Gröbner bases of ideals cogenerated by Pfaffians

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Abstract
We characterise the class of one-cogenerated Pfaffian ideals whose natural generators form a Gröbner basis with respect to any anti-diagonal term-order. We describe their initial ideals as well as the associated simplicial complexes, which turn out to be shellable and thus Cohen-Macaulay. We also provide a formula for computing their multiplicity.

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Introduction

Let $X = (X_{ij})$ be a skew-symmetric $n \times n$ matrix of indeterminates. By [DeP], the polynomial ring $R = K[X] := K[X_{ij}: 1 \le i < j \le n]$, K being a field, is an algebra with straightening law (ASL for short) on the poset $P(X)$ of all Pfaffians of X with respect to the natural partial order defined in [DeP]. Given any subset of $P(X)$, the ideal of R it generates is called a Pfaffian ideal. The special case of Pfaffian ideals $I_{2r}(X)$ generated by the subset $P_{2r}(X)$ of $P(X)$ consisting of all Pfaffians of size $2r$ has been studied extensively ([A], [KL], [Ma]). These ideals belong to a wider family of Pfaffian ideals called one-cogenerated or simply cogenerated. A cogenerated Pfaffian ideal of R is an ideal generated by all Pfaffians of $P(X)$ of any size which are not bigger than or equal to a fixed Pfaffian α. We denote it by $I_\alpha(X) := (\beta \in P(X): \beta \not\ge \alpha)$. Clearly, if the size of α is $2t$, then all Pfaffians of size bigger than $2t$ are in $I_\alpha(X)$. The ring $R_\alpha(X) = K[X]/I_\alpha(X)$ inherits the ASL structure from $K[X]$, by means of which one is able to prove that $R_\alpha(X)$ is a Cohen-Macaulay normal domain, and characterise Gorensteiness, as performed in [D]. In [D2] a formula for the a-invariant of $R_\alpha(X)$ is also given.

Our attention will focus on the properties of cogenerated Pfaffian ideals and their Gröbner bases (G-bases for short) w.r.t. anti-diagonal term orders, which are natural in this setting. By [Ku, Theorem 4.14] and, independently, by [HT, Theorem 5.1], the set $P_{2r}(X)$ is a G-basis for the ideal $I_{2r}(X)$. In a subsequent remark the authors ask whether their result can be extended to any cogenerated Pfaffian ideal. This question is very natural, and in the analogue cases of ideals of minors of a generic matrix and of a symmetric matrix the answer is affirmative, as proved respectively in [HT] and [C]. Quite surprisingly the answer is negative (see Example 2.1) and that settles the starting point of our investigation. The aim of this paper is to characterise cogenerated Pfaffians ideals whose natural generators are a G-basis w.r.t. any anti-diagonal term order in terms of their cogenerator. We call such ideals G-Pfaffian ideals. In Section

1 we set some notation, recall some basic notions of standard monomial theory (cf. [BV]), among which that of standard tableau, and describe the Knuth-Robinson-Schensted correspondence (KRS for short) introduced and studied in [K], since this is the main tool used to prove results of this kind. KRS has been first used by Sturmfels [St] to compute G-bases of determinantal ideals (see also [BC], [BC2]) and it has been applied in [HT] to the study of Pfaffian ideals of fixed size. It turns out that the original KRS is not quite right for our purposes, therefore the first part of Section 2 is devoted to the analysis of a modification that can be applied to d-tableaux in a smart way. In the remaining part of the section we state our main result, cf. Theorem 2.2, by characterising the class of G-Pfaffian ideals. This is performed by proving the two implications separately in Theorem 2.8 and Proposition 2.9 by means of what we call BKRS. In Section 3, Proposition 3.3, we describe the initial ideals of such ideals and in Corollary 3.4 their minimal set of generators. Since these ideals are squarefree, we also study their associated simplicial complexes. By describing faces and facets of the associated simplicial complex, Proposition 3.5 and Theorem 3.6 resp., we are able to prove that these complexes are pure, cf. Corollary 3.8, and simplicial balls, whereas they are not simplicial spheres (see Corollary 3.9). Furthermore, in Proposition 3.10 we provide a formula for computing their multiplicity. Finally, in Proposition 3.12, we prove shellability, which yields that the simplicial complexes associated with G-Pfaffian ideals are Cohen-Macaulay as well. The interested reader can find other recent developments in the study of Pfaffian ideals in [RSh] (cf. Remark 2.10 (iv)) and [JW] (see the end of the last section).

1 Standard monomial theory for Pfaffians and KRS

Let $X = (X_{ij})$ be a skew-symmetric $n \times n$ matrix of indeterminates and let $R = K[X] := K[X_{ij} \colon 1 \leq i < j \leq n]$ the polynomial ring over the field K. The Pfaffian $\alpha = \alpha(A)$ of a skew-symmetric sub-matrix A of X with row and column indexes $a_1 < \ldots < a_{2t}$ is denoted by $[a_1, \ldots, a_{2t}]$. We say that α is a $2t$-*Pfaffian* and that the *size* of α is $2t$. Let now $P(X)$ be the set of all Pfaffians of X and let us recall the definition of partial order on X as introduced in [DeP]. Let $\alpha = [a_1, \ldots, a_{2t}], \beta = [b_1, \ldots, b_{2s}] \in P(X)$. Then

$$\alpha \leq \beta \quad \text{if and only if} \quad t \geq s \text{ and } a_i \leq b_i \text{ for } i = 1, \ldots, 2s.$$

Definition 1.1. Let $\alpha \in P(X)$. The ideal of R cogenerated by α is the ideal

$$I_\alpha(X) := (\beta \in P(X) \colon \beta \not\geq \alpha).$$

We observe that the ideal of R generated by $P_{2r}(X)$, the set of all Pfaffians of size $2r$, is nothing but $I_\alpha(X)$, where $\alpha = [1, \ldots, 2r-2]$. We recall that a *standard monomial* of R is a product $\alpha_1 \cdot \ldots \cdot \alpha_h$ of Pfaffians with $\alpha_1 \leq \ldots \leq \alpha_h$. Since R is an ASL on $P(X)$, standard monomials form a basis of R as a K-vector space and, since $I_\alpha(X)$ is an order ideal, the ring $R_\alpha(X)$ inherits the ASL structure by that of R:

Proposition 1.2. *The standard monomials $\alpha_1 \cdot \ldots \cdot \alpha_h$ with $\alpha_1 \leq \ldots \leq \alpha_h$ and $\alpha_1 \not\geq \alpha$ form a K-basis of $I_\alpha(X)$.*

