Algorithmic simplification of knot diagrams: new moves and experiments

Carlo PETRONIO   Adolfo ZANELLATI

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Abstract

This note has an experimental nature and contains no new theorems.
We introduce certain moves for classical knot diagrams that for all the very many examples we have tested them on give a *monotonic complete simplification*. A *complete simplification* of a knot diagram D is a sequence of moves that transform D into a diagram D' with the minimal possible number of crossings for the isotopy class of the knot represented by D. The simplification is *monotonic* if the number of crossings never increases along the sequence. Our moves are certain $Z_{1,2,3}$ generalizing the classical Reidemeister move $R_{1,2,3}$, and another one C (together with a variant $\widetilde{C}$) aimed at detecting whether a knot diagram can be viewed as a connected sum of two easier ones.

We present an accurate description of the moves and several results of our implementation of the simplification procedure based on them, publicly available on the web.

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This paper describes constructions and experimental results originally due to the second named author, that were formalized, put into context, and generalized in collaboration with the first named author. No new theorem is proved.

We introduce new combinatorial moves $Z_{1,2,3}$ on knot diagrams, together with another move C and a variant $\widetilde{C}$ of C, that perform very well in the task of simplifying diagrams. Namely, as we have checked by implementing the moves in [29] and applying them to a wealth of examples, the moves are apparently very efficient in carrying out a *monotonic complete simplification*. Here by *complete simplification* of a knot diagram D we mean a sequence of moves that transform D into a diagram D' with the minimal possible number of crossings for the isotopy class of the knot represented by D. The

simplification is *monotonic* if the number of crossings never increases along the sequence.

The moves $Z_{1,2,3}$ extend the classical Reidemeister moves $R_{1,2,3}$, while C and $\widetilde{C}$ are aimed at detecting whether a diagram can be viewed as the connected sum of two easier ones. In practice, the simplification procedure based on the moves $Z_{1,2,3}$ only is already quite powerful, since it allows for instance to monotonically untangle most of the known hard diagrams of the unknot. However the moves $Z_{1,2,3}$ only are not sufficient in general, as an example provided to us by the referee shows, but this example is easily dealt with using the move C. We do not know whether the use of all the moves $Z_{1,2,3}$, C and $\widetilde{C}$ allows a monotonic complete simplification of any knot diagram, but if this were the case one would have an algorihtm to compute the crossing number of a knot, and in particular to detect knottedness.

In the history of knot theory quite some energy has been devoted to the problem of effectively detecting whether or not a knot diagram represents the unknot, and more generally of computing the crossing number of a link starting from an arbitrary diagram representing it. The point here is that, using only Reidemeister moves, one may have to temporarily increase the number of crossings before reducing the link to a minimal crossing diagram. This phenomenon occurs both for the unknot and for more general knots and links, as explained below. Many solutions of the unknotting problem, with various degrees of suitability for actual implementation, have been obtained over time, see Birman and Hirsch [1], Galatolo [12], Hass-Lagarias [14], Dynnikov [9] (building on Birman and Menasco [2, 3] and Cromwell [7]), and Lackenby [24]. Dynnikov's technique also leads to the solution of other computational problems in link theory (see also Kazantsev [20] for a further development), but apparently not a method for calculating the crossing number. The result of Lackenby provides an upper bound on the number of Reidemeister moves needed to untangle a diagram of the unknot, so it is a fundamental one from a theoretical viewpoint, but it is of impractical use. See also [4, 11, 10] for some contributions more focused on algorithmic efficiency.

Concerning the crossing number, a vast and deep literature exists on it, see for instance [25, 21, 13, 5, 17, 18, 28] and the recent fundamental results of Lackenby [22, 23], but no effective algorithm to compute it is available to our knowledge. We note that the achievement of Coward and Lackenby [6] yields a theoretical algorithmic procedure to decide whether two diagrams represent the same link, so in principle it allows one to compute the crossing number in an indirect way, namely by comparing a diagram to *all* those

having fewer crossings. The moves introduced in this paper are not proved to actually yield an algorithm to compute the crossing number in general, but in practice they seem to work very effectively. We mention here that the simplification algorithm based on our moves $Z_{1,2,3}$, C and $\widetilde{C}$ applies not only to knots but also to multi-component links, but our implementation [29] is currently restricted to knots, so we have no evidence of efficiency for links so far.

The structure of the paper is as follows. In Section 1 we introduce the three moves $Z_{1,2,3}$ that extend the classical Reidemeister moves (see, *e.g.*, [27]), and we describe an algorithm based on these moves aimed at simplifying link diagrams. Then in Section 2 we describe many experimental applications of our algorithm to famously hard knot diagrams, displaying its remarkable practical performance. Finally, in Section 3 we introduce the further moves C and $\widetilde{C}$, showing that they sometimes make the simplification task much faster, and, more importantly, verifying that in some cases they allow simplifications that the moves $Z_{1,2,3}$ only are not capable of realizing.

1 Generalized Reidemeister moves and the reduced simplification algorithm

In this section we introduce the moves $Z_{1,2,3}$ and we describe the diagram simplification algorithm based on them.

Notation for links and the crossing number We will define our moves in a formal abstract way, and we will also illustrate them pictorially. To do this, we define a *link* L to be a tame embedding $\ell : \bigsqcup_{i=1}^{p} S_i^1 \to S^3$, where each S_i^1 is a circle, $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and ℓ avoids ∞, with ℓ viewed up to isotopy in S^3. We then define a *link diagram* D to be an immersion $p : \bigsqcup_{i=1}^{p} S_i^1 \to S^2$ with normal double points (crossings) as only singularities, where $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and p avoids ∞, together with the usual under-over indication at each crossing (no specific notation is employed for this

indication). Here D is viewed up to isotopy on S^2 and it determines a link $[D]$.

We denote by $c(D)$ the number of crossings of a diagram D, and by $c(L)$ the minimum of $c(D)$ over the diagrams D with $[D] = L$. This minimum is called the *crossing number* of L. If $[D] = L$ and $c(D) = c(L)$ we say that D is *minimal*. The general aim of this paper is to describe combinatorial moves capable of completely simplifying any given diagram D in a monotonic fashion, namely transforming D into a minimal diagram D' with $[D'] = [D]$ without ever increasing c.

Graphic conventions A link diagram D is drawn as usual, and in our figures of a move μ we always adopt the following conventions:

- A thin solid line is part of D;

- A slightly thicker line is also part of D, highlighted for the role it plays in the definition of μ;

- A very thick line represents a portion of the plane where several strands of D can appear and cross each other; often these thick lines are drawn so as to merely suggest the possible behaviors of D, the exact conditions to be met being described in the formal definition;

- A dashed line is always transverse to D and not part of D, with its ends (if any) on D but not at crossings;

- A gray shading is used to highlight a planar region playing a role in the definition of μ.

If $\alpha \subset S_i^1$ is a closed arc we call *extension* of $p(\alpha)$ a curve $p(\widetilde{\alpha})$ where $\widetilde{\alpha}$ is an arc that contains α in its interior. A *germ of extension* is one of the two components of $p(\widetilde{\alpha} \setminus \alpha)$, viewed up to inclusion.

The move Z_1 Let $\alpha \subset S_i^1$ be a closed arc such that $p|_\alpha$ is a simple closed curve, and let Ω be the component of $S^2 \setminus p(\alpha)$ that does not contain the germs of extensions of $p(\alpha)$. Let $\beta_1, \ldots, \beta_N$ be the components (each an arc or a circle) of $p^{-1}(\Omega)$, and suppose we can assign them labels $\lambda_1, \ldots, \lambda_N$ in $\{U, O\}$ so that:

- If β_j is an arc then $p(\beta_j)$ is over $p(\alpha)$ at both ends if $\lambda_j = O$, and under $p(\alpha)$ if $\lambda_j = U$;

- If $\lambda_j = U$ and $\lambda_k = O$ then $p(\beta_j)$ is under $p(\beta_k)$ where they cross.
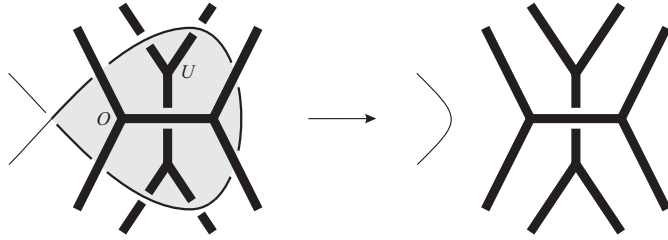
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Figure 1: The thick arc on D is $p(\alpha)$, and the gray region is Ω.

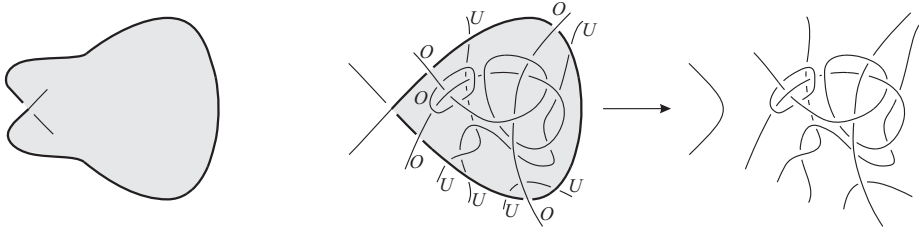Figure 2: Left: the wrong choice of Ω for Z_1. Right: an instance of Z_1.

We then call Z_1 the move that consists of collapsing α to a point (see Fig. 1). Of course Z_1 preserves the link type, and the Reidemeister move R_1 is a special case of Z_1.

Figure 2-left illustrates the *wrong* choice of Ω as a component of $S^2 \backslash p(\alpha)$. An application of Z_1 is given in Fig. 2-right.

The move Z_2 For $\lambda = U, O$ let $\alpha_\lambda \subset S^1_{i_\lambda}$ be a closed arc such that $p|_{\alpha_\lambda}$ is a simple curve. Suppose that $p(\alpha_U)$ and $p(\alpha_O)$ have the same ends at two crossings of D, and that $p(\alpha_U)$ is under $p(\alpha_O)$ at both. Set $\alpha = \alpha_U \cup \alpha_O$, let Ω be one of the components of $S^2 \setminus p(\alpha)$, and assume that Ω does not contain any of the four germs of extensions of $p(\alpha)$. Let $\beta_1, \ldots, \beta_N$ be the components (each a arc or a circle) of $p^{-1}(\Omega)$, and suppose it is possible to assign them labels $\lambda_1, \ldots, \lambda_N$ in $\{U, O\}$ so that precisely the same conditions as in the definition of the move Z_1 are met. Under these assumptions we call Z_2 the move that consists of interchanging $p|_{\alpha_U}$ and $p|_{\alpha_O}$ and pulling apart their ends (see Fig. 3). Of course Z_2 preserves the link type, and the Reidemeister move R_2 is a special case of Z_2.

