

Diffeological connections on diffeological vector pseudo-bundles

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Abstract

We consider one possible definition of a diffeological connection on a diffeological vector pseudo-bundle. It is different from the one proposed in [7] and is in fact simpler, since it is obtained by a straightforward adaption of the standard definition of a connection as an operator on the space of all smooth sections. One aspect prominent in the diffeological context has to do with the choice of an appropriate substitute for tangent vectors and smooth vector fields, since there are not yet standard counterparts for these notions. In this respect we opt for the simplest possibility; since there is an established notion of the (pseudo-)bundle of differential forms on a diffeological space, we take the corresponding dual pseudo-bundle to play the role of the tangent bundle. Smooth vector fields are then smooth sections of this dual pseudo-bundle; this is one reason why we devote a particular attention to the space of smooth sections of an arbitrary diffeological vector pseudo-bundle (one curiosity is that it might easily turn out to be infinite-dimensional, even when the pseudo-bundle itself has a trivial finite-dimensional vector bundle as the underlying map). We concentrate a lot on how this space interacts with the gluing construction for diffeological vector pseudo-bundles (described in [10]). We then deal with the same question for the proposed notion of a diffeological connection.

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Introduction

Diffeology can be seen as a way to extend the field of application of differential geometry (or of differential calculus, according to some). There have been, and are, other attempts to do this; some of these approaches are summarized in [16]. Diffeological spaces first appeared in [14], [15]; a lot of fundamental concepts, such as the underlying topology, called *D-topology*, and the counterpart of the fibre bundle, among others, were developed in [6]. A recent and comprehensive source on the field of diffeology is [7].

From a certain (necessarily simplistic, but still interesting) point of view, diffeology can be seen as a way to consider any given function as a smooth one — and then see what happens. This is essentially the notion of a *diffeology generated by a given plot*; what becomes for instance of the usual \mathbb{R} if we consider the modulus $|x|$ as a smooth function *into* it? One immediate answer (there would be of course more intricate ones) is that no linear function on it is smooth then (except the zero one); and this is just the most basic of examples. This is the kind of a straightforward (it can be said, naive) approach that we opt for in this paper.

The notion of a connection A certain preliminary notion of a diffeological connection is sketched out in [7]. Our approach is different from one therein, but it is very much straightforward. A usual connection on a smooth vector bundle $E \rightarrow M$ over a smooth manifold M can be defined as a smooth operator $C^\infty(M, E) \rightarrow C^\infty(M, T^*M \otimes E)$, that is linear and obeys the Leibnitz rule. For all objects that appear in its definition, there are well-established diffeological counterparts, with the bundle of *diffeological differential 1-forms* $\Lambda^1(X)$ over a diffeological space X (see [7] again, although it is not the original source) taking the place of the cotangent bundle. Thus, the definition-by-analogy of a diffeological connection on a *diffeological vector pseudo-bundle* $\pi : V \rightarrow X$ is an obvious matter; it suffices to substitute X for M , V for E , and consider diffeological forms instead of sections of the cotangent bundle. A few minor details need to be explained (which we do), and it also should be specified that the covariant derivatives are taken with respect to sections of the dual pseudo-bundle $(\Lambda^1(X))^*$, which for us plays the

role of the tangent bundle (of which there is not yet a standard theory in diffeology). However, covariant derivatives is the only place where we need tangent vectors.

Most of what we do is devoted to constructing connections on pseudo-bundles obtained by *diffeological gluing* (see [9]). To this end we first dedicate significant attention to the behavior of the spaces of sections under gluing. Thus, if $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ are two pseudo-bundles, and $\pi_1 \cup_{(\tilde{f}, f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \rightarrow X_1 \cup_f X_2$ is the result of their gluing (see below for the precise definition), the space of sections $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ is a smooth surjective image of a subset of the direct product $C^\infty(X_1, V_1) \times C^\infty(X_2, V_2)$ (Section 2). We use this to show that if V_1 and V_2 are both endowed with connections, and these connections satisfy a specified *compatibility condition*, then there is an induced connection on $V_1 \cup_{\tilde{f}} V_2$. If, finally, V_1 and V_2 are endowed with *pseudo-metrics* g_1 and g_2 (these are diffeological counterparts of Riemannian metrics) that are well-behaved with respect to each other, and the two connections on V_1 and V_2 are compatible with these pseudo-metrics, then they, again, induce a connection on $V_1 \cup_{\tilde{f}} V_2$; this resulting connection is compatible with \tilde{g} , a pseudo-metric determined by g_1 and g_2 .

Diffeological gluing A large part of our approach consists in establishing how the above-listed components behave with respect to the operation of *diffeological gluing*. On the level of underlying sets, this is the standard operation of topological gluing; the resulting space is endowed with a diffeology that is probably the finest sensible one: it naturally includes the diffeologies on the factors, and not much else. One disadvantage of this notion is that this is a pretty weak diffeology, that loses (or risks losing) sight of some natural aspects of the underlying space; for instance, the obvious gluing diffeology on the union of the coordinate axes in \mathbb{R}^2 is weaker than the subset diffeology relative to its inclusion into \mathbb{R}^2 , see [18] (on the other hand, the concept of gluing may provide a natural framework for treating objects such as manifolds with corners, see below for more detail). As of now, we view this notion of the gluing diffeology as more of a precursor to a coarser one, with more involved properties — but still as a useful testing ground for the constructions that we are considering.

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1 The main notions

Here we briefly recall the notions of diffeology that appear throughout this paper, in the form in which they appear in [7].

1.1 Diffeological spaces and diffeologies

The central object for diffeology is a set X endowed with a *diffeological structure*, which is a collection of maps from usual domains to X ; three natural conditions must be satisfied.

Definition 1.1. ([15]) A *diffeological space* is a pair (X, \mathcal{D}_X) where X is a set and \mathcal{D}_X is a specified collection, also called the *diffeology* of X or its *diffeological structure*, of maps $U \rightarrow X$ (called *plots*) for each open set U in \mathbb{R}^n and for each $n \in \mathbb{N}$, such that for all open subsets $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ the following three conditions are satisfied:

1. (The covering condition) Every constant map $U \rightarrow X$ is a plot;
2. (The smooth compatibility condition) If $U \rightarrow X$ is a plot and $V \rightarrow U$ is a smooth map (in the usual sense) then the composition $V \rightarrow U \rightarrow X$ is also a plot;
3. (The sheaf condition) If $U = \cup_i U_i$ is an open cover and $U \rightarrow X$ is a set map such that each restriction $U_i \rightarrow X$ is a plot then the entire map $U \rightarrow X$ is a plot as well.

A standard example of a diffeological space is a standard manifold whose diffeology consists of all usual smooth maps into it, but many others, and quite exotic, examples can be found.

Since a diffeological structure on a given set is defined as a collection of maps (into the set), two diffeologies can be compared with respect to inclusion. One says that a diffeology \mathcal{D}' is **finer** than another diffeology \mathcal{D} if $\mathcal{D}' \subset \mathcal{D}$, whereas \mathcal{D} is said to be **coarser** than \mathcal{D}' . The finest of all possible diffeologies is called **discrete**, and the coarsest one is said to be the **coarse** (or **indiscrete**) diffeology. Furthermore, many standard diffeologies are defined as the finest/coarsest diffeology with a given property P (see [7] for the existence issues and other details).

Generated diffeology Let X be any set, and let $A = \{U_i \rightarrow X\}_{i \in I}$ be a set of maps into it, where each map is defined on a domain of some \mathbb{R}^{m_i} . The **diffeology generated by A** is the finest diffeology on X that contains all maps in A as plots.

Smooth maps, pushforwards and pullbacks Let X and Y be two diffeological spaces, and let $f : X \rightarrow Y$ be a set map. It is a **smooth map** if for every plot p of X the composition $f \circ p$ is a plot of Y . Suppose now that only X is endowed with a diffeology; then Y can be endowed with the so-called **pushforward diffeology** with respect to f , which is the finest diffeology for which f is smooth. If, on the contrary, Y carries a diffeology and X does not, then it can be given a natural one, the so-called **pullback diffeology**, defined as the coarsest diffeology for which f is smooth. These are reciprocally inverse notions, in the sense that locally a plot p_Y of a pushforward diffeology has form $p_Y = f \circ p_X$ for some p_X a plot of X , and the pullback diffeology includes all maps p_X for which p_Y is a plot (of Y).

Inductions and subductions Let X and Y be diffeological spaces, and $f : X \rightarrow Y$ be a smooth map. We say that f is an **induction** if it is injective and the pushforward by f of the diffeology of X coincides with the subset diffeology on $f(X)$ (since f is smooth, this pushforward diffeology is always contained in the aforementioned subset diffeology, but in general it is properly contained in it). We say that f is a **subduction** if it is surjective, and the diffeology of Y coincides with the pushforward of the diffeology of X by f ; once again, in general it is strictly contained in it.

Subset diffeology and quotient diffeology Each and every subset $Y \subseteq X$ of a diffeological space X carries a natural diffeology, called **the subset diffeology**, composed essentially of all plots of X whose image is wholly contained in Y . Likewise, the quotient of X by any equivalence relation \sim carries the **quotient diffeology**, defined as the pushforward of the diffeology of X by the natural projection $X \rightarrow X/\sim$.

The disjoint union diffeology and the product diffeology Let X_1, \dots, X_n be diffeological spaces. The **disjoint union diffeology** on the disjoint union of them is the finest diffeology such that for each $i = 1, \dots, n$ the natural injection $X_i \rightarrow X$; is smooth; the **product diffeology** on their direct product is the coarsest diffeology such that for each $i = 1, \dots, n$ the natural projection $\pi_i : X = X_1 \times \dots \times X_n \rightarrow X_i$ is smooth.

Functional diffeology If X and Y are two diffeological spaces, $C^\infty(X, Y)$ stands for the set of all smooth maps $X \rightarrow Y$; this set carries a natural diffeology called the **functional diffeology**, defined as the coarsest diffeology such that the *evaluation map*, $\text{EV} : C^\infty(X, Y) \times X \rightarrow Y$, defined by $\text{EV}(f, x) = f(x)$ is smooth.

1.2 Diffeological vector spaces

We consider next the notion of a diffeological vector space and the related ones; see [7], and also [17] and [19], for further details.

The definition A **diffeological vector space** (over \mathbb{R}) is a vector space V endowed with a **vector space diffeology**, that is, any diffeology for which the following two maps are smooth: the addition map $V \times V \rightarrow V$, where $V \times V$ carries the product diffeology, and the scalar multiplication map $\mathbb{R} \times V \rightarrow V$, where \mathbb{R} has the standard diffeology and $\mathbb{R} \times V$ carries the product diffeology. One concept that we will

make frequent use of is that of the *vector space diffeology generated* by a given set A of maps. This is the smallest diffeology that is both a vector space diffeology and that contains the set A .

Vector space diffeologies are, relatively speaking, rather large ones. For instance, the finest vector space diffeology on \mathbb{R}^n is its standard diffeology (the finest of all diffeologies is the discrete one). A vector space endowed with its finest vector space diffeology is called a **fine vector space**.¹

Subspaces and quotients Any usual vector subspace of a diffeological vector space is naturally a diffeological vector space for the subset diffeology. The same is true for any quotient of a vector space, which is automatically assumed to carry the quotient diffeology.

Linear maps and smooth linear maps If the diffeology of a given diffeological vector space is not the standard one, not every linear map on it is smooth. As an example, it suffices to take any $V = \mathbb{R}^n$ and endow it with the vector space diffeology generated by a single plot $\mathbb{R} \ni x \mapsto |x|e_n \in V$. Then the last element of the canonical dual basis, $e^n : V \rightarrow \mathbb{R}$, is not a diffeologically smooth map (as a map into the standard \mathbb{R}).

Diffeological dual The **diffeological dual** V^* of a diffeological vector space V ([17], [19]) is the space of all smooth linear maps with values into the standard \mathbb{R} , endowed with the corresponding functional diffeology. With respect to this diffeology, V^* is itself a diffeological vector space and, if $\dim(V) < \infty$, it is a standard space (unless V itself is standard, the dimension of V^* is strictly smaller than that of V).

Scalar products and pseudo-metrics A scalar product on a diffeological vector space is a smooth non-degenerate symmetric bilinear form $V \times V \rightarrow \mathbb{R}$ (for the product diffeology on $V \times V$ and the standard diffeology on \mathbb{R}). However, a scalar product in this sense rarely exists on diffeological vector spaces; in particular, among the finite-dimensional ones, scalar products exist only on those diffeomorphic to the standard \mathbb{R}^n (see [7]). In general, the maximal rank of a smooth symmetric bilinear form on V is equal to the dimension if its diffeological dual V^* ; a smooth symmetric semidefinite positive bilinear form that achieves this rank (there is always one) is called a **pseudo-metric** on V .

Direct sum of diffeological vector spaces This is the usual direct sum endowed with the *product* diffeology. It is interesting to note that frequently enough, decompositions of a given diffeological vector space V as a direct sum of two of its subspaces do not give back the original diffeology of V , meaning that the product diffeology corresponding to the subset diffeologies on the summands can be strictly finer than the original diffeology of V . If the opposite is true, then we call such a decomposition **smooth**, and say that any of its factors **splits off smoothly** in V .

Tensor product of diffeological vector spaces Given a finite collection V_1, \dots, V_n of diffeological vector spaces, their usual tensor product $V_1 \otimes \dots \otimes V_n$ is endowed with the **tensor product diffeology** (see [17], [19]). The tensor product diffeology is defined as the quotient diffeology corresponding to the usual representation of $V_1 \otimes \dots \otimes V_n$ as the quotient of the free product $V_1 \times \dots \times V_n$ (that is endowed with the finest vector space diffeology on the free product containing the product, *i.e.* the direct sum, diffeology on $V_1 \oplus \dots \oplus V_n$) by the kernel of the universal map onto $V_1 \otimes \dots \otimes V_n$.

1.3 Diffeological vector pseudo-bundles

The notion of a diffeological vector pseudo-bundle appeared initially in [6] as a partial instance of *diffeological fibre bundle* (see also [7], Chapter 8), then in [17] under the name of a *regular vector bundle*, and finally in [2] under the name of a *diffeological vector space over X* . We use the term *diffeological vector pseudo-bundle*, in order to emphasize that frequently it is not really a bundle (it is not required to be locally trivial), and also to avoid confusion with individual diffeological vector spaces, something which might happen with the term adopted in [2] when both concepts appear simultaneously.

¹We mostly treat finite-dimensional spaces here, for which a fine space is just a standard \mathbb{R}^n of appropriate dimension.

1.3.1 Properties

We recall the notion of a diffeological vector pseudo-bundle, along with the main properties.

The definition A **diffeological vector pseudo-bundle** is a smooth surjective map $\pi : V \rightarrow X$ between two diffeological spaces V and X such that for each $x \in X$ the pre-image $\pi^{-1}(x)$ carries a vector space structure. The corresponding addition and scalar multiplication maps, as well as the zero section $s_0 : X \rightarrow V$, must be smooth for the appropriate diffeologies.

More precisely, the addition on each fibre induces a map $V \times_X V \rightarrow V$. It should be smooth for the subset diffeology on $V \times_X V$ as a subset of $V \times V$, which itself is considered with the product diffeology. Likewise, scalar multiplication induces a map $\mathbb{R} \times V \rightarrow V$, which must be smooth for the standard diffeology on \mathbb{R} and the corresponding product diffeology on $\mathbb{R} \times V$.

Non-local triviality The only true difference of the above definition with respect to the usual one of a vector bundle is thus the absence of the usual requirement of the existence of local trivializations. Indeed, a diffeological pseudo-bundle may not be locally trivial in the diffeological sense (nor in the topological one); this matter was already discussed in [6] (see also the more recent [7]). One specific example, which has the merit of arising in an independent context (thus indicating that the interest in the notion is not a purely aprioristic one), is due to Christensen-Wu [2]. The internal tangent bundle, described therein, of the union of the coordinate axes in \mathbb{R}^2 (as a diffeological space, this union carries the subset diffeology inherited from the standard diffeology of \mathbb{R}^2) is not locally trivial.

The operations All the usual operations on vector bundles (direct sum, tensor product, taking the dual bundle) are for diffeological vector pseudo-bundles as well, although, due to the absence of local trivializations, they are defined in a slightly different manner. Most of the material on this matter appeared first (to my knowledge) in [17]; some additional considerations appear in [9].

- **Sub-bundles and quotient pseudo-bundles.** A sub-bundle Z of a diffeological vector pseudo-bundle $\pi : V \rightarrow X$ is any subset of V such that its intersection with any fibre is a vector subspace of the latter. Endowed with the subset diffeology and the restriction of π to it, it becomes itself a diffeological vector pseudo-bundle with the base space X . Note that any collection of vector subspaces of fibres of V , one per fibre, is a sub-bundle.

If we fix now a sub-bundle $Z \subset V$, there is a natural fibrewise quotient $W = V/Z$ that is called the quotient pseudo-bundle. This is indeed a diffeological vector pseudo-bundle for the quotient diffeology. The corresponding subset diffeology on each fibre $(V/Z)_x$ of the quotient V/Z is the same as the quotient diffeology relative to the subset diffeology on V_x .

- **Direct sum.** Given two diffeological vector pseudo-bundles $\pi_1 : V_1 \rightarrow X$, $\pi_2 : V_2 \rightarrow X$ over the same base space X , their direct sum pseudo-bundle is given by the subset $V_1 \times_X V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2, \pi_1(v_1) = \pi_2(v_2)\} \subset V_1 \times V_2$. It carries the subset diffeology relative to the product diffeology on $V_1 \times V_2$ and is endowed with the obvious fibrewise operations of addition and scalar multiplication, which are smooth. Moreover, the subset diffeology of the fibre at any $x \in X$ is the same as that of $\pi_1^{-1}(x) \oplus \pi_2^{-1}(x)$.
- **Tensor product.** The tensor product of two vector pseudo-bundles $\pi_1 : V_1 \rightarrow X$ and $\pi_2 : V_2 \rightarrow X$ can be constructed as the quotient pseudo-bundle of the free product pseudo-bundle $V_1 \times_X V_2 \rightarrow X$, with the quotient being taken over the sub-bundle consisting of the kernels of the universal maps on each fibre: $\pi_1^{-1}(x) \times \pi_2^{-1}(x) \rightarrow \pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$. The diffeology is then the corresponding quotient diffeology and the fibre at any $x \in X$ is the diffeological tensor product $\pi_1^{-1}(x) \otimes \pi_2^{-1}(x)$ of the corresponding fibres.
- **Dual pseudo-bundle.** For any diffeological vector pseudo-bundle $\pi : V \rightarrow X$ there is the corresponding dual pseudo-bundle such that the fibre at any $x \in X$ is the diffeological dual $(\pi^{-1}(x))^*$, so we can write $V^* := \cup_{x \in X} V_x^*$. The diffeology on V^* can be characterized as follows: a map $p : \mathbb{R}^l \supset U \rightarrow V^*$ is a plot for the dual bundle diffeology on V^* if and only if for every plot

$q : \mathbb{R}^{l'} \supset U' \rightarrow V$ of V the map $Y' \rightarrow \mathbb{R}$ defined on $Y' = \{(u, u') | \pi^*(p(u)) = \pi(q(u')) \in X\} \subset U \times U'$ and acting by $(u, u') \mapsto p(u)(q(u'))$, is smooth for the subset diffeology of $Y' \subset \mathbb{R}^{l+l'}$ and the standard diffeology of \mathbb{R} . The corresponding subset diffeology on each fibre V_x^* is its usual functional diffeology as the dual space of the diffeological vector space V_x .

1.3.2 Assembling a pseudo-bundle from simpler components: diffeological gluing

This operation, considered in some detail in [9], mimicks the usual topological gluing, with which it coincides on the level of underlying topological spaces. The space comes with the standard choice of diffeology, called the **gluing diffeology**. As we mentioned in the Introduction, this is in some sense the finest diffeology that it makes sense to consider.

The gluing of two diffeological spaces Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supset Y \rightarrow X_2$ be a map smooth for the subset diffeology of Y .² Then there is a usual topological gluing of X_1 to X_2 along f (symmetric if f is injective), defined as

$$X_1 \cup_f X_2 = (X_1 \sqcup X_2) / \sim, \text{ where } X_1 \supset Y \ni y \sim f(y) \in X_2.$$

This space is endowed with the quotient diffeology of the disjoint union diffeology on $X_1 \sqcup X_2$.

The inductions i_1 and i_2 There are two natural inclusions into the space $X_1 \cup_f X_2$, whose ranges cover it. These are given by the maps $i_1 : (X_1 \setminus Y) \hookrightarrow (X_1 \sqcup X_2) \rightarrow X_1 \cup_f X_2$, where the second arrow stands for the natural projection onto the quotient space, and $i_2 : X_2 \hookrightarrow (X_1 \sqcup X_2) \rightarrow X_1 \cup_f X_2$. They are clearly bijective; furthermore, they are inductions (see [10]). The images $i_1(X_1 \setminus Y)$ and $i_2(X_2)$ are disjoint and yield a covering of $X_1 \cup_f X_2$, which is useful for constructing maps on $X_1 \cup_f X_2$.

Plots of the gluing diffeology on $X_1 \cup_f X_2$ The plots of $X_1 \cup_f X_2$ admit the following local description. Let $p : U \rightarrow X_1 \cup_f X_2$ be a plot; then for every $u \in U$ there is a neighborhood $U' \subset U$ of u such that the restriction of p on U' lifts to a plot $p' : U' \rightarrow (X_1 \sqcup X_2)$. Since locally every plot of $X_1 \sqcup X_2$ is a plot of either X_1 or X_2 , up to restricting it to a connected component U'' of U' , there exists either a plot $p_1 : U'' \rightarrow X_1$ or a plot $p_2 : U'' \rightarrow X_2$ such that $p|_{U''}$ lifts to, respectively, p_1 or p_2 . Furthermore, if $p|_{U''}$ lifts to p_2 then its actual form is $p|_{U''} = i_2 \circ p_2$, whereas if it lifts to p_1 , its actual form is then as follows:

$$p|_{U''}(u'') = \begin{cases} i_1(p_1(u'')) & \text{if } p_1(u'') \in X_1 \setminus Y, \\ i_2(f(p_1(u''))) & \text{if } p_1(u'') \in Y. \end{cases}$$

The gluing of two pseudo-bundles The operation of gluing of two pseudo-bundles consists in performing twice the gluing of diffeological spaces, once for the total spaces, and the second time for the base spaces; the two gluing maps must be consistent with each other for the result to be a pseudo-bundle. Specifically, let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let $f : X_1 \supset Y \rightarrow X_2$ be a smooth map, and let $\tilde{f} : \pi_1^{-1}(Y) \rightarrow \pi_2^{-1}(f(Y))$ be any smooth lift of f such that the restriction of \tilde{f} on each fibre $\pi_1^{-1}(y)$ for $y \in Y$ is linear. Consider the diffeological spaces $V_1 \cup_{\tilde{f}} V_2$ and $X_1 \cup_f X_2$; since \tilde{f} is a lift of f , the pseudo-bundle projections π_1 and π_2 yield a well-defined map, denoted by $\pi_1 \cup_{(\tilde{f}, f)} \pi_2$, from $V_1 \cup_{\tilde{f}} V_2$ to $X_1 \cup_f X_2$. Furthermore, by the linearity assumption on \tilde{f} (see [9]), the map

$$\pi_1 \cup_{(\tilde{f}, f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \rightarrow X_1 \cup_f X_2$$

is itself a diffeological vector pseudo-bundle.