A natural way to represent monomials, i.e. products of Pfaffians, is by the use of tableaux. Given $\alpha_i = [a_{1i}, a_{2i}, \dots, a_{t_i i}]$ for $i = 1, \dots, h$, one identifies a monomial $\alpha_1 \cdot \ldots \cdot \alpha_h$ with the tableau $T = |\alpha_1|\alpha_2|\ldots|\alpha_h|$, whose i-th column is filled with the indexes of the Pfaffian α_i. Clearly, such a tableau has two properties: the size of all of its columns is even, i.e. T is a *d-tableau*, and each column is a strictly increasing sequence of integers. We recall also that the *shape* of T is the vector $(\lambda_1, \dots, \lambda_t)$ where λ_j is the number of entries in the j^{th} row of T; the *length* of T is simply the number of entries of the first column and it is denoted by length(T). Finally, T is said to be *standard* if the elements in every row form a weakly increasing sequence. For instance, the standard monomial $\alpha_1\alpha_2\alpha_3 = [1,2,3,4][2,3,4,5][2,6]$ is encoded into the standard tableau $T = |\alpha_1|\alpha_2|\alpha_3|$, as shown in Figure 1.

1	2	2
2	3	6
3	4	
4	5	

Figure 1: $T = |\alpha_1|\alpha_2|\alpha_3|$.

Thus, the tableau T is a standard d-tableau with shape $(3,3,2,2)$ and length$(T) = 4$. Obviously, if T is a d-tableau of shape $(\lambda_1, \dots, \lambda_t)$, t is even and $\lambda_1 = \lambda_2, \dots, \lambda_{t-1} = \lambda_t$. Also observe that a monomial is standard if and only if the corresponding tableau is standard.

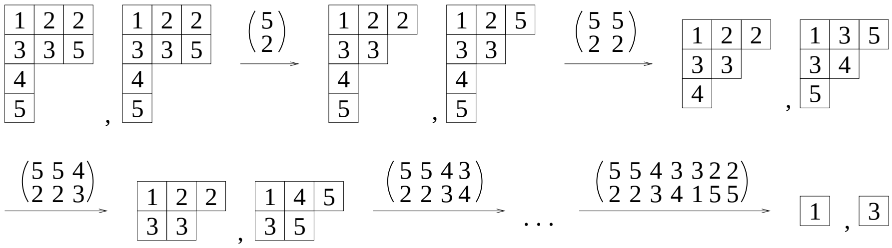
A very effective tool in studying G-bases of order ideals is the KRS correspondence. For the reader's sake, we recall now the original KRS (cf. [K]) as it is used in [HT] to prove that the $2r$-Pfaffians are a G-bases of the ideal they generate. For more information on KRS the reader is also referred to [F]. KRS is a bijection between the set of pairs of standard tableaux, which correspond naturally to standard monomials in the case of minors of a generic matrix, and ordinary monomials. Let (T_1, T_2) be an ordered pair of standard tableaux of the same shape (a *standard bi-tableau* for short) with k elements each. One first associates with (T_1, T_2) a two-lined array $\begin{pmatrix} u_1 & u_2 & \dots & u_k \\ v_1 & v_2 & \dots & v_k \end{pmatrix}$ which satisfies the conditions $(\bullet)$: $u_1 \geq \ldots \geq u_k$ and $v_i \leq v_{i+1}$ if $u_i = u_{i+1}$. Such an array can in turn be identified with the monomial $f = \Pi_{i=1}^k X_{v_i u_i}$ of R. The correspondence between tableaux and arrays relies on the **delete** procedure we describe below.

delete: It applies to a standard tableau T and an element u, which is a corner of T, in the following way. Remove u and set it in place of the first (strictly) smaller element of the above row going from right to left. Use the newly removed element in the same way, until an element v is taken away from the first row of T. The result is a pair (u, v) and a tableau T' with exactly one element less than T.

Now, we describe the KRS.

KRS: Let (T_1, T_2) be a bi-tableau. Take the largest element u_1 of T_1 with largest column index and remove it from T_1, obtaining a smaller tableau T_1'. Apply **delete** to T_2 and the element of T_2 which is in the same position of u_1 in T_1, obtaining an element v_1 and a smaller tableau T_2'. Notice that this can be done because u_1 is placed in a corner of T_1 and that T_1 and T_2 have the same shape. The first column of the resulting array is thus given by u_1 and v_1. Proceed in this way, starting again with the bi-tableau (T_1', T_2'), until all of the elements are removed and the full sequence is achieved. By [K], the latter fulfils the desired conditions $(\bullet)$.

Example 1.3. Let $T = |\alpha_1|\alpha_2|\alpha_3|$, with $\alpha_1 = [1,3,4,5]$, $\alpha_2 = [2,3]$ and $\alpha_3 = [2,5]$. We apply the above procedure to the bi-tableau (T,T):



and, thus, $\mathrm{KRS}(T,T) = X_{25}X_{25}X_{34}X_{43}X_{13}X_{52}X_{52}X_{31} = X_{25}^2X_{34}^2X_{13}^2$.

In [HT] such a correspondence is used in the case of Pfaffians in the following manner. Given a standard monomial, one considers its corresponding tableau T and applies KRS to the standard bi-tableau of type (T,T) obtaining the monomial $f = \mathrm{KRS}(T,T)$. We recall now the following definition.

Definition 1.4. The *width* of a monomial $f = \Pi_{i=1}^k X_{v_i u_i}$, with $u_1 \geq u_2 \geq \cdots \geq u_k$, is the length of the longest increasing subsequence of $v_1, \dots, v_k$ and it is denoted by $\mathrm{width}(f)$.

For instance, the longest increasing sequence in the previous example is $2,3,4,5$, therefore the width of the monomial is 4. By applying [K, Theorem 3] one has that $f = g^2$, where the essential data is contained in g, and the square appears because T is used "twice". This is not an inconvenience in studying ideals generated by Pfaffians of a fixed size, since the crucial point in the argument is the equality $\mathrm{length}(T) = \mathrm{width}(f) = 2\,\mathrm{width}(g)$.
In our case the same holds (cf. Lemma 2.6) but it is not sufficient to gather the information we need. Therefore we shall use a modified version of KRS that produces directly g as an output and carries information on the indeterminates of g as well. This is taken care of in the next section.

2 A characterisation of G-Pfaffian ideals

Throughout this section and in the rest of the paper we shall consider *anti-diagonal* term orders on R. We recall that a term order is said to be anti-diagonal if the initial monomial of the Pfaffian $[a_1, ..., a_{2t}]$ is its main anti-diagonal (*adiag* for short), i.e.

$$\mathrm{in}([a_1, ..., a_{2t}]) = X_{a_1 a_{2t}} X_{a_2 a_{2t-1}} \cdot \ldots \cdot X_{a_t a_{t+1}}.$$

The aim of this section is to characterise what we call *G-Pfaffian* ideals, i.e. one-cogenerated ideals of Pfaffians whose natural generators are a G-bases w.r.t. such term orders. This is not always the case as it is shown in the following example.