Remark 1.1. In the description of both the moves Z_1 and Z_2, the labels for the β_j's that are arcs are fixed by the first condition, while both labels
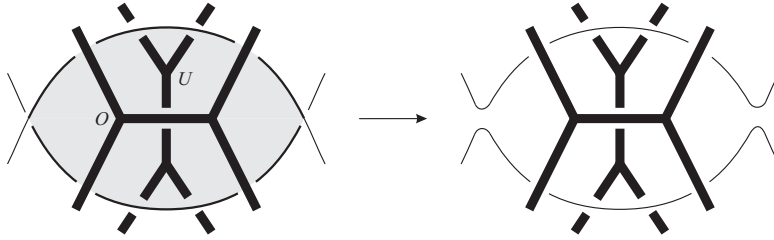
Figure 3: The move Z_2. The thick arcs on D are $p(\alpha_{U,O})$, and the gray region is Ω.

U and O should be tested for each β_j being a circle, verifying whether some choice allows the second condition to be met.

The move Z_3 Let $\alpha \subset S_i^1$ be a closed arc such that $p|_\alpha$ is a simple curve with its ends not being crossings of D. Suppose that $p(\alpha)$ is over (respectively, under) at all the crossings it contains. Let γ be a simple arc in S^2 with the same ends as $p(\alpha)$ but otherwise disjoint from it, and transverse to D (including at its ends). Then the move Z_3 consists of replacing $p|_\alpha$ by γ and stipulating that γ is over (respectively, under) at all the crossings it contains. Figure 4 shows this move for the case of an overarc and displays a small subtlety: two slightly different situations can arise, namely the two ends of γ can lie on the same side of $p(\alpha)$, as in Fig. 4-top, or on opposite sides, as in Fig. 4-bottom. In other words, if Ω is one of the two components of $S^2 \setminus (p(\alpha) \cup \gamma)$, we see that Ω can contain an even (0 or 2) or odd (1) number of germs of extensions of $p(\alpha)$ —the two even cases are the same up to the choice of Ω.

We will now introduce some restrictions on Z_3. To this we note that for a split link $L_1 \sqcup L_2$ we have $c(L_1 \sqcup L_2) = c(L_1) + c(L_2)$, so we can rule out this case and always assume that the regions of $S^2 \setminus D$ are topological discs. We then denote by $\widehat{D}$ the dual planar graph, and we stipulate that the move Z_3 can be applied only if both the following hold:

- (*maximality*) α is a maximal overarc or underarc of D;

- (*minimality*) γ defines a minimal path in the graph $\widehat{D}$ dual to D.

Note that the minimality condition is met even if γ does cross D at all; in this case, the induced path in $\widehat{D}$ is a point.

Remark 1.2. As shown in Fig. 5, a Reidemeister move R_3 can always be realized as a Z_3 (but perhaps not satisfying maximality and minimality).
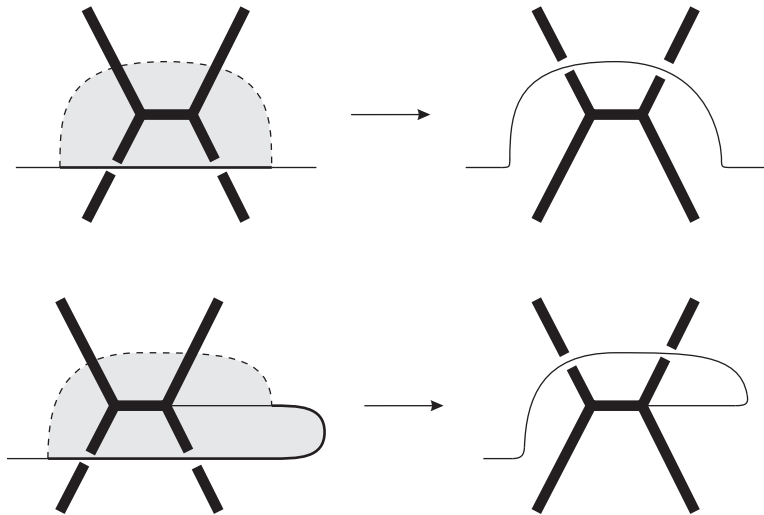
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Figure 4: Move Z_3 for an overarc, in its two possible versions. The thick arc is $p(\alpha)$, and the gray region Ω is actually not necessary to define the move (see the text).

Figure 5: An R_3 move is a Z_3 move.

The reduced simplification algorithm A move $D \to D'$ on link diagrams is called *decreasing* if $c(D') < c(D)$, *horizontal* if $c(D') = c(D)$, and *non-increasing* if $c(D') \leqslant c(D)$. A sequence $D = D_0 \to D_1 \to \ldots \to D_k \to D'$ of moves is called *monotonic* if each move $D_{i-1} \to D_i$ is non-increasing, and *strictly monotonic* if each $D_{i-1} \to D_i$ is decreasing.

We now describe the reduced version of the algorithm implemented in [29] (see Section 3 for the complete version):

Procedure $\mathcal{P}$ If there is a sequence $D = D_0 \to D_1 \to \ldots \to D_k \to D'$ with each $D_{i-1} \to D_i$ a horizontal Z_3 move and $D_k \to D'$ a decreasing $Z_{1,2,3}$ move, then apply the procedure to D'. If not, output D.

Two remarks about this procedure are in order:

- While applying procedure $\mathcal{P}$ to any given D one can restrict to sequences of horizontal Z_3 moves $D = D_0 \to D_1 \to \ldots \to D_k$ such that the D_i's are pairwise distinct, so a finite number of such sequences must be inspected;

- After one sequence $D = D_0 \to D_1 \to \ldots \to D_k \to D'$ of moves as in the definition of $\mathcal{P}$ is found, the moves are immediately performed and $\mathcal{P}$ proceeds with D', so D is never considered again.

We say that $\mathcal{P}$ succeeds on a link diagram D if it allows a complete simplification of D, namely, if it turns D into a diagram realizing the crossing number of $[D]$. Note that by definition this simplification is always monotonic, and it is strictly monotonic if no horizontal Z_3 move is employed.

Remark 1.3. By dropping the minimality and maximality restrictions on Z_3, one could have a potentially more general simplification procedure, but in practice this does not lead to any advantage. Similarly, one could inspect all the possible initial sequences $D = D_0 \to D_1 \to \ldots \to D_k \to D'$, and recursively proceed for all the diagrams D' thus obtained, but again this has no practical effect.

Remark 1.4. In practice procedure $\mathcal{P}$ has proved extremely efficient in bringing to their minimal status many complicated knot diagrams, see Section 2. In a previous version of this paper we were putting forward the conjecture that $\mathcal{P}$ would successfully do this for *every* diagram, but now *we know this conjecture is false*, because the referee provided us with a 120-crossing diagram of the unknot that does not untangle via non-increasing $Z_{1,2,3}$ moves, shown in Fig. 27 below. In fact, when we wrote the first version

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of the paper, we already had in our algorithm a further move C, described in Section 3 below. We knew examples where C allowed a quicker monotonic simplification than the $Z_{1,2,3}$ only, but we decided not to include C in the description of the algorithm because we had no example in which C was essential. The 120-crossing diagram is precisely such an example, because after one move C it untangles very quickly via $Z_{1,2,3}$. The enhanced procedure that uses also C will be described in Section 3.

2 Experimental evidence

In this section we explain the practical performance of the reduced procedure $\mathcal{P}$ on many knot diagrams that are known to be hard, and particularly on many diagrams of the unknot that do not monotonically untangle via Reidemeister moves. We will provide full details for some examples, while we will confine ourselves to the essential information for other cases, referring the reader to [29]. This website contains a publicly accessible implementation of our algorithm, with documentation and instructions. In particular, our software allows the user to input diagrams and simplify them.

We emphasize here that we know that $\mathcal{P}$ does not always succeed (see Remark 1.4 above) and that [29] already implements the complete procedure described below in Section 3.

Technical notes We have implemented [29] procedure $\mathcal{P}$ using Visual Basic 6, at 32 bits, under Windows XP on a 2010 laptop, and most of the time it takes is actually absorbed by the graphic handling of the diagrams —the "quick simplification" procedure typically runs much faster. For practical reasons our implementation is currently limited to the case of knots, but the extension to the case of multi-component links, possibly even split ones, presents no theoretical difficulties and will be carried out in the future.

We will begin by showing in full detail how the famous Culprit, Goeritz and Thistlethwaite diagrams of the unknot (see for instance [14, 16, 19]) untangle in a strictly monotonic and very quick way using our $Z_{1,2,3}$ moves. Then we will treat the Hass-Nowik knots [15], showing that the moves $Z_{1,2,3}$ untangle them in linear time (while the Reidemeister moves do so only quadratically). Next, we will describe the performance of our algorithm on many other diagrams of the unknot, mentioning in particular cases where a monotonic untangling via $Z_{1,2,3}$ moves succeeds but a strictly monotonic untangling impossible, and even cases where more than one consecutive horizontal Z_3 move is unavoidable. Finally, we will turn to non-trivial (and even

composite) knots, showing that our reduced algorithm is also successful (and extremely efficient) in bringing their diagrams to a minimal crossing number status. We single out here the Kazantsev knot diagram [20], with 23 crossings, which is intractable using only the Reidemeister moves, while our procedure $\mathcal{P}$ monotonically reduces it to its minimal status with 17 crossings via 6 moves of type Z_3 (one of which is horizontal). To conclude we will provide two examples of diagrams of non-trivial knots that require more than one consecutive horizontal Z_3 move to reach a minimal status.

The Culprit knot In Fig. 6 we show how procedure $\mathcal{P}$ applies to the Culprit knot. In the top-left corner of the picture we show the original Culprit diagram, then on its right the identification of a Z_3 move that applies to it, and below it the application of this move. The rest of the picture is similarly organized, with one more Z_3 move and one Z_1 move that completely untangle the diagram. We make the following remarks:

- No horizontal Z_3 move is needed in this case;

- The second Z_3 move applied could equivalently be described as a Z_2 (and, actually, an R_2) move; moreover, two other Z_2 moves could be applied instead of it, see Fig. 7, and both would also quickly lead to the untangling; in all the subsequent examples we will refrain from showing alternative simplification moves, even when many exist;

- The region Ω to which the final Z_1 move is applied is unbounded.

The Goeritz knot The Goeritz knot of Fig. 8-top/left is untangled as shown in the rest of the figure, via four Z_3 moves (none of which horizontal) and one Z_1 (applied to an unbounded Ω). Here and elsewhere we consider a knot diagram to be untangled if it has fewer than three crossings, because any diagram of a non-trivial knot has more than two.

The Thistlethwaite knot The Thistlethwaite knot of Fig. 9-top/left is untangled as shown in the rest of the figure, via three $Z_{1,2,3}$ moves. Again all the moves strictly decrease the number of vertices, and Z_1 is applied to an unbounded Ω.