Notation The total and the base space of the new pseudo-bundle are both results of a diffeological gluing, so everything we have said about this operation applies to each of them. In particular, there are the two pairs of standard inductions, which are denoted, as before, by i_1, i_2 for the base space $X_1 \cup_f X_2$ and by j_1, j_2 for the total space $V_1 \cup_{\tilde{f}} V_2$, that is, $j_1 : (V_1 \setminus \pi_1^{-1}(Y)) \hookrightarrow (V_1 \sqcup V_2) \rightarrow V_1 \cup_{\tilde{f}} V_2$ and $j_2 : V_2 \hookrightarrow (V_1 \sqcup V_2) \rightarrow V_1 \cup_{\tilde{f}} V_2$.

²Otherwise it can be any map, but we will often have to assume that f is a diffeomorphism with its image, although its domain of definition could be any type of set.

Why gluing? Although we will not be able to really consider this in the present paper, we briefly mention how the gluing procedure provides a natural context for notions such as δ -functions. What we mean by this is the following. Let $X_1 \subset \mathbb{R}^2$ be the x -axis, let $X_2 = \{(0,1)\}$, and let the gluing map $f : Y = \{(0,0)\} \rightarrow \{(0,1)\}$ be the obvious map. Let $p : \mathbb{R} \rightarrow X_1 \cup_f X_2$ be the map defined by $p(x) = i_1(x,0)$ for $x \neq 0$ and $p(0) = i_2(0,1)$. By definition of the gluing diffeology, this is a plot of $X_1 \cup_f X_2$ (and so an instance of a smooth function into it). We now can observe that p can be seen as the δ -function $\mathbb{R} \rightarrow \mathbb{R}$, by projecting both X_1 and X_2 onto the y -axis of their ambient \mathbb{R}^2 . More precisely, let $\text{pr}_y : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection of \mathbb{R}^2 onto the y -axis, that is, $\text{pr}_y(x,y) = y$, and let $h : X_1 \cup_f X_2 \rightarrow \mathbb{R}$ be given by

$$h(\tilde{x}) = \begin{cases} \text{pr}_y(i_1^{-1}(\tilde{x})) & \text{if } \tilde{x} \in i_1(X_1 \setminus Y), \\ \text{pr}_y(i_2^{-1}(\tilde{x})) & \text{if } \tilde{x} \in i_2(X_2). \end{cases}$$

As follows from the definitions of i_1 and i_2 , this defines h on the entire $X_1 \cup_f X_2$. Observe finally that the composition $h \circ p$ is indeed the usual δ -function, *i.e.*, the function δ given by $\delta(x) = 0$ if $x \neq 0$ and $\delta(0) = 1$.

1.3.3 Gluing of pseudo-bundles and the vector bundles operations

The gluing is usually well-behaved with respect to the usual operations on vector pseudo-bundles (see [9], [10]), with the one exception being the operation of taking the dual pseudo-bundle.

The dual pseudo-bundles Given a gluing along a pair (\tilde{f}, f) , with a smoothly invertible f , of two pseudo-bundles $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$, we can obtain the following two pseudo-bundles:

$$(\pi_1 \cup_{(\tilde{f}, f)} \pi_2)^* : (V_1 \cup_{\tilde{f}} V_2)^* \rightarrow X_1 \cup_f X_2 \text{ and } \pi_2^* \cup_{(\tilde{f}^*, f)} \pi_1^* : V_2^* \cup_{\tilde{f}^*} V_1^* \rightarrow X_2 \cup_{f^{-1}} X_1.$$

In general, they are not diffeomorphic (see the above two references); on the other hand, they are so if \tilde{f}^* (but maybe not \tilde{f} itself) is a diffeomorphism (see [11]).

Direct sum If we are given two diffeological vector pseudo-bundles, $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$, and a gluing of them along (\tilde{f}, f) , as well as two other pseudo-bundles $\pi'_1 : V'_1 \rightarrow X_1$ and $\pi'_2 : V'_2 \rightarrow X_2$, with the gluing along (\tilde{f}', f) , we can obtain

$$\left(\pi_1 \cup_{(\tilde{f}, f)} \pi_2 \right) \oplus \left(\pi'_1 \cup_{(\tilde{f}', f)} \pi'_2 \right) : \left(V_1 \cup_{\tilde{f}} V_2 \right) \oplus \left(V'_1 \cup_{\tilde{f}'} V'_2 \right) \rightarrow X_1 \cup_f X_2$$

and

$$(\pi_1 \oplus \pi'_1) \cup_{\tilde{f} \oplus \tilde{f}'} (\pi_2 \oplus \pi'_2) : (V_1 \oplus V'_1) \cup_{\tilde{f} \oplus \tilde{f}'} (V_2 \oplus V'_2) \rightarrow X_1 \cup_f X_2.$$

It is easy (although tedious) to check that they are diffeomorphic via a diffeomorphism covering the identity map on the bases ([9], [10]).

Tensor product Likewise, there are two pseudo-bundles that constructed using the operation of the tensor product:

$$(\pi_1 \cup_{(\tilde{f}, f)} \pi_2) \otimes (\pi'_1 \cup_{(\tilde{f}', f)} \pi'_2) : (V_1 \cup_{\tilde{f}} V_2) \otimes (V'_1 \cup_{\tilde{f}'} V'_2) \rightarrow X_1 \cup_f X_2$$

and

$$(\pi_1 \otimes \pi'_1) \cup_{(\tilde{f} \otimes \tilde{f}', f)} (\pi_2 \otimes \pi'_2) : (V_1 \otimes V'_1) \cup_{(\tilde{f} \otimes \tilde{f}', f)} (V_2 \otimes V'_2) \rightarrow X_1 \cup_f X_2.$$

These are diffeomorphic as well.

1.4 Diffeological counterparts of Riemannian metrics

A **pseudo-metric** on a diffeological vector pseudo-bundle $\pi : V \rightarrow X$ is a smooth section of the pseudo-bundle $\pi^* \otimes \pi^* : V^* \otimes V^* \rightarrow X$, *i.e.*, a smooth map $g : X \rightarrow V^* \otimes V^*$ such that for all $x \in X$ the value $g(x)$ is a symmetric form of rank $\dim((\pi^{-1}(x))^*)$, with all the eigenvalues non-negative; in other words, it is a pseudo-metric on the diffeological vector space $\pi^{-1}(x)$. Such a pseudo-metric obviously exists on any

trivial diffeological pseudo-bundle; the gluing construction for pseudo-metrics, that we briefly summarize below, allows to put them on a number of more complicated pseudo-bundles.

Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing of the former to the latter such that f is a diffeomorphism. Suppose that each of V_1, V_2 admits a pseudo-metric; let g_i be a chosen pseudo-metric on V_i for $i = 1, 2$. We say that g_1 and g_2 are **compatible** (with the gluing along (\tilde{f}, f)) if for every $y \in Y$ and for all $v_1, v_2 \in \pi_1^{-1}(y)$ we have

$$g_1(y)(v_1, v_2) = g_2(f(y))(\tilde{f}(v_1), \tilde{f}(v_2)).$$

Assuming now that g_1 and g_2 are compatible, we define a map $\tilde{g} : X_1 \cup_f X_2 \rightarrow (V_1 \cup_{\tilde{f}} V_2)^* \otimes (V_1 \cup_{\tilde{f}} V_2)^*$ by setting:

$$\tilde{g}(x)(\cdot, \cdot) = \begin{cases} g_1(i_1^{-1}(x))(j_1^{-1}(\cdot), j_1^{-1}(\cdot)) & \text{for } x \in i_1(X_1 \setminus Y) \\ g_2(i_2^{-1}(x))(j_2^{-1}(\cdot), j_2^{-1}(\cdot)) & \text{for } x \in i_2(X_2). \end{cases}$$

This turns out to be a pseudo-metric on $V_1 \cup_{\tilde{f}} V_2$ (see [10] for more details).

Finally, if $\pi : V \rightarrow X$ is a pseudo-bundle endowed with a pseudo-metric g then V^* can sometimes be endowed with the **dual pseudo-metric** g^* , which is defined as follows. Let $\Phi_g : V \rightarrow V^*$ be the usual **pairing map**, *i.e.*, one defined by $\Phi_g(v)(\cdot) = g(\pi(v))(v, \cdot)$. This map is always smooth but in general is not invertible, unless we consider it as defined on a specific smaller sub-bundle of V (called its characteristic sub-bundle); even in this latter case, it is not clear whether its inverse is always smooth. The map $g^* : X \rightarrow V^{**} \otimes V^{**}$ is given by the identity

$$g^*(x)(\Phi_g(v_1), \Phi_g(v_2)) = g(x)(v_1, v_2);$$

since the map Φ_g is surjective and its kernel is contained in the collection of the isotropic subspaces of the fibres, g^* is always well-defined. However, by the above remark on Φ_g not necessarily having a smooth (right) inverse, we cannot be certain that g^* is always smooth; when necessary, we impose this as an assumption.

1.5 Diffeological 1-forms

For diffeological spaces, there exists a rather well-developed theory of differential k -forms on them (see [7], Chapter 6, for a detailed exposition); we recall the main notions for the case $k = 1$.

1.5.1 Differential 1-forms and differentials of functions

Here are the basic definitions.

A diffeological 1-form A differential 1-form on a diffeological space X is defined by assigning to each plot $p : \mathbb{R}^k \supset U \rightarrow X$ a (usual) differential 1-form $\omega(p)(u) = f_1(u)du_1 + \dots + f_k(u)du_k \in \Lambda^1(U)$ such that this assignment satisfies the following compatibility condition. If $q : U' \rightarrow X$ is another plot of X such that there exists a usual smooth map $F : U' \rightarrow U$ with $q = p \circ F$ then $\omega(q)(u') = F^*(\omega(p)(u))$. The definition of a diffeological k -form is actually the same, except that the differential forms assigned to the domains of plots are (obviously) k -forms.

The differential of a function Let X be a diffeological space, and let $f : X \rightarrow \mathbb{R}$ be a diffeologically smooth function on it; recall that this means that for every plot $p : U \rightarrow X$ the composition $f \circ p : U \rightarrow \mathbb{R}$ is smooth in the usual sense, therefore $d(f \circ p)$ is a differential form on U . It is quite easy to see that the assignment $p \mapsto d(f \circ p) =: \omega(p)$ is a differential 1-form on X ; indeed, let $g : V \rightarrow U$ be a smooth function. The smooth compatibility condition $\omega(p \circ g) = g^*(\omega(p))$ is then equivalent to $d((f \circ p) \circ g) = g^*(d(f \circ p))$, *i.e.*, to a standard property of usual differential forms.

1.5.2 The space $\Omega^1(X)$ of 1-forms

The set of all differential 1-forms on X is denoted by $\Omega^1(X)$; it carries a natural functional diffeology with respect to which it becomes a diffeological vector space. The addition and the scalar multiplication operations, that make $\Omega^1(X)$ into a vector space, are given pointwise (at points in the domains of plots). The functional diffeology on $\Omega^1(X)$ is characterized by the following condition:

- a map $q : U' \rightarrow \Omega^1(X)$ is a plot of $\Omega^1(X)$ if and only if for every plot $p : U \rightarrow X$ the map $U' \times U \rightarrow \Omega^1(\mathbb{R}^n)$ given by $(u', u) \mapsto (q(u')(p))(u)$ is smooth, where $U \subset \mathbb{R}^n$.

The expression $q(u')(p)$ stands for the 1-form on the domain of definition of p , *i.e.*, the domain U , that the differential 1-form $q(u')$ on X assigns to the plot p of X .

1.5.3 The pseudo-bundle of k -forms $\Lambda^k(X)$

There is a natural quotienting of $\Omega^k(X)$, which gives, at every point $x \in X$, the set of all distinct values, at x , of differential k -forms on X ; this set is denoted by $\Lambda_x^k(X)$. The collection of all $\Lambda_x^k(X)$, for each $x \in X$, yields the pseudo-bundle $\Lambda^k(X)$ (we describe it for a generic $k \geq 1$, since the description is exactly the same for all k).

The fibre $\Lambda_x^k(X)$ Let X be a diffeological space, and let x be a point of it. A plot $p : U \rightarrow X$ is *centered at x* if $U \ni 0$ and $p(0) = x$. Let \sim_x be the following equivalence relation: two k -forms $\alpha, \beta \in \Omega^k(X)$ are equivalent, $\alpha \sim_x \beta$, if and only if, for every plot p centered at x , we have $\alpha(p)(0) = \beta(p)(0)$. The class of α for the equivalence relation \sim_x is called **the value of α at the point x** and is denoted by α_x . The set of all the values at the point x , for all k -forms on X , is denoted by $\Lambda_x^k(X)$:

$$\Lambda_x^k(X) = \Omega^k(X) / \sim_x = \{\alpha_x \mid \alpha \in \Omega^k(X)\}.$$

Any element $\alpha \in \Lambda_x^k(X)$ is said to be a **k -form of X at the point x** (and x is said to be the *basepoint* of α). The space $\Lambda_x^k(X)$ is then called the **space of k -forms of X at the point x** .

The space $\Lambda_x^k(X)$ as a quotient of $\Omega^k(X)$ Two k -forms α and β have the same value at the point x if and only if their difference vanishes at this point: $(\alpha - \beta)_x = 0$. The set $\{\alpha \in \Omega^k(X) \mid \alpha_x = 0_x\}$ of the k -forms of X vanishing at the point x is a vector subspace of $\Omega^k(X)$; furthermore,

$$\Lambda_x^k(X) = \Omega^k(X) / \{\alpha \in \Omega^k(X) \mid \alpha_x = 0_x\}.$$

In particular, as a quotient of a diffeological vector space by a vector subspace, each space $\Lambda_x^k(X)$ is naturally a diffeological vector space; the addition and the scalar multiplication on $\Lambda_x^k(X)$ are well-defined for any choice of representatives.

The pseudo-bundle $\Lambda^k(X)$ k -forms The **pseudo-bundle of k -forms over X** , denoted by $\Lambda^k(X)$, is the union of all spaces $\Lambda_x^k(X)$, that is,

$$\Lambda^k(X) = \bigcup_{x \in X} \Lambda_x^k(X) = \{(x, \alpha) \mid \alpha \in \Lambda_x^k(X)\}.$$

It has an obvious structure of a pseudo-bundle over X and is endowed with the diffeology that is the pushforward of the product diffeology on $X \times \Omega^k(X)$ by the projection

$$\pi^{\Omega, \Lambda} : X \times \Omega^k(X) \rightarrow \Lambda^k(X)$$

acting by $\pi^{\Omega, \Lambda}(x, \alpha) = (x, \alpha_x)$. Note that for this diffeology the natural projection

$$\pi^\Lambda : \Lambda^k(X) \rightarrow X$$

is a local subduction. Moreover, each subspace $(\pi^\Lambda)^{-1}(x)$ is smoothly isomorphic to $\Lambda_x^k(X)$.

The plots of the pseudo-bundle $\Lambda^k(X)$ A map $p : U \ni u \mapsto (p_1(u), p_2(u)) \in \Lambda^k(X)$ defined on some domain U in some \mathbb{R}^m is a plot of $\Lambda^k(X)$ if and only if the following two conditions are fulfilled:

1. The map p_1 is a plot of X ;
2. For all $u \in U$ there exists an open neighborhood U' of u and a plot $q : U' \rightarrow \Omega^k(X)$ such that for all $u' \in U'$ we have $p_2(u') = q(u')(p_1)(u')$.

In other words, a plot of $\Lambda^k(X)$ is locally represented by a pair, consisting of a plot of X and a plot of $\Omega^k(X)$ (with the same domain of definition).

1.6 Diffeological forms and diffeological gluing

Here we recall the main facts regarding the behavior of diffeological forms under gluing; these are established in [12]).

1.6.1 The space of forms $\Omega^1(X_1 \cup_f X_2)$

Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a smooth map. Let

$$\pi : X_1 \sqcup X_2 \rightarrow X_1 \cup_f X_2$$

be the quotient projection. We recall first the image of the pullback map (see 6.38 in [7])

$$\pi^* : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1 \sqcup X_2) \cong \Omega^1(X_1) \times \Omega^1(X_2).$$

Note that π^* is injective but typically not surjective.

Denote by $\Omega_f^1(X_1)$ the subset of the so-called f -invariant forms, where a form $\omega_1 \in \Omega^1(X_1)$ is called **f -invariant** if for any f -equivalent plots p' and p'' we have $\omega_1(p') = \omega_1(p'')$; two plots p' and p'' are **f -equivalent** if they have the same domain of definition U and for any $u \in U$ the inequality $p'(u) \neq p''(u)$ implies that $p'(u), p''(u) \in Y$ and $f(p'(u)) = f(p''(u))$; thus, by definition,

$$\Omega_f^1(X_1) = \{\omega_1 \in \Omega^1(X_1) \mid \omega_1(p') = \omega_1(p'') \text{ whenever } p', p'' \text{ are } f\text{-equivalent}\}.$$

The set $\Omega_f^1(X_1)$ is a vector subspace of $\Omega^1(X_1)$; if f is injective then $\Omega_f^1(X_1) = \Omega^1(X_1)$. If now $\omega_1 \in \Omega_f^1(X_1)$ and $\omega_2 \in \Omega^1(X_2)$ then ω_1 and ω_2 are called **compatible** if for every plot p_1 of the subset diffeology on Y we have that $\omega_1(p_1) = \omega_2(f \circ p_1)$; the first term ω_1 of any compatible pair is always f -invariant.

Theorem 1.2. ([12]) *The space $\Omega^1(X_1 \cup_f X_2)$ is diffeomorphic to the space $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$, where*

$$\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) := \{(\omega_1, \omega_2) \mid \omega_1 \in \Omega_f^1(X_1), \omega_2 \in \Omega^1(X_2), \omega_1 \text{ and } \omega_2 \text{ are compatible}\}$$

is considered with the subset diffeology relative to its inclusion into $\Omega^1(X_1) \times \Omega^1(X_2)$ endowed with the product diffeology.

If $\Omega^1(X_1) \times \Omega^1(X_2)$ is considered with the algebraic structure of the direct sum $\Omega^1(X_1) \oplus \Omega^1(X_2)$ (that it naturally carries via its identification with $\Omega^1(X_1 \cup_f X_2)$; the direct sum structure on $\Omega^1(X_1) \times \Omega^1(X_2)$ corresponds to the vector space structure on $\Omega^1(X_1 \cup_f X_2)$) then $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$ is a vector subspace, and the diffeomorphism whose existence is claimed in Theorem 1.2 is an isomorphism. This diffeomorphism can be explicitly described as follows.

Let $\omega_1 \in \Omega_f^1(X_1)$ be an f -invariant form on X_1 , and let $\omega_2 \in \Omega^1(X_2)$ be a form on X_2 compatible with ω_1 . Then there is a natural induced form on $X_1 \cup_f X_2$, that is denoted by $\omega_1 \cup_f \omega_2$ and that is defined as follows. Let p be a plot of $X_1 \cup_f X_2$ with a connected domain of definition, and let p_i be its lift to a plot of X_i for $i = 1$ or $i = 2$. Then we set

$$(\omega_1 \cup_f \omega_2)(p) = \begin{cases} \omega_1(p_1) & \text{if } p = \pi \circ p_1, \\ \omega_2(p_2) & \text{if } p = \pi \circ p_2. \end{cases}$$

The form $\omega_1 \cup_f \omega_2$ is well-defined, because every plot of $X_1 \cup_f X_2$ with a connected domain of definition lifts either to a plot of X_1 or one of X_2 , and there can be at most one lift to a plot of X_2 , while all lifts to X_1 of the same plot are f -equivalent (recall that ω_1 is required to be f -invariant). We have the following statement (it is a direct consequence of [12], Theorem 3.9 and Theorem 3.1).

Theorem 1.3. *The assignment $(\omega_1, \omega_2) \mapsto \omega_1 \cup_f \omega_2$ yields a vector space diffeomorphism*

$$\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \cong \Omega^1(X_1 \cup_f X_2).$$

The inverse of this diffeomorphism filters through the pullback map $\pi^ : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1 \sqcup X_2)$.*

Finally, a further observation can be made on the structure of the space $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$ (see [12]).

Theorem 1.4. *The projection $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \rightarrow \Omega^1(X_2)$ is always a surjective map. If f is a subduction onto $f(Y)$ then the projection $\Omega_f^1(X_1) \times_{\text{comp}} \Omega^1(X_2) \rightarrow \Omega_f^1(X_1)$ is also surjective.*

1.6.2 The fibrewise structure of the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$

We now consider the pseudo-bundle $\Lambda^1(X_1 \cup_f X_2)$. We almost always assume that f is a diffeomorphism, although something can be said regarding the case when it is not (see [12]).

Compatibility of elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ To give the desired description, we need a certain compatibility notion for elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$. This compatibility is relative to the map f and applies to elements of fibres over the domain of gluing. We define the subset $\Lambda_{x_1}^1(X_1) \times_{comp} \Lambda_{x_2}^1(X_2)$ as the following quotient:

$$\Lambda_{x_1}^1(X_1) \times_{comp} \Lambda_{x_2}^1(X_2) := (\Omega^1(X_1) \times_{comp} \Omega^1(X_2)) / ((\Omega^1(X_1) \times_{comp} \Omega^1(X_2)) \cap (\Omega_{x_1}^1(X_1) \times \Omega_{x_2}^1(X_2))).$$

This implies the following notion of compatibility for two individual elements of $\Lambda_y^1(X_1)$ and $\Lambda_{f(y)}^1(X_2)$ is then immediately obvious.

Definition 1.5. *Let $y \in Y$. Two cosets $\omega_1 + \Omega_y^1(X_1)$ and $\omega_2 + \Omega_{f(y)}^1(X_2)$ are said to be **compatible** if for any $\omega'_1 \in \Omega_y^1(X_1)$ and for any $\omega'_2 \in \Omega_{f(y)}^1(X_2)$ the forms $\omega_1 + \omega'_1$ and $\omega_2 + \omega'_2$ are compatible.*

This is also equivalent to

$$(\omega_1 + \Omega_y^1(X_1)) \times_{comp} (\omega_2 + \Omega_{f(y)}^1(X_2)) = (\omega_1 + \Omega_y^1(X_1)) \times (\omega_2 + \Omega_{f(y)}^1(X_2)).$$

The individual fibres of $\Lambda^1(X_1 \cup_f X_2)$ Assuming that f is a diffeomorphism, the fibres of $\Lambda^1(X_1 \cup_f X_2)$ can be fully described. It turns out that any of them is diffeomorphic to either a fibre of one of the factors or to the direct sum of such. This distinction depends on whether the fibre is at a point of the domain of gluing or outside of it.