Example 2.1. Let X be a 6×6 skew symmetric matrix of indeterminates, and let $I_\alpha(X)$ be the Pfaffian ideal cogenerated by $\alpha = [1,2,4,5]$. The natural generators of $I_\alpha(X)$ are $[1,2,3,4,5,6]$, $[1,2,3,4]$, $[1,2,3,5]$, $[1,2,3,6]$ whose leading terms are $[1,6][2,5][3,4]$, $[1,4][2,3]$, $[1,5][2,3]$, $[1,6][2,3]$ respectively. Therefore, the element $[1,2,3,4][1,5] - [1,2,3,5][1,4]$ belongs to $I_\alpha(X)$ but its initial term, which is $[1,5][2,4][1,3]$, is not divisible by any of the leading terms of the generators.

The main result of this section is stated in the following theorem.

Theorem 2.2. *The natural generators of $I_\alpha(X)$ form a G-basis of $I_\alpha(X)$ w.r.t. any anti-diagonal term order if and only if $\alpha = [a_1, \dots, a_{2t}]$, with $a_i = a_{i-1} + 1$ for $i = 3, \dots, 2t-1$.*

For the purpose of proving the theorem, we use the following result about KRS, which is valid for any KRS correspondence. This is essentially due to Sturmfels [St]. From now on we identify, with some abuse of notation, standard monomials with standard tableaux.

Lemma 2.3. *Let $I \subset R$ be an ideal and let B be a K-basis of I consisting of standard tableaux. Let S be a subset of I such that for all $T \in B$ there exists $s \in S$ such that $\mathrm{in}(s) | \mathrm{KRS}(T)$. Then S is a G-basis of I and $\mathrm{in}(I) = \mathrm{KRS}(I)$.*

Proof. See for instance that of [BC, Lemma 2.1]. ▲

Now we describe the KRS correspondence we are going to use through the rest of the paper. This is a bijection between standard d-tableaux and ordinary monomials, as introduced in [Bu]. This variant, which we denote by BKRS, makes a different use of the **delete** procedure.
BKRS: Consider the largest element of T with largest column index, we say u_1, and its upper neighbour u'. Remove u_1 from T and call the resulting tableau T'. Apply **delete** to T' and u' to produce the element v_1. The output is (u_1, v_1) and the tableau T'', and the first step is concluded. Evidently, one has that $u_1 > v_1$. Now we can start again with the tableau T'' and proceeding in this fashion provides the sought after two-lined array.
Furthermore, as it has been shown in [Bu, Section 2], the above array is ordered lexicographically and the correspondence is $1:1$. In this manner, one obtains a bijection between d-tableaux and two-lined arrays satisfying (•), which in turn can be identified with monomials of R.

Example 2.4. We compute $\mathrm{BKRS}(T)$, where T is as in Example 1.3:

$$\begin{array}{|c|c|c|}\hline 1&2&2\\\hline 3&3&5\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array} \xrightarrow{\begin{pmatrix}5\\2\end{pmatrix}} \begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1} 5\\\cline{1-1}\end{array} \xrightarrow{\begin{pmatrix}5&5\\2&2\end{pmatrix}} \begin{array}{|c|c|}\hline 1&3\\\hline 3&4\\\hline\end{array} \xrightarrow{\begin{pmatrix}5&5&4\\2&2&3\end{pmatrix}} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$$

and, thus, $\mathrm{BKRS}(T) = X_{25}^2 X_{34} X_{13}$.

The fundamental connection between KRS and BKRS is yielded by the following proposition.

Proposition 2.5. *Let T be a standard d-tableau. If $g = \mathrm{BKRS}(T)$ and $f = \mathrm{KRS}(T,T)$, then $f = g^2$.*

Proof. See the proof in [Bu, pag. 20]. ▲

As a consequence, one has the following result.

Lemma 2.6. *Let f be the image of a standard d-tableau T by the* BKRS *correspondence. Then* $\mathrm{width}(f) = \mathrm{length}(T)/2$.

Proof. The proof follows by combining the results in [HT, Section 5] and Proposition 2.5. ▲

For instance, in the last example it can be immediately seen that length$(T)/2 = 2$, which is the length of the sequence $2, 3$. The following remark about the BKRS procedure will be useful in the next proof.

Remark 2.7. An element of the first column of a d-tableau T is moved when all of the elements of the first column below it have been moved (and the bottom one deleted). In fact, an element of the first column of T is moved only when replaced by its lower neighbour. To state this clearly, let a be the element of T in position $(i, 1)$ and b the element of the $(i + 1)^{st}$ row which is moved in position $(i, 1)$. Then $a < b$ and b is smaller than any other element of the i^{th} row. Since T is standard, b must belong to the first column.

Now that we have set our tools properly, our next task is to show that, for the class of cogenerated Pfaffian ideals described in Theorem 2.2, the natural generators form a G-basis.

Theorem 2.8. *Let $\alpha = [a_1, \dots, a_{2t}]$, with $a_i = a_{i-1} + 1$ for $i = 3, \dots, 2t - 1$. The natural generators of $I_\alpha(X)$ form a G-basis of $I_\alpha(X)$.*