The Hass-Nowik knots The diagram of the unknot shown in Fig. 10-top has $7n - 1$ crossings and was shown in [15] to require at least $2n^2 + 3n - 2$ Reidemeister moves to reduce to the trivial diagram. In the rest of Fig. 10 we
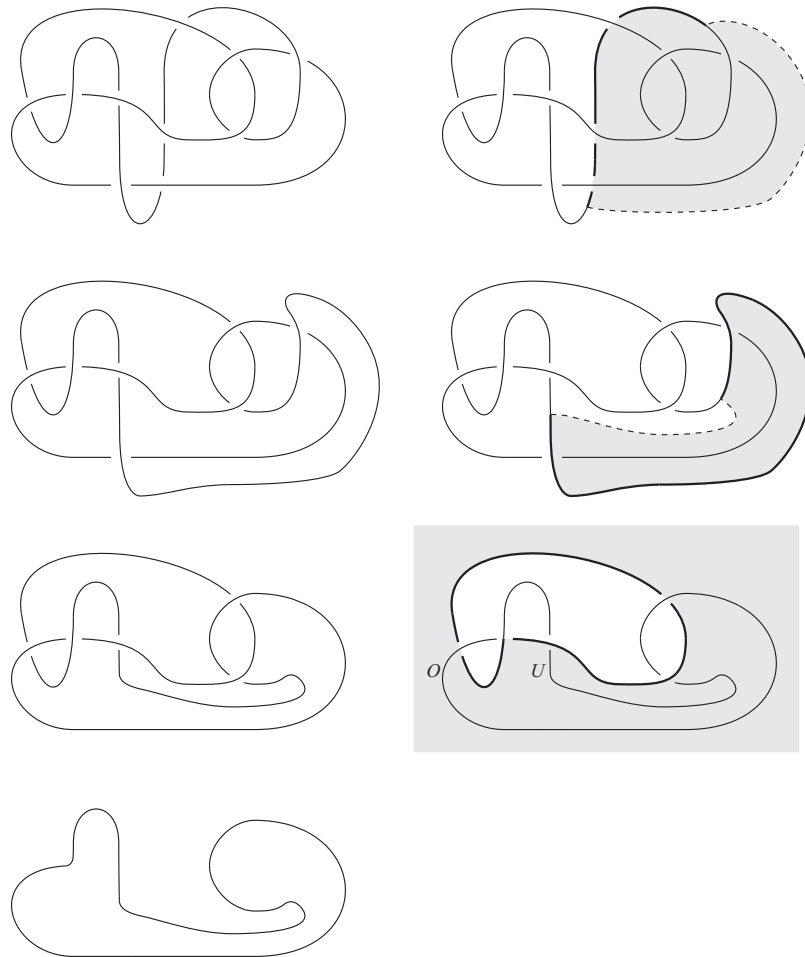
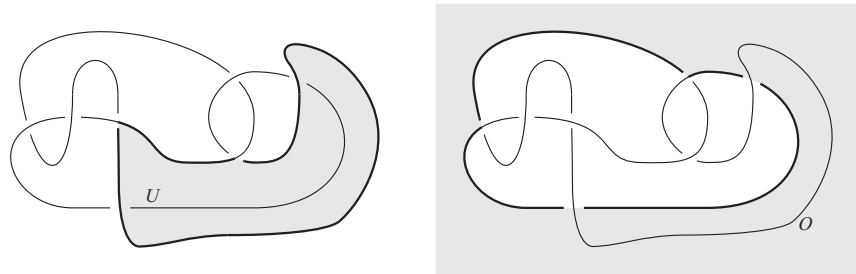Figure 6: Monotonic untangling of the Culprit knot via two Z_3 and one Z_1 moves.

Figure 7: Alternative Z_2 moves one could apply to the Culprit knot after the first Z_3.

show how to apply $2n-1$ moves of type Z_2, thus reducing to $3n+1$ crossings, and then in Fig. 11 we show a move Z_1 that removes $2n+1$ crossings, after which an obvious sequence of n moves R_1 untangles the diagram. This shows that while $2n^2 + 3n - 2$ Reidemeister moves are necessary, $3n$ (strictly decreasing) moves $Z_{1,2,3}$ suffice.

Necessity of horizontal moves In Figure. 12-top/left we show a certain diagram K_{31} of the unknot, also considered by Dynnikov [8]. Direct inspection shows (and our software [29] confirms) that no decreasing $Z_{1,2,3}$ move applies to this diagram. However, as shown in the same picture, after a horizontal Z_3 move a decreasing move is possible. As illustrated in [29], a sequence of 8 further $Z_{1,2,3}$ moves (all decreasing) leads to a complete untanglement of the diagram.

More hard diagrams of the unknot The reader will find in [29] full details on the following examples, all of which refer to diagrams of the unknot that do not monotonically untangle via Reidemeister moves:

- A diagram apparently due to Kauffman, with 9 crossings, monotonically untangled by $\mathcal{P}$ via 3 moves;

- The "monster" diagram (a name apparently also due to Kauffman), with 10 crossings, that untangles via 3 moves;

- A certain K_{12} diagram, with 12 crossings, that untangles via 4 moves;

- The two Ochiai diagrams, with 13 and 16 crossings respectively, that untangle via 5 and 6 moves;
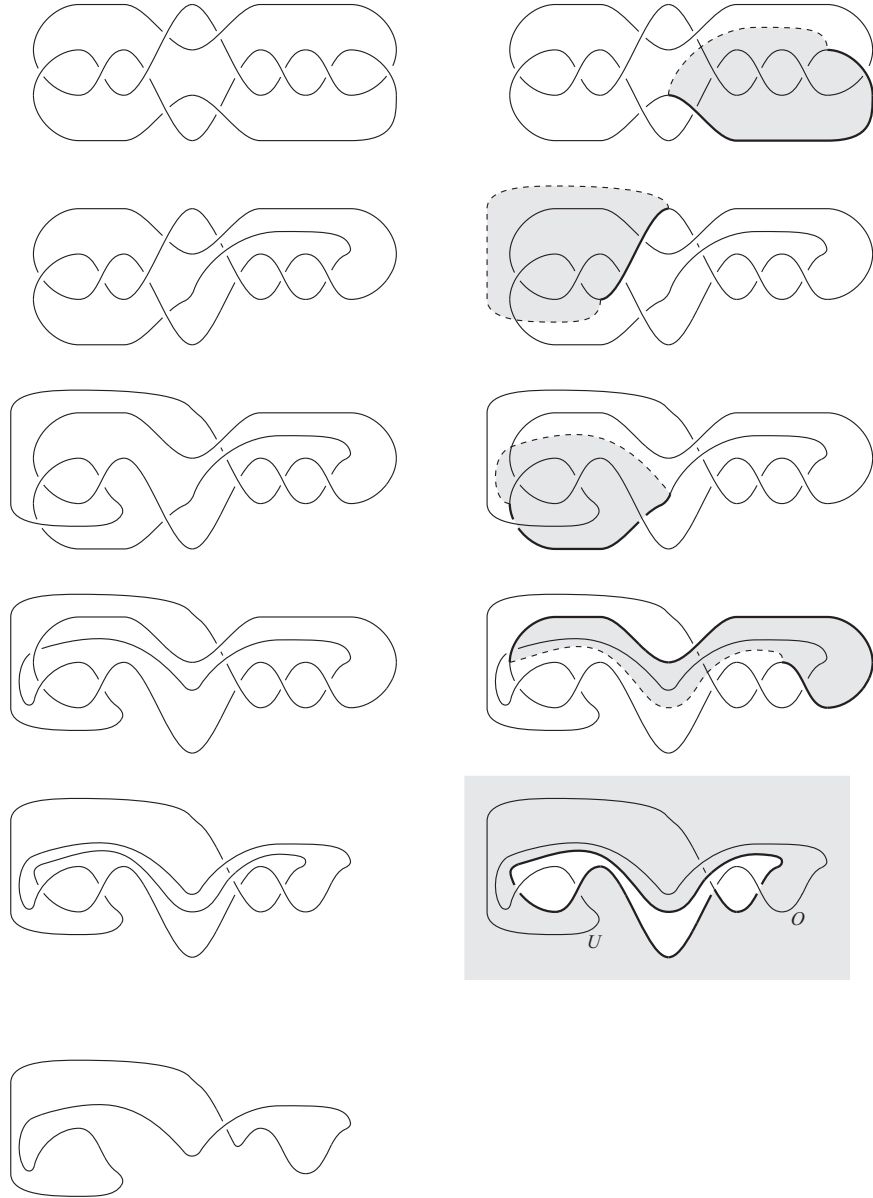
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Figure 8: Monotonic untangling of the Goeritz knot via four Z_3 and one Z_1 moves.
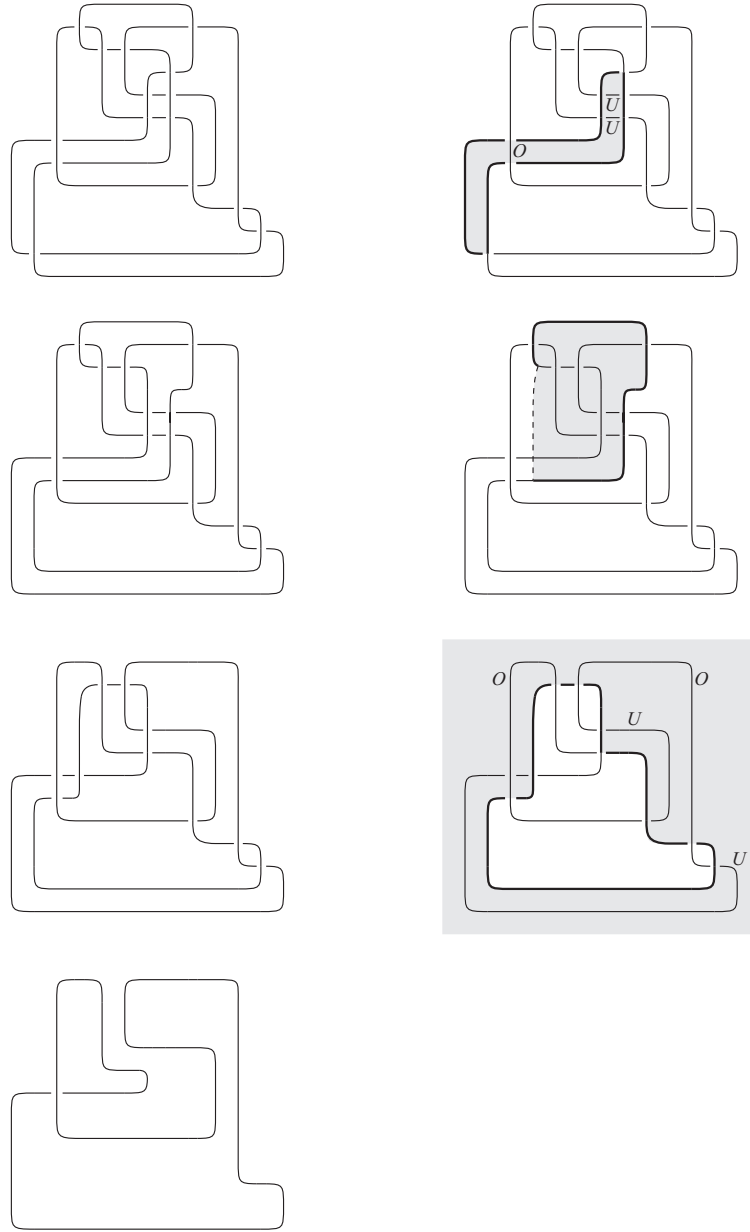
Figure 9: Monotonic untangling of the Thistlethwaite knot via Z_2, Z_3 and Z_1.
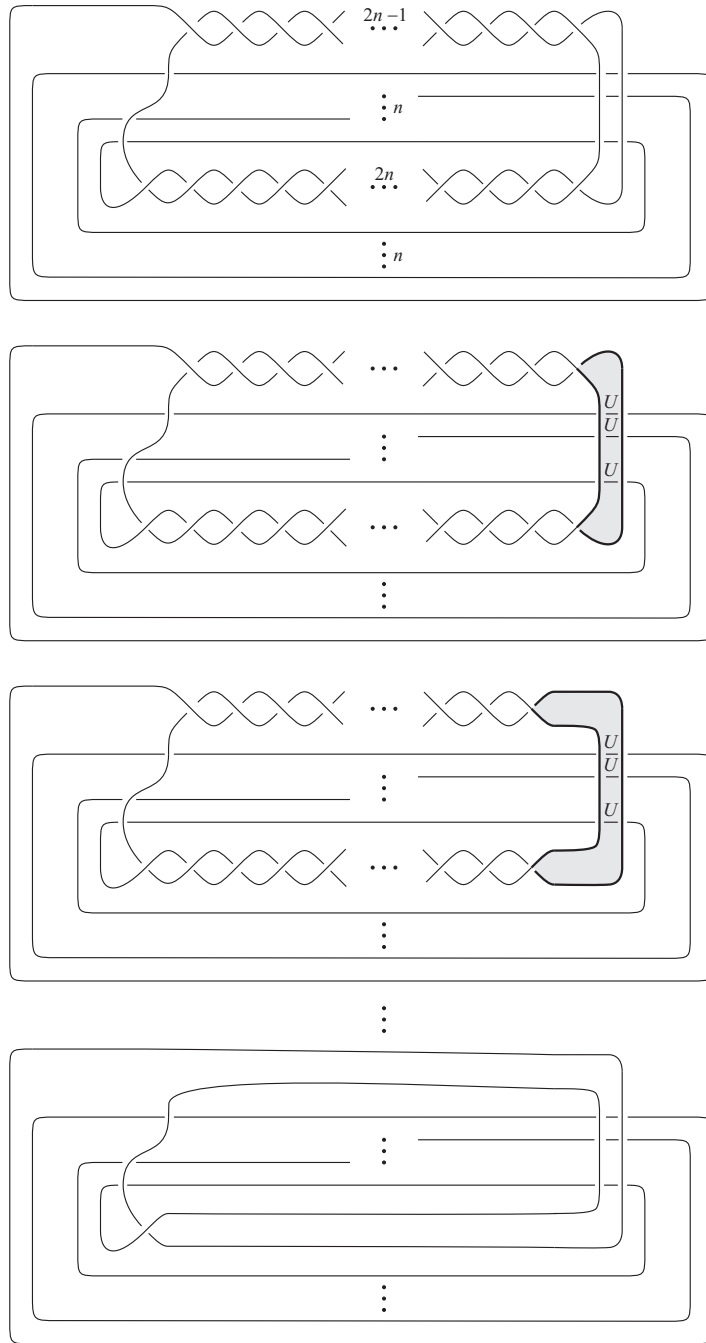
Figure 10: A diagram of the unknot that untangles quadratically using the Reidemeister moves and linearly using the moves $Z_{1,2,3}$: the initial sequence of Z_2 moves.
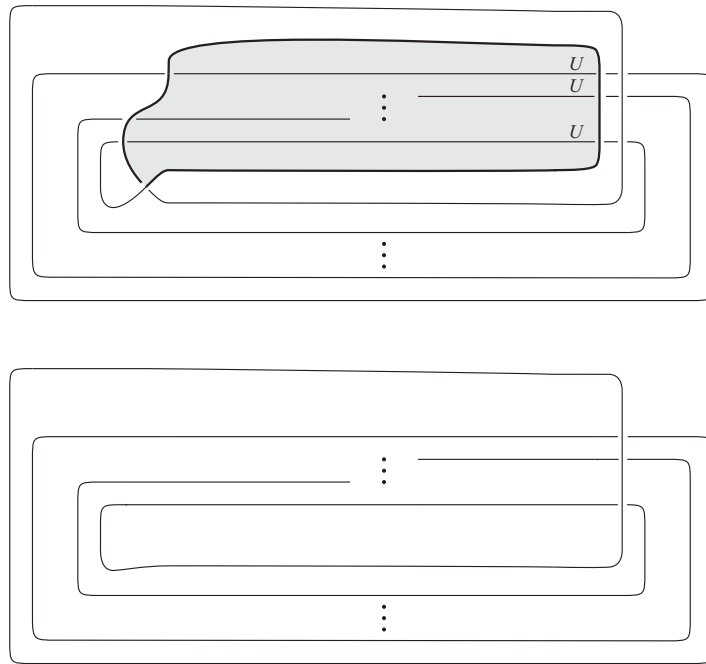
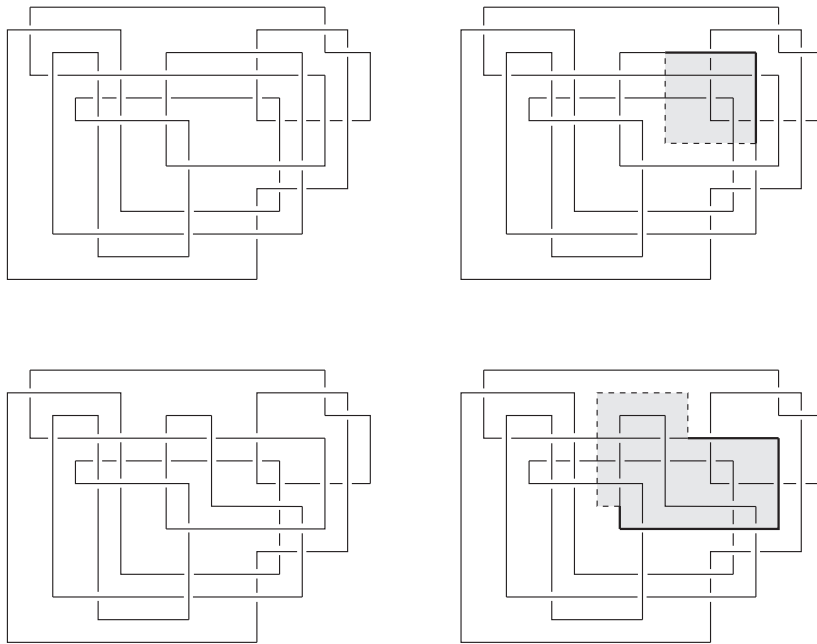Figure 11: A Z_1 move on the last diagram of Fig. 10, after which a sequence of R_1 moves gives the trivial diagram.