Theorem 1.6. ([12]) *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism of its domain with its image. Then*

1. *For any $x_1 \in i_1(X_1 \setminus Y)$ there is a diffeomorphism*

$$\Lambda_{x_1}^1(X_1 \cup_f X_2) \cong \Lambda_{i_1^{-1}(x_1)}^1(X_1).$$

2. *For any $x_2 \in i_2(X_2 \setminus f(Y))$ there is a diffeomorphism*

$$\Lambda_{x_2}^1(X_1 \cup_f X_2) \cong \Lambda_{i_2^{-1}(x_2)}^1(X_2).$$

3. *For any $y \in Y$ there is a diffeomorphism*

$$\Lambda_{i_2(f(y))}^1(X_1 \cup_f X_2) \cong \Lambda_y^1(X_1) \oplus_{comp} \Lambda_{f(y)}^1(X_2).$$

Example 1.7. *Let X_1 and X_2 be two diffeological spaces, and let $x_i \in X_i$ be a point, for $i = 1, 2$; let $f : \{x_1\} \rightarrow \{x_2\}$ be the obvious map. Then $X_1 \cup_f X_2$ is the usual wedge $X_1 \vee X_2$ of X_1 and X_2 . Since any diffeological form assigns the zero value to any constant plot, any two forms on, respectively, X_1 and X_2 , are automatically compatible. Therefore*

$$\Omega^1(X_1 \cup_f X_2) = \Omega^1(X_1 \vee X_2) \cong \Omega^1(X_1) \times \Omega^1(X_2).$$

On the other hand, the fibre of $\Lambda^1(X_1 \vee X_2)$ at any point $x'_i \in X_i \subset X_1 \vee X_2$ is $\Lambda_{x'_i}^1(X_i)$, except for the wedge point $x = [x_1] = [x_2]$, where it is the direct product $\Lambda_{x_1}^1(X_1) \times \Lambda_{x_2}^1(X_2)$.

The corresponding decomposition of $\Lambda^1(X_1 \cup_f X_2)$ We still assume that f is a diffeomorphism. For practical purposes, the following description of $\Lambda^1(X_1 \cup_f X_2)$ is quite useful.

Theorem 1.8. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \supseteq Y \rightarrow X_2$ be a diffeomorphism of its domain with its image. Let $\pi^\Lambda : \Lambda^1(X_1 \cup_f X_2) \rightarrow X_1 \cup_f X_2$, $\pi_1^\Lambda : \Lambda^1(X_1) \rightarrow X_1$, and $\pi_2^\Lambda : \Lambda^1(X_2) \rightarrow X_2$ be the pseudo-bundle projections. Then*

$$\Lambda^1(X_1 \cup_f X_2) \cong \cup_{x_1 \in X_1 \setminus Y} \Lambda_{x_1}^1(X_1) \bigcup \cup_{y \in Y} \left(\Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2) \right) \bigcup \cup_{x_2 \in X_2 \setminus f(Y)} \Lambda_{x_2}^1(X_2),$$

where \cong has the following meaning:

- the set $\cup_{x_1 \in X_1 \setminus Y} \Lambda_{x_1}^1(X_1) \subset \Lambda^1(X)$ is identified with $(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y))$ and with $(\pi_1^\Lambda)^{-1}(X_1 \setminus Y)$. This identification is a diffeomorphism for the subset diffeologies relative to the inclusions

$$(\pi^\Lambda)^{-1}(i_1(X_1 \setminus Y)) \subset \Lambda^1(X_1 \cup_f X_2) \quad \text{and} \quad (\pi_1^\Lambda)^{-1}(X_1 \setminus Y) \subset \Lambda^1(X_1);$$

- likewise, the set $\cup_{x_2 \in X_2 \setminus f(Y)} \Lambda_{x_2}^1(X_2)$ is identified with $(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y)))$ and with $(\pi_2^\Lambda)^{-1}(X_2 \setminus f(Y))$, with the identification being again a diffeomorphism for the subset diffeologies relative to the inclusions

$$(\pi^\Lambda)^{-1}(i_2(X_2 \setminus f(Y))) \subset \Lambda^1(X_1 \cup_f X_2) \quad \text{and} \quad (\pi_2^\Lambda)^{-1}(X_2 \setminus f(Y));$$

- finally, the set $\cup_{y \in Y} \left(\Lambda_y^1(X_1) \oplus_{\text{comp}} \Lambda_{f(y)}^1(X_2) \right)$ is given the direct sum diffeology as the appropriate (determined by compatibility) subset of the result of the direct of the following two restricted pseudo-bundles:

$$\pi_1^\Lambda|_{(\pi_1^\Lambda)^{-1}(Y)} : (\pi_1^\Lambda)^{-1}(Y) \rightarrow Y \quad \text{and} \quad f^{-1} \circ \pi_2^\Lambda|_{(\pi_2^\Lambda)^{-1}(f(Y))} : (\pi_2^\Lambda)^{-1}(f(Y)) \rightarrow Y.$$

This direct sum subset can also be identified with $(\pi^\Lambda)^{-1}(i_2(f(Y)))$ and given the subset diffeology relative to the inclusion

$$(\pi^\Lambda)^{-1}(i_2(f(Y))) \subset \Lambda^1(X_1 \cup_f X_2);$$

once again, this identification is a diffeomorphism for the above direct sum diffeology and the just-mentioned subset diffeology.

We should note that the three diffeologies indicated in the theorem do not fully describe the diffeology of $\Lambda^1(X_1 \cup_f X_2)$ (for the obvious reason, if they did then $\Lambda^1(X_1 \cup_f X_2)$ would be disconnected in the underlying D-topology, which in general it does not have to be). In the next section we consider other types of plots of $\Lambda^1(X_1 \cup_f X_2)$.

1.6.3 The diffeology of $\Lambda^1(X_1 \cup_f X_2)$, and the maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$

It is ultimately a consequence of the properties of the gluing diffeology (even if $\Lambda^1(X_1 \cup_f X_2)$ is not obtained by any kind of gluing between $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$) that every plot $p : U \rightarrow \Lambda^1(X_1 \cup_f X_2)$ locally can be represented in the form

$$p(u) = p_1^\Omega(u) \cup_f p_2^\Omega(u) + \Omega_{\pi^\Lambda(p(u))}^1(X_1 \cup_f X_2),$$

where $p_1^\Omega : U \rightarrow \Omega^1(X_1)$ and $p_2^\Omega : U \rightarrow \Omega^1(X_2)$ are plots of $\Omega^1(X_1)$ and $\Omega^1(X_2)$ respectively, such that for any $u \in U$ the forms $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible. Furthermore, the composition $\pi^\Lambda \circ p$ with the pseudo-bundle projection $\pi^\Lambda : \Lambda^1(X_1 \cup_f X_2) \rightarrow X_1 \cup_f X_2$ lifts to either a plot of X_1 or one of X_2 .

The definition of $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ Consider $\Lambda^1(X_1 \cup_f X_2)$ as a diffeological quotient of

$$(X_1 \sqcup X_2) \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)) \cong$$

$$\cong (X_1 \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))) \sqcup (X_2 \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))),$$

therefore $(\pi^\Lambda)^{-1}(i_1(X_1) \cup f(i_2(Y)))$ is the quotient of $X_1 \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$ by

$$\bigcup_{x \in X_1 \setminus Y} \{x\} \times (\Omega_x^1(X_1) \times_{\text{comp}} \Omega^1(X_2)) \cup \bigcup_{y \in Y} \{y\} \times (\Omega_y^1(X_1) \times_{\text{comp}} \Omega_{f(y)}^1(X_2)).$$

Likewise, $(\pi^\Lambda)^{-1}(i_2(X_2))$ is the quotient of $X_2 \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2))$ by

$$\bigcup_{y' \in f(Y)} \{y'\} \times (\Omega_{f^{-1}(y')}^1(X_1) \times_{\text{comp}} \Omega_{y'}^1(X_2)) \cup \bigcup_{x \in X_2 \setminus f(Y)} \{x\} \times (\Omega^1(X_1) \times_{\text{comp}} \Omega_x^1(X_2)).$$

Let now

$$\rho_1 : X_1 \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)) \rightarrow X_1 \times \Omega^1(X_1) \text{ and}$$

$$\rho_2 : X_2 \times (\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)) \rightarrow X_2 \times \Omega^1(X_2)$$

be the maps each of which acts on X_i by identity and on $\Omega^1(X_1) \times_{\text{comp}} \Omega^1(X_2)$ by the projection onto the first and the second factor respectively. These maps obviously preserve the corresponding spaces of vanishing 1-forms, therefore their pushforwards to the appropriate subsets of $\Lambda^1(X_1 \cup_f X_2)$ are well-defined. Thus, they induce well-defined and smooth maps

$$\tilde{\rho}_1^\Lambda : \Lambda^1(X_1 \cup_f X_2) \supset (\pi^\Lambda)^{-1}(i_1(X_1) \cup f(i_2(Y))) \rightarrow \Lambda^1(X_1) \text{ and}$$

$$\tilde{\rho}_2^\Lambda : \Lambda^1(X_1 \cup_f X_2) \supset (\pi^\Lambda)^{-1}(i_2(X_2)) \rightarrow \Lambda^1(X_2).$$

These maps allow for the following characterization of the diffeology of $\Lambda^1(X_1 \cup_f X_2)$.

Theorem 1.9. ([12]) *The diffeology of $\Lambda^1(X_1 \cup_f X_2)$ is the coarsest one such that both $\tilde{\rho}_i^\Lambda$ are smooth.*

Compatibility and $\tilde{\rho}_i^\Lambda$ being subductions Let $i : Y \hookrightarrow X_1$ and $j : f(Y) \hookrightarrow X_2$ be the natural inclusions, and consider the pullback maps $i^* : \Omega^1(X_1) \rightarrow \Omega^1(Y)$ and $f^* \circ j^* : \Omega^1(X_2) \rightarrow \Omega^1(Y)$. Denote by \mathcal{D}_1^Ω and \mathcal{D}_2^Ω , respectively, the pushforward of the diffeology of $\Omega^1(X_1)$ by i^* and the pushforward of the diffeology of $\Omega^1(X_2)$ by $f^* \circ j^*$.

Lemma 1.10. *Let $p_1^\Omega : U \rightarrow \Omega^1(X_1)$ be a plot of $\Omega^1(X_1)$. Up to replacing U with its smaller sub-domain, there exists a plot $p_2^\Omega : U \rightarrow \Omega^1(X_2)$ such that $p_1^\Omega(u)$ and $p_2^\Omega(u)$ are compatible for all $u \in U$ if and only if $i^* \circ p_1^\Omega \in \mathcal{D}_2^\Omega$.*

The analogous statement for an arbitrary plot p_2^Ω of $\Omega^1(X_2)$ is also true. From the two put together, the following can be easily obtained.

Theorem 1.11. ([12]) *The maps $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$ are subductions if and only if $\mathcal{D}_1^\Omega = \mathcal{D}_2^\Omega$.*

1.6.4 Putting a pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$

Assume that both $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ admit pseudo-metrics; this, in particular, implies that they both have finite-dimensional fibres. Therefore by Theorem 1.8 $\Lambda^1(X_1 \cup_f X_2)$ has finite-dimensional fibres as well. It turns out that in some cases there is an induced pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$.

The compatibility of pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ In the case of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ the notion of compatibility is essentially one of the invariance with respect to $\tilde{\rho}_1^\Lambda$ and $\tilde{\rho}_2^\Lambda$.

Definition 1.12. *Let g_1^Λ and g_2^Λ be pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. They are said to be **compatible**, if for all $y \in Y$ and for all $\omega, \mu \in (\pi^\Lambda)^{-1}(i_2(f(y)))$ we have*

$$g_1^\Lambda(y)(\tilde{\rho}_1^\Lambda(\omega), \tilde{\rho}_1^\Lambda(\mu)) = g_2^\Lambda(f(y))(\tilde{\rho}_2^\Lambda(\omega), \tilde{\rho}_2^\Lambda(\mu)).$$

Remark 1.13. *The compatibility condition for pseudo-metrics can be equivalently stated as, for any $y \in Y$ and for any two compatible pairs (ω', ω'') and (μ', μ'') , where $\omega', \mu' \in (\pi_1^\Lambda)^{-1}(y)$ and $\omega'', \mu'' \in (\pi_2^\Lambda)^{-1}(f(y))$, we have*

$$g_1^\Lambda(y)(\omega', \mu') = g_2^\Lambda(f(y))(\omega'', \mu'').$$

The induced pseudo-metric on $\Lambda^1(X_1 \cup_f X_2)$ Let g_1^Λ and g_2^Λ be two compatible, in the sense of Definition 1.12, pseudo-metrics on $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$ respectively. We can then obtain a pseudo-metric g^Λ on $\Lambda^1(X_1 \cup_f X_2)$, which is defined by setting:

$$g^\Lambda(x)(\cdot, \cdot) = \begin{cases} g_1^\Lambda(i_1^{-1}(x))(\tilde{\rho}_1^\Lambda(\cdot), \tilde{\rho}_1^\Lambda(\cdot)) & \text{if } x \in i_1(X_1 \setminus Y) \\ g_2^\Lambda(i_2^{-1}(x))(\tilde{\rho}_2^\Lambda(\cdot), \tilde{\rho}_2^\Lambda(\cdot)) & \text{if } x \in i_2(X_2 \setminus f(Y)) \\ \frac{1}{2}(g_1^\Lambda(f^{-1}(i_2^{-1}(x)))(\tilde{\rho}_1^\Lambda(\cdot), \tilde{\rho}_1^\Lambda(\cdot)) + g_2^\Lambda(i_2^{-1}(x))(\tilde{\rho}_2^\Lambda(\cdot), \tilde{\rho}_2^\Lambda(\cdot))) & \text{if } x \in i_2(f(Y)) \end{cases}$$

That this is indeed a pseudo-metric follows from the compatibility of g_1^Λ and g_2^Λ . Indeed, this condition ensures that the evaluation of g^Λ on any triple of plots with connected domains of definition coincides with the evaluation of either g_1^Λ or g_2^Λ on another specific triple of plots. Thus, it is automatically smooth, and other properties of a pseudo-metric easily follow.

2 Sections of diffeological pseudo-bundles

In this section we consider the space $C^\infty(X, V)$ of smooth sections of a given finite-dimensional diffeological vector pseudo-bundle. For non-standard diffeologies on one or both of X and V , this space may easily be of infinite dimension; immediately below we provide an example of this. On the other hand, when it is the spaces themselves that are non-standard (say, they are not topological manifolds), the space of sections might be finite-dimensional, as we illustrate via the study of the behavior of the space of sections under gluing of pseudo-bundles, where most of the effort has to be spent on the case when the gluing is performed along a non-diffeomorphism f . In this regard, we obtain the answer in the most general case, showing that the space of sections of the result of gluing is always a smooth surjective image of a subspace of the direct product of the spaces of sections of the factors (in particular, the finiteness of the dimension is preserved, *i.e.*, the existence of local bases, meaning that if the spaces of sections of the factors are finite-dimensional then so is the space of sections of the result of gluing).

Observe that we discuss in fact only the case of global sections, because this is not really different from restricting ourselves to the local case. Indeed, in the natural topology underlying a diffeological structure, the so-called D-topology (introduced in [6]), the open sets can be of any form. What this implies at the moment is that there is no fixed local shape for diffeological pseudo-bundle, or, said differently, any diffeological vector pseudo-bundle can appear as the restriction of a larger pseudo-bundle to a D-open (open in D-topology) neighborhood of a fibre.

The final conclusion of this section is that there is a natural smooth surjective map (a subduction, in fact)

$$\mathcal{S} : C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2),$$

where $C_{(f, \tilde{f})}^\infty(X_1, V_1) \leq C^\infty(X_1, V_1)$ is the subspace of the so-called (f, \tilde{f}) -invariant sections (these are sections s such that having $f(y) = f(y')$ implies that $\tilde{f}(s(y)) = \tilde{f}(s(y'))$), and $C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2)$ is the subset of the direct product $C_{(f, \tilde{f})}^\infty(X_1, V_1) \times C^\infty(X_2, V_2)$ that consists of all *compatible* pairs (a pair (s_1, s_2) is compatible if $\tilde{f} \circ s_1 = s_2 \circ f$ wherever defined). The map \mathcal{S} is an instance of a more general procedure of gluing compatible maps concurrently with gluing of their domains and their ranges (see [10] for the general case; the map \mathcal{S} is described in detail below). Notice also that

$$C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) = C^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2),$$

that is, if (s_1, s_2) is a compatible pair then s_1 is necessarily (f, \tilde{f}) -invariant. Also, \mathcal{S} is a diffeomorphism if f and \tilde{f} are so.

2.1 The space $C^\infty(X, V)$ over $C^\infty(X, \mathbb{R})$

In the case of diffeological pseudo-bundles, the space of all smooth sections $C^\infty(X, V)$ may have infinite dimension over $C^\infty(X, \mathbb{R})$ when normally we would not expect it. To begin our consideration of the subject, we provide a simple example of this.

Example 2.1. Let $\pi : V \rightarrow X$ be the projection of $V = \mathbb{R}^3$ onto its first coordinate; thus, X is \mathbb{R} , which we endow with the standard diffeology. Endow V with the pseudo-bundle diffeology generated by the plot $\mathbb{R}^2 \ni (u, v) \mapsto (u, 0, |v|)$; this diffeology is a product diffeology for the decomposition $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ into the direct product of the standard \mathbb{R} with \mathbb{R}^2 carrying the vector space diffeology generated by the plot $v \mapsto (0, |v|)$. In this case the space $C^\infty(X, V)$ of smooth sections of the pseudo-bundle π has infinite dimension over $C^\infty(X, \mathbb{R}) = C^\infty(\mathbb{R}, \mathbb{R})$; let us explain why.

Proof. Since the diffeology of X is standard, the ring $C^\infty(X, \mathbb{R})$ includes the usual smooth functions $\mathbb{R} \rightarrow \mathbb{R}$ only. The diffeology of V is actually a vector space diffeology generated by the plot $(u, v) \mapsto (u, 0, |v|)$; an arbitrary plot of it has therefore the form

$$\mathbb{R}^{l+m+n} \supseteq U \ni (x, y, z) \mapsto (f_1(x), f_2(y), g_0(z) + g_1(z)|h_1(z)| + \dots + g_k(z)|h_k(z)|),$$

where U is a domain, and $f_1 : \mathbb{R}^l \subseteq U_x \rightarrow \mathbb{R}$, $f_2 : \mathbb{R}^m \supseteq U_y \rightarrow \mathbb{R}$ and $g_0, g_1, \dots, g_k, h_1, \dots, h_k : \mathbb{R}^n \supseteq U_z \rightarrow \mathbb{R}$ are some ordinary smooth functions. Hence any smooth section $s \in C^\infty(X, V)$ has (at least locally) form

$$s(x) = (x, f(x), g_0(x) + g_1(x)|h_1(x)| + \dots + g_k(x)|h_k(x)|)$$

for some ordinary smooth functions $f, g_0, g_1, \dots, g_k, h_1, \dots, h_k : \mathbb{R} \supseteq U \rightarrow \mathbb{R}$; and *vice versa* every such expression corresponds locally to a smooth section $X \rightarrow V$ (and can be extended, by a standard partition-of-unity argument, to a section in $C^\infty(X, V)$). Since g_i and h_i are any smooth functions at all, and they can be in any finite number, for any finite arbitrarily long collection $x_1, \dots, x_k \in \mathbb{R}$ there is a diffeologically smooth section s that, seen as a usual map $\mathbb{R} \rightarrow \mathbb{R}^3$, is non-differentiable precisely at the points x_1, \dots, x_k (and smooth outside of them). Thus, it is impossible that all such sections be linear combinations over $C^\infty(\mathbb{R}, \mathbb{R})$ of the same finite set of (at least continuous) functions $\mathbb{R} \rightarrow \mathbb{R}^3$. \square

Our main interest thus is when the space of sections turns out to be finite-dimensional. We thus concentrate, in the sections that follow, on the behavior of this space under the operation of gluing.

2.2 The space $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ relative to $C^\infty_{(f, \tilde{f}}(X_1, V_1)$ and $C^\infty(X_2, V_2)$

Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two finite-dimensional diffeological vector pseudo-bundles, let (\tilde{f}, f) be a pair of smooth maps that defines a gluing between them, and let $Y \subset X_1$ be the domain of definition of f . Consider the three corresponding spaces of smooth sections, *i.e.*, the spaces $C^\infty(X_1, V_1)$, $C^\infty(X_2, V_2)$, and $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$. The latter space can be reconstructed from the former two by using the notion of gluing of compatible smooth maps, as it appears in [10].

2.2.1 Compatible sections

Consider a pair $\varphi_1 : X_1 \rightarrow Z_1$ and $\varphi_2 : X_2 \rightarrow Z_2$ of smooth maps between some diffeological spaces that are, in turn, endowed with fixed smooth maps $f : X_1 \supset Y \rightarrow X_2$ and $g : \varphi_1(Y) \rightarrow Z_2$. We say that φ_1 and φ_2 are **(f, g) -compatible** if $g \circ \varphi_1 = \varphi_2 \circ f$ wherever defined. This allows to define an obvious map

$$\varphi_1 \cup_{(f, g)} \varphi_2 : X_1 \cup_f X_2 \rightarrow Z_1 \cup_g Z_2,$$

which is smooth for the gluing diffeologies on $X_1 \cup_f X_2$ and $Z_1 \cup_g Z_2$ (see [10], Proposition 4.4). A pair of sections s_1, s_2 of two diffeological pseudo-bundles is then a particular instance of maps φ_i , with Z_i being V_i , with the role of g being played by \tilde{f} . Two such sections are **compatible** if $\tilde{f} \circ s_1 = s_2 \circ f$ on the whole domain of definition.

2.2.2 Compatibility and (f, \tilde{f}) -invariant sections

Let $s_1 \in C^\infty(X_1, V_1)$ be such that there exists a section $s_2 \in C^\infty(X_2, V_2)$ compatible with it, that is, for all $y \in Y$ we have $\tilde{f}(s_1(y)) = s_2(f(y))$; let $y' \in Y$ be a point such that $f(y) = f(y')$. The compatibility condition implies then that

$$\tilde{f}(s_1(y)) = s_2(f(y)) = s_2(f(y')) = \tilde{f}(s_1(y'));$$

thus, although $s_1(y)$ and $s_1(y')$ do not have to coincide, their images under \tilde{f} necessarily do. This justifies the following definition, and an easy lemma that follows it.