Proof. By Proposition 1.2 and Lemma 2.3 it is enough to prove that, given a d-tableau T whose first column β is in $I_\alpha(X)$, there exists a Pfaffian in $I_\alpha(X)$ whose initial term divides BKRS(T).
Let now $\beta = [b_1, \dots, b_{2s}] \in I_\alpha(X)$. Thus, $\beta \not\geq \alpha$ and one of the following must hold true: (i) $b_1 < a_1$; (ii) $b_2 < a_2$; (iii) $s = t$ and $b_{2t} < a_{2t}$; (iv) $s > t$.
We consider each case separately.
(i): b_1 is the smallest element of T, therefore by BKRS it is paired with a bigger element, we say c. Thus, $\text{in}(I_\alpha(X)) \ni \text{in}([b_1, c]) = [b_1, c] | \text{BKRS}(T)$, as desired.
(ii): Without loss of generality we may exclude the trivial case when T is just the one-columned tableau $|\beta|$. Suppose there is an element e which, when deleted, pushes the element b_2 into the first row in place of an element, we say c, which is thus paired with e. By Remark 2.7 we know that e belonged to the first column of T. As a consequence, if d is paired with b_2, we have $c < b_2 < d < e$. Therefore, since $b_2 < a_2$, $[c, b_2, d, e] \in I_\alpha(X)$ and its leading term divides BKRS(T).
(iii): The element b_{2t} is the last element of the first column of T and its row index is even. Let T_k be the tableau occurring during the computation of BKRS(T) with the property that the biggest entry of T_k with largest column index is b_{2t}. Let T_{k+1} be the next tableau occurring in the procedure. Finally, let $f_k = \text{BKRS}(T_k)$ and $f_{k+1} = \text{BKRS}(T_{k+1})$. Evidently, $f_k = f_{k+1} X_{i_0 b_{2t}}$ for some $i_0 < b_{2t}$. Now, length$(T_k) = 2t$ and length$(T_{k+1}) = 2t - 2$, therefore by Lemma 2.6, width$(f_k) = t$ and width$(f_{k+1}) = t - 1$. Thus, f_{k+1} is divided by a monomial $X_{i_1 j_1} \cdot \ldots \cdot X_{i_{t-1} j_{t-1}}$, with $i_1 < \ldots < i_{t-1} < j_{t-1} < \ldots < j_1$ and, since the width of f_k is one more than that of f_{k+1}, $i_0 < i_1$ and $j_1 < b_{2t}$. Now, $[i_0, \dots, i_{t-1}, j_{t-1}, \dots, j_1, b_{2t}]$ is an element of $I_\alpha(X)$, its initial term divides f_k and BKRS(T), as desired.
(iv) Suppose now that $s > t$. If $f = \text{BKRS}(T)$, then width$(f) = \text{length}(T)/2 = s$ by Lemma 2.6. Thus, there exists a $2s$-Pfaffian whose initial term divides BKRS(T), but all $2s$-Pfaffians are in $I_\alpha(X)$. This concludes the proof of the last case and of the theorem. ▲

It is somehow surprising that the ideals satisfying the conditions of Theorem 2.2 are indeed the only ones endowed with this property. We prove this fact next. In the proof, we shall use the following standard expansion formula for Pfaffians: Given a $m \times m$ skew-symmetric submatrix $A = (a_{ij})$ of X, we denote by $A(i,j)$ the submatrix of A obtained by deleting the i^{th} and j^{th} row and column. Fixed an index $1 \leq i \leq m$, we have

$$\alpha(A) = \sum_{j=1}^{m} (-1)^{i+j+1} \sigma(i,j) a_{ij} \alpha(A(i,j)) \tag{2.1}$$

where $\sigma(i,j)$ is the sign of $j - i$.

Proposition 2.9. *Let $\alpha = [a_1, \ldots, a_{2t}] \in P(X)$ and set $a_{2t+1} := +\infty$, $i := \min\{k \geq 2\colon a_k + 1 < a_{k+1}\}$. If $i < 2t - 1$, then the natural generators of $I_\alpha(X)$ are not a G-basis for $I_\alpha(X)$.*

Proof. We prove that there exists an element which belongs to $I_\alpha(X)$ and whose initial term is not divisible by the initial term of any Pfaffian in $I_\alpha(X)$. In order to do so, we need to distinguish two cases.
Case 1: i is even.
We start by observing that $i+2$ is even and $\leq 2t$ since $i < 2t-1$. We thus may let $\beta_1 := [a_1, a_2, \ldots, a_i, a_i+1, a_{i+1}]$, $\gamma_1 := [a_1, a_{i+2}]$, $\beta_2 := [a_1, a_2, \ldots, a_i, a_i + 1, a_{i+2}]$, $\gamma_2 := [a_1, a_{i+1}]$ and consider the element $\beta_1\gamma_1 - \beta_2\gamma_2$. Expanding β_1 and β_2 along the a_1^{st} row by means of (2.1), we obtain that β_1 is the alternating sum of $[a_1, a_{i+1}][a_2, \ldots, a_i, a_i + 1]$, $[a_1, a_i + 1][a_2, \ldots, a_i, a_{i+1}]$ and terms which do not contain either $[a_1, a_{i+1}]$ or $[a_1, a_i+1]$. Similarly, β_2 is the alternating sum of $[a_1, a_{i+2}][a_2, \ldots, a_i, a_i+1]$, $[a_1, a_i+1][a_2, \ldots, a_i, a_{i+2}]$ and terms which do not contain $[a_1, a_{i+2}]$. We observe that $[a_1, a_{i+2}]$ is the largest indeterminate which appears in β_i, γ_i, $i = 1, 2$, and $[a_1, a_{i+2}] > [a_1, a_{i+1}] > [a_1, a_i+1] > [a_1, a_i] > \ldots$. A quick verification on the sign of the summands shows that a simplification occurs and it turns out that

$$\begin{aligned} \text{in}(\beta_1\gamma_1 - \beta_2\gamma_2) &= \text{in}([a_1, a_{i+2}][a_1, a_i + 1][a_2, \ldots, a_i, a_{i+1}]) \\ &= [a_1, a_{i+2}][a_1, a_i + 1][a_2, a_{i+1}] \, \text{in}[a_3, \ldots, a_i]. \end{aligned}$$

This is an element of $\text{in}(I_\alpha(X))$, since $\beta_1, \beta_2 \in I_\alpha(X)$, and we identify it with the array

$$\begin{pmatrix} a_{i+2} & a_{i+1} & a_i + 1 & a_i & \ldots & a_{\frac{i+4}{2}} \\ a_1 & a_2 & a_1 & a_3 & \ldots & a_{\frac{i+2}{2}} \end{pmatrix}. \tag{2.2}$$

We now search for all Pfaffians f such that $\text{in}(f)$ divides this monomial and show that they are not in $I_\alpha(X)$. Our task is reduced to merely considering all Pfaffians that one can build choosing sequences of growing indexes in the second row of (2.2) and the reverse of the corresponding sequence which is determined in the first row of (2.2) by this choice. It is immediate to see that no Pfaffian of size $i+4$ can be built this way. Moreover, the only such Pfaffian of size $i+2$ is $\overline{\alpha} := [a_1, a_2, \ldots, a_i, a_{i+1}, a_{i+2}] > \alpha$. As for those of size i, the only one which is not a sub-Pfaffian of $\overline{\alpha}$ is $\beta = [a_1, a_3, \ldots, a_i, a_i + 1] > \alpha$. Since all the other Pfaffians are sub-Pfaffians of $\overline{\alpha}$ or of β, and thus bigger than α, the proof of this case is complete.
Case 2: i is odd.
Since $i < 2t-1$, $i+3 \leq 2t$. Thus, we may let $\beta_1 := [a_1, a_2, \ldots, a_i, a_i + 1, a_{i+1}, a_{i+3}]$, $\gamma_1 := [a_2, a_{i+2}]$, $\beta_2 := [a_1, a_2, \ldots, a_i, a_i + 1, a_{i+2}, a_{i+3}]$, $\gamma_2 := [a_2, a_{i+1}]$. Recalling that $a_i + 1 < a_{i+1} < a_{i+2}$, by computing as before we obtain

$$\text{in}(\beta_1\gamma_1 - \beta_2\gamma_2) = [a_1, a_{i+3}][a_2, a_{i+2}][a_2, a_i + 1] \, \text{in}[a_3, \ldots, a_i, a_{i+1}],$$

its corresponding array being

$$\left(\begin{array}{ccccccc} a_{i+3} & a_{i+2} & a_{i+1} & a_i+1 & a_i & \dots & a_{\frac{i-3}{2}} \\ a_1 & a_2 & a_3 & a_2 & a_4 & \dots & a_{\frac{i+3}{2}} \end{array} \right),$$

and, by repeating the arguments of the previous case, we reach the conclusion in a similar fashion. ▲