Figure 12: A diagram of the unknot that untangles in a monotonic but not in a strictly monotonic way (first two of a sequence of 10 untangling $Z_{1,2,3}$ moves).

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Figure 13: By inserting a positive full twist of the seven strands entering the shadowed lower box in the picture, and a negative full twist of the seven strands entering the shadowed upper box in the picture, we get a diagram of the unknot with 204 crossings.

- A diagram with 32 crossings, apparently due to Freedman, that untangles via 8 moves;

- A satellite diagram of the unknot with 64 crossings, apparently due to Hass, monotonically untangled by procedure $\mathcal{P}$ via 31 moves;

- The *Haken Gordian knot*, with 141 crossings, untangled via 53 moves;

- The *Haken satellite knot*, with 188 crossings, untangled via 68 moves.

A big unknot We show in Fig. 13 a diagram of the unknot provided to us by the referee (in fact a variation of the diagram of Fig. 27 below). Our reduced procedure untangles it via a strictly monotonic sequence of 80 moves of type $Z_{1,2,3}$.
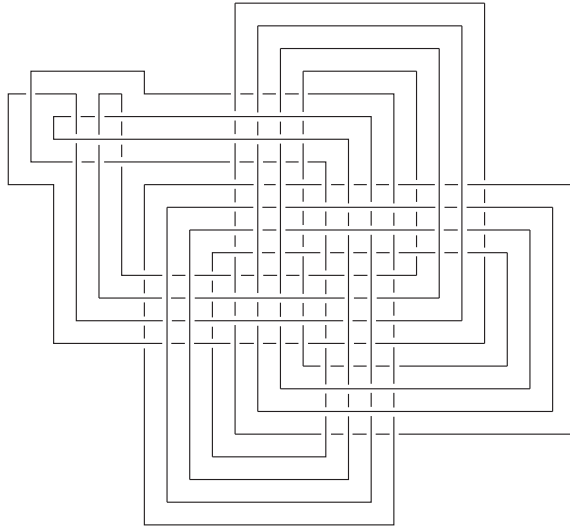
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Figure 14: A diagram of the unknot with 138 crossings.

Multiple horizontal moves For all the diagrams mentioned so far, the simplification was achieved through a sequence of non-increasing $Z_{1,2,3}$ moves with *at most one* consecutive horizontal Z_3. We describe here a first examples showing that multiple horizontal Z_3's may be required. It was provided to us by the referee, and it is shown in Figure 14. Our reduced algorithm manages to untangle it using a sequence of 37 moves of type $Z_{1,2,3}$, three of which horizontal Z_3's, including two consecutive ones at level 137. See also Fig. 24 below.

Quick simplifications We mention here two more diagrams of the unknot, both with 7 crossings, one from [26] and one known as the "nasty" diagram. They both simplify monotonically also via Reidemeister moves, but their untanglement via moves $Z_{1,2,3}$ is particularly efficient: only one Z_1 suffices for the first diagram, while a Z_2 and a Z_1 suffice for the second one.

Simplification of diagrams of non-trivial knots In Fig. 15-left we show a diagram [26] of the trefoil, and the rest of the picture proves that it reduces to the usual minimal diagram via one Z_1 move, followed by an obvious R_1 and an obvious R_2 (not illustrated). We should mention that this diagram is also monotonically reduced to a minimal one using Reidemeister moves only, but six of them are required.
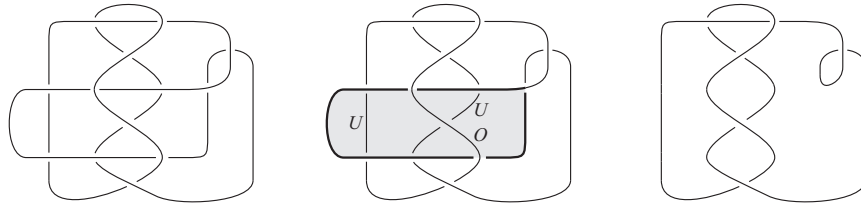
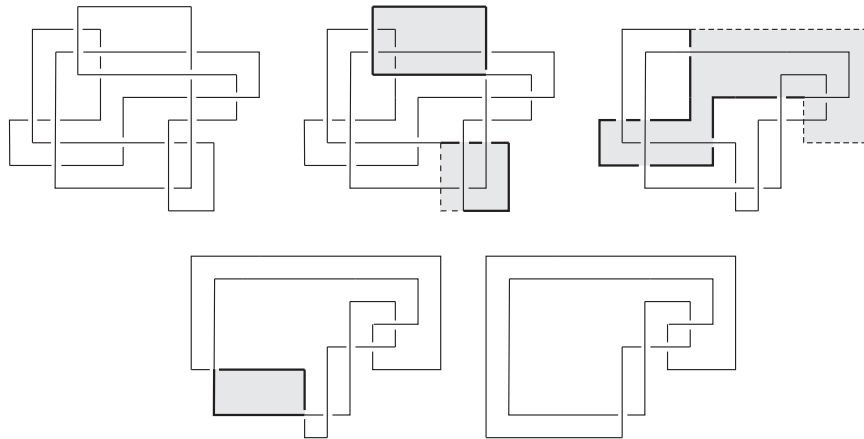19

Figure 15: Simplification of a diagram of the trefoil.