Definition 2.2. Let $\pi_1 : V_1 \rightarrow X_1$ be a diffeological vector pseudo-bundle, let W and Z be any two diffeological spaces, let $f : Y \rightarrow Z$ be a smooth map defined on a subset Y of X_1 , and let $\tilde{f} : \pi_1^{-1}(Y) \rightarrow W$ be a lift of f to V_1 . A section $s_1 \in C^\infty(X_1, V_1)$ of this pseudo-bundle is called (f, \tilde{f}) -invariant if for any $y, y' \in Y$ such that $f(y) = f(y')$ (in Z) we have that $\tilde{f}(s_1(y)) = \tilde{f}(s_1(y'))$ (in W). A function $h \in C^\infty(X_1, \mathbb{R})$ is called f -invariant if $h(y) = h(y')$ for all $y, y' \in Y$ such that $f(y) = f(y')$.

The lemma below follows immediately from what has been said prior to the definition.

Lemma 2.3. Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let (\tilde{f}, f) be a gluing between them, and let $s_1 \in C^\infty(X_1, V_1)$ be such that there exists $s_2 \in C^\infty(X_2, V_2)$ compatible with s_1 . Then s_1 is (f, \tilde{f}) -invariant.

Thus, we only need to take into consideration (f, \tilde{f}) -invariant sections $X_1 \rightarrow V_1$. The set of all of them is denoted by

$$C_{(f, \tilde{f})}^\infty(X_1, V_1) = \{s \in C^\infty(X_1, V_1) \mid s \text{ is } (f, \tilde{f})\text{-invariant}\}.$$

Let us now consider some properties of this set.

Proposition 2.4. The set $C_{(f, \tilde{f})}^\infty(X_1, V_1)$ is closed with respect to the summation and multiplication by f -invariant functions.

Proof. Given two (f, \tilde{f}) -invariant sections $s_1, s_2 \in C^\infty(X_1, V_1)$ and an f -invariant function $h \in C^\infty(X_1, \mathbb{R})$, we need to show that $s_1 + s_2$ and hs_1 are again (f, \tilde{f}) -invariant. Let $y, y' \in Y$ be such that $f(y) = f(y')$. Then it follows from the linearity of \tilde{f} on each fibre in its domain of definition that $\tilde{f}((hs)(y)) = \tilde{f}(h(y)s(y)) = h(y)\tilde{f}(s(y)) = h(y')\tilde{f}(s(y')) = \tilde{f}((hs)(y'))$, as wanted. Furthermore, by assumption $\tilde{f}(s_1(y)) = \tilde{f}(s_1(y'))$ and $\tilde{f}(s_2(y)) = \tilde{f}(s_2(y'))$. Since the restriction of \tilde{f} on any fibre is linear, we have

$$\tilde{f}((s_1 + s_2)(y)) = \tilde{f}(s_1(y) + s_2(y)) = \tilde{f}(s_1(y)) + \tilde{f}(s_2(y)) = \tilde{f}(s_1(y')) + \tilde{f}(s_2(y')) = \tilde{f}((s_1 + s_2)(y')),$$

which completes the proof. \square

2.2.3 The map $\mathcal{S} : C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$

The notion of compatibility of sections allows us to define the (partial) operation of gluing for smooth sections of the pseudo-bundles π_1 and π_2 , through which we define the map announced in the title of the section.

The section $s_1 \cup_{(f, \tilde{f})} s_2$ of $\pi_1 \cup_{(\tilde{f}, f)} \pi_2$ Let s_1 and s_2 be two compatible smooth sections of the pseudo-bundles π_1 and π_2 respectively. We define a section $s_1 \cup_{(f, \tilde{f})} s_2 : X_1 \cup_f X_2 \rightarrow V_1 \cup_{\tilde{f}} V_2$ of the pseudo-bundle $\pi_1 \cup_{(\tilde{f}, f)} \pi_2$ by imposing

$$(s_1 \cup_{(f, \tilde{f})} s_2)(x) = \begin{cases} s_1(i_1^{-1}(x)) & \text{for } x \in i_1(X_1 \setminus Y), \text{ and} \\ s_2(i_2^{-1}(x)) & \text{for } x \in i_2(X_2). \end{cases}$$

This turns out to be a smooth section of $\pi_1 \cup_{(\tilde{f}, f)} \pi_2$, as follows from Proposition 4.4 in [10].

The induced map \mathcal{S} into $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ Consider the direct product $C^\infty(X_1, V_1) \times C^\infty(X_2, V_2)$; let

$$C^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) = \{(s_1, s_2) \mid s_i \in C^\infty(X_i, V_i), s_1, s_2 \text{ are } (f, \tilde{f})\text{-compatible}\}.$$

The latter set is endowed with the subset diffeology relative to its inclusion into $C^\infty(X_1, V_1) \times C^\infty(X_2, V_2)$ (which in turn has the product diffeology coming from the functional diffeologies on each $C^\infty(X_i, V_i)$). Notice that by Lemma 2.3

$$C^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) = C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2).$$

The map

$$\mathcal{S} : C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$$

is defined by

$$\mathcal{S}(s_1, s_2) = s_1 \cup_{(f, \tilde{f})} s_2;$$

it has the following property.

Theorem 2.5. ([10]) *The map \mathcal{S} is smooth, for the subset diffeology on $C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2)$ and the functional diffeology on $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$.*

2.2.4 Pseudo-bundles operations and the map \mathcal{S}

Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two finite-dimensional diffeological vector pseudo-bundles, with (\tilde{f}, f) being a gluing between them, and let $\pi'_1 : V'_1 \rightarrow X_1$ and $\pi'_2 : V'_2 \rightarrow X_2$ be two other pseudo-bundles, with the same base spaces, with (\tilde{f}', f) also a gluing between them.

- ([10], Proposition 4.7) Let $s_i \in C^\infty(X_i, V_i)$ for $i = 1, 2$ be (f, \tilde{f}) -compatible sections, and let $h_i \in C^\infty(X_i, \mathbb{R})$ be such that $h_2(f(y)) = h_1(y)$ for all $y \in Y$. Then

$$(h_1 \cup_f h_2) \left(s_1 \cup_{(f, \tilde{f})} s_2 \right) = (h_1 s_1) \cup_{(f, \tilde{f})} (h_2 s_2).$$

- ([10], Proposition 4.8) Let $s_i \in C^\infty(X_i, V_i)$ and $s'_i \in C^\infty(X_i, V'_i)$ be such that s_1, s_2 are (f, \tilde{f}) -compatible, while s'_1, s'_2 are (f, \tilde{f}') -compatible. Then

$$\left(s_1 \cup_{(f, \tilde{f})} s_2 \right) \otimes \left(s'_1 \cup_{(f, \tilde{f}')} s'_2 \right) = (s_1 \otimes s'_1) \cup_{(f, \tilde{f} \otimes \tilde{f}')} (s_2 \otimes s'_2).$$

In addition to these, we now prove that \mathcal{S} is additive with respect to the direct sum structure on $C^\infty(X_1, V_1) \times C^\infty(X_2, V_2)$, of which $C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2)$ is then a subspace.

Lemma 2.6. *If $s_1, t_1 \in C^\infty(X_1, V_1)$ and $s_2, t_2 \in C^\infty(X_2, V_2)$ are such that both (s_1, s_2) and (t_1, t_2) are (f, \tilde{f}) -compatible pairs, then also $(s_1 + t_1, s_2 + t_2)$ is a (f, \tilde{f}) -compatible pair, and*

$$(s_1 + t_1) \cup_{(f, \tilde{f})} (s_2 + t_2) = s_1 \cup_{(f, \tilde{f})} s_2 + t_1 \cup_{(f, \tilde{f})} t_2.$$

Proof. Let $y \in Y$; then

$$\tilde{f}(s_1(y) + t_1(y)) = \tilde{f}(s_1(y)) + \tilde{f}(t_1(y)) = s_2(f(y)) + t_2(f(y)),$$

so $s_1 + t_1$ and $s_2 + t_2$ are (f, \tilde{f}) -compatible. Now, by definition

$$\begin{aligned} \left((s_1 + t_1) \cup_{(f, \tilde{f})} (s_2 + t_2) \right) (x) &= \begin{cases} (s_1 + t_1)(i_1^{-1}(x)) \\ (s_2 + t_2)(i_2^{-1}(x)) \end{cases} = \begin{cases} s_1(i_1^{-1}(x)) + t_1(i_1^{-1}(x)) \\ s_2(i_2^{-1}(x)) + t_2(i_2^{-1}(x)) \end{cases} = \\ &= \begin{cases} s_1(i_1^{-1}(x)) \\ s_2(i_2^{-1}(x)) \end{cases} + \begin{cases} t_1(i_1^{-1}(x)) \\ t_2(i_2^{-1}(x)) \end{cases} = (s_1 \cup_{(f, \tilde{f})} s_2)(x) + (t_1 \cup_{(f, \tilde{f})} t_2)(x), \end{aligned}$$

where in each two-part formula the first line applies to $x \in i_1(X_1 \setminus Y)$ and the second line, to $x \in i_2(X_2)$. The final equality that we obtain is precisely the first item in the statement of the lemma, so we are done. \square

2.3 The map \mathcal{S} is a subduction

In this section we show that \mathcal{S} is a subduction of the space $C_{(f, \tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2)$ onto the space $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$.

2.3.1 Gluing along diffeomorphisms

Clearly in this case $C^\infty(X_1, V_1) = C^\infty_{(f, \tilde{f})}(X_1, V_1)$. Furthermore, it is rather easy to show that \mathcal{S} is actually a diffeomorphism; we need a preliminary statement first.

Lemma 2.7. *The maps $\tilde{i}_1 : X_1 \rightarrow X_1 \cup_f X_2$ and $\tilde{j}_1 : V_1 \rightarrow V_1 \cup_{\tilde{f}} V_2$ defined as the compositions of the natural inclusions into $X_1 \sqcup X_2$ and $V_1 \sqcup V_2$ with the corresponding quotient projections, are diffeomorphisms with their images.*

Proof. That \tilde{i}_1 and \tilde{j}_1 are bijections with their respective images is immediately obvious. Furthermore, they are always smooth, since they are compositions of two smooth maps. Finally, their inverses are smooth by the definition of the gluing diffeology as a pushforward one (the assumption that f and \tilde{f} are diffeomorphisms is only needed for the existence of these inverses). \square

We are now ready to prove the following.

Proposition 2.8. *If both \tilde{f} and f are diffeomorphisms of their domains with their images, the map \mathcal{S} is a diffeomorphism*

$$C^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2).$$

Proof. The inverse of \mathcal{S} is obtained by assigning to each section $s \in C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ the pair

$$(\tilde{j}_1^{-1} \circ s|_{\tilde{i}_1(X_1)} \circ \tilde{i}_1, j_2^{-1} \circ s|_{i_2(X_2)} \circ i_2),$$

where $\tilde{i}_1 : X_1 \rightarrow X_1 \cup_f X_2$ and $\tilde{j}_1 : V_1 \rightarrow V_1 \cup_{\tilde{f}} V_2$ are the just-mentioned inclusions of X_1 and V_1 into $X_1 \cup_f X_2$ and $V_1 \cup_{\tilde{f}} V_2$ respectively. It follows that $\tilde{j}_1^{-1} \circ s|_{\tilde{i}_1(X_1)} \circ \tilde{i}_1 =: s_1 \in C^\infty(X_1, V_1)$, whereas $j_2^{-1} \circ s|_{i_2(X_2)} \circ i_2 =: s_2 \in C^\infty(X_2, V_2)$ holds even without extra assumptions.

Let us formally check that s_1 and s_2 are compatible. Let $y \in Y$; then $\tilde{f}(s_1(y)) = \tilde{f}(\tilde{j}_1^{-1}(s(\tilde{i}_1(y))))$, and $s_2(f(y)) = j_2(s(i_2(f(y))))$. Since $\tilde{i}_1(y) = i_2(f(y))$ by construction, we have $\tilde{f}(s_1(y)) = \tilde{f}(\tilde{j}_1^{-1}(s(i_2(f(y))))))$, and it suffices to note that $\tilde{f} \circ \tilde{j}_1^{-1} = j_2$ on the entire $\pi_2^{-1}(i_2(f(Y)))$. \square

2.3.2 The pseudo-bundle $\pi_1^{(\tilde{f}, f)} : V_1^{\tilde{f}} \rightarrow X_1^f$ of (\tilde{f}, f) -equivalence classes

In the case when f and \tilde{f} are not diffeomorphisms, we need an auxiliary construction, that of the pseudo-bundle mentioned in the title of the section. Its base space and its total space are obtained from X_1 and V_1 respectively by natural quotientings, given by f and \tilde{f} , and the pseudo-bundle projection is induced by π_1 .

The spaces X_1^f and $V_1^{\tilde{f}}$ The base space X_1^f is defined as the diffeological quotient X_1/\sim^f , where the equivalence relation \sim^f is given by

$$y_1 \sim^f y_2 \Leftrightarrow f(y_1) = f(y_2).$$

Likewise, the space $V_1^{\tilde{f}}$ is the quotient of V_1 by the equivalence relation $\sim^{\tilde{f}}$ that is analogous to \sim^f and is given by

$$v_1 \sim^{\tilde{f}} v_2 \Leftrightarrow \tilde{f}(v_1) = \tilde{f}(v_2).$$

The two quotient projections are denoted respectively by $\chi_1^f : X_1 \rightarrow X_1^f$ and by $\chi_1^{\tilde{f}} : V_1 \rightarrow V_1^{\tilde{f}}$. The space X_1^f is endowed with the map $f_\sim : \chi_1^f(Y) \rightarrow X_2$, and the space $V_1^{\tilde{f}}$, with the map $\tilde{f}_\sim : \chi_1^{\tilde{f}}(\pi_1^{-1}(Y)) \rightarrow V_2$. These are the pushforwards of, respectively, f and \tilde{f} by the quotient projections χ_1^f and $\chi_1^{\tilde{f}}$. If either of f , \tilde{f} is a subduction then the corresponding induced map f_\sim or \tilde{f}_\sim is a diffeomorphism with its image.

The map $\pi_1^{(\tilde{f}, f)}$ We now show that the induced pseudo-bundle projection $V_1^{\tilde{f}} \rightarrow X_1^f$ is indeed a pseudo-bundle.

Lemma 2.9. *There is a well-defined smooth map $\pi_1^{(\tilde{f}, f)} : V_1^{\tilde{f}} \rightarrow X_1^f$ such that*

$$\chi_1^f \circ \pi_1 = \pi_1^{(\tilde{f}, f)} \circ \chi_1^{\tilde{f}}.$$

Furthermore, for any $x \in X_1^f$ the pre-image $(\pi_1^{(\tilde{f}, f)})^{-1}(x) \subset V_1^{\tilde{f}}$ inherits from V_1 the structure of a diffeological vector space, with respect to which the corresponding restriction of \tilde{f}_\sim , when it is defined, is a linear map.

Proof. That $\pi_1^{(\tilde{f}, f)}$ is uniquely defined by the condition given, follows from $\chi_1^{\tilde{f}}$ being surjective, and that it is well-defined follows from $\sim^{\tilde{f}}$ being a fibrewise equivalence relation. That the pre-image, in $V_1^{\tilde{f}}$, of any point $x \in X_1^f$ under the map $\pi_1^{(\tilde{f}, f)}$ inherits from V_1 a (smooth) vector space structure is obvious from the following considerations: over a point not in $\chi_1^f(Y)$, it coincides with the corresponding fibre of V_1 itself, while over $x \in \chi_1^f(Y)$ it coincides with the quotient of $\pi_1^{-1}(x) \subset V_1$ over the kernel of $\tilde{f}|_{\pi_1^{-1}(x)}$. For the same reason, the induced map \tilde{f}_\sim is linear on each fibre where it is defined, *i.e.*, on $(\pi_1^{(\tilde{f}, f)})^{-1}(x)$ with $x \in \chi_1^f(Y)$. \square

The following is then an immediate consequence.

Corollary 2.10. *The map $\pi_1^{(\tilde{f}, f)} : V_1^{\tilde{f}} \rightarrow X_1^f$ is a diffeological vector pseudo-bundle, and the pair (\tilde{f}_\sim, f_\sim) defines a gluing of it to the pseudo-bundle $\pi_2 : V_2 \rightarrow X_2$. Furthermore, the pseudo-bundle*

$$\pi_1^{(\tilde{f}, f)} \cup_{(\tilde{f}_\sim, f_\sim)} \pi_2 : V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2 \rightarrow X_1^f \cup_{f_\sim} X_2$$

is diffeomorphic to the pseudo-bundle

$$\pi_1 \cup_{(\tilde{f}, f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \rightarrow X_1 \cup_f X_2.$$

In particular, there is a diffeomorphism

$$C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2) \cong C^\infty(X_1^f \cup_{f_\sim} X_2, V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2).$$

Proof. It is evident from the definition of diffeological gluing that $X_1^f \cup_{f_\sim} X_2 \cong X_1 \cup_f X_2$; it remains to notice that the same kind of diffeomorphism between $V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2$ and $V_1 \cup_{\tilde{f}} V_2$ is fibre-to-fibre relative to, respectively, the projections $\pi_1^{(\tilde{f}, f)} \cup_{(\tilde{f}_\sim, f_\sim)} \pi_2$ and $\pi_1 \cup_{(\tilde{f}, f)} \pi_2$. \square

The space of sections $C^\infty(X_1^f \cup_{f_\sim} X_2, V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2)$ By the general definition, sections $s_1^f \in C^\infty(X_1^f, V_1^{\tilde{f}})$ and $s_2 \in C^\infty(X_2, V_2)$ are compatible if

$$\tilde{f}_\sim(s_1^f(y)) = s_2(f_\sim(y)) \text{ for all } y \in \pi_1^f(Y).$$

The advantage of considering the reduced pair $(X_1^f, V_1^{\tilde{f}})$ lies in the presentation, resulting from Corollary 2.10, of the pseudo-bundle $\pi_1 \cup_{(\tilde{f}, f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \rightarrow X_1 \cup_f X_2$ as one obtained by gluing along a pair of diffeomorphisms (thus always possible, as long as we assume that both f and \tilde{f} are subductions onto their respective images). The following is a consequence of Proposition 2.8 and the above corollary.

Proposition 2.11. *Assume that f and \tilde{f} are subductions. Then*

$$C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2) \cong C^\infty(X_1^f, V_1^{\tilde{f}}) \times_{\text{comp}} C^\infty(X_2, V_2),$$

where the compatibility is with respect to the maps (f_\sim, \tilde{f}_\sim) .

2.3.3 The map $\mathcal{S}_1 : C_{(f,\tilde{f})}^\infty(X_1, V_1) \rightarrow C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}})$ and its properties

To make use of Proposition 2.11, we need to relate the space $C^\infty(X_1^f, V_1^{\tilde{f}})$ to the initial space $C_{(f,\tilde{f})}^\infty(X_1, V_1)$. To do so, we consider the map

$$\mathcal{S}_1 : C_{(f,\tilde{f})}^\infty(X_1, V_1) \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}})$$

acting by

$$\mathcal{S}_1 : C_{(f,\tilde{f})}^\infty(X_1, V_1) \ni s_1 \mapsto s_1^f \in C^\infty(X_1^f, V_1^{\tilde{f}}) \text{ such that } s_1^f \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s_1.$$

The map \mathcal{S}_1 is well-defined The definition of \mathcal{S}_1 that we have given above is an indirect one, so we must check that it is well given.

Lemma 2.12. *For every $s_1 \in C_{(f,\tilde{f})}^\infty(X_1, V_1)$ there exists and is unique $s_1^f \in C^\infty(X_1^f, V_1^{\tilde{f}})$ such that $s_1^f \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s_1$.*

Proof. The definition of s_1^f is as follows: for any given point $x \in X_1^f$, let $x' \in X_1$ be any point (existing by surjectivity of χ_1^f) such that $\chi_1^f(x') = x$; define $s_1^f(x)$ to be $s_1^f(x) = \chi_1^{\tilde{f}}(s_1(x'))$. Let us show that the definition is well-posed; let $x'' \in X_1$ be another point such that $\chi_1^f(x'') = x$. We need to show that $\chi_1^{\tilde{f}}(s_1(x')) = \chi_1^{\tilde{f}}(s_1(x''))$. Since $\chi_1^f(x'') = x = \chi_1^f(x')$ is equivalent to $f(x') = f(x'')$, by (f, \tilde{f}) -invariance of s_1 we obtain that $\tilde{f}(s_1(x')) = \tilde{f}(s_1(x''))$. This in turn is equivalent to $\chi_1^{\tilde{f}}(s_1(x')) = \chi_1^{\tilde{f}}(s_1(x''))$, therefore s_1^f is well-defined (and is obviously a map that goes $X_1^f \rightarrow V_1^{\tilde{f}}$).

The smoothness of s_1^f follows from the expression that defines it. Specifically, if $p^f : U \rightarrow X_1^f$ is a plot then, assuming that U is small enough, there exists a plot $p : U \rightarrow X_1$ such that $p^f = \chi_1^f \circ p$. Therefore $s_1^f \circ p^f = \chi_1^{\tilde{f}} \circ s_1 \circ p$. The latter is a plot of $V_1^{\tilde{f}}$, since s_1 is smooth as a map $X_1 \rightarrow V_1$, and $\chi_1^{\tilde{f}}$ is smooth, because the diffeology of $V_1^{\tilde{f}}$ is the pushforward of that of V_1 by it. \square

We thus obtain the following statement.

Corollary 2.13. *The map \mathcal{S}_1 is well-defined as a map $C_{(f,\tilde{f})}^\infty(X_1, V_1) \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}})$.*

The map \mathcal{S}_1 is linear Recall that $C_{(f,\tilde{f})}^\infty(X_1, V_1)$ has the structure of a module over the ring of f -invariant functions; likewise, $C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}})$ has the structure of a module over the ring of f_\sim -invariant functions. The map \mathcal{S}_1 respects these two structures, as the next statement shows.