Remarks and Examples 2.10. **(i)** Let us consider an interesting subclass of G-Pfaffian ideals. Let $\alpha = [1, 2, \dots, 2t-1, b]$. Then $I_\alpha(X)$ is generated by the $2t$-Pfaffians indexed in the first $b-1$ rows and columns of X and by all $(2t+2)$-Pfaffians. Note that, despite the fact that there are several expansion formulas for Pfaffians (cf. for instance [S], [Ku]), some of which resembling well-known expansion formulas for minors, it is not possible to expand all of the $2t$-Pfaffians by means of those in the first $b-1$ rows and columns only. Thus, even in this simple case, the minimal set of generators of $I_\alpha(X)$ contains Pfaffians of different sizes.
(ii) Let $I_\alpha(X)$ and $I_\beta(X)$ be G-Pfaffian ideals. By Lemma 2.3 it is easy to prove (see also [BC2]) that the generators of $I_\alpha(X)$ and of $I_\beta(X)$ together form a G-basis of $I_\alpha(X) + I_\beta(X)$ w.r.t. any anti-diagonal term order (note that, in general, $I_\alpha(X) + I_\beta(X)$ is not a cogenerated ideal).
(iii) The G-Pfaffian ideals we considered in **(i)** belong to the class of generalised ladder Pfaffian ideals, as introduced in [DGo]. By using linkage Gorla, Migliore and Nagel [GoMiN] proved that the natural generators of such ideals form a G-basis w.r.t. any anti-diagonal term order. Observe that the only generalised ladder ideals among G-Pfaffian ideals are those considered in **(i)**.
(iv) In [RSh] a class of Pfaffian ideals, containing cogenerated ideals, is considered. By using an approach which involves Schubert varieties, the authors describe initial ideals of the ideals in this class. In Remark 1.9.1 they emphasise the fact that the generators are not a G-basis w.r.t. the orders that they consider.

3 Multiplicity and shellability

We start this section by proving an easy but useful reduction.

Proposition 3.1. *Let X, R_X, α, $I_\alpha(X)$ be an anti-symmetric matrix of indeterminates (X_{ij}) of size n, the ring $K[X]$, a Pfaffian $[a_1, \dots, a_{2t}] \in P(X)$ and the Pfaffian ideal of R_X cogenerated by α respectively. Moreover, let X', $R_{X'}$, β and $I_\beta(X')$ an anti-symmetric matrix of indeterminates (X'_{hk}) of size $n-a_1+1$, the ring $K[X']$, the Pfaffian $[1, a_2-a_1+1, \dots, a_{2t}-a_1+1]$ and the Pfaffian ideal of $R_{X'}$ cogenerated by β respectively. Then*

$$R_X/I_\alpha(X) \simeq R_{X'}/I_\beta(X').$$

Proof. If $a_1 = 1$ there is nothing to prove. Thus, let us assume that $a_1 > 1$ and observe that $X_{ij} \in I_\alpha(X)$, for all $1 \le i \le a_1 - 1$ and these are the only indeterminates contained therein. The reader can easily see that the desired isomorphism is yielded by modding out these indeterminates and by a change of coordinates that preserves the poset structure. ▲

As a consequence, in order to study numerical invariants of $I_\alpha(X)$, we may without loss of generality assume that $a_1 = 1$.

Remark 3.2. *Computational issue.* The previous proposition also makes the computation of bigger examples, which are extremely resource intensive, possible.

Next, we describe initial monomial ideals of G-Pfaffian ideals.
Having in mind what initial monomials are w.r.t. anti-diagonal term orders, we say that a monomial $X_{i_1j_1} \cdot \ldots \cdot X_{i_tj_t}$ is a t-adiag if $i_1 < i_2 < \ldots < i_t < j_t < j_{t-1} < \ldots < j_1$. Let $\alpha = [1, a, \ldots, a + 2t - 3, b] \in P_{2t}(X)$. All the $(2t + 2)$-Pfaffians belong to $I_\alpha(X)$. All other Pfaffians in $I_\alpha(X)$ are of size $\leq 2t$ and are of type $[c, d, *, \ldots, *]$ with $c < d \leq a - 1$ or $[e_1, \ldots, e_{2t}]$ with $e_{2t} \leq b - 1$. Now observe that, for any Pfaffian $[c, d, *, \ldots, *, e, f]$, one has $\mathrm{in}([c, d, e, f]) | \mathrm{in}([c, d, *, \ldots, *, e, f])$ and, since the generators form a G-basis, then $I_\alpha(X)$ is generated by

$$\{[c, d] : c < d \leq a-1\} \cup \{[c, d, e, f] : c < d \leq a-1\} \cup \{[e_1, \ldots, e_{2t}] : e_{2t} \leq b-1\}.$$

In this way we have proven the following result.

Proposition 3.3. *The ideal* $\mathrm{in}(I_\alpha(X))$ *is generated by*
(i) all of the indeterminates in the first $a-1$ rows and columns (region A);
(ii) all of the 2-adiags in the first $a-1$ rows (region $A \cup B \cup C$);
(iii) all of the t-adiags in the first $b-1$ rows and columns (region $A \cup B \cup D$);
(iv) all of the $(t+1)$-adiags.

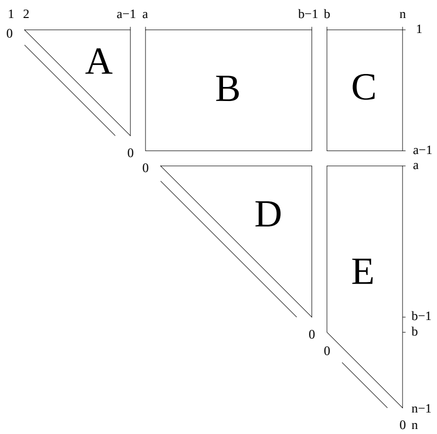


Figure 2: X_+ and its regions.

Corollary 3.4. *The minimal set of generators of* $\mathrm{in}(I_\alpha(X))$ *is given by:*
(i) all of the indeterminates in A;
(ii) all of the 2-adiags in $B \cup C$;
(iii) all of the t-adiags in $B \cup D$ with at most one indeterminate in B;
(iv) all of the $(t+1)$-adiags outside of A, with at most one indeterminate in B and at most $t-1$ indeterminates in $B \cup D$.

Let $X_+ := \{(i, j) | 1 \leq i < j \leq n\}$. In the rest of this section, with some abuse of notation, we shall identify X_{ij} with (i, j). Our next task is to describe the simplicial complex Δ_α associated with $I_\alpha(X)$, which is the family of all subsets Z of X_+ such that the corresponding monomial $\Pi_{(i,j)\in Z} X_{ij}$ does not belong to $\text{in}(I_\alpha(X))$. The *modus operandi* is the same as in the classical cases: we decompose a set Z into disjoint chains, in a way similar to that of the "light and shadow" procedure described in [V] and used in [HT]. Our proof is based on the use of two lights, say a "sunlight" coming from the lower-left side and a "moonlight" coming from the upper-right side, and a mixed use of such decompositions. Let us describe this in a more precise manner. Given a set $Z \subset X_+$, we let

$$\delta(Z) := \{(i,j) \in Z \colon \nexists (i',j') \in Z \text{ with } i' > i, j' < j\},$$
$$\delta'(Z) := \{(i,j) \in Z \colon \nexists (i',j') \in Z \text{ with } i' < i, j' > j\}.$$

We also let $Z_1 := \delta(Z)$ and $Z_1' := \delta'(Z)$. For $h > 1$, we let

$$Z_h := \delta(Z \setminus \cup_{k<h} Z_k) \quad \text{and} \quad Z_h' := \delta'(Z \setminus \cup_{k<h} Z_k').$$

We thus obtain two decompositions of $Z = \cup_{h=1}^{r} Z_h$ and $Z = \cup_{k=1}^{s} Z_k'$, as disjoint chains, the first one corresponding to the sunlight shadows and the second one to the moonlight shadows. Note that $r = s$, since the number of components in such a decomposition only depends on the maximal length of an anti-chain, i.e. an adiag, contained in Z; note also that if $P \in Z_i$ (resp. Z_j') there is an adiag of length i (resp. j) starting (resp. ending) in P. In the classical cases, the use of one light only is sufficient for describing the faces of the complex, but this does not work in our case. Therefore, we use a mixed decomposition: given a subset Z of X_+ we decompose it as

$$Z = Z_1' \cup Z_1 \cup Z_2 \cup \ldots \cup Z_r,$$

where $Z_1' = \delta'(Z)$ and $Z_1 \cup \ldots \cup Z_r$ is the sunlight decomposition of $Z \setminus Z_1'$. Furthermore, if $Z_{t-1} \neq \emptyset$, we let $F \subset B$ be the subset of points in X_+ with column index larger than the smallest column index of a point in Z_{t-1}. We can now describe the faces of Δ_α.

Proposition 3.5. *With the above notation, $Z \in \Delta_\alpha$ iff*
- $Z \cap A = \emptyset$;
- $Z \setminus Z_1' \subseteq D \cup E$;
- $r \leq t-1$;
- if $r = t-1$ and $Z_1' \cap (B \cup D) \neq \emptyset$ then $Z_1' \cap F = \emptyset$.

Proof. $\Leftarrow$: We need to show that none of the forbidden adiags, i.e. those described in Proposition 3.3, are in Z. If there were a 2-adiag of Z contained in $B \cup C$, there would exist a point $P \in (B \cup C) \setminus Z_1'$. Thus $P \in (Z \setminus Z_1') \setminus (D \cup E)$ and the second condition would be violated. If there were a $(t+1)$-adiag of Z in $X_+ \setminus A$ then $r \geq t$, since there would be a t-adiag in $Z \setminus Z_1'$. Suppose now that there exists a t-adiag d of points of Z in $B \cup D$: d cannot be contained in D, since this would imply $r \geq t$. Thus $Z_1' \cap (B \cup D) \neq \emptyset$ and there exists a $(t-1)$-adiag in D, which implies $r = t-1$. We thus can assume that $Z_1' \cap F = \emptyset$. But, if we let P_i be the points of d such that $P_i \in Z_i$ with $i = 1, \ldots, t-1$ and $P_t \in Z_1'$, we would also have $P_t \in Z_1' \cap F$, which is a contradiction. Now the conclusion is straightforward by a use of Proposition 3.3.
$\Rightarrow$: Let $Z = Z_1' \cup Z_1 \cup Z_2 \cup \ldots \cup Z_r \in \Delta_\alpha$. Clearly, $Z \cap A = \emptyset$. If there existed $P \in (Z \setminus Z_1') \setminus (D \cup E)$, we would have $P \in (B \cup C) \setminus Z_1'$. Thus, there would exist $Q \in Z_1' \cap (B \cup C)$ so that Q, P formed a 2-adiag in $B \cup C$, and this takes care of the second condition. Suppose now $r \geq t$. Then there exists a t-adiag $P_1, \ldots, P_t \in D \cup E$ with $P_t \notin Z_1'$. Therefore, we may prolong the t-adiag to a

$(t+1)$-adiag in $X_+ \setminus A$, which is not possible. Now we only have to show that, if $r = t-1$ and $Z'_1 \cap (B \cup D) \neq \emptyset$ then $Z'_1 \cap F = \emptyset$. If this were not the case, we could form a t-adiag in $B \cup D$ taking one point in each Z_i, $i = 1, \ldots, t-1$ and one point in $Z'_1 \cap F$. Again, this is not possible and we are done. ▲

Next, we describe the facets of Δ_α. We denote a saturated chain of X_+ with starting point Q and ending point P simply by QP and call it, as is common in the literature, a *path*. In what follows $\sqcup$ denotes a union of non-intersecting paths.

Theorem 3.6. *Let $Q = (1, a)$, $Q_i = (a, a+2i-1)$ for $i = 1, \ldots, t-2$ and $P_j = (n-2j+1, n)$ for $j = 1, \ldots, t$. Furthermore, let $Q_{t-1} = (a, k)$, $Q^h = (h, b)$, $P_{hk} = (h, k)$, with $h, k \in \mathbb{N}$. Then*

$$Z \text{ is a facet of } \Delta_\alpha \quad \text{iff} \quad Z = (QP_{hk} \sqcup Q^h P_t) \sqcup_{i=1}^{t-1} Q_i P_i,$$

for some $h \in \{1, \ldots, a-1\}$ and $k \in \{a+2t-3, \ldots, b-1\}$.

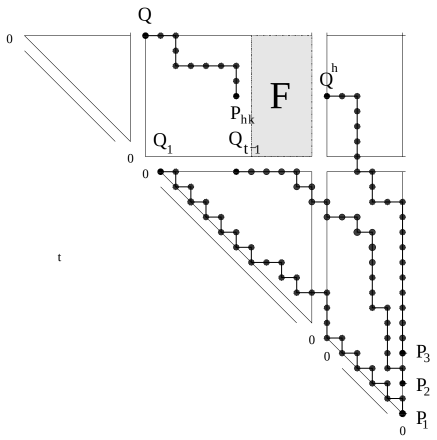


Figure 3: A facet of Δ_α with $t = 3$.

Before we proceed with the proof of the theorem, we need a couple of preparatory results. First, it might be useful to recall that, if $n = 2m+i$ with $i \in \{0, 1\}$, then the longest adiag contained in X_+ has length m. Therefore, being $b-a \geq 2t-2$, D contains the $(t-1)$-adiag of the points $R_1 = (a, b-1), R_2 = (a+1, b-2), \ldots, R_{t-1} = (a+t-2, b-t+1)$.

Lemma 3.7. *Let $Z = Z'_1 \cup Z_1 \cup \ldots \cup Z_r$ be a facet of Δ_α. Then, $r = t-1$.*

Proof. For $i = 1, \ldots, t-1$ let R_i be the points described above. By contradiction, we let $r < t-1$ and we shall prove that Z is not maximal. If $R_i \in Z$ for all $i = 1, \ldots, t-1$, R_i all belong to different components of Z and this implies that $R_1 \in Z'_1$. By definition of Z'_1, we have $Z \cap C = \emptyset$. Now we let $R := (a-1, b) \in C$ and the reader is left with the task to verify by means of Proposition 3.5 that $Z \cup \{R\}$ is a face of Δ_α. Else, if there exists $R_{i_0} \notin Z$, again by Proposition 3.5, $Z \subsetneq Z \cup \{R_{i_0}\} \in \Delta_\alpha$. In both cases the maximality of Z is contradicted and the lemma is proven. ▲

Proof of Theorem 3.6. It is immediately verified by means of Proposition 3.5 that the union Z of such chains is a face of Δ_α. Also, it is easy to see that, given any point in $P \in X_+ \setminus Z$, $Z \cup \{P\}$ is not a face of Δ_α, therefore Z is a facet. Conversely, let Z be a facet. By Lemma 3.7, we may decompose $Z = Z'_1 \cup Z_1 \cup \ldots Z_{t-1}$ and observe that, by maximality, each Z_i must end exactly in P_i, for $i = 1, \ldots, t-1$ and Z'_1 in P_t. Moreover, $Z'_1 \subseteq B \cup C \cup E$: if there were a point $P \in Z'_1 \cap D$ then, by Proposition 3.5, $Z'_1 \cap F = \emptyset$. Evidently, a point of Z_{t-1} does not belong to Z'_1 thus a point of Z_{t-1} with smallest column index is in the shadow of a point $P' \in Z'_1 \cap C$, so that we would have a 2-adiag P', P contained in the same component Z'_1, which is a contradiction. Now, again by maximality, $Z'_1 \cap B \neq \emptyset$, therefore, $Z'_1 \cap F = \emptyset$ and Z'_1 can be seen as a path from Q to a point $P_{hk} = (h, k)$ with $h \in \{1, \ldots, a-1\}$ and $k \leq b-1$ together with a path from a point $Q^{h'} = (h', b)$ to P_t. Since Z cannot contain 2-adiags in $B \cup C$, $h' \geq h$ and maximality forces $h' = h$ and $Q' = P_{hk}$. Finally, we consider $Z \setminus Z'_1$ and we follow the strategy in [HT, Theorem 5.4] to obtain that $Z_1, \ldots, Z_{t-2}$ are paths starting in Q_i, $i = 1, \ldots, t-2$. Therefore $k \geq a + 2t - 3$ and Z_{t-1} turns out to be a path from (a, k), with $k \in \{a + 2t - 3, \ldots, b-1\}$. Now it is sufficient to collect all the data concerning the starting and ending points of the components to reach the conclusion of this implication and of the proof. ▲

As a straightforward consequence, all of the facets of Δ_α have the same cardinality, i.e. are of maximal dimension.

Corollary 3.8. *Let α be a G-Pfaffian and Δ_α its associated simplicial complex. Then, Δ_α is pure.*

Proof. By Proposition 3.1 we may assume that the first entry of α is 1. By the previous theorem, we can compute the cardinality of the facets of Δ_α. We have $k + h - a$ points in QP_{hk}, $2n - 2t - h - b + 2$ points in $Q^h P_t$, $2n - 2t + 4 - a - k$ points in $Q_{t-1}P_{t-1}$ and $\sum_{i=1}^{t-2}(2n - 2a - 4i + 3) = (t-2)(2n - 2a - 2t + 5)$ points in the other $t-2$ chains. Altogether these numbers add up to $d := 2nt - 1 - b - 2(t-1)a - (2t-3)(t-1)$, which evidently does not depend on h and k, therefore the complex is pure of dimension $d - 1$. ▲

We observe that, since $\dim K[\Gamma] = \dim \Gamma + 1$ for any simplicial complex Γ, the dimension computed in the above proof is the one predicted in [D, (1.1)].

Corollary 3.9. *Δ_α is a simplicial ball and not a simplicial sphere.*

Proof. By means of Theorem 3.6, one can easily verify that every sub-maximal face of Δ_α is contained at most into two distinct facets of Δ_α. Moreover there is at least one sub-maximal face which is contained exactly in one facet. By [Bj et al., 4.7.22] these facts imply the assertion. ▲

Now we calculate the multiplicity of $R_\alpha(X)$ for a G-Pfaffian α. For the sake of notational simplicity, we now modify slightly the notation introduced in Theorem 3.6 and we let $Q_t = (h, b)$. Furthermore, let $A_{hk} = (a_{ij}^{hk})$ be the $t \times t$ matrix with entries

$$a_{ij}^{hk} = \binom{x_{P_j} + y_{P_j} - x_{Q_i} - y_{Q_i}}{x_{P_j} - x_{Q_i}} - \binom{x_{P_j} + y_{P_j} - x_{Q_i} - y_{Q_i}}{x_{P_j} - y_{Q_i}},$$

where x_p and y_p denote the coordinates of a point P in X_+. Observe that only the last two rows and columns of A_{hk} are really dependent on h, k. By Proposition 3.1, it is sufficient to provide a formula for a reduced G-Pfaffian.

Proposition 3.10. *Let $\alpha = [1, a, \ldots, a+2t-3, b]$. Then*

$$e(R_\alpha(X)) = \sum_{\substack{h=1,\ldots,a-1 \\ k=a+2t-3,\ldots,b-1}} \binom{h+k-a-1}{h-1} \det A_{hk}.$$

Proof. Since Δ_α is pure by Corollary 3.8, the multiplicity is just the cardinality of the set of all facets. By Theorem 3.6, these are $QP_{hk} \sqcup_{i=1}^{t} Q_i P_i$, with $h \in \{1, \ldots, a-1\}, k \in \{a+2t-3, \ldots, b-1\}$. Thus, the number we are seeking for is $\sum_{h,k} r(h,k)s(h,k)$ where $r(h,k)$ counts all saturated chains from Q to P_{hk} and $s(h,k)$ counts all disjoint unions of t saturated chains $Q_i P_i$. It is well known that $r(h,k) = \binom{x_{P_{hk}} - x_Q + y_{P_{hk}} - y_Q}{x_{P_{hk}} - x_Q}$. Moreover, arguing as in [GV] yields that $s(h,k) = \det(b_{ij})$ where b_{ij} is the number of saturated paths from Q_i to P_j. Since all of these paths are contained in X_+, by [M, Chap. 1, Theorem 1], $a_{ij}^{hk} = b_{ij}$ for all $i,j = 1, \ldots, t$ and the proof is complete. ▲

Example 3.11. The multiplicity of the ring $R_\alpha(X)$, where X is a 15×15 skew-symmetric matrix and $\alpha = [4, 8, 9, 12]$ is 50752, as it can be easily computed using the previous proposition. For doing so, we consider the Pfaffian $[1, 5, 6, 9]$, with $n = 12$, $t = 2$, $a = 5$ and $b = 9$, and compute the 12 determinants of the 2×2 matrices A_{hk} where $h = 1, \ldots, 4$ and $k = 6, 7, 8$.

We are now in a position of proving that Δ_α is shellable and, thus, Cohen-Macaulay.

Proposition 3.12. *Δ_α is shellable.*

Proof. Given $x = (u,v) \in X_+$, we set $\mathcal{R}_x := \{(i,j) \in X_+ \colon i < u, j > v\}$, $\overline{\mathcal{R}}_x := \{(i,j) \in X_+ \colon i \le u, j \ge v\}$, $\mathcal{L}_x := \{(i,j) \in X_+ \colon i > u, j < v\}$, and $\overline{\mathcal{L}}_x := \{(i,j) \in X_+ \colon i \le u, j \ge v\}$. Furthermore, if $S \subseteq X_+$, we let $\mathcal{R}_S := \cup_{x \in S} \mathcal{R}_x$, $\overline{\mathcal{R}}_S := \cup_{x \in S} \overline{\mathcal{R}}_x$, $\mathcal{L}_S := \cup_{x \in S} \mathcal{L}_x$ and $\overline{\mathcal{L}}_S := \cup_{x \in S} \mathcal{L}_x$. Finally, we let $x^L := (u+1, v-1)$. Our first task is to find a partial order on the set of the facets of Δ_α. Let $F = Z_1 \cup \ldots \cup Z_{t-1} \cup Z_t$ and $F' = Z'_1 \cup \ldots \cup Z'_{t-1} \cup Z'_t$ be decompositions of two facets F and F' of Δ_α, where we renamed the first components to Z_t and Z'_t for reason of notation. Now we may let

$$F \succeq F' \quad \text{if and only if} \quad Z_i \subseteq \overline{\mathcal{R}}_{Z'_i} \quad \text{for all} \quad i+1, \ldots, t.$$

Next, we extend $\preceq$ to a total order on the set of facets of Δ_α. Thus, for the purpose of proving that Δ_α is shellable, we need to show that, given any two facets $F \succ F'$, there exists $x \in F \setminus F'$ and a third facet F'' such that $F \succ F''$ and $F \setminus F'' = \{x\}$. Let $F \succ F'$ be two given facets decomposed as above. Since $F \not\preceq F'$, we may consider the least integer i such that $Z'_i \not\subseteq \overline{\mathcal{R}}_{Z_i}$, with $i \le t-1$ or $i = t$. In case $i \le t-1$, if $y \in Z'_i \setminus \overline{\mathcal{R}}_{Z_i}$ then $y \in \mathcal{L}_{Z_i}$. Now we pick an element x such that $y \in \mathcal{L}_x$ and $\overline{\mathcal{R}}_x \cap Z_i = \{x\}$ by taking an upper right corner of the chain Z_i which is in $\mathcal{R}_y$, cf. Figure 4. Finally let $F'' := (F \cup \{x^L\}) \setminus \{x\}$. It is easy to verify that $x^L \in X_+ \setminus F$ and that F'' is a facet of Δ_α with the requested properties. In fact, $x^L \in F$ would imply $x^L \in Z_{i-1} \cap \overline{\mathcal{R}}_y$. Since $y \in Z'_i \subseteq \mathcal{R}_{Z'_{i-1}}$, y is also in $\overline{\mathcal{R}}_{Z_{i-1}}$ because, by minimality of i, $Z'_{i-1} \subseteq \overline{\mathcal{R}}_{Z_{i-1}}$. Finally, this implies $x^L \in \mathcal{R}_{Z_{i-1}}$ and, consequently, $x^L \notin Z_{i-1}$, which is a contradiction.
In the other case, we let $Q^h = (h,b) \in Z_t$ and start by arguing as before to find y. In constructing x, if we can choose a point of F other than Q^h, we do it and the proof runs as in the previous case. Otherwise, we set $F'' := (F \cup \{P_{h+1k}\}) \setminus \{Q^h\}$ and are left with the task to prove that F'' is a facet, which is equivalent to say

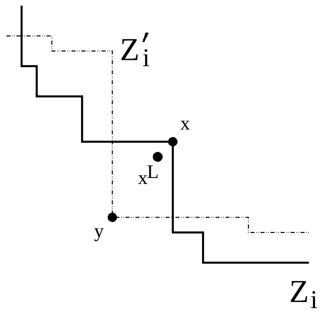


Figure 4: Constructing a shelling on Δ_α.

that $P_{h+1k} \notin F$. Since P_{hk} is in Z_t, if P_{h+1k} belonged to F, it would be a point of Z_{t-1} and $h+1=a$. Therefore $y \in Z'_t \cap \mathcal{L}_{Q^h} \subseteq (B \cup C \cup E) \cap \mathcal{L}_{Q^h} = \emptyset$, which is the desired contradiction. ▲

In light of the results we obtained, it is now very natural to look for a new term order, w.r.t. which the generators of any cogenerated ideal form a G-basis. An interesting non anti-diagonal term order is found in [JW], for which the result is proven for Pfaffians of fixed size and a complete description of the associated simplicial complexes is given. It does not extend however to our setting (cf. again Example 2.1).

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