forms vanishing at x has the following structure:

$$\Omega^1_x(X_1^f) := \bigcap_{\tilde{x} \in (\pi_1^f)^{-1}(x)} (\Omega^1_f)_{\tilde{x}}(X_1).$$

Assume for simplicity that $(\pi_1^f)^{-1}(x) = \{\tilde{x}_1, \tilde{x}_2\}$; thus, we have $\Omega^1_x(X_1^f) = (\Omega^1_f)_{\tilde{x}_1}(X_1) \cap (\Omega^1_f)_{\tilde{x}_2}(X_1)$, and so

$$\Lambda^1_x(X_1^f) = \Omega^1(X_1^f)/\Omega^1_x(X_1^f) \cong \Omega^1_f(X_1)/\left((\Omega^1_f)_{\tilde{x}_1}(X_1) \cap (\Omega^1_f)_{\tilde{x}_2}(X_1)\right).$$

We observe that, in general, the latter quotient is not identified with either $(\Lambda^1_f)_{\tilde{x}_1}(X_1)$ or $(\Lambda^1_f)_{\tilde{x}_2}(X_1)$. But both of them can be described as surjective images of $\Omega^1_f(X_1)/\Big((\Omega^1_f)_{\tilde{x}_1}(X_1) \cap (\Omega^1_f)_{\tilde{x}_2}(X_1)\Big)$. This is due to the following simple property of quotients of (abstract) groups: if A is a group and $B, C \leqslant A$ are two normal subgroups of it then $A/B \cong (A/(B \cap C)) / (B/(B \cap C))$ and $A/C \cong (A/(B \cap C)) / (C/(B \cap C))$. This indicates that, if f is not injective, the fibres of $\Lambda^1(X_1^f)$ are larger than those of $\Lambda^1_f(X_1)$; we end our discussion with this observation.

8.2 Remarks on $\Omega^k(X_1 \cup_f X_2)$ and $\Lambda^k(X_1 \cup_f X_2)$ with $k > 1$

Throughout the paper, we have only been speaking about 1-forms. However, except when considering specifically differentials of functions, we did not use the forms being degree one anywhere; in particular, the results obtained remain valid for any $k \geqslant 1$.

References

[1] J.D. Christensen – G. Sinnamon – E. Wu, *The D-topology for diffeological spaces*, arXiv.math 1302.2935v4.

[2] J.D. Christensen – E. Wu, *Tangent spaces and tangent bundles for diffeological spaces*, arXiv:1411.5425v1.

[3] P. Iglesias-Zemmour, *Fibrations difféologiques et homotopie*, Thèse de doctorat d'État, Université de Provence, Marseille, 1985.

[4] P. Iglesias-Zemmour, *Diffeology*, Mathematical Surveys and Monographs, 185, AMS, Providence, 2013.

[5] Y. Karshon — J. Watts, *Basic forms and orbit spaces: a diffeological approach*, to appear in SIGMA.

[6] E. Pervova, *Multilinear algebra in the context of diffeology*, arXiv:1504.08186v2.

[7] E. Pervova, *Diffeological Clifford algebras and pseudo-bundles of Clifford modules*, arXiv:1505.06894v2.

[8] E. Pervova, *On the notion of scalar product for finite-dimensional diffeological vector spaces*, arXiv:1507.03787v1.

[9] E. Pervova, *Diffeological vector pseudo-bundles*, Topology and Its Applications **202** (2016), pp. 269-300.

[10] E. Pervova, *Diffeological vector pseudo-bundles, and diffeological pseudo-metrics on them as substitutes for Riemannian metrics*, arXiv:1601.00170v1.

[11] J.M. Souriau, *Groups différentiels*, Differential geometrical methods in mathematical physics (Proc. Conf., Aix-en-Provence/Salamanca, 1979), Lecture Notes in Mathematics, 836, Springer, (1980), pp. 91-128.

[12] J.M. Souriau, *Groups différentiels de physique mathématique*, South Rhone seminar on geometry, II (Lyon, 1984), Astérisque 1985, Numéro Hors Série, pp. 341-399.

[13] A. Stacey, *Comparative smootheology*, Theory Appl. Categ., 25(4) (2011), pp. 64-117.

[14] M. Vincent, *Diffeological differential geometry*, Master Thesis, University of Copenhagen, 2008.

[15] J. Watts, *Diffeologies, Differential Spaces, and Symplectic Geometry*, PhD Thesis, 2012, University of Toronto, Canada.

[16] E. Wu, *Homological algebra for diffeological vector spaces*, arXiv:1406.6717v1.

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