Figure 16: Simplification of a diagram of the figure-eight knot using a move Z_1 and a simultaneous Z_3, then a Z_3 and lastly a Z_2 (actually, an R_2).

Something similar happens for the diagram from [26] of the figure-eight knot, shown in Fig. 16. Via Reidemeister moves, this diagram reduces to 4 crossings monotonically, but only in a dozen passages, while procedure $\mathcal{P}$ completely simplifies it in 4 moves (in various different ways). Note that in Fig. 16 we show on the same diagram the result of a move and the identification of the next one, and we will do the same henceforth.

The Kazantsev knot A rather complicated non-trivial knot we treat comes from [20] and is shown in Fig. 17. This diagram has 23 crossings and cannot be monotonically simplified via Reidemeister moves, whereas an application of procedure $\mathcal{P}$ with 6 moves of type Z_3 (one of which is horizontal, and actually an R_3) reduces it to its minimal status with 17 crossings.
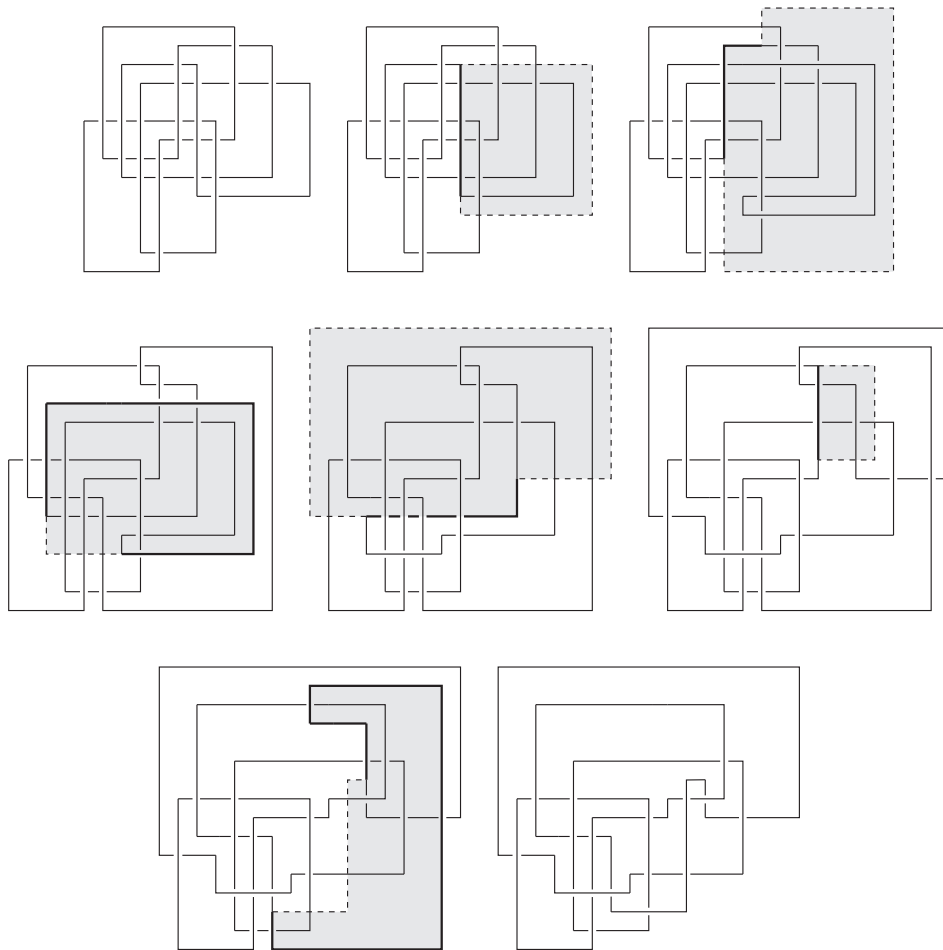
Figure 17: (Non-strictly) monotonic simplification via Z_3 of the Kazantsev knot.
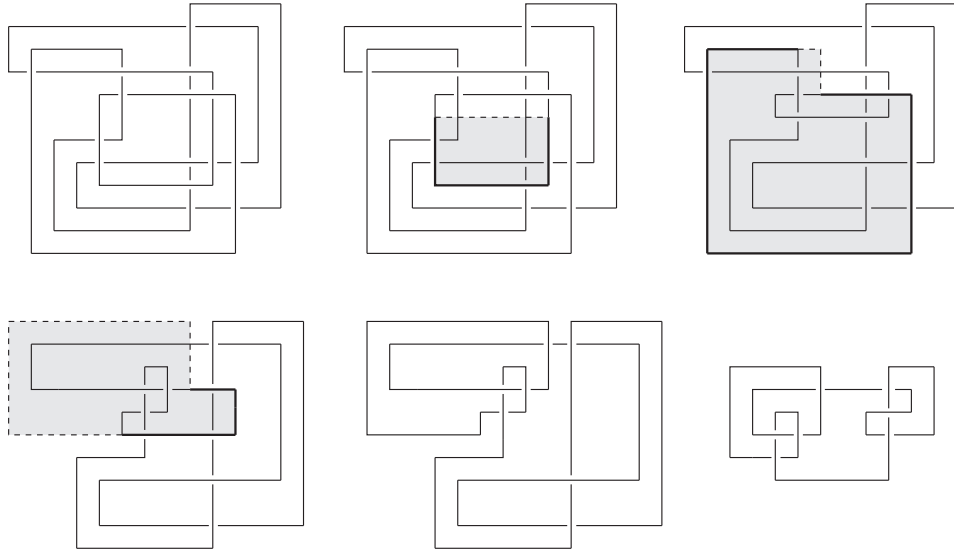
Figure 18: Simplification of a connected sum of a trefoil and a figure-eight knot.

A composite knot   Another interesting example of the efficiency of procedure $\mathcal{P}$ is illustrated in Fig. 18. Here we show a diagram with 15 crossings that cannot be monotonically simplified via Reidemeister moves, while three strictly decreasing Z_3's lead to a minimal diagram, that actually identifies the knot as the connected sum of a trefoil and a figure-eight knot (the last diagram is a mere redrawing of the previous one, to make the connected sum structure more apparent).

Two big non-trivial knots   We conclude by showing two diagrams of non-trivial knots that the reduced procedure $\mathcal{P}$ brings to a minimal status but only using more than one consecutive horizontal Z_3. The first diagram was suggested to us by Malik Obeidin, and it is the diagram shown in Fig. 19. The same figure shows how to transform it into a minimal diagram via two horizontal Z_3 moves followed by a decreasing one. The second diagram may be found in [29]. It has 32 crossings and $\mathcal{P}$ turns it into a minimal diagram with 28 via a sequence of 10 moves, and more precisely:

- Two horizontal Z_3 moves at level 32;

- One decreasing Z_2 move leading to level 30;
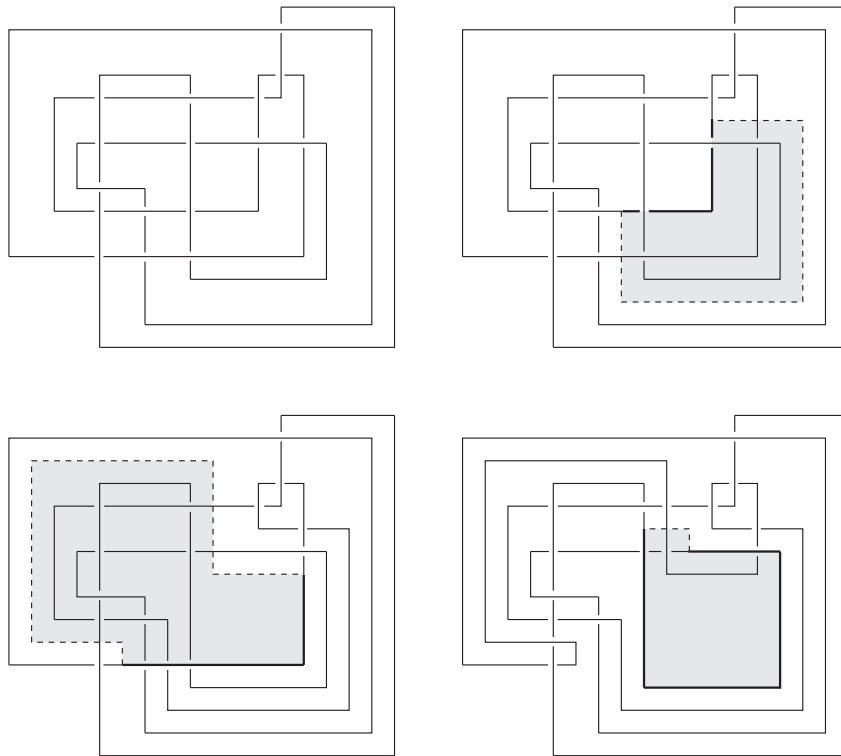
- Three horizontal Z_2 moves at level 30;

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Figure 19: A 16-crossing diagram and two horizontal Z_3 moves applied to it, after which there exists a Z_3 decreasing c by 1 (outcome of the move not shown).

- One decreasing move Z_3 leading to level 29;

- Two horizontal moves Z_3 at level 29;

- One decreasing move Z_3 leading to level 28.

3 Detection of composite links and the complete algorithm

In this section we introduce a further move C (and a variant $\widetilde{C}$ of it) that applied to a diagram D of a link L gives diagrams D_0, D_1 of links L_0, L_1 such that L is the connected sum of L_0 and L_1. Moreover D_0 and D_1 have strictly fewer crossings than D. This allows to break the task of simplifying D into the same task separately for D_0 and D_1. In particular:

- If via moves $Z_{1,2,3}$ we can show that both $[D_0]$ and $[D_1]$ are trivial, we can conclude that $[D]$ is also trivial;

- If via moves $Z_{1,2,3}$ we can show that one of D_0 or D_1 is trivial and we can transform the other one into a minimal D', we can conclude that $c([D]) = c(D')$;

- If via moves $Z_{1,2,3}$ we can transform D_0 or D_1 into minimal D'_0 and D'_1, *assuming that the conjectural [22] additivity of the crossing number is true*, we can conclude that $c([D]) = c(D'_0) + c(D'_1)$.

Our software [29] already contains the implementation of the moves C and $\widetilde{C}$ and performs the simplification of diagrams using them (but we recall that it currently handles knots only). The software can be asked by the user to look for a move C or $\widetilde{C}$, while in the quick simplification procedure it only uses these moves if it gets stuck with the $Z_{1,2,3}$.

The move C   Let $p : \overset{p}{\underset{i=1}{\sqcup}} S^1_i \to S^2$ be the immersion associated to a link diagram D. Let $\Omega \subset S^2$ be a tame topological disc with $\partial\Omega$ transverse to D, and $\alpha, \beta_1, \ldots, \beta_N$ be the components of $p^{-1}(\Omega)$. Suppose that α is an arc and that one can assign labels $\lambda_1, \ldots, \lambda_N$ in $\{U, O\}$ to $\beta_1, \ldots, \beta_N$ so that (see Fig. 20):

- If $\lambda_j = O$ then $p(\beta_j)$ is over $p(\alpha)$ where they cross;

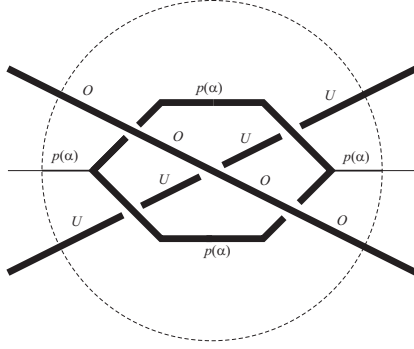- If $\lambda_j = U$ then $p(\beta_j)$ is under $p(\alpha)$ where they cross;

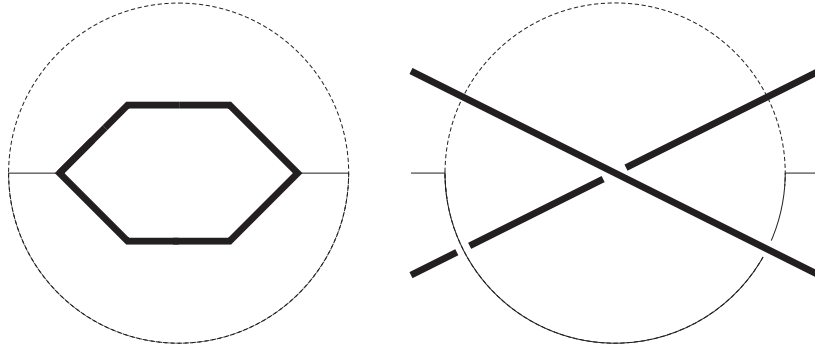Figure 20: A diagram D to which a move C applies.

Figure 21: The diagrams D_0 and D_1 produced by C.

- If $\lambda_j = O$ and $\lambda_k = U$ then $p(\beta_j)$ is over $p(\beta_k)$ where they cross.

Let γ be any of the two halves of $\partial\Omega$ into which the ends of $p(\alpha)$ split it. Given these data, we call C the move that replaces D by the pair of diagrams (D_0, D_1), where (see Fig. 21):

- D_0 is $p(\alpha)$ union γ with crossings as in D (none on γ);

- D_1 is D minus $p(\alpha)$ with crossings as in D, union γ, with γ under any β_j with $\lambda_j = O$, and over any β_j with $\lambda_j = U$.

Proposition 3.1. $[D] = [D_0] \# [D_1]$.

Proof. The assumptions imply that $[D]$ can be realized as a link L in $\mathbb{R}^3$ such that $L \cap (\Omega \times \mathbb{R})$ is the disjoint union of three portions:

- One in $\Omega \times (-\varepsilon, \varepsilon)$, with projection $p(\alpha)$;

- One in $\Omega \times (1 - \varepsilon, 1 + \varepsilon)$, with projection $\bigcup\{p(\beta_i) : \lambda_i = O\}$;

- One in $\Omega \times (-1 - \varepsilon, -1 + \varepsilon)$, with projection $\bigcup\{p(\beta_i) : \lambda_i = U\}$.

So the sphere $\partial(\Omega \times [-\varepsilon, \varepsilon])$ meets L transversely at two points, whence it allows to express L as $L_0 \# L_1$, and of course $L_j = [D_j]$. □

Remark 3.2. If in the definition of the move C we have that α is not an arc but one of the circles S_i^1 then L is the split link $[D_0] \sqcup [D_1]$, where D_0 is $p(\alpha)$ and D_1 is D with $p(\alpha)$ removed, both with crossings as in D.

We now note the following:

- If the ends of $p(\beta_i)$ separate the ends of $p(\alpha)$ on $\partial\Omega$ then $p(\beta_i)$ crosses $p(\alpha)$ at least once.

We also stipulate that the move C can only be applied if the following further assumption is met (otherwise we can replace Ω by a smaller disc):

- If the ends of $p(\beta_i)$ do not separate the ends of $p(\alpha)$ on $\partial\Omega$ then $p(\beta_i)$ crosses $p(\alpha)$ at least twice.

The remark and the assumption just made easily imply the following:

Proposition 3.3.
- $c(D_0) + c(D_1) \leqslant c(D)$;

- $c(D_0) < c(D)$ *unless* $p^{-1}(\Omega) = \alpha$ *and* D *has no crossings outside* Ω, *in which case* $D_0 = D$.

- $c(D_1) < c(D)$ *unless* $p(\alpha)$ *has no self-crossing and* $p(\beta_j)$ *crosses* $p(\alpha)$ *as many times as it crosses* γ, *in which case* $D_1 = D$.

Of course we do not apply the move C in any of the situations described in the last two items of this proposition. An example of application of C is provided in Figg. 22 and 23, and a first example showing that C is useful is described in Fig. 24, where we prove that it applies to the 138-crossing diagram of the unknot already considered in Fig. 14. The two resulting diagrams, shown in Fig. 25 are then both reduced via moves $Z_{1,2,3}$ to the trivial diagram in a handful of steps. Another diagram for which C is useful, even if not essential, is that of Fig. 18, as shown in Fig. 26.

We now describe an example provided to us by the referee showing that the move C is actually essential. Figure 27 displays a diagram with 120 crossings, which in fact represents the unknot, but our reduced algorithm
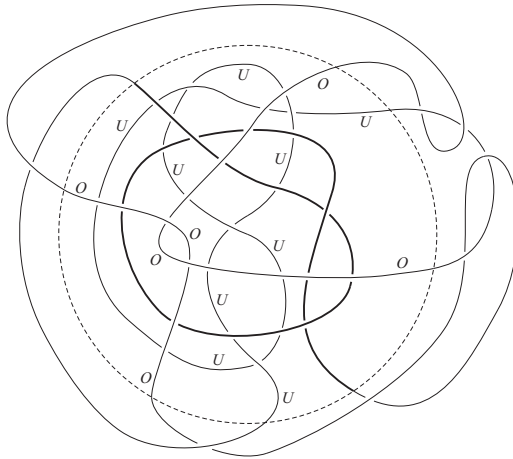
Figure 22: A move C applies to this diagram D. The disc Ω is the bounded region with the dashed circle as boundary, and $p(\alpha)$ is the slightly thicker arc.
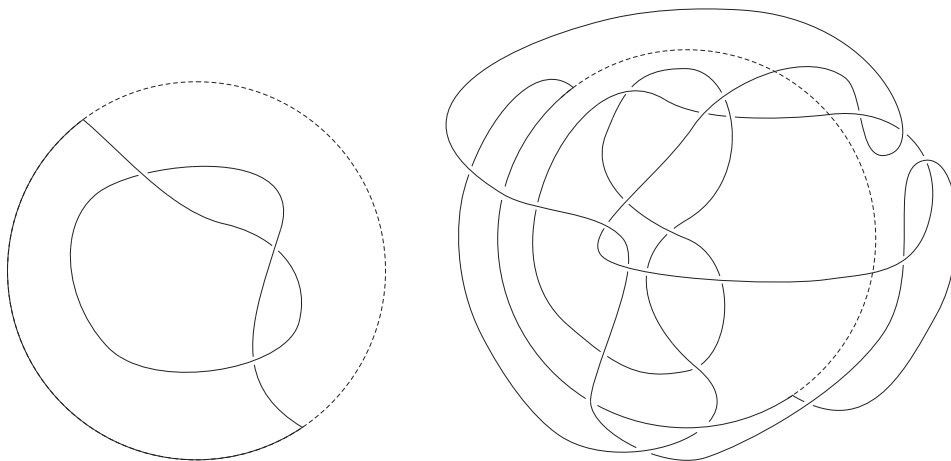

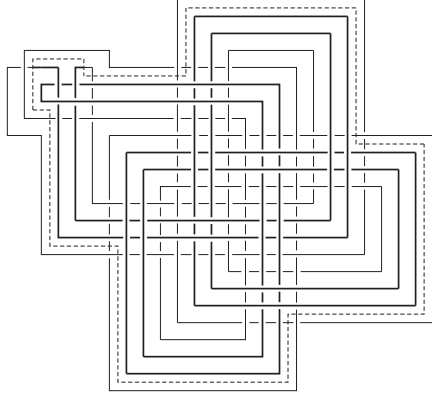Figure 23: Applyting C to the D of Fig. 22 we get these D_0 and D_1.

Figure 24: A move C applies to the diagram of Fig. 14. The disc Ω is the bounded one with the dashed curve as boundary, $p(\alpha)$ is the slightly thicker arc inside Ω, and all the other arcs in Ω have label U.
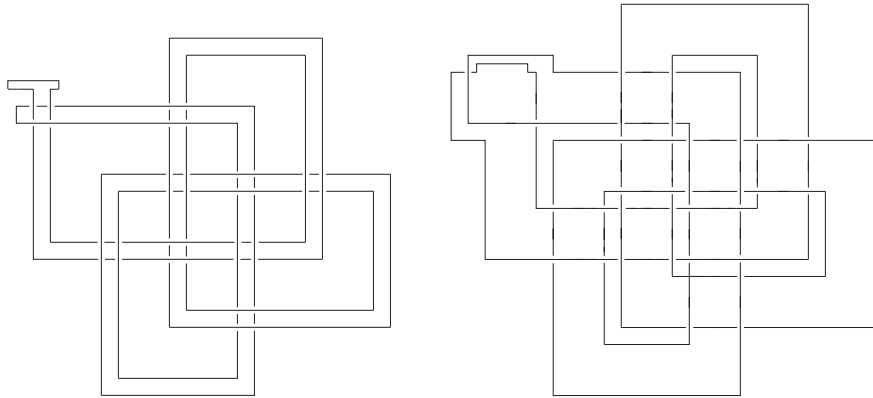

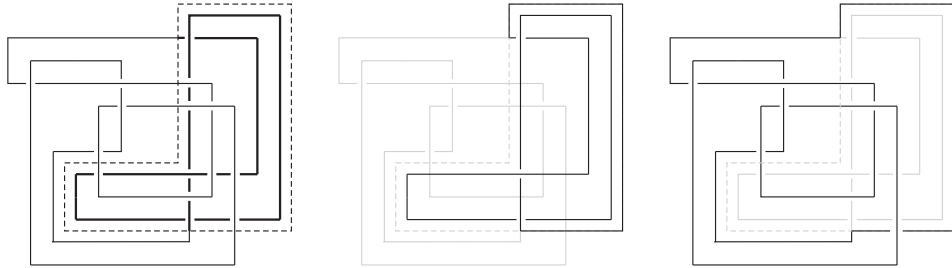Figure 25: Diagrams produced by the move C of Fig. 24.

Figure 26: Left: a move C applies to the diagram of Fig. 18. The disc Ω is the bounded one with the dashed curve as boundary, $p(\alpha)$ is the slightly thicker arc inside Ω, and all the other arcs in Ω have label O. Center and right: the diagrams produced by C are the standard one of the trefoil and one that becomes the standard one for the figure-eight after a more R_2.
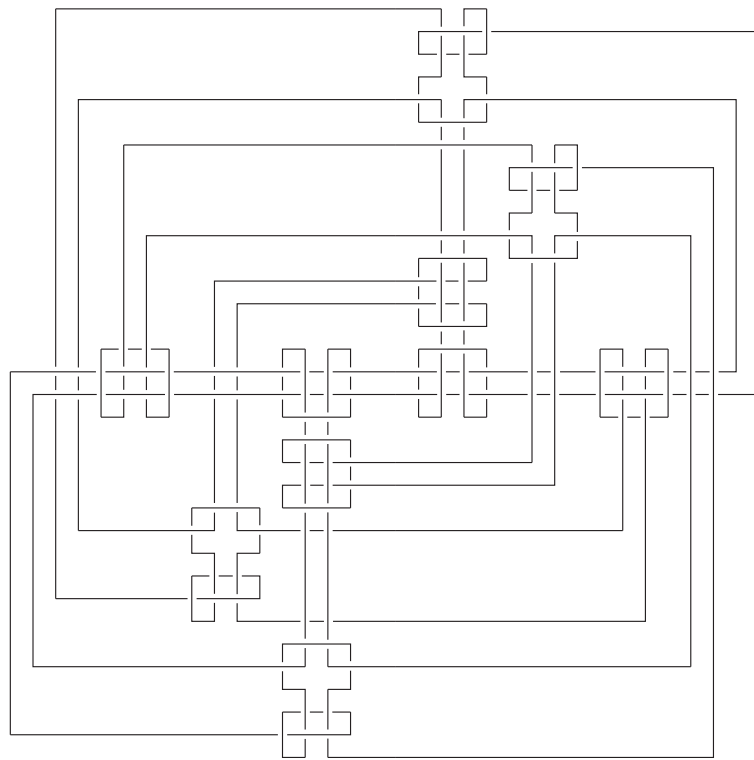

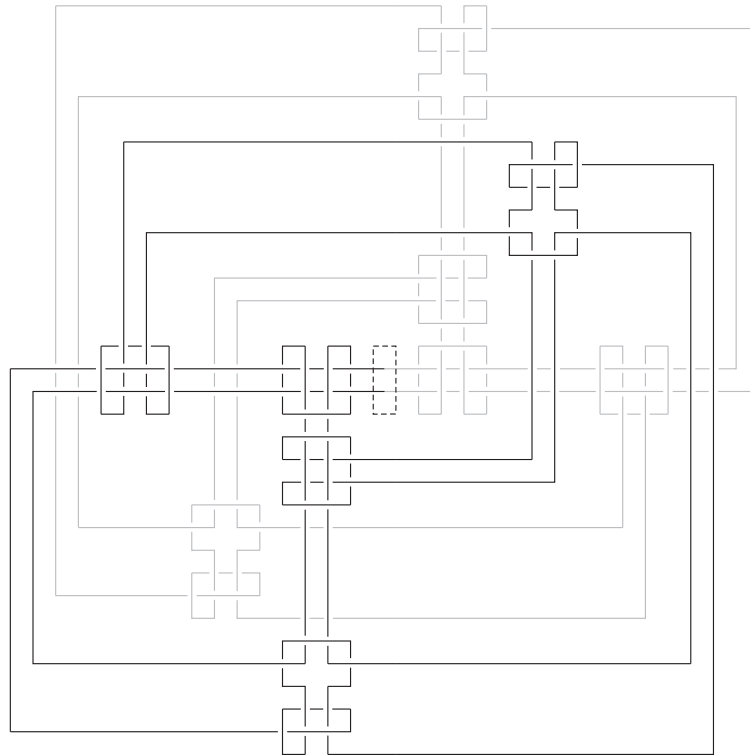Figure 27: A diagram of the unknot with 120 crossings.

Figure 28: A move C applies to the diagram of Fig. 27. The disc Ω is the unbounded one with the dashed curve as boundary, and actually two interpretations of the move are possible: either one declares the black arc in Ω to be $p(\alpha)$ and the gray one to have label U, or one declares the gray arc to be $p(\alpha)$ and the black one to have label O. The result is the same in either case.
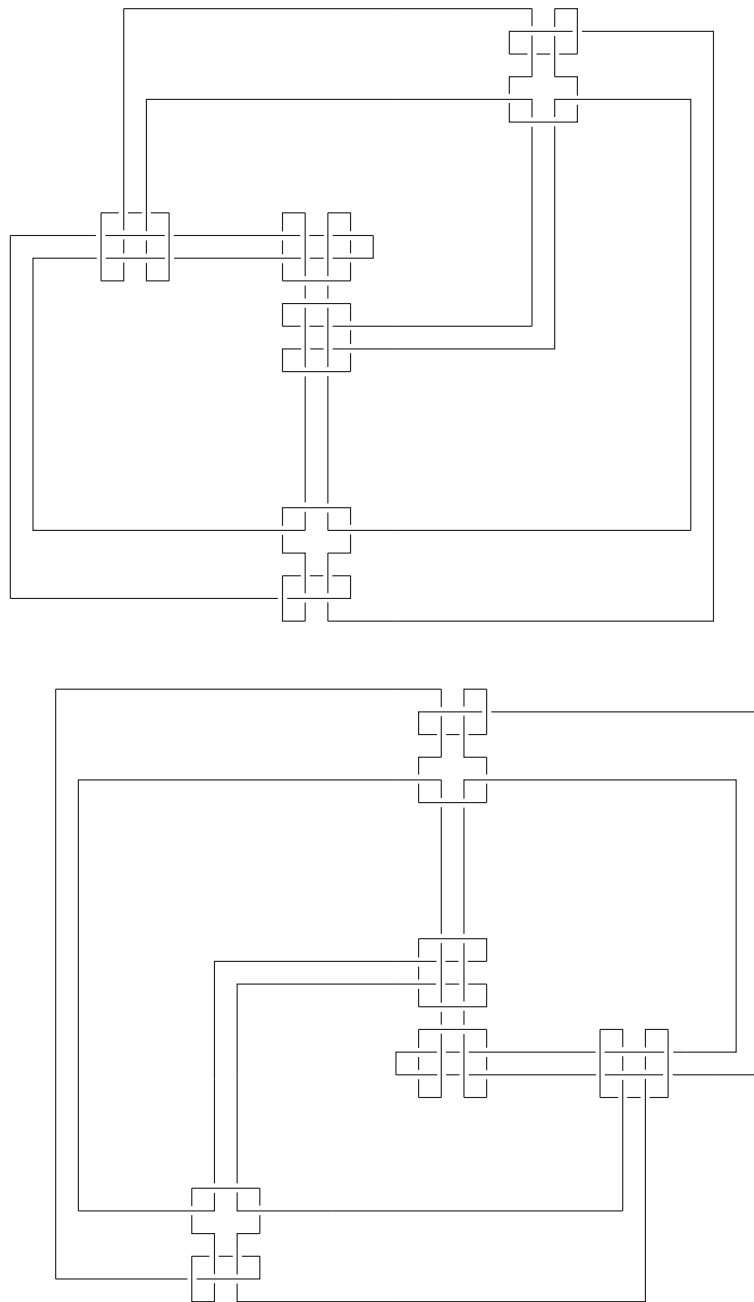
Figure 29: The move C of Fig. 28 gives these diagrams that untangle very easily.
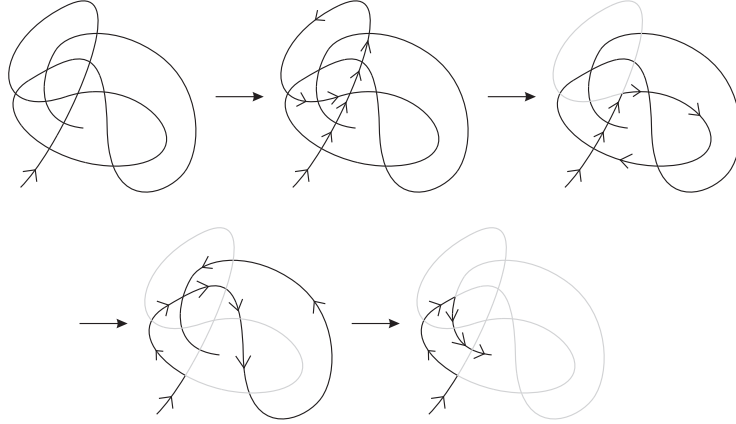
Figure 30: Straightening a curve by removing curls as soon as they are born.

(using monotonic $Z_{1,2,3}$ moves only) was unable to untangle it. On the other hand, our software found a C-move that applies to the diagram, as shown in Fig. 28, after which we get the diagrams of Fig. 29, which are both very quickly untangled via $Z_{1,2,3}$ moves.

Straightening of a curve   Before we can describe the announced variant $\widetilde{C}$ of the move C, we need to give an easy definition:

Definition 3.4. If c is any tame curve in the plane with normal crossings only and distinct ends, we denote by $\sigma(c)$ and call *straightening* of c any simple curve contained in c and having the same ends as c.

Remark 3.5. $\sigma(c)$ is not unique but finitely many possibilities exist, and one can be constructed algorithmically by following c from one of its ends and erasing any curl as soon as it is born, see Fig. 30.

The move $\widetilde{C}$ Let $p : \overset{p}{\underset{i=1}{\sqcup}} S^1_i \to S^2$ be the immersion associated to a link diagram D. Let Ω and $\widetilde{\Omega}$ be tame topological discs in S^2, with Ω contained in the interior of $\widetilde{\Omega}$, so the closure of $\widetilde{\Omega} \setminus \Omega$ is an annulus Θ. Suppose $\partial\Omega$ and $\partial\widetilde{\Omega}$ are transverse to D and the following happens (see Fig. 31):

- There exists an arc $\alpha \subset S^1_i$ such that $p(\alpha) \subset \Theta$ has one end on each component of $\partial\Theta$;

- The preimage $p^{-1}\left(\widetilde{\Omega}\right)$ has components $\widetilde{\beta}_0, \beta_1, \ldots, \beta_N$ with $\widetilde{\beta}_0 \supset \alpha$;
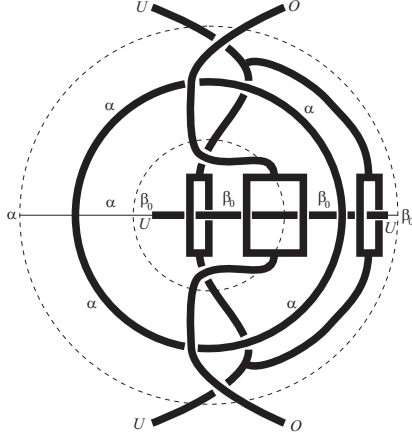
Figure 31: A diagram D to which the move $\widetilde{C}$ applies.

- If $\beta_0 = \widetilde{\beta}_0 \setminus \alpha$ one can choose labels $\lambda_0, \lambda_1, \ldots, \lambda_N$ in $\{O, U, \}$ so that:

  - If $\lambda_j = O$ then $p(\beta_j)$ is over $p(\alpha)$ where they cross;
  - If $\lambda_j = U$ then $p(\beta_j)$ is under $p(\alpha)$ where they cross;
  - If $\lambda_j = O$ and $\lambda_k = U$ then $p(\beta_j)$ is over $p(\beta_k)$ where they cross.

Given these data, we call C the move that replaces D by the pair of diagrams (D_0, D_1), where (see Fig. 32):

- D_0 is $p(\alpha) \cup \sigma(\beta_0)$ union one of the two halves into which the ends of β_0 split $\partial\widetilde{\Omega}$, with all the crossings as in D;

- D_1 is $(D \setminus p(\alpha)) \cup \sigma(p(\alpha))$, with all the crossings as in D.

Proposition 3.6. $[D] = [D_0] \# [D_1]$.

Proof. Suppose the label λ_0 of β_0 is U. The assumptions then imply that $[D]$ can be realized as a link L in $\mathbb{R}^3$ such that $L \cap \left(\widetilde{\Omega} \times \mathbb{R}\right)$ is the disjoint union of 4 portions:

- One in $\Theta \times (-\varepsilon, \varepsilon)$, with projection $p(\alpha)$;

- One in $\widetilde{\Omega} \times (1 - \varepsilon, 1 + \varepsilon)$, with projection $\bigcup\{p(\beta_i) : \lambda_i = O\}$;

- One in $\widetilde{\Omega} \times (-1 - \varepsilon, -1 + \varepsilon)$, with projection $\bigcup\{p(\beta_i) : \lambda_i = U\}$;
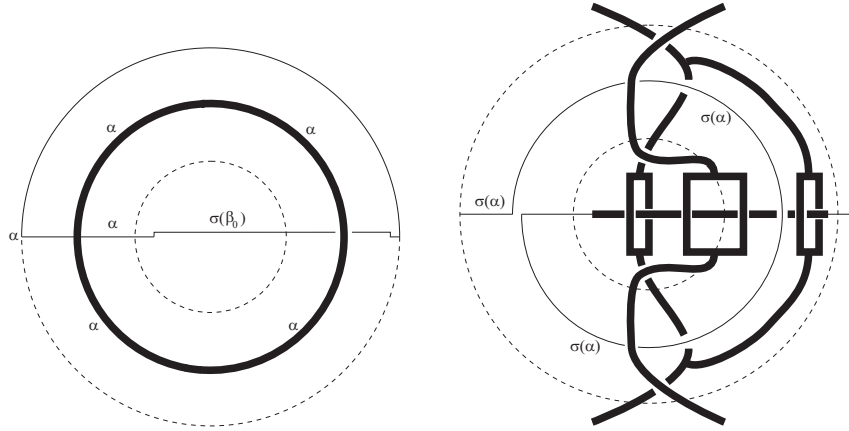
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Figure 32: The diagrams D_0 and D_1.

- A vertical arc $P \times [-1, 0]$ with $P \in \partial\Omega$.

So the sphere $\partial(\Omega \times [-\varepsilon, \varepsilon])$ meets L transversely at two points, whence it allows to express L as $L_0 \# L_1$, and of course $L_j = [D_j]$. ☐

Remark 3.7.
- In D_0 we used $\sigma(\beta_0)$ to join the ends of $p(\alpha)$ to have a definite procedure, but any simple arc joining them in Θ would work, provided its crossings with $p(\alpha)$ have the same type as those of β_0;

- Similarly, in D_1 we could replace $\sigma(p(\alpha))$ with any simple arc in Θ having the same ends, provided its crossings with the β_i's have the same type as those of $p(\alpha)$.

An example of move $\widetilde{C}$ is shown in Figg. 33 and 34; note that no move C exists in this case. As for the move C, the construction implies that $c(D_0) + c(D_1) \leqslant c(D)$. Moreover, $c(D_0) < c(D)$ and $c(D_1) < c(D)$ except in situations, that one can easily avoid, where one of the D_i's is actually D.

We conclude by remarking that we do not know whether $\widetilde{C}$ is actually essential: all the the knot diagrams we have used so far as tests for our algorithm were completely simplified in a monotonic fashion using the moves $Z_{1,2,3}$ and C only. (For the diagram of Fig. 33 a C appears after some $Z_{1,2,3}$.)

References

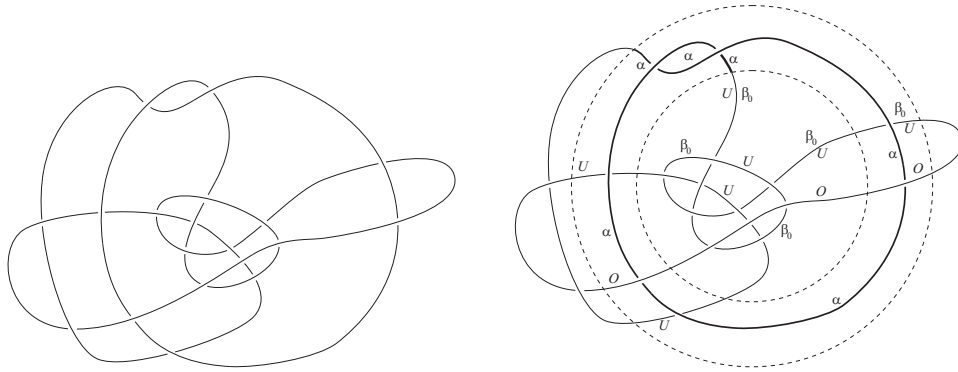[1] J. Birman, M. D. Hirsch, *A new algorithm for recognizing the unknot*, Geom. Top. **2** (1998), 175-220.

Figure 33: A diagram and the identification of a move $\widetilde{C}$ that applies to it.
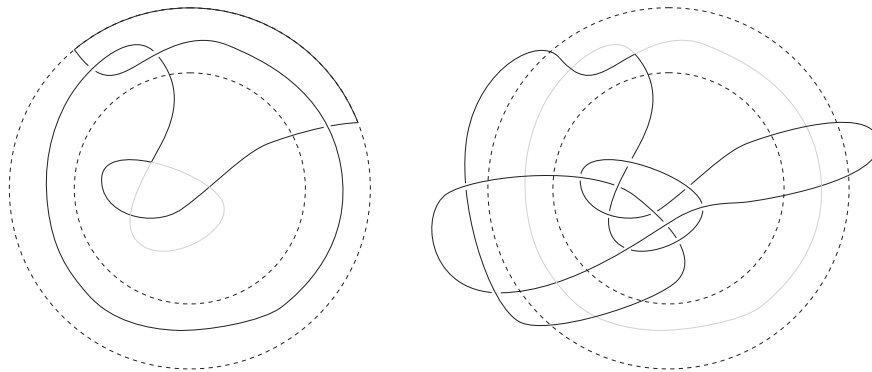

Figure 34: Diagrams resulting from the move $\widetilde{C}$.

[2] J. Birman, W. Menasco, *Studying links via closed braids IV: Composite links and split links*, Invent. Math. **102** (1990), 115-139.

[3] J. Birman, W. Menasco, *Studying links via closed braids V: Closed braid representatives of the unlink*, Trans. Amer. Math. Soc. **329** (1992), 585-606.

[4] H. Chuichiro, H. Miwa, *Unknotting number and number of Reidemeister moves needed for unlinking*, Topology Appl. **159** (2012), 1467-1474.

[5] J.-W. Chung, X.-S. Lin, *On the bridge number of knot diagrams with minimal crossings*, Math. Proc. Camb. Philos. Soc. **137** (2004), 617-632.

[6] A. Coward, M. Lackenby, *An upper bound on Reidemeister moves*, Amer. J. Math. **136** (2014), 1023-1066.

[7] P. Cromwell, *Embedding knots and links in an open book I: Basic properties*, Topology Appl. **64** (1995), 37-58.

[8] I. A. Dynnikov, *Three-page link presentation and an untangling algorithm*, In: "Proc. of the International Conference Low-Dimensional Topology and Combinatorial Group Theory, Chelyabinsk, July 31-August 7, 1999," Kiev, 2000, pp. 112-130.

[9] I. A. Dynnikov, *Arc-presentations of links. Monotonic simplification*, Fundam. Math. **190** (2006), 29-76.

[10] A. Enrich, L. Kauffman, *Unknotting unknots*, Amer. Math. Monthly **121** (2014), 379-390.

[11] B. Foley, *Automated Reidemeister moves: a numerical approach to the unknotting problem*, To appear in "Proceedings of Midstates Conference for Undergraduate Research in Computer Science and Mathematics".

[12] S. Galatolo, *On a problem of effective knot theory*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **9** (1998), 299-306.

[13] H. Gruber, *Estimates for the minimal crossing number*, `arXiv:math/0303273v3`.

[14] J. Hass, J. Lagarias, *The number of Reidemeister moves needed for unknotting*, J. Amer. Math. Soc. **14** (2001), 399-428 (electronic).

[15] J. Hass, T. Nowik, *Unknot diagrams requiring a quadratic number of Reidemeister moves to untangle*, Discrete Computut. Geom. **44** (2010), 91-95.

[16] A. Henrich, L. H. Kauffman, *Unknotting unknots*, Amer. Math Monthly **121** (2014), 379-390.

[17] J. Hoste, *The enumeration and classification of knots and links*, In: "The Handbook of Knot Theory," (W. W. Menasco, M. B. Thistlethwaite, eds.), Elsevier, Amsterdam, 2005, pp. 209-232.

[18] J. Hoste, P. D. Shanahan, *Boundary slopes of 2-bridge links determine the crossing number*, Kobe J. Math. **24** (2007), 21-39.

[19] L. H. Kauffman, S. Lambropoulou, *Hard unknots and collapsing tangles*, In: "Introductory Lectures on Knot Theory" (L. H. Kauffman et al., eds.) Series on Knots and Everything, Vol. 46, World Scientific, Hackensack, NJ, 2012, pp. 187-247.

[20] A. Kazantsev, *Monotonic simplification of arc-presentations of some satellite knots*, Russ. Math. Surv. **65** (2010), 785-787.

[21] S. King, *Crossing number of links formed by edges of a triangulation*, J. Knot Theory Ramifications **12** (2003), 281-286.

[22] M. Lackenby, *The crossing number of composite knots*, J. Topol. **2** (2009), 747-768.

[23] M. Lackenby, *The crossing number of satellite knots*, Algebr. Geom. Topol. **14** (2014), 2379-2409.

[24] M. Lackenby, *A polynomial upper bound on Reidemeister moves*, `arXiv:1302.0180v2`, to appear in Ann. of Math.

[25] Y. Ohyama, *On the minimal crossing number and the braid index of links*, Canad. J. Math. **45** (1993), 117-131.

[26] V. V. Prasolov, A. B. Sossinsky, "Knots, Links, Braids and 3-Manifolds. An Introduction to the New Invariants in Low-Dimensional Topology," Translations of Mathematical Monographs Vol. 154, American Mathematical Society, Providence, RI, 1997.

[27] D. ROLFSEN, "Knots and Links," Publish or Perish, Berkeley, CA, 1976.

[28] A. STOIMENOW, *Tait's conjectures and odd crossing number of amphicheiral knots*, Bull. Amer. Math. Soc. (NS) **45** (2008), 285-291.

[29] A. ZANELLATI, `www.zanellati.it`, personal web page.

adolfo@zanellati.it

Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 PISA – Italy
petronio@dm.unipi.it