Theorem 2.14. *The map \mathcal{S}_1 is additive, that is, for any two sections $s_1, s'_1 \in C_{(f,\tilde{f})}^\infty(X_1, V_1)$ we have*

$$\mathcal{S}_1(s_1 + s'_1) = \mathcal{S}_1(s_1) + \mathcal{S}_1(s'_1).$$

Furthermore, if $h : X_1 \rightarrow \mathbb{R}$ is an f -invariant function and $h^f : X_1^f \rightarrow \mathbb{R}$ is defined by $h^f = h \circ \chi_1^f$ then

$$\mathcal{S}_1(hs_1) = h^f \mathcal{S}_1(s_1).$$

Proof. Let $s_1, s'_1 \in C_{(f,\tilde{f})}^\infty(X_1, V_1)$ be two sections. The images $\mathcal{S}_1(s_1)$, $\mathcal{S}_1(s'_1)$, and $\mathcal{S}_1(s_1 + s'_1)$ are defined by the following identities:

$$\mathcal{S}_1(s_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s_1, \quad \mathcal{S}_1(s'_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s'_1, \quad \mathcal{S}_1(s_1 + s'_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ (s_1 + s'_1).$$

We obviously have

$$\begin{aligned} \mathcal{S}_1(s_1 + s'_1) \circ \chi_1^f &= \chi_1^{\tilde{f}} \circ (s_1 + s'_1) = \chi_1^{\tilde{f}} \circ s_1 + \chi_1^{\tilde{f}} \circ s'_1 = \\ &= \mathcal{S}_1(s_1) \circ \chi_1^f + \mathcal{S}_1(s'_1) \circ \chi_1^f = (\mathcal{S}_1(s_1) + \mathcal{S}_1(s'_1)) \circ \chi_1^f, \end{aligned}$$

that is,

$$\mathcal{S}_1(s_1 + s'_1) \circ \chi_1^f = (\mathcal{S}_1(s_1) + \mathcal{S}_1(s'_1)) \circ \chi_1^f.$$

Since χ_1^f is surjective, we have $\mathcal{S}_1(s_1 + s'_1) = \mathcal{S}_1(s_1) + \mathcal{S}_1(s'_1)$, as wanted. As for the second statement of the theorem, using again $\mathcal{S}_1(s_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s_1$, and the linearity of \tilde{f} , we obtain

$$\mathcal{S}_1(hs_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ (hs_1) = h(\chi_1^{\tilde{f}} \circ s_1) = (h^f \circ \chi_1^f)(\chi_1^{\tilde{f}} \circ s_1) = (h^f \circ \chi_1^f)(\mathcal{S}_1(s_1) \circ \chi_1^f) = (h^f \mathcal{S}_1(s_1)) \circ \chi_1^f.$$

Again by surjectivity of χ_1^f , we obtain that $\mathcal{S}_1(hs_1) = h^f \mathcal{S}_1(s_1)$, which completes the proof. \square

The map \mathcal{S}_1 is smooth We finally show that the map \mathcal{S}_1 is smooth for the functional diffeologies on $C_{(f,\tilde{f})}^\infty(X_1, V_1)$ and $C^\infty(X_1^f, V_1^{\tilde{f}})$.

Theorem 2.15. *The map \mathcal{S}_1 is a smooth map $C_{(f,\tilde{f})}^\infty(X_1, V_1) \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}})$.*

Proof. Observe that the functional diffeology on $C_{(f,\tilde{f})}^\infty(X_1, V_1)$ is the subset diffeology with respect to its inclusion into $C^\infty(X_1, V_1)$. Let $q : U \rightarrow C_{(f,\tilde{f})}^\infty(X_1, V_1)$ be a plot; recall that this means that for any plot $p : U' \rightarrow X_1$ the map $(u, u') \mapsto q(u)(p(u'))$ is smooth as a map from $U \times U'$ to V_1 (that is, it is a plot of V_1). Let us consider the composition $\mathcal{S}_1 \circ q$; we need to show that it is a plot of $C^\infty(X_1^f, V_1^{\tilde{f}})$, and so it should satisfy the analogous condition.

Let $p^f : U' \rightarrow X_1^f$ be a plot of X_1^f ; by the definition of the diffeology of the latter, there is a plot p of X_1 such that $p^f = \chi_1^f \circ p$. Thus, we have

$$(u, u') \mapsto (\mathcal{S}_1 \circ q)(u)(p(u')) = (\mathcal{S}_1(q(u)) \circ \chi_1^f)(p(u')) = (\chi_1^{\tilde{f}} \circ q(u))(p(u')) = \chi_1^{\tilde{f}}(q(u)(p(u'))),$$

and since $(u, u') \mapsto q(u)(p(u'))$ is a plot of V_1 by assumption, the resulting map $(u, u') \mapsto \chi_1^{\tilde{f}}(q(u)(p(u')))$ is a plot of $V_1^{\tilde{f}}$ by the definition of its diffeology, whence the claim. \square

In what follows we will show that, while \mathcal{S}_1 may not be injective, it is always surjective, which allows the space of sections $C^\infty(X_1^f, V_1^{\tilde{f}})$ to act as a substitute for the space $C_{(f,\tilde{f})}^\infty(X_1, V_1)$.

2.3.4 The quotient pseudo-bundle $\pi_1^{V_1/\text{Ker}(\tilde{f})} : V_1(\tilde{f}) \rightarrow X_1$

This auxiliary pseudo-bundle allows to consider the surjectivity of \mathcal{S}_1 ; this reasoning is straightforward. Let $\text{Ker}(\tilde{f})$ be the sub-bundle of V_1 formed by the union of the following subspaces in $\pi_1^{-1}(x)$: the subspace $\ker(\tilde{f}|_{\pi_1^{-1}(x)})$ if $x \in Y$, and the zero subspace otherwise. It is endowed with the subset diffeology relative to the inclusion $\text{Ker}(\tilde{f}) \subseteq V_1$ and with the restriction $\pi_1|_{\text{Ker}(\tilde{f})} =: \pi_1^{\text{ker}}$ of π_1 ; with respect to these it is a diffeological vector pseudo-bundle (see [9]). The sub-bundle thus obtained is called the **kernel of \tilde{f}** .

Consider the corresponding quotient pseudo-bundle with the total space $V_1/\text{Ker}(\tilde{f}) =: V_1(\tilde{f})$, the base space X_1 , and the induced pseudo-bundle projection denoted by $\pi_1^{V_1/\text{Ker}(\tilde{f})} : V_1(\tilde{f}) \rightarrow X_1$. The usual quotient projection $V_1 \rightarrow V_1/\text{Ker}(\tilde{f}) = V_1(\tilde{f})$ will be denoted by $\chi_1^{V_1(\tilde{f})}$. This quotient projection covers the identity map on X_1 , that is,

$$\pi_1 = \pi_1^{V_1/\text{Ker}(\tilde{f})} \circ \chi_1^{V_1(\tilde{f})}.$$

Lemma 2.16. *There is a smooth surjective pseudo-bundle map $\chi_1^0 : V_1(\tilde{f}) \rightarrow V_1^{\tilde{f}}$ covering the map $\chi_1^f : X_1 \rightarrow X_1^f$.*

Proof. This follows from the construction of $V_1(\tilde{f})$ and that of $V_1^{\tilde{f}}$. Indeed, $V_1^{\tilde{f}}$ is the quotient of $V_1(\tilde{f})$ by the following equivalence relation. Let \tilde{f}_0 be the pushforward of \tilde{f} to the quotient $V_1(\tilde{f})$. The space $(V_1(\tilde{f}))^{\tilde{f}_0}$, defined as the quotient of $V_1(\tilde{f})$ by the equivalence relation $v_1 \sim^0 v_2 \Leftrightarrow \tilde{f}_0(v_1) = \tilde{f}_0(v_2)$, is then precisely the space $V_1^{\tilde{f}}$. \square

Remark 2.17. The pseudo-bundle map $(\chi_1^{\tilde{f}}, \chi_1^f)$ filters through the pseudo-bundle map $(\chi_1^{V_1(\tilde{f})}, Id_{X_1})$, that is, we have

$$\chi_1^{\tilde{f}} = \chi_1^0 \circ \chi_1^{V_1(\tilde{f})}.$$

Furthermore, there is an induced gluing of $\pi_1^{V_1/Ker(\tilde{f})} : V_1(\tilde{f}) \rightarrow X_1$ to $\pi_2 : V_2 \rightarrow X_2$ along the maps (\tilde{f}_0, f) that yields the pseudo-bundle

$$\pi_1^{V_1/Ker(\tilde{f})} \cup_{(\tilde{f}_0, f)} \pi_2 : V_1(\tilde{f}) \cup_{\tilde{f}_0} V_2 \rightarrow X_1 \cup_f X_2.$$

2.3.5 \mathcal{S}_1 may not be injective

We now indicate the reason why \mathcal{S}_1 may not be injective, although for reasons of brevity we do not provide a complete treatment of any specific example (which is easy to find anyway).

Observation 2.18. Let s_1 and s'_1 be two sections in $C^\infty(X_1, V_1)$ such that $\chi_1^{\tilde{f}} \circ s_1 = \chi_1^{\tilde{f}} \circ s'_1$; suppose that there exists $x \in X_1$ such that $s_1(x) \neq s'_1(x)$. Since by assumption $\chi_1^{\tilde{f}}(s_1(x)) = \chi_1^{\tilde{f}}(s'_1(x))$, and $\chi_1^{\tilde{f}}$ is injective outside of $\pi_1^{-1}(Y)$, we conclude that $x \in Y$, and that $s_1(x) - s'_1(x)$ belongs to the fibre at x of $Ker(\tilde{f})$. This is easily seen to be a vice versa.

A concrete example could be obtained by assuming that $Ker(\tilde{f})$ splits off as a smooth direct summand in the pseudo-bundle V_1 and is such that there exists a smooth non-zero section $s_0 : X \rightarrow Ker(\tilde{f})$. These assumptions suffice for \mathcal{S}_1 to be non-injective. More precisely, for any section $s_1 \in C^\infty_{(f, \tilde{f})}(X_1, V_1)$ we have that $s_1 + s_0 \in C^\infty_{(f, \tilde{f})}(X_1, V_1)$.

Indeed, if $y, y' \in Y \subset X_1$ are such that $f(y) = f(y')$, then by assumption and by linearity of \tilde{f}

$$\tilde{f}(s_1(f(y))) = \tilde{f}(s_1(f(y'))) \Rightarrow \tilde{f}(s_1(f(y)) + s_0(f(y))) = \tilde{f}(s_1(f(y))).$$

Similarly,

$$\tilde{f}(s_1(f(y')) + s_0(f(y'))) = \tilde{f}(s_1(f(y'))) \Leftrightarrow \tilde{f}((s_1 + s_0)(f(y))) = \tilde{f}((s_1 + s_0)(f(y')));$$

in particular, $\mathcal{S}_1(s_1 + s_0)$ is well-defined. It remains to observe that by Theorem 2.14

$$\mathcal{S}_1(s_1 + s_0) = \mathcal{S}_1(s_1) + \mathcal{S}_1(s_0),$$

and since $\mathcal{S}_1(s_0)$ is the zero section, this is equal to $\mathcal{S}_1(s_1)$. Since $s_1 + s_0 \neq s_1$ by the choice of s_0 , we see that \mathcal{S}_1 is not injective.

2.3.6 Surjectivity of \mathcal{S}_1 : the case of the trivial $Ker(\tilde{f})$

We treat the case of the trivial $Ker(\tilde{f})$ separately, since for obvious reasons it is possible to obtain stronger statements in this case. Indeed, the assumption that $Ker(\tilde{f})$ is trivial implies that $V_1(\tilde{f}) = V_1$, and allows to define, for any given section $s \in C^\infty(X_1^f, V_1^{\tilde{f}})$, its pullback via the map \mathcal{S}_1 to a well-defined and unique section $X_1 \rightarrow V_1$. This pullback, denoted by $\mathcal{S}_1^{-1}(s)$, is given by the following formula:

$$\mathcal{S}_1^{-1}(s)(x) = \begin{cases} (\chi_1^{\tilde{f}})^{-1}(s(\chi_1^f(x))) & \text{for } x \in X_1 \setminus Y \\ (\chi_1^{\tilde{f}}|_{\chi_1^{-1}(x)})^{-1}(s(\chi_1^f(x))) & \text{for } x \in Y. \end{cases}$$

Since under the present assumption the restriction of \tilde{f} on each individual fibre in its domain is injective, the restriction of $\chi_1^{\tilde{f}}$ on any fibre in V_1 is injective as well. It is also obvious that the map $\mathcal{S}_1^{-1}(s)$ thus obtained is (f, \tilde{f}) -invariant. We need to verify is that it is smooth as a map $X_1 \rightarrow V_1$.

Lemma 2.19. The map $\mathcal{S}_1^{-1}(s) : X_1 \rightarrow V_1$ is smooth for every $s \in C^\infty(X_1^f, V_1^{\tilde{f}})$.

Proof. Let $p : U \rightarrow X_1$ be a plot of X_1 ; we need to show that $u \mapsto \mathcal{S}_1^{-1}(s)(p(u))$ is a plot of V_1 . By definition of a pushforward diffeology, this is equivalent, for U small enough, to $u \mapsto \chi_1^{\tilde{f}}(\mathcal{S}_1^{-1}(s)(p(u)))$ being a plot of $V_1^{\tilde{f}}$. By an easy calculation we obtain

$$\chi_1^{\tilde{f}}(\mathcal{S}_1^{-1}(s)(p(u))) = s(\chi_1^f(p(u))).$$

Since χ_1^f is smooth by construction, $\chi_1^f \circ p$ is a plot of X_1^f , and since s is smooth by assumption, $s \circ \chi_1^f \circ p$ is a plot of $V_1^{\tilde{f}}$, whence the claim. \square

Lemma 2.19 yields a well-defined inverse map

$$\mathcal{S}_1^{-1} : C^\infty(X_1^f, V_1^{\tilde{f}}) \rightarrow C_{(f, \tilde{f})}^\infty(X_1, V_1).$$

Moreover, we have the following statement.

Theorem 2.20. *Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing between them such that \tilde{f} is injective on each fibre in its domain of definition. Then \mathcal{S}_1^{-1} is smooth as a map $C^\infty(X_1^f, V_1^{\tilde{f}}) \rightarrow C_{(f, \tilde{f})}^\infty(X_1, V_1)$.*

Proof. Let $q : U \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}})$ be a plot of $C^\infty(X_1^f, V_1^{\tilde{f}})$; recall that, as for any functional diffeology, this means that for any plot $p^f : U' \rightarrow X_1^f$ of X_1^f the corresponding evaluation map $(u, u') \mapsto q(u)(p^f(u'))$ is a plot of $V_1^{\tilde{f}}$. Let us show that the evaluation map corresponding to $\mathcal{S}_1^{-1} \circ q$ is a plot of V_1 .

Let $p : U' \rightarrow X_1$ be a plot of X_1 . As in the previous proof, up to restricting U and U' as necessary, it would be sufficient to prove that $(u, u') \mapsto \chi_1^{\tilde{f}}((\mathcal{S}_1^{-1} \circ q)(u)(p(u')))$ is a plot of $V_1^{\tilde{f}}$. By definition of \mathcal{S}_1^{-1} we have

$$\chi_1^{\tilde{f}}((\mathcal{S}_1^{-1} \circ q)(u)(p(u'))) = q(u)(\chi_1^f(p(u'))).$$

Since $\chi_1^f \circ p$ is a plot of X_1^f by construction, the resulting map is a plot of $V_1^{\tilde{f}}$ by the assumption on q , whence the claim. \square

The following is now an obvious conclusion.

Corollary 2.21. *Under the assumptions of Theorem 2.20, the spaces of sections $C_{(f, \tilde{f})}^\infty(X_1, V_1)$ and $C^\infty(X_1^f, V_1^{\tilde{f}})$ are diffeomorphic.*

2.3.7 Surjectivity of \mathcal{S}_1 in the case when $\text{Ker}(\tilde{f})$ is non-trivial

We first assume for simplicity that f is injective, so $X_1^f = X_1$ and $V_1(\tilde{f}) = V_1^{\tilde{f}}$. Notice also that under this assumption \mathcal{S}_1 is determined by the simpler condition $\mathcal{S}_1(s') = \chi_1^{\tilde{f}} \circ s'$.

Proposition 2.22. *Let (\tilde{f}, f) be such that f is injective. Then for every smooth section $s : X_1 \rightarrow V_1(\tilde{f})$ there exists a smooth (f, \tilde{f}) -invariant section $s' : X_1 \rightarrow V_1$ such that $\mathcal{S}_1(s') = s$.*

Proof. Since by assumption V_1 has finite-dimensional fibres only, we can choose an arbitrary direct sum decomposition $V_1 = V_1^0 \oplus \text{Ker}(\tilde{f})$. The direct sum complement V_1^0 thus chosen is also a sub-bundle, but if the decomposition is not smooth then the diffeology of V_1 is coarser than the respective direct sum diffeology. Also, having fixed such a decomposition, for every section $s : X_1 \rightarrow V_1(\tilde{f})$ there is a well-defined pullback of it to a (non-smooth *a priori*) section $X_1 \rightarrow V_1$.

Since $V_1(\tilde{f}) = V_1/\text{Ker}(\tilde{f})$, we can write its elements as cosets $v + \text{Ker}(\tilde{f})$. The map $\chi_1^{\tilde{f}}$ then has form $\chi_1^{\tilde{f}}(v) = v + \text{Ker}(\tilde{f})$, and every plot of $V_1(\tilde{f})$ has form $\chi_1^{\tilde{f}} \circ p$ for some plot p of V_1 . Now, if $s : X_1 \rightarrow V_1(\tilde{f})$ is a smooth section, then for any given $x \in X_1$ we can denote by $t(s)(x)$ the unique element of V_1^0 contained in the coset $s(x)$. The map $t(s)$ thus defined is a section $X_1 \rightarrow V_1$.

To show that $t(s)$ is smooth as a map $X_1 \rightarrow V_1$, let $q : U \rightarrow X_1$ be a plot of X_1 . We need to show that $u \mapsto t(s)(q(u))$ is a plot of V_1 . This is equivalent to showing that there exists a sub-domain U' of

U such that on this sub-domain $u \mapsto \chi_1^{\tilde{f}}(t(s)(q(u)))$ is a plot of $V_1(\tilde{f})$. But we have by construction that $\chi_1^{\tilde{f}}(t(s)(q(u))) = s(q(u))$ on the whole U . Since by assumption s is smooth as a map $X_1 \rightarrow V_1(\tilde{f})$, we have that $u \mapsto s(q(u))$ is a plot of $V_1(\tilde{f})$. The map $t(s)$ is thus the section s' we were looking for; in particular, it is clearly (f, \tilde{f}) -invariant. \square

Example 2.23. Let $V_1 = \mathbb{R} \times \mathbb{R}^2$, with the first factor carrying the standard diffeology and the second, the vector space diffeology generated by $u \mapsto |u|(e_y + e_z)$; let X_1 be the standard \mathbb{R} identified with the first factor, so the second factor is the fibre. Let \tilde{f} be defined over the whole X_1 (so on the entire V_1), and let it act by $(x, y, z) \mapsto (x, 0, z)$; we may assume it to take values in some $V_2 = \mathbb{R} \times \mathbb{R}$, where the first factor is the standard \mathbb{R} identified with the corresponding base space and the second is \mathbb{R} with the vector space diffeology generated by $u \mapsto |u|e_z$. Thus, \tilde{f} is smooth, and $V_1(\tilde{f})$ can actually be identified with V_2 . It is convenient to represent both of them by the subset $\{(x, 0, z)\}$ of \mathbb{R}^3 .

Observe that every section $X_1 \rightarrow V_1(\tilde{f})$ is a linear combination with coefficients that are usual smooth functions in x of sections of form $x \mapsto (x, 0, |g(x)|)$ (where again g is a usual smooth function). It is then obvious that every such map lifts to the section $X_1 \rightarrow V_1$ that is given by $x \mapsto (x, |g(x)|, |g(x)|)$.

Corollary 2.21 and Proposition 2.22 allow to show that \mathcal{S}_1 is always surjective.

Theorem 2.24. Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing of the former to the latter such that f and \tilde{f} are subductions onto their respective images. Then the map \mathcal{S}_1 is surjective as a map $C_{(f, \tilde{f})}^\infty(X_1, V_1) \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}})$.

Proof. Recall that the pseudo-bundle map $(\chi_1^{\tilde{f}} : V_1 \rightarrow V_1^{\tilde{f}}, \chi_1^f : X_1 \rightarrow X_1^f)$ filters through the pseudo-bundle maps $(\chi_1^{V_1(\tilde{f})} : V_1 \rightarrow V_1(\tilde{f}), \text{Id}_{X_1} : X_1 \rightarrow X_1)$ and $(\chi_1^0 : V_1(\tilde{f}) \rightarrow V_1^{\tilde{f}}, \chi_1^f : X_1 \rightarrow X_1^f)$. Accordingly, \mathcal{S}_1 decomposes into the following composition of maps.

Let $\mathcal{S}_1^{V_1(\tilde{f})} : C_{(f, \tilde{f})}^\infty(X_1, V_1) \rightarrow C^\infty(X_1, V_1(\tilde{f}))$ be the map defined by

$$\mathcal{S}_1^{V_1(\tilde{f})}(s) = \chi_1^{V_1(\tilde{f})} \circ s$$

(it coincides with \mathcal{S}_1 if f and \tilde{f} are such that $V_1(\tilde{f}) = V_1^{\tilde{f}}$). Let $\mathcal{S}_1^0 : C_{(f, \tilde{f}_0)}^\infty(X_1, V_1(\tilde{f})) \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}})$ be the map defined by

$$\mathcal{S}_1^0(s) \circ \chi_1^f = \chi_1^0 \circ s.$$

We claim, first of all, that

$$\mathcal{S}_1 = \mathcal{S}_1^0 \circ \mathcal{S}_1^{V_1(\tilde{f})}.$$

Indeed,

$$\left(\mathcal{S}_1^0 \circ \mathcal{S}_1^{V_1(\tilde{f})}\right)(s) = \mathcal{S}_1^0\left(\chi_1^{V_1(\tilde{f})} \circ s\right),$$

and the latter satisfies the identity

$$\mathcal{S}_1^0\left(\chi_1^{V_1(\tilde{f})} \circ s\right) \circ \chi_1^f = \chi_1^0 \circ \chi_1^{V_1(\tilde{f})} \circ s = \chi_1^{\tilde{f}} \circ s$$

by Lemma 4.22. Since $\chi_1^{\tilde{f}} \circ s = \mathcal{S}_1(s) \circ \chi_1^f$, we get that $\mathcal{S}_1(s) = (\mathcal{S}_1^0 \circ \mathcal{S}_1^{V_1(\tilde{f})})(s)$ for any (f, \tilde{f}) -invariant section $s : X_1 \rightarrow V_1$.

Now, by Corollary 2.21 the map \mathcal{S}_1^0 is a diffeomorphism between $C_{(f, \tilde{f}_0)}^\infty(X_1, V_1(\tilde{f}))$ and $C^\infty(X_1^f, V_1^{\tilde{f}})$. It thus suffices to show that $\mathcal{S}_1^{V_1(\tilde{f})}$ maps $C_{(f, \tilde{f})}^\infty(X_1, V_1)$ onto $C_{(f, \tilde{f}_0)}^\infty(X_1, V_1(\tilde{f}))$. This is obtained by first applying Proposition 2.22 where instead of f we consider Id_{X_1} and instead of \tilde{f} , the quotient map $\chi_1^{V_1(\tilde{f})}$. The proposition then guarantees that every section $X_1 \rightarrow V_1(\tilde{f})$ pulls back to a $(\text{Id}_{X_1}, \chi_1^{V_1(\tilde{f})})$ -invariant section $X_1 \rightarrow V_1$. We thus need to check that any (f, \tilde{f}_0) -invariant section admits a pullback that is (f, \tilde{f}) -invariant; and this easily follows from $\tilde{f} = \tilde{f}_0 \circ \chi_1^{V_1(\tilde{f})}$, i.e., from the very definition of \tilde{f}_0 . Thus, as a map $C_{(f, \tilde{f})}^\infty(X_1, V_1) \rightarrow C_{(f, \tilde{f}_0)}^\infty(X_1, V_1(\tilde{f}))$, the map $\mathcal{S}_1^{V_1(\tilde{f})}$ is onto, which completes the proof. \square

2.3.8 \mathcal{S}_1 is a subduction

We have just seen (Theorem 2.24) that if \tilde{f} and f are subductions then \mathcal{S}_1 is surjective. We now show that a stronger statement is true: under the same assumption, \mathcal{S}_1 is a subduction itself.

Theorem 2.25. *Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing of the former to that latter such that both \tilde{f} and f are subsections onto their images. Then the map \mathcal{S}_1 is a subduction of $C_{(f, \tilde{f})}^\infty(X_1, V_1)$ onto $C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}})$.*

Proof. We need to show that every plot $q^{f, \tilde{f}}$ of $C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}})$ locally has form $\mathcal{S}_1 \circ q$ for some plot q of $C_{(f, \tilde{f})}^\infty(X_1, V_1)$. Thus, let $q^{f, \tilde{f}} : U \rightarrow C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}})$ (we will assume that U is small enough, as needed); this means that for any plot $p^f : U' \rightarrow X_1^f$ of X_1^f the usual evaluation map $(u, u') \mapsto q^{f, \tilde{f}}(u)(p^f(u'))$ is a plot of $V_1^{\tilde{f}}$. Now we also assume that U' is small enough so that $p^f = \chi_1^f \circ p$ for some plot p of X_1 .

As shown in the proof of Proposition 2.22, the map \mathcal{S}_1 admits a right inverse, depending on the choice of a decomposition of V into a direct sum with $\text{Ker}(\tilde{f})$. Let $(\mathcal{S}_1)^{-1}$ be any fixed choice of a right inverse; define a map $q : U \rightarrow C_{(f, \tilde{f})}^\infty(X_1, V_1)$ by setting $q = (\mathcal{S}_1)^{-1} \circ q^{f, \tilde{f}} \circ \chi_1^f$. By the usual definition, this is a plot if, up to further restricting U , we have that, for any given plot $p : U' \rightarrow X_1$ of X_1 , the following is a plot of V_1 :

$$(u, u') \mapsto (\mathcal{S}_1)^{-1}(q^{f, \tilde{f}}(u))(p(u')).$$

Now, if we assume U and U' to be small enough, this is a plot of V_1 if and only if the following is a plot of $V_1^{\tilde{f}}$:

$$(u, u') \mapsto \chi_1^{\tilde{f}}((\mathcal{S}_1)^{-1}(q^{f, \tilde{f}}(u))(p(u'))).$$

Recalling now the definition of $(\mathcal{S}_1)^{-1}$, we get that

$$\chi_1^{\tilde{f}}((\mathcal{S}_1)^{-1}(q^{f, \tilde{f}}(u))(p(u'))) = q^{f, \tilde{f}}(u)(\chi_1^f(p(u'))),$$

which is the value of the evaluation of $q^{f, \tilde{f}}(u)$ on the plot $\chi_1^f \circ p$ of X_1^f . Therefore it is a plot of $V_1^{\tilde{f}}$, due to $q^{f, \tilde{f}}$ being a plot of $C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}})$, so we conclude that q is indeed a plot of $C_{(f, \tilde{f})}^\infty(X_1, V_1)$. Since $q^{f, \tilde{f}} = \mathcal{S}_1 \circ q$ by construction, and it was arbitrarily chosen, we obtain the claim. \square

2.3.9 \mathcal{S}_1 preserves compatibility

The only item that is still lacking for relating the pseudo-bundle $\pi_1 \cup_{(\tilde{f}, f)} \pi_2 : V_1 \cup_{\tilde{f}} V_2 \rightarrow X_1 \cup_f X_2$ to its reduced version $\pi_1^{(\tilde{f}, f)} \cup_{(\tilde{f}_\sim, f_\sim)} \pi_2 : V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2 \rightarrow X_1 \cup_{f_\sim} X_2$ is a description of the interaction of the map \mathcal{S}_1 with the two compatibility conditions (one relative to (\tilde{f}, f) and the other to (\tilde{f}_\sim, f_\sim)). We provide it in this section.

Proposition 2.26. *For a given gluing (\tilde{f}, f) of a pseudo-bundle $\pi_1 : V_1 \rightarrow X_1$ to another pseudo-bundle $\pi_2 : V_2 \rightarrow X_2$, assume that both \tilde{f} and f are subductions, and let $s_i \in C^\infty(X_i, V_i)$ for $i = 1, 2$. If s_1 and s_2 are (f, \tilde{f}) -compatible then $\mathcal{S}_1(s_1)$ and s_2 are (f_\sim, \tilde{f}_\sim) -compatible.*

Recall that s_1 being (f, \tilde{f}) -compatible with some s_2 implies it being (f, \tilde{f}) -invariant, so the expression $\mathcal{S}_1(s_1)$ makes sense.

Proof. The (f, \tilde{f}) -compatibility of s_1 and s_2 means precisely that for all $y \in Y$ we have $\tilde{f}(s_1(y)) = s_2(f(y))$; we need to show that $\tilde{f}_\sim(\mathcal{S}_1(s_1)(\chi_1^f(y))) = s_2(f_\sim(\chi_1^f(y)))$. By definition $f_\sim(\chi_1^f(y)) = f(y)$ and $\mathcal{S}_1(s_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s_1$, so the desired condition is equivalent to $\tilde{f}_\sim(\chi_1^{\tilde{f}}(s_1(y))) = s_2(f(y))$. It remains to notice that $\tilde{f}_\sim(\chi_1^{\tilde{f}}(s_1(y))) = \tilde{f}(s_1(y))$ by definition of \tilde{f}_\sim , so the (f_\sim, \tilde{f}_\sim) -compatibility does follow from the (f, \tilde{f}) -compatibility of s_1 and s_2 . \square

The inverse of Proposition 2.26 is true as well.

Proposition 2.27. *Let $s_1 \in C_{(f,\tilde{f})}^\infty(X_1, V_1)$ and $s_2 \in C^\infty(X_2, V_2)$ be two sections such that $\mathcal{S}_1(s_1)$ and s_2 are (f_\sim, \tilde{f}_\sim) -compatible. Then s_1 and s_2 are (f, \tilde{f}) -compatible.*

Proof. The proof is the same as the previous one, just going in the opposite direction. Let $\chi_1^f(y)$ be a point in the domain of f_\sim ; the assumption of (f_\sim, \tilde{f}_\sim) -compatibility means precisely that

$$\tilde{f}_\sim(\mathcal{S}_1(s_1)(\chi_1^f(y))) = s_2(f_\sim(\chi_1^f(y))).$$

Recall that $f_\sim \circ \chi_1^f = f$ by definition, so the right-hand side coincides with $s_2(f(y))$. Since \mathcal{S}_1 is defined by the identity $\mathcal{S}_1(s_1) \circ \chi_1^f = \chi_1^{\tilde{f}} \circ s_1$, the left-hand side becomes $\tilde{f}_\sim(\chi_1^{\tilde{f}}(s_1(y)))$. Since furthermore $\tilde{f}_\sim \circ \chi_1^{\tilde{f}} = \tilde{f}$ (by the definition of the map \tilde{f}_\sim), the left-hand side is then equal to $\tilde{f}_\sim(s_1(y))$. We thus have

$$\tilde{f}_\sim(s_1(y)) = \tilde{f}_\sim(\mathcal{S}_1(s_1)(\chi_1^f(y))) = s_2(f_\sim(\chi_1^f(y))) = s_2(f(y)),$$

i.e., that s_1 and s_2 are (f, \tilde{f}) -compatible. □

Putting the two propositions together, we obtain the following result.

Corollary 2.28. *Suppose that both \tilde{f} and f are subductions. Then $(\mathcal{S}_1, \text{Id}_{C^\infty(X_2, V_2)})$ is well-defined and surjective as a map $C_{(f,\tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C_{(f_\sim, \tilde{f}_\sim)}^\infty(X_1^f, V_1^{\tilde{f}}) \times_{\text{comp}} C^\infty(X_2, V_2)$.*

2.3.10 The space $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ is a smooth surjective image of $C_{(f,\tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \subseteq C^\infty(X_1, V_1) \times C^\infty(X_2, V_2)$

We now collect the results of the current section into the final statement, which is as follows.

Theorem 2.29. *Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing of the former pseudo-bundle to the latter, such that both \tilde{f} and f are subductions onto their respective images. The map \mathcal{S} is a subduction of $C_{(f,\tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2)$ onto $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$.*

Proof. It suffices to recall the diffeomorphism $C^\infty(X_1^f \cup_{f_\sim} X_2, V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2) \cong C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ of Proposition 2.11, which for the moment we denote by \tilde{F} . By Theorem 2.5 we have two versions of the map \mathcal{S} , one for the original pseudo-bundle, and one for its restricted version:

$$\mathcal{S} : C_{(f,\tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2) \quad \text{and}$$

$$\mathcal{S}^{(f,\tilde{f})} : C^\infty(X_1^f, V_1^{\tilde{f}}) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1^f \cup_{f_\sim} X_2, V_1^{\tilde{f}} \cup_{\tilde{f}_\sim} V_2),$$

that by the same theorem are smooth. By Corollary 2.28 there is a well-defined and factor-to-factor map

$$(\mathcal{S}_1, \text{Id}_{C^\infty(X_2, V_2)}) : C_{(f,\tilde{f})}^\infty(X_1, V_1) \times_{\text{comp}} C^\infty(X_2, V_2) \rightarrow C^\infty(X_1^f, V_1^{\tilde{f}}) \times_{\text{comp}} C^\infty(X_2, V_2),$$

i.e., one that acts as \mathcal{S}_1 on the first factor and as the identity map on the second factor. Observing now that

$$\mathcal{S} = \tilde{F} \circ \mathcal{S}^{(f,\tilde{f})} \circ (\mathcal{S}_1, \text{Id}_{C^\infty(X_2, V_2)}),$$

it follows from Proposition 2.8, implying that $\mathcal{S}^{(f,\tilde{f})}$ is a diffeomorphism, and Theorem 2.25 that \mathcal{S} is a subduction, which completes the proof. □

3 Diffeological connections: the verbatim extension

One can define a diffeological connection by the minimal possible extension of the standard definition of a Riemannian connection. The resulting notion is then as follows.

Definition 3.1. *Let $\pi : V \rightarrow X$ be a finite-dimensional diffeological vector pseudo-bundle, and let $C^\infty(X, V)$ be the space of its smooth sections. A **connection** on this pseudo-bundle is a smooth linear operator*

$$\nabla : C^\infty(X, V) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V),$$

which satisfies the Leibnitz rule, that is, for every function $f \in C^\infty(X, \mathbb{R})$ and for every section $s \in C^\infty(X, V)$ we have

$$\nabla(fs) = df \otimes s + f\nabla s.$$

We need to explain first of all why this definition is well-posed. The meaning of the question is as follows. Although, as already mentioned, the differentials of functions are well-defined in the diffeological context, they are elements of $\Omega^1(X)$, while for the statement of the Leibnitz rule we need them to be sections of $\Lambda^1(X)$. For this reason the meaning of df is one of the section given by

$$df : x \mapsto \pi^{\Omega, \Lambda}(x, df)$$

(we keep the same symbol for both df an element of $\Omega^1(X)$ and df a section of $\Lambda^1(X)$). Having specified this, the definition is well-posed.

3.1 An example for a nonstandard pseudo-bundle

Let us describe first of all an example of a diffeological connection that is not a standard connection on a smooth manifold.

The pseudo-bundle and its gluing presentation We consider the pseudo-bundle $\pi : V \rightarrow X$, where X and V are the following subsets of \mathbb{R}^3 :

$$X = \{xy = 0, z = 0\}, \quad V = \{xy = 0\},$$

and π is the restriction to V of the standard projection of \mathbb{R}^3 onto xy -coordinate plane. Each fibre $\pi^{-1}(x, y, 0)$ of V is endowed with the vector space structure of the usual \mathbb{R} relative to the third coordinate (keeping the first two fixed):

$$(x, y, z_1) + (x, y, z_2) = (x, y, z_1 + z_2), \quad \lambda(x, y, z) = (x, y, \lambda z) \text{ for } \lambda \in \mathbb{R}.$$

The diffeologies on V and X are gluing diffeologies coming from their presentations as

$$X = X_1 \cup_f X_2, \quad X_1 = \{y = z = 0\}, \quad X_2 = \{x = z = 0\} \quad \text{and} \quad V = V_1 \cup_{\tilde{f}} V_2, \quad V_1 = \{y = 0\}, \quad V_2 = \{x = 0\},$$

where the gluing maps f and \tilde{f} are the restrictions of the identity map $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ to their domains of definition; these domains of definition are, the origin $\{(0, 0, 0)\}$ for f , and the z -axis $\{(0, 0, z)\}$ for \tilde{f} . The four spaces X_1, X_2, V_1, V_2 carry the subset diffeology relative to their inclusions into \mathbb{R}^3 , and the gluing diffeologies on X and V correspond to those; notice that these gluing diffeologies are strictly weaker than their subset diffeologies relative to \mathbb{R}^3 (see [18]). We denote the restrictions of π to V_1 and to V_2 by π_1 and π_2 respectively.

Pseudo-metrics on $\pi : V \rightarrow X$ We obtain a pseudo-metric on $\pi : V \rightarrow X$ by gluing two compatible pseudo-metrics on $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ respectively. We denote them by g_1 and g_2 respectively and define them to be $g_1(x, 0, 0) = h_1(x)dz^2$ and $g_2(0, y, 0) = h_2(y)dz^2$, where $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$ are usual smooth functions; they obviously need to be everywhere positive. The compatibility condition for them takes form $h_1(0) = h_2(0)$. Assuming this, we obtain a pseudo-metric \tilde{g} on V defined by

$$\tilde{g}(x, y, 0) = \begin{cases} h_1(x)dz^2, & \text{if } y = 0, \\ h_2(y)dz^2, & \text{if } x = 0. \end{cases}$$

The standard connections on the factors The two factors $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ are both diffeomorphic to the standard trivial bundle $\mathbb{R}^2 \rightarrow \mathbb{R}$, and so can be seen as the usual tangent bundles $TX_1 \cong T\mathbb{R} \cong TX_2$. Thus, g_1 and g_2 are Riemannian metrics on them, and we can consider the usual Levi-Civita connections ∇^1 and ∇^2 on them. Their Christoffel symbols are $\Gamma_{11}^1(g_1) = \frac{h_1'(x)}{2h_1(x)}$ and $\Gamma_{11}^1(g_2) = \frac{h_2'(y)}{2h_2(y)}$. The formulae for ∇^1 and ∇^2 therefore are

$$\nabla^1(x, 0, 1) = \frac{h_1'(x)}{2h_1(x)} dx \otimes (x, 0, 1) \quad \text{and} \quad \nabla^2(0, y, 1) = \frac{h_2'(y)}{2h_2(y)} dy \otimes (0, y, 1)$$

and in full form

$$\nabla^1(x, 0, s_1(x)) = \frac{h_1'(x)(s_1'(x) + s_1(x))}{2h_1(x)} dx \otimes (x, 0, 1), \quad \nabla^2(0, y, s_2(y)) = \frac{h_2'(y)(s_2'(y) + s_2(y))}{2h_2(y)} dy \otimes (0, y, 1).$$

The resulting connection In this specific case it is actually quite straightforward to assemble a connection on V out of ∇^1 and ∇^2 . The explicit formula is as follows:

$$\nabla(x, y, s(x, y)) = \begin{cases} \frac{h_1'(x)(\frac{\partial s}{\partial x}(x, 0) + s(x, 0))}{2h_1(x)} dx \otimes (x, 0, 1) & \text{if } y = 0, \\ \frac{h_2'(y)(\frac{\partial s}{\partial y}(0, y) + s(0, y))}{2h_2(y)} dy \otimes (0, y, 1) & \text{if } x = 0, \\ \left(\frac{h_1'(0)(\frac{\partial s}{\partial x}(0, 0) + s(0, 0))}{2h_1(0)} dx + \frac{h_2'(0)(\frac{\partial s}{\partial y}(0, 0) + s(0, 0))}{2h_2(0)} dy \right) \otimes (0, 0, 1) & \text{if } x = y = 0. \end{cases}$$

Here s can be, in particular, any smooth two-variable function; however, more generally it is a formal pair of functions s_1 (in variable x) and s_2 (in variable y) such that $s_1(0) = s_2(0)$.

Observations on the example The example just made give a rough idea of how one can obtain a connection on $V_1 \cup_{\bar{f}} V_2$ out of two given connections on V_1 and V_2 . On the other hand, it does not give a complete picture; indeed, the simplicity of the domain of gluing on the base spaces ensures that we do not have to impose any conditions on ∇^1 and ∇^2 , although later on we will see that a certain compatibility condition is needed.

3.2 Covariant derivatives

The usual notion of the covariant derivative of a section $s \in C^\infty(M, E)$ along a smooth vector field $X \in C^\infty(M, TM)$ extends easily to smooth sections $s \in C^\infty(X, V)$ of a diffeological vector pseudo-bundle. It suffices to specify that such derivatives are with respect to smooth sections of the pseudo-bundle $(\Lambda^1(X))^*$.

Definition 3.2. *Let $\pi : V \rightarrow X$ be a finite-dimensional diffeological vector pseudo-bundle, let $\nabla : C^\infty(X, V) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V)$ be a diffeological connection on it, and let $t \in C^\infty(X, (\Lambda^1(X))^*)$ be a smooth section of the dual pseudo-bundle $(\Lambda^1(X))^*$. Let $s \in C^\infty(X, V)$; the **covariant derivative** of s along t is the section $\nabla s(t) = \nabla_t s$.*

Lemma 3.3. *For any $t \in C^\infty(X, (\Lambda^1(X))^*)$ and for any $s \in C^\infty(X, V)$ we have $\nabla_t s \in C^\infty(X, V)$.*

Proof. This is obvious, since the diffeology on $(\Lambda^1(X))^*$, as on any dual pseudo-bundle, is defined so that the evaluation functions $x \mapsto t(x)(\alpha^i(x))$ be smooth. \square

We thus conclude that if ∇ and t are as above, ∇_t is well-defined as an operator $C^\infty(X, V) \rightarrow C^\infty(X, V)$. We furthermore have the following.

Theorem 3.4. *For any $t \in C^\infty(X, (\Lambda^1(X))^*)$ the map $\nabla_t : C^\infty(X, V) \rightarrow C^\infty(X, V)$ given by $s \mapsto \nabla_t s$ is smooth for the functional diffeology on $C^\infty(X, V)$.*

Proof. Let $p : U \rightarrow C^\infty(X, V)$ be a plot of $C^\infty(X, V)$. By the properties of a functional diffeology, the map $U \times X \rightarrow V$ given by $(u, x) \mapsto p(u)(x)$ is smooth, which also implies that for any plot $q : U' \rightarrow X$ the map $U \times U' \rightarrow V$ acting by $(u, u') \mapsto p(u)(q(u'))$ is a plot of V .

In order to prove that ∇_t is smooth, we need to show that $u \mapsto \nabla_t p(u)$ is a plot of $C^\infty(X, V)$. Since ∇ is smooth as a map $C^\infty(X, V) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V)$, its composition with any given plot p of $C^\infty(X, V)$ is a plot of $C^\infty(X, \Lambda^1(X) \otimes V)$. This composition has form $u \mapsto \nabla p(u)$. It remains to notice that $u \mapsto \nabla_t(p(u))$ is the evaluation of it on the constant plot of $(\Lambda^1(X))^*$ with value t , which implies that it is a plot of $C^\infty(X, V)$, as wanted. \square

There are also the expected linearity properties, stated below.

Theorem 3.5. *The operator $t \mapsto \nabla_t$ is $C^\infty(X, \mathbb{R})$ -linear, that is, $\nabla_{t_1+t_2} = \nabla_{t_1} + \nabla_{t_2}$ and $\nabla_{f \cdot t} = f \nabla_t$ for any smooth function $f : X \rightarrow \mathbb{R}$.*

Proof. This is a direct consequence of the definitions. \square

3.3 Compatibility with a pseudo-metric

The usual notion of compatibility of a connection with a Riemannian metric extends trivially to the diffeological context. Let X be a diffeological space such that $\Lambda^1(X)$ has only finite-dimensional fibres and admits a pseudo-metric. Assume furthermore that there is a choice of a pseudo-metric g^Λ on $\Lambda^1(X)$ such that the induced bilinear form g^{Λ^*} on $(\Lambda^1(X))^*$ is also a pseudo-metric. Finally, let ∇ be a diffeological connection on $(\Lambda^1(X))^*$.

Definition 3.6. *The connection ∇ is said to be **compatible with the pseudo-metric** g^{Λ^*} if for every two smooth sections s, t of the pseudo-bundle $(\pi^\Lambda)^* : (\Lambda^1(X))^* \rightarrow X$ we have that*

$$d(g^{\Lambda^*}(s, t)) = g^{\Lambda^*}(\nabla s, t) + g^{\Lambda^*}(s, \nabla t),$$

where for every 1-form $\omega \in \Lambda^1(X)$ we set by definition $g^{\Lambda^*}(\omega \otimes s, t) = g^{\Lambda^*}(s, \omega \otimes t) = \omega \cdot g^{\Lambda^*}(s, t)$.

4 Pseudo-bundle operations and diffeological connections

The usual connections are well-behaved with respect to the standard operations, such those of direct sum, tensor product, or taking dual, on smooth vector bundles. In this section we show that the same is true of diffeological connections in the case of direct sums and tensor products, while the situation is more complicated for dual pseudo-bundles.

4.1 Direct sum

Let $\pi_1 : V_1 \rightarrow X$ and $\pi_2 : V_2 \rightarrow X$ be two diffeological vector pseudo-bundles over the same base space X . Suppose that each of them can be endowed with a connection; let

$$\nabla^1 : C^\infty(X, V_1) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V_1) \quad \text{and} \quad \nabla^2 : C^\infty(X, V_2) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V_2).$$

Consider the direct sum pseudo-bundle $\pi_1 \oplus \pi_2 : V_1 \oplus V_2 \rightarrow X$; let

$$\text{pr}_{V_1} : V_1 \oplus V_2 \rightarrow V_1 \quad \text{and} \quad \text{pr}_{V_2} : V_1 \oplus V_2 \rightarrow V_2$$

be the standard direct sum projections, and let

$$\text{Incl}_{V_1} : V_1 \cong V_1 \oplus \{0\} \hookrightarrow V_1 \oplus V_2 \quad \text{and} \quad \text{Incl}_{V_2} : V_2 \cong \{0\} \oplus V_2 \hookrightarrow V_1 \oplus V_2$$

be the obvious inclusions. These maps are smooth by the definition of the diffeology on a direct sum of pseudo-bundles (see [17]).

Definition 4.1. *The direct sum of the connections ∇^1 and ∇^2 is the operator*

$$\nabla^1 \oplus \nabla^2 : C^\infty(X, V_1 \oplus V_2) \rightarrow C^\infty(X, \Lambda^1(X) \otimes (V_1 \oplus V_2))$$

defined as follows. Let $s \in C^\infty(X, V_1 \oplus V_2)$ be an arbitrary section; denote by $s_1 := \text{pr}_{V_1} \circ s$ and $s_2 := \text{pr}_{V_2} \circ s$. We define

$$(\nabla^1 \oplus \nabla^2)s = (\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_1}) \circ (\nabla^1 s_1) + (\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_2}) \circ (\nabla^2 s_2).$$

The final sum is of course taken in $\Lambda^1(X) \otimes (V_1 \oplus V_2)$. The following then is an easy analogue of the standard fact.

Proposition 4.2. *Let X be a diffeological space, let $\pi_1 : V_1 \rightarrow X$ and $\pi_2 : V_2 \rightarrow X$ be two diffeological vector pseudo-bundles over it, and let ∇^1 and ∇^2 be connections on V_1 and V_2 respectively. Then $\nabla^1 \oplus \nabla^2$ is well-defined and is a connection on $V_1 \oplus V_2$.*

Proof. The linearity property and the Leibnitz rule are established exactly as in the standard case, so we do not spell that out. What we really need to prove is that $\nabla^1 \oplus \nabla^2$ is well-defined as a map into $C^\infty(X, \Lambda^1(X) \otimes (V_1 \oplus V_2))$, and that it is smooth as a map $C^\infty(X, V_1 \oplus V_2) \rightarrow C^\infty(X, \Lambda^1(X) \otimes (V_1 \oplus V_2))$ for the respective functional diffeologies. Now, the former amounts to showing that for every section $s \in C^\infty(X, V_1 \oplus V_2)$ we have that $(\nabla^1 \oplus \nabla^2)s$ is a smooth map $X \rightarrow \Lambda^1(X) \otimes (V_1 \oplus V_2)$.

Let $p : U \rightarrow X$ be a plot of X ; then $s \circ p$ is a plot of $V_1 \oplus V_2$. We can assume that U is small enough so that $s \circ p = q_1 \oplus q_2$ for $q_i : U \rightarrow V_i$ a plot of V_i . Observe also that $s_i = \text{pr}_{V_i} \circ s \in C^\infty(X, V_i)$ and that by construction $s \circ p = q_1 \oplus q_2 = (s_1 \circ p) \oplus (s_2 \circ p)$. This allows us to obtain the following form for $(\nabla^1 \oplus \nabla^2)s \circ p$:

$$(\nabla^1 \oplus \nabla^2)s \circ p = (\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_1}) \circ (\nabla^1 s_1)p + (\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_2}) \circ (\nabla^2 s_2)p.$$

Now, $(\nabla^1 s_1)p$ is a plot of $\Lambda^1(X) \otimes V_1$ and $(\nabla^2 s_2)p$ is a plot of $\Lambda^1(X) \otimes V_2$, hence $(\nabla^1 \oplus \nabla^2)s \circ p$ is a plot of $\Lambda^1(X) \otimes (V_1 \oplus V_2)$, and since p is (a restriction of) any plot, this means that $(\nabla^1 \oplus \nabla^2)s$ is smooth.

We now need to show that $\nabla^1 \oplus \nabla^2 : C^\infty(X, V_1 \oplus V_2) \rightarrow C^\infty(X, \Lambda^1(X) \otimes (V_1 \oplus V_2))$ is smooth. Let $q : U \rightarrow C^\infty(X, V_1 \oplus V_2)$ be a plot for the functional diffeology on $C^\infty(X, V_1 \oplus V_2)$. Observe that each $u \mapsto \text{pr}_{V_i} \circ q(u)$ is a plot of $C^\infty(X, V_i)$ for $i = 1, 2$; write q_i for $\text{pr}_{V_i} \circ q$. We then have by construction that

$$(\nabla^1 \oplus \nabla^2)q(u) = (\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_1}) \circ (\nabla^1 q_1(u)) + (\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_2}) \circ (\nabla^2 q_2(u)).$$

It remains to notice that each $u \mapsto \nabla^i q_i(u)$ is by assumption plots of $C^\infty(X, \Lambda^1(X) \otimes V_i)$, and that the post-composition with the fixed map $\text{Id}_{\Lambda^1(X)} \otimes \text{Incl}_{V_i}$ induces a smooth map $C^\infty(X, \Lambda^1(X) \otimes V_i) \rightarrow C^\infty(X, \Lambda^1(X) \otimes (V_1 \oplus V_2))$, to conclude that $u \mapsto (\nabla^1 \oplus \nabla^2)q(u)$ is a plot of the latter, which yields the final claim. \square

4.2 Tensor product

The case of the tensor product is analogous. Let again X be a diffeological space, and let $\pi_1 : V_1 \rightarrow X$ and $\pi_2 : V_2 \rightarrow X$ be two diffeological vector pseudo-bundles over it. Consider the corresponding tensor product pseudo-bundle $\pi_1 \otimes \pi_2 : V_1 \otimes V_2 \rightarrow X$. Let also ∇^1 and ∇^2 be connections on V_1 and V_2 respectively.

Definition 4.3. *The tensor product of the connections ∇^1 and ∇^2 is the operator*

$$\nabla^\otimes : C^\infty(X, V_1 \otimes V_2) \rightarrow C^\infty(X, \Lambda^1(X) \otimes V_1 \otimes V_2)$$

given by

$$\nabla^\otimes := \nabla^1 \otimes \text{Id}_{C^\infty(X, V_2)} + \text{Id}_{C^\infty(X, V_1)} \otimes \nabla^2.$$

The following then is a complete analogue of both the standard statement and of Proposition 4.2, so we omit the proof.

Proposition 4.4. *Let X be a diffeological space, let $\pi_1 : V_1 \rightarrow X$ and $\pi_2 : V_2 \rightarrow X$ be two diffeological vector pseudo-bundles over it, and let ∇^1 and ∇^2 be connections on V_1 and V_2 respectively. Then ∇^\otimes is well-defined and is a connection on $V_1 \otimes V_2$.*

4.3 Connections by duality

Regarding the case of the pseudo-bundle dual to one equipped with a connection, we only give some preliminary indications of potential problems, essentially leaving the question in the open. Recall first the standard notion. Let $E \rightarrow M$ be a usual smooth vector bundle of finite rank, and let ∇ be a connection on E . The dual connection ∇^* on E^* is then determined, by setting for an arbitrary local basis $\{e_i\}$ of E and the corresponding dual basis $\{e^i\}$, that

$$0 = d(e^i, e_j) = (\nabla^* e^i, e_j) + (e^i, \nabla e_j).$$

Consider now the differences in the diffeological case. There seem to be two main items involved in the existence of these differences: one is the potential absence of local triviality, the other is the fact that pseudo-metrics are not Riemannian metrics, that is, they do not in general endow fibres with scalar products. The implication of the latter point is that the dual (obtained as usual via the corresponding pairing map) of a local basis in general is not a basis; furthermore, it is not immediately clear whether one can always be extracted from it. Below we elaborate on these two points.

4.3.1 The local triviality issues

The best possible counterpart, in the case of pseudo-bundles, for the notion of locality is the range of a plot. Indeed, there is not a standard local model for diffeological spaces (and therefore neither there is for pseudo-bundles), akin to, in the case of usual smooth vector bundles, the trivial bundle over an open disc: an open neighborhood (more precisely, a D-open one) can be of any shape.

In the finite-dimensional case, a pseudo-bundle $\pi : V \rightarrow X$ being locally trivial along a given plot $p : U \rightarrow X$ is equivalent precisely to the existence of sections s_1, \dots, s_n such that for every smooth local section $s \in C^\infty(p(U), \pi^{-1}(p(U)))$ there exist $f_1, \dots, f_n \in C^\infty(p(U), \mathbb{R})$ such that for all $u \in U$ we have $s(p(u)) = f_1(p(u))s_1(p(u)) + \dots + f_n(p(u))s_n(p(u))$. The true (potential) difference with respect to the standard case is that in the diffeological case, the number of sections needed to form a basis over a single $p(U)$ may vary.

4.3.2 Taking into account the *a priori* degeneracy of pseudo-metrics

Here is a naive explanation of the possible problem. Let $\pi : V \rightarrow X$ be a diffeological vector pseudo-bundle, and let g be a pseudo-metric on it. As already mentioned, it is sufficient to assume that it admits a global basis; let $\{s_i : X \rightarrow V\}_{i=1}^n$ be one. The standard construction of the dual basis yields

$$V^* \ni s^i(x), \quad s^i(x)(\cdot) = g(x)(s_i(x), \cdot) \text{ for all } x \in X.$$

What *a priori* may happen is that for a specific element s_i of the initial basis the corresponding $s^i(x)$ will be non-zero for some x , and that it will be the zero function for other x .

Lemma 4.5. *Let $\pi : V \rightarrow X$ be a finite-dimensional diffeological vector pseudo-bundle that admits a pseudo-metric and a global basis $\{s_i\}$ of smooth sections. Then the following two conditions are equivalent:*

1. *There exists a smooth basis $\{s_i\}$ of $C^\infty(X, V)$ such that the collection $\{s^i\}$ of the duals of all s_i 's contains a global basis of smooth sections of $\pi^* : V^* \rightarrow X$, and*
2. *The characteristic sub-bundle V_0 of V splits off as a smooth direct summand in V .*

Proof. Assume 1. Let g be a pseudo-metric on V , and suppose that the ordering of $\{s_i\}$ is such that s^1, \dots, s^m with $m \leq n$ form a basis of V^* . Since $g(x)$ is a pseudo-metric in the sense of diffeological vector spaces, there is a smooth direct sum decomposition of the fibre V_x of V at x as $V_x = (V_0)_x \oplus (V_1)_x$, where $(V_0)_x \leq V_x$ is the characteristic subspace, and $(V_1)_x$ is its smooth direct complement. By definition $\bigcup_{x \in X} (V_0)_x$ is the characteristic sub-bundle V_0 of V ; since each $(V_1)_x$ is a vector subspace in the fibre, the union $V_1 := \bigcup_{x \in X} (V_1)_x$ of all of them, endowed with the subset diffeology, is a sub-bundle of V . Finally, $V_0 \oplus V_1$ has the same underlying space as V , however *a priori* its diffeology could be strictly weaker than that of V ; what we wish to show now is that these two diffeologies actually coincide (that is, that $V = V_0 \oplus V_1$ is a smooth decomposition).

Let $\text{pr}_i : V \rightarrow V_i$ for $i = 0, 1$ be the two projections associated to this direct sum decomposition. Consider $s'_1 = \text{pr}_0 \circ s_1, \dots, s'_m = \text{pr}_0 \circ s_m$; these are sections of V_0 . Moreover, their being smooth sections is equivalent to what we wish to establish, V_0 splitting off as a smooth direct summand.

Observe that $\Phi_g(s'_i) = s^i$ for all $i = 1, \dots, m$. Since the restriction of Φ_g onto V_0 is bijective onto V^* , we have that $s'_i = \Phi_g^{-1}(s^i)$, so our desired claim is equivalent to Φ_g^{-1} being smooth, and this is equivalent to g^* being a pseudo-metric. This last is quite evident from the assumption in 1.

Assume now 2. Let $V = V_0 \oplus V_1$ be the smooth decomposition that exists by this assumption; recall that each fibre of the sub-bundle V_1 is an isotropic subspace of the relevant $g(x)$. It now suffices to choose any smooth basis $\{s_1, \dots, s_m\}$ for $C^\infty(X, V_0)$ and any smooth basis $\{s_{m+1}, \dots, s_n\}$ for $C^\infty(X, V_1)$; putting the two together, we obtain a smooth basis $\{s_1, \dots, s_n\}$ satisfying the desired property. Indeed, by construction s^{m+1}, \dots, s^n are all zero functions; furthermore, s^1, \dots, s^m form a basis of V^* due to $\Phi_g|_{V_0}$ being a bijection with V^* . \square

What may happen if 2 is not satisfied is this. Let $\{s_i\}_{i=1}^n$ be any smooth basis of $C^\infty(X, V)$, and suppose for simplicity that all fibres of V^* have dimension $n-1$. It might then happen that all $s^i = \Phi_g \circ s_i$ are non-zero as maps $X \rightarrow V$, but for every $x \in X$ there is an index $i(x) \in \{1, \dots, n\}$ such that $\{s^1(x), \dots, s^n(x)\} \setminus \{s^{i(x)}(x)\}$ is a basis of the vector space $(\pi^*)^{-1}(x) \subset V^*$, and the function $X \ni x \mapsto i(x)$ is a non-constant one. It is not quite clear whether this really represents an obstacle to constructing the dual connection via the standard procedure; for the moment we limit ourselves to the description of the situation as just given, leaving the rest for future work.

5 Diffeological gluing and connections

Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing between them. Suppose furthermore that each of them can be endowed with a diffeological connection; let ∇^1 and ∇^2 be connections on V_1 and V_2 respectively. In this section we consider how, under specific assumptions on these connections, we can obtain a connection on $V_1 \cup_{\tilde{f}} V_2$; the assumptions necessary take form, once again, of an appropriate compatibility notion.

5.1 The pullback map f^* between the sub-bundles of $\Lambda^1(X_2)$ and $\Lambda^1(X_1)$

Any connection on $V_1 \cup_{\tilde{f}} V_2$ has the form of an operator $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2) \rightarrow C^\infty(X_1 \cup_f X_2, \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2))$. In order to describe such an operator in terms of two operators of form $C^\infty(X_1, V_1) \rightarrow C^\infty(X_1, \Lambda^1(X_1) \otimes V_1)$ and $C^\infty(X_2, V_2) \rightarrow C^\infty(X_2, \Lambda^1(X_2) \otimes V_2)$, we are going to need an appropriate notion of a pullback map between certain subsets of $\Lambda^1(X_2)$ and $\Lambda^1(X_1)$.

5.1.1 $f^* : \Lambda^1(X_2) \rightarrow \Lambda^1(X_1)$ for a smooth map $f : X_1 \rightarrow X_2$

The case when f is defined on the entire X_1 is the simpler one; we consider it first. Recall that we already have the notion of a pullback map $f^* : \Omega^1(X_2) \rightarrow \Omega^1(X_1)$. together with f^{-1} it gives a pseudo-bundle map between the trivial bundles $X_2 \times \Omega^1(X_2)$ and $X_1 \times \Omega^1(X_1)$, which acts (in an obvious manner) by

$$(f^{-1}, f^*)(x_2, \omega_2) = (f^{-1}(x_1), f^*\omega_2).$$

Proposition 5.1. *Let X_1 and X_2 be two diffeological spaces, and let $f : X_1 \rightarrow X_2$ be a diffeomorphism. Then $f^* : \Omega^1(X_2) \rightarrow \Omega^1(X_1)$ induces a well-defined pullback map $\Lambda^1(X_2) \rightarrow \Lambda^1(X_1)$.*

Proof. Let $x_2 \in X_2$, and let $\omega_2 \in \Omega^1(X_2)$ be a form vanishing at x_2 . We wish to know whether $f^*\omega_2$ vanishes at $f^{-1}(x_2)$. Consider a plot p of X_1 centered at this point; then trivially $f \circ p$ is a plot of X_2 centered at x_2 . Furthermore, we have

$$f^*\omega_2(p)(f^{-1}(x_2)) = \omega_2(f \circ p)(x_2) = 0.$$

Since p and x_2 are arbitrary, we conclude that $f^*(\Omega^1_{x_2}(X_2)) \subseteq \Omega^1_{f^{-1}(x_2)}(X_1)$. In fact, since f is a diffeomorphism, we can apply the analogous reasoning to f^{-1} , obtaining

$$f^*(\Omega^1_{x_2}(X_2)) = \Omega^1_{f^{-1}(x_2)}(X_1).$$

It follows from what has been established in the previous paragraph it is obvious that f^* yields a pseudo-bundle map between the two sub-bundles (of $X_2 \times \Omega^1(X_2)$ and $X_1 \times \Omega^1(X_1)$ respectively) consisting of vanishing forms:

$$X_2 \times \Omega^1(X_2) \supseteq \left(\bigcup_{x_2 \in X_2} \{x_2\} \times \Omega_{x_2}^1(X_2) \right) \rightarrow \left(\bigcup_{x_1 \in X_1} \{x_1\} \times \Omega_{x_1}^1(X_1) \right) \subseteq X_1 \times \Omega^1(X_1);$$

this pseudo-bundle map covers f^{-1} . Therefore f^* descends to a well-defined map on the quotient pseudo-bundles

$$(X_2 \times \Omega^1(X_2)) / \left(\bigcup_{x_2 \in X_2} \{x_2\} \times \Omega_{x_2}^1(X_2) \right) \rightarrow (X_1 \times \Omega^1(X_1)) / \left(\bigcup_{x_1 \in X_1} \{x_1\} \times \Omega_{x_1}^1(X_1) \right).$$

It remains to recall our prior observation that these quotients are precisely the corresponding Λ^1 -bundles, that is,

$$(X_2 \times \Omega^1(X_2)) / \left(\bigcup_{x_2 \in X_2} \{x_2\} \times \Omega_{x_2}^1(X_2) \right) \cong \Lambda^1(X_2), \quad (X_1 \times \Omega^1(X_1)) / \left(\bigcup_{x_1 \in X_1} \{x_1\} \times \Omega_{x_1}^1(X_1) \right) \cong \Lambda^1(X_1),$$

whence the claim. \square

The final conclusion is that the pullback map f^* is well-defined as a map $f^* : \Lambda^1(X_2) \rightarrow \Lambda^1(X_1)$; we do not introduce a separate notation for it, since it will always be clear from the context whether we mean the pullback map defined between the $\Omega^1(X_i)$'s or the $\Lambda^1(X_i)$'s.

5.1.2 The map f^* for $f : X_1 \supset Y \rightarrow f(Y) \subseteq X_2$

Let us now consider the general case. Suppose that f is defined on a proper subset of X_1 , and that its image is an *a priori* proper subset of X_2 . There is of course again a well-defined pullback map f^* but it is not defined on the whole $\Lambda^1(X_2)$. We shall relate the domain and the range of f^* to certain subsets of $\Lambda^1(X_2)$ and $\Lambda^1(X_1)$, and use it for an alternative description of the compatibility of elements of $\Lambda^1(X_i)$, in a form suitable for defining subsequently the compatibility of connections on pseudo-bundles over X_1 and X_2 .

The properties of the pullback map $f^* : \Lambda^1(f(Y)) \rightarrow \Lambda^1(Y)$ Considering Y and $f(Y)$ as diffeological spaces for their natural subset diffeologies, it follows from Proposition 5.1 that there is the pullback map

$$f^* : \Lambda^1(f(Y)) \rightarrow \Lambda^1(Y);$$

its precursor is the pullback map $f^* : \Omega^1(f(Y)) \rightarrow \Omega^1(Y)$. There is a natural commutativity between these two versions of f^* expressed by

$$\pi_Y^{\Omega, \Lambda} \circ (f^{-1}, f^*) = f^* \circ \pi_{f(Y)}^{\Omega, \Lambda},$$

where the f^* on the left is the Ω^1 -version of the pullback map, while f^* on the right is the Λ^1 -version, and where

- $\pi_Y^{\Omega, \Lambda} : Y \times \Omega^1(Y) \rightarrow \Lambda^1(Y)$ is the defining projection of $\Lambda^1(Y)$, and
- $\pi_{f(Y)}^{\Omega, \Lambda} : f(Y) \times \Omega^1(f(Y)) \rightarrow \Lambda^1(f(Y))$ is the defining projection of $\Lambda^1(f(Y))$.

The two compositions are defined on $f(Y) \times \Omega^1(f(Y))$.

The pullback map f^* and the compatibility of elements of $\Omega^1(X_1)$ and $\Omega^1(X_2)$ Consider now the natural inclusions $i : Y \hookrightarrow X_1$ and $j : f(Y) \hookrightarrow X_2$; these give rise to the pullback maps $i^* : \Omega^1(X_1) \rightarrow \Omega^1(Y)$ and $j^* : \Omega^1(X_2) \rightarrow \Omega^1(f(Y))$ (note that in general they may not be surjective).

Lemma 5.2. *The map $(i^{-1}, i^*) : i(Y) \times \Omega^1(X_1) \rightarrow Y \times \Omega^1(Y)$ descends to a well-defined map $i_\Lambda^* : \Lambda^1(X_1) \supset (\pi_1^\Lambda)^{-1}(Y) \rightarrow \Lambda^1(Y)$; in particular,*

$$\pi_Y^{\Omega, \Lambda} \circ (i^{-1}, i^*) = i_\Lambda^* \circ \pi_1^{\Omega, \Lambda}|_{i(Y) \times \Omega^1(X_1)}.$$

Proof. It suffices to show that i^* preserves the vanishing of 1-forms. Let $y \in Y$ and let $\omega_1 \in \Omega_y^1(X_1)$. We need to show that $i^*\omega_1$ vanishes at y , so let $p : U \rightarrow Y$ be a plot centered at y , $p(0) = y$. Let us calculate $(i^*\omega_1)(p)(0) = \omega_1(i \circ p)(0) = 0$, because by assumption ω_1 vanishes at $y/i(y)$ and $i \circ p$ is obviously a plot of X_1 centered at y . Thus, $i^*(\Omega_y^1(X_1)) \subseteq \Omega_y^1(Y)$, whence the claim. \square

A completely analogous statement is also true for the other factor.

Lemma 5.3. *The map $(j^{-1}, j^*) : j(f(Y)) \times \Omega^1(X_2) \rightarrow f(Y) \times \Omega^1(f(Y))$ descends to a well-defined map $j_\Lambda^* : \Lambda^1(X_2) \supset (\pi_2^\Lambda)^{-1}(f(Y)) \rightarrow \Lambda^1(f(Y))$ such that*

$$\pi_{f(Y)}^{\Omega, \Lambda} \circ (j^{-1}, j^*) = j_\Lambda^* \circ \pi_2^{\Omega, \Lambda}|_{j(f(Y)) \times \Omega^1(X_2)}.$$

Recall now ([12]) that ω_1 and ω_2 are compatible if and only if

$$i^*\omega_1 = f^*(j^*\omega_2),$$

where f^* is the Ω^1 -version of the pullback map, $f^* : \Omega^1(f(Y)) \rightarrow \Omega^1(Y)$. Let $y \in Y$ be arbitrary, and let $\alpha_1 = \omega_1 + \Omega_y^1(X_1) \in \Lambda_y^1(X_1)$ and $\alpha_2 = \omega_2 + \Omega_{f(y)}^1(X_2) \in \Lambda_{f(y)}^1(X_2)$ be two compatible elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$. The compatibility condition for such elements means that any pair (ω'_1, ω'_2) , where $\omega'_1 \in \alpha_1$ and $\omega'_2 \in \alpha_2$, is a compatible one, that is, by the aforementioned criterion

$$i^*(\omega'_1) = f^*(j^*(\omega'_2)) \text{ for all } \omega'_1 \in \alpha_1 \text{ and } \omega'_2 \in \alpha_2.$$

Proposition 5.4. *Two elements $\alpha_1 \in \Lambda_y^1(X_1)$ and $\alpha_2 \in \Lambda_{f(y)}^1(X_2)$ are compatible if and only if the following is true:*

$$i_\Lambda^*\alpha_1 = f^*(j_\Lambda^*\alpha_2).$$

Proof. This follows from Lemmas 5.2 and 5.2. Indeed, by construction there exist $\omega_1 \in \Omega^1(X_1)$ and $\omega_2 \in \Omega^1(X_2)$ such that $\alpha_1 = \pi_1^{\Omega, \Lambda}(y, \omega_1)$ and $\alpha_2 = \pi_2^{\Omega, \Lambda}(f(y), \omega_2)$; furthermore, α_1 and α_2 are compatible if and only if any two such ω_1 and ω_2 are compatible. By Lemma 5.2, Lemma 5.2, and the construction of the pullback map $f^* : \Omega^1(f(Y)) \rightarrow \Omega^1(Y)$ we then have

$$i_\Lambda^*\alpha_1 = \pi_Y^{\Omega, \Lambda}(y, i^*\omega_1) \text{ and } f^*(j_\Lambda^*\alpha_2) = f^*(\pi_{f(Y)}^{\Omega, \Lambda}(f(y), j^*\omega_2)) = \pi_Y^{\Omega, \Lambda}(y, f^*(j^*\omega_2)).$$

These expressions are equal for all choices of $\omega_i \in \alpha_i$ if and only if α_1^* and α_2^* are compatible, by the definition of compatibility of elements of $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$, and the aforementioned criterion of compatibility of elements of $\Omega^1(X_1)$ and $\Omega^1(X_2)$; this completes the proof. \square

Proposition 5.4 provides our main criterion for compatibility of elements in $\Lambda^1(X_1)$ and $\Lambda^1(X_2)$, in the form suitable for defining compatible connections (which is one of our main goals); we do this in the section immediately below.

5.2 The induced connection on $V_1 \cup_{\tilde{f}} V_2$

Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, and let (\tilde{f}, f) be a gluing between them such that both \tilde{f} and f are diffeomorphisms of their domains with their images. Given a connection ∇^1 on V_1 and a connection ∇^2 on V_2 , we might be able to obtain out of them an induced connection on $V_1 \cup_{\tilde{f}} V_2$; for this to be feasible, the two connections must be subject to some restrictions, which are expressed by the appropriate compatibility notion. After describing this notion, we provide the construction of the induced connection, proving that it is indeed a connection.

5.2.1 The definition of compatible connections

The idea behind the compatibility notion for connections ∇^1 and ∇^2 on V_1 and V_2 is as follows. Let $s_1 \in C^\infty(X_1, V_1)$ and $s_2 \in C^\infty(X_2, V_2)$; let $y \in Y$. Then $(\nabla^1 s_1)(y) = \sum \alpha^i \otimes v_i$ for some $\alpha^i \in \Lambda_y^1(X_1)$ and $v_i \in V_1$; likewise, $(\nabla^2 s_2)(f(y)) = \sum_j \beta^j \otimes w_j$ for $\beta^j \in \Lambda_{f(y)}^1(X_2)$ and $w_j \in V_2$. Now, $(\nabla^1 s_1)(y)$ and $(\nabla^2 s_2)(f(y))$ can be easily identified with certain elements of

$$\left(\Lambda_y^1(X_1) \oplus \Lambda_{f(y)}^1(X_2) \right) \otimes \left(V_1 \cup_{\tilde{f}} V_2 \right)_{i_2(f(y))};$$

this direct sum contains the corresponding fibre of $\Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$ as a (generally proper) subspace.

Definition 5.5. Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let f and \tilde{f} be maps defining a gluing of the former to the latter, each of which is a diffeomorphism of its domain with its image, and let Y be the domain of definition of f . Let ∇^1 be a connection on V_1 , and let ∇^2 be a connection on V_2 . We say that ∇^1 and ∇^2 are **compatible** if for any pair $s_1 \in C^\infty(X_1, V_1)$ and $s_2 \in C^\infty(X_2, V_2)$ of compatible sections, and for any $y \in Y$, we have

$$\left((i_\Lambda^* \otimes \tilde{f}) \circ (\nabla^1 s_1) \right) (y) = \left((f^* j_\Lambda^*) \otimes Id_{V_2} \right) \circ (\nabla^2 s_2) (f(y)).$$

We can now better formulate our reason for defining the compatibility of connections in the way we just, by stating the following.

Proposition 5.6. Let ∇^1 and ∇^2 be compatible connections on V_1 and V_2 respectively. Then for any compatible sections $s_1 \in C^\infty(X_1, V_1)$ and $s_2 \in C^\infty(X_2, V_2)$ and for any $y \in Y$ we have

$$\left((Id_{\Lambda_y^1(X_1)} \otimes \tilde{f}) \circ (\nabla^1 s_1) \right) (y) \oplus (\nabla^2 s_2)(f(y)) \in \left(\Lambda_y^1(X_1) \oplus_{comp} \Lambda_{f(y)}^1(X_2) \right) \otimes V_2.$$

Proof. The statement of the proposition expresses the fact that two elements $\alpha_1 \in \Lambda_y^1(X_1)$ and $\alpha_2 \in \Lambda_{f(y)}^1(X_2)$ are compatible if and only if $i_\Lambda^* \alpha_1 = f^* (j_\Lambda^* \alpha_2)$, and this is the content of Proposition 5.4. \square

5.2.2 The induced connection ∇^\cup

Let the two pseudo-bundles V_1 and V_2 be endowed with connections ∇^1 and ∇^2 , and assume that these connections are compatible in the sense of Definition 5.5. We shall first describe the connection on $V_1 \cup_{\tilde{f}} V_2$ induced by them and then prove that it is, indeed, a connection. Recall that all throughout we assume that all gluings are along diffeomorphisms.

The definition of ∇^\cup Let $x \in i_2(f(Y))$, and let $\text{Incl}_{\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1)}$ and $\text{Incl}_{\Lambda_{i_2^{-1}(x)}^1(X_2)}$ stand for the two standard inclusions

$$\text{Incl}_{\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1)} : \Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \cong \Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \oplus \{0\} \hookrightarrow \Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \oplus \Lambda_{i_2^{-1}(x)}^1(X_2),$$

$$\text{Incl}_{\Lambda_{i_2^{-1}(x)}^1(X_2)} : \Lambda_{i_2^{-1}(x)}^1(X_2) \cong \{0\} \oplus \Lambda_{i_2^{-1}(x)}^1(X_2) \hookrightarrow \Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1) \oplus \Lambda_{i_2^{-1}(x)}^1(X_2).$$

The connection ∇^\cup is then defined as follows.

Definition 5.7. Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let (\tilde{f}, f) be a gluing between them, and let ∇^1 and ∇^2 be compatible connections on V_1 and V_2 respectively. The **induced connection** ∇^\cup on $V_1 \cup_{\tilde{f}} V_2$ is the operator defined as follows. Let $s \in C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ be a section. Since f and \tilde{f} are diffeomorphisms, it has a unique presentation of form $s = s_1 \cup_{(f, \tilde{f})} s_2$ for $s_1 \in C^\infty(X_1, V_1)$ and $s_2 \in C^\infty(X_2, V_2)$. Then

$$(\nabla^\cup s)(x) = \begin{cases} \begin{aligned} & \left((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1 \right) \left((\nabla^1 s_1)(i_1^{-1}(x)) \right) & \text{for } x \in i_1(X_1 \setminus Y), \\ & \left((\tilde{\rho}_2^\Lambda)^{-1} \otimes j_2 \right) \left((\nabla^2 s_2)(i_2^{-1}(x)) \right) & \text{for } x \in i_2(X_2 \setminus f(Y)), \end{aligned} \\ \left(\text{Incl}_{\Lambda_{f^{-1}(i_2^{-1}(x))}^1(X_1)} \otimes (j_2 \circ \tilde{f}) \right) \left((\nabla^1 s_1)(f^{-1}(i_2^{-1}(x))) \right) \oplus \\ \oplus \left(\text{Incl}_{\Lambda_{i_2^{-1}(x)}^1(X_2)} \otimes j_2 \right) \left((\nabla^2 s_2)(i_2^{-1}(x)) \right) & \text{for } x \in i_2(f(Y)). \end{cases}$$

Proof that ∇^\cup is a connection Two items need to be checked: one, that ∇^\cup is well-defined as a map

$$C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2) \rightarrow C^\infty(X_1 \cup_f X_2, \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)),$$

and two, that it is smooth for the functional diffeologies on these two spaces.

Lemma 5.8. *For every section $s \in C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ and for every $x \in X_1 \cup_f X_2$ we have $(\nabla^\cup s)(x) \in \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$.*

Proof. We shall consider separately the cases when $x \in i_1(X_1 \setminus Y)$, $x \in i_2(X_2 \setminus f(Y))$, and $x \in i_2(f(Y))$; the former two are actually analogous, so it suffices to treat just one of them. Let $x \in i_1(X_1 \setminus Y)$. Then by construction

$$(\nabla^\cup s)(x) = ((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1) ((\nabla^1 s_1)(i_1^{-1}(x))).$$

Since ∇^1 is a connection on V_1 , we have that $(\nabla^1 s_1)(i_1^{-1}(x)) \in \Lambda^1(X_1) \otimes V_1$. Its image under the map $(\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1$ belongs to $\Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$ by the definition of this map. As just mentioned, the case of $x \in i_2(X_2 \setminus f(Y))$ is completely analogous.

Let $x \in i_2(f(Y))$. To abbreviate the lengthy expression for $(\nabla^\cup s)(x)$, let us write $y := f^{-1}(i_2^{-1}(x))$ and $y' = i_2^{-1}(x)$. Since the expression for $(\nabla^\cup s)(x)$ involves both $\nabla^1 s_1$ and $\nabla^2 s_2$, and s_1 and s_2 are compatible, we can draw the desired conclusion from Proposition 5.6. \square

Thus, $\nabla^\cup s$ is always well-defined as a map $X_1 \cup_f X_2 \rightarrow \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$. Next, we need to show that it is actually smooth.

Lemma 5.9. *For every section $s \in C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ the section $\nabla^\cup s : X_1 \cup_f X_2 \rightarrow \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$ is smooth, that is, $\nabla^\cup s \in C^\infty(X_1 \cup_f X_2, \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2))$.*

Proof. Showing that $\nabla^\cup s$ is smooth amounts to showing that for any arbitrary plot $p : U \rightarrow X_1 \cup_f X_2$ of $X_1 \cup_f X_2$ the composition $(\nabla^\cup s) \circ p$ is a plot of $\Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$. As usual, we can assume that U is connected, so that p lifts to either a plot p_1 of X_1 or to a plot p_2 of X_2 ; accordingly, for any $u \in U$ either

$$p(u) = \begin{cases} i_1(p_1(u)) & \text{if } p_1(u) \in X_1 \setminus Y, \\ i_2(f(p_1(u))) & \text{if } p_1(u) \in Y \end{cases} \quad \text{or } p(u) = i_2(p_2(u)).$$

Assume first that p lifts to p_1 . Then

$$(\nabla^\cup s)(p(u)) = \begin{cases} ((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1) ((\nabla^1 s_1)(p_1(u))) & \text{if } p_1(u) \in X_1 \setminus Y, \\ \left(\text{Incl}_{\Lambda^1_{f^{-1}(i_2^{-1}(x))}(X_1)} \otimes (j_2 \circ \tilde{f}) \right) ((\nabla^1 s_1)(p_1(u))) \oplus \\ \oplus \left(\text{Incl}_{\Lambda^1_{i_2^{-1}(x)}(X_2)} \otimes j_2 \right) ((\nabla^2 s_2)(f(p_1(u)))) & \text{if } p_1(u) \in Y. \end{cases}$$

By Theorem 1.9 and the definition of the tensor product of diffeological vector pseudo-bundles, to check that this is a plot of $\Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$, it suffices to check that its composition with $\tilde{\rho}_1^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}$ is a plot of $\Lambda^1(X_1) \otimes (V_1 \cup_{\tilde{f}} V_2)$ and that the composition with $\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}$, where defined, is smooth as a map into $\Lambda^1(X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$, for the *subset* diffeology on $p_1^{-1}(Y) \subseteq U$.

The composition of $(\nabla^\cup s) \circ p$ with $\tilde{\rho}_1^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}$ has form $(\text{Id}_{\Lambda^1(X_1)} \otimes j_1) \circ (\nabla^1 s_1) \circ p_1$ at points of $p_1^{-1}(X_1 \setminus Y)$. Since over $i_2(f(Y))$ the map $\tilde{\rho}_1^\Lambda$ acts by the projection of the direct sum $\Lambda^1_{f^{-1}(x)}(X_1) \oplus \Lambda^1_{f^{-1}(x)}(X_2)$ onto its first factor, for $u \in p_1^{-1}(Y)$ this composition has form

$$\begin{aligned} & \left(\tilde{\rho}_1^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2} \right) \circ (\nabla^\cup s) \circ p = \\ & = \left(\tilde{\rho}_1^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2} \right) \circ \left(\text{Incl}_{\Lambda^1_{f^{-1}(i_2^{-1}(x))}(X_1)} \otimes (j_2 \circ \tilde{f}) \right) \circ (\nabla^1 s_1) \circ p_1 = (\text{Id}_{\Lambda^1(X_1)} \otimes (j_2 \circ \tilde{f})) \circ (\nabla^1 s_1) \circ p_1. \end{aligned}$$

Thus, the complete form of the composition under consideration is

$$\left(\tilde{\rho}_1^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}\right) \circ (\nabla^\cup s) \circ p = \begin{cases} (\text{Id}_{\Lambda^1(X_1)} \otimes j_1) \circ (\nabla^1 s_1) \circ p_1 & \text{for } u \text{ such that } p_1(u) \in X_1 \setminus Y, \\ (\text{Id}_{\Lambda^1(X_1)} \otimes (j_2 \circ \tilde{f})) \circ (\nabla^1 s_1) \circ p_1 & \text{for } u \text{ such that } p_1(u) \in Y. \end{cases}$$

Since $(\nabla^1 s_1) \circ p_1$ is a plot of $\Lambda^1(X_1) \otimes V_1$ by assumption, it suffices to recall that $\begin{cases} j_1 \\ j_2 \circ \tilde{f} \end{cases}$ is a smooth inclusion of V_1 into $V_1 \cup_{\tilde{f}} V_2$.

Let us now consider the composition $(\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\nabla^\cup s) \circ p$. This is defined only for u such that $p_1(u) \in Y$; using the definition of $\tilde{\rho}_2^\Lambda$, the restriction of this composition to $p_1^{-1}(Y) \subseteq U$ has form

$$\left((\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\nabla^\cup s) \circ p\right)|_{p_1^{-1}(Y)} = (\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\text{Incl}_{i_2^{-1}(x)}^{\Lambda^1(X_2)} \otimes j_2) \circ (\nabla^2 s_2) \circ (f \circ p_1).$$

We need to show that this is a plot relative to the subset diffeology on $p_1^{-1}(Y)$, that is, if $p'_1 : U' \rightarrow U$ is a usual smooth map whose range is contained in $p_1^{-1}(Y)$ then $(\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\nabla^\cup s) \circ (p \circ p'_1)$ must be a plot of $\Lambda^1(X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$. We then have

$$(\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\nabla^\cup s) \circ (p \circ p'_1) = (\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\text{Incl}_{i_2^{-1}(x)}^{\Lambda^1(X_2)} \otimes j_2) \circ (\nabla^2 s_2) \circ (f \circ p_1 \circ p'_1),$$

and it suffices to observe that $f \circ p_1 \circ p'_1$ is a plot of X_2 , since by assumption f is smooth, $p_1 \circ p'_1$ is a plot of X_1 by the axioms of diffeology, and its range is contained in Y by construction. Thus, it follows from the assumption on ∇^2 that $(\tilde{\rho}_2^\Lambda \otimes \text{Id}_{V_1 \cup_{\tilde{f}} V_2}) \circ (\nabla^\cup s) \circ (p \circ p'_1)$ is indeed a plot of $\Lambda^1(X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$, as wanted, which completes the treatment of the case when p lifts to a plot of X_1 .

If p lifts to a plot p_2 of X_2 , the proof is completely analogous, so we avoid spelling it out, ending the proof with this remark. \square

We shall check next the standard linearity properties of ∇^\cup .

Lemma 5.10. *The operator ∇^\cup is linear and satisfies the Leibnitz rule.*

Proof. All maps, as well as operations, involved in the construction of ∇^\cup are fibrewise additive, so the additivity of ∇^\cup is obvious. Let us check that ∇^\cup satisfies the Leibnitz rule. Let $h \in C^\infty(X_1 \cup_f X_2, \mathbb{R})$, and let $s \in C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$. Define $h_1 \in C^\infty(X_1, \mathbb{R})$ and $h_2 \in C^\infty(X_2, \mathbb{R})$ by

$$h_1(x_1) = \begin{cases} h(i_1(x_1)) & \text{if } x_1 \in X_1 \setminus Y, \\ h(i_2(f(x_1))) & \text{if } x_1 \in Y \end{cases}, \quad \text{and } h_2(x_2) = h(i_2(x_2)) \text{ for all } x_2 \in X_2.$$

Notice that this corresponds to the presentation of h as $h = h_1 \cup_f h_2$, already mentioned in Section 2.2.4. Recall also that by Theorem 2.29 s admits a presentation as $s = s_1 \cup_{(f, \tilde{f})} s_2$ for some $s_1 \in C^\infty(X_1, V_1)$ and $s_2 \in C^\infty(X_2, V_2)$, that in our present case (of gluing along two diffeomorphisms) are also uniquely defined. Finally, recall from Section 2.2.4 that

$$hs = (h_1 \cup_f h_2)(s_1 \cup_{(f, \tilde{f})} s_2) = (h_1 s_1) \cup_{(f, \tilde{f})} (h_2 s_2).$$

By assumption ∇^1 is a connection, so we have that $\nabla^1(h_1 s_1) = dh_1 \otimes s_1 + h_1(\nabla^1 s_1)$, and likewise, ∇^2 being a connection as well, we have that $\nabla^2(h_2 s_2) = dh_2 \otimes s_2 + h_2(\nabla^2 s_2)$. Thus, it suffices to check that

$$(\tilde{\rho}_1^\Lambda)^{-1}(dh_1(x)) = dh(i_1(x)) \text{ for all } x \in X_1 \setminus Y \quad \text{and} \quad (\tilde{\rho}_2^\Lambda)^{-1}(dh_2(x)) = dh(i_2(x))$$

to obtain the desired equality $\nabla^\cup(hs) = dh \otimes s + h(\nabla^\cup s)$. Let us consider the first of these equalities, in its equivalent form $\tilde{\rho}_1^\Lambda(dh(i_1(x))) = dh_1(x)$.

Recall that, as a section of $\Lambda^1(X_1 \cup_f X_2)$, the differential dh is defined by $dh(x) = \pi^{\Omega, \Lambda}(x, dh)$ for all $x \in X_1 \cup_f X_2$, where dh on the right stands for the element of $\Omega^1(X_1 \cup_f X_2)$ given by $dh(p) = d(h \circ p)$ for any plot p of $X_1 \cup_f X_2$. Thus, we can also write $dh(x) = dh + \Omega_x^1(X_1 \cup_f X_2)$. Likewise, dh_1 , as a section of $\Lambda^1(X_1)$, is given by $dh_1(x_1) = \pi_1^{\Omega, \Lambda}(x_1, dh_1)$, with, on the right, $dh_1 \in \Omega^1(X_1)$ being

given by $dh_1(p_1) = d(h_1 \circ p_1)$ for any plot p_1 of X_1 , and equivalently, $dh_1 : X_1 \rightarrow \Lambda^1(X_1)$ is given by $dh_1(x_1) = dh_1 + \Omega_{x_1}^1(X_1)$. Recalling now the standard induction $i_1 : X_1 \setminus Y \hookrightarrow X_1 \cup_f X_2$, and its extension $\tilde{i}_1 : X_1 \hookrightarrow X_1 \cup_f X_2$ that exists by the assumption on f , we have the pullback map $\tilde{i}_1^* : \Omega^1(X_1 \cup_f X_2) \rightarrow \Omega^1(X_1)$.

We now claim that \tilde{i}_1^* is a lift of $\tilde{\rho}_1^\Lambda$, *i.e.*, that the following is true:

$$\tilde{\rho}_1^\Lambda \circ \pi^{\Omega, \Lambda} = \pi_1^{\Omega, \Lambda} \circ (\tilde{i}_1^{-1}, \tilde{i}_1^*)$$

wherever this expression makes sense, that is, on the direct product $X_1 \times \Omega^1(X_1 \cup_f X_2)$. Putting everything together, we obtain

$$\tilde{\rho}_1^\Lambda(dh(\tilde{i}_1(x))) = \tilde{\rho}_1^\Lambda(\pi^{\Omega, \Lambda}(\tilde{i}_1(x), dh)) = \pi_1^{\Omega, \Lambda}(x, \tilde{i}_1^*(dh)) = \pi_1^{\Omega, \Lambda}(x, dh_1) = dh_1(x),$$

where we only need to check the equality $(\tilde{i}_1^*(dh)) = dh_1$, where $dh \in \Omega^1(X_1 \cup_f X_2)$ and $dh_1 \in \Omega^1(X_1)$. Indeed, let p_1 be a plot of X_1 ; then $(\tilde{i}_1^*(dh))(p_1) = dh(\tilde{i}_1 \circ p_1) = d(h \circ \tilde{i}_1 \circ p_1)$ by definition. Since $dh_1(p_1) = d(h_1 \circ p_1)$ and $h \circ \tilde{i}_1 = h_1$, we immediately obtain the desired conclusion. We have in fact obtained slightly more, namely, that the equalities stated hold on the entire domain of definition of $\tilde{\rho}_1^\Lambda$, that is, we have

$$\tilde{\rho}_1^\Lambda(dh(\tilde{i}_1(x))) = dh_1(x) \text{ for all } x \in X_1.$$

Observe furthermore that the case of $i_2(x)$ for $x \in X_2$ is treated in exactly the same manner, so we have that

$$\tilde{\rho}_2^\Lambda(dh(i_2(x))) = dh_2(x) \text{ for all } x \in X_2.$$

Let us now confront the two sides of the equality in the Leibnitz rule, considering

$$\nabla^\cup(hs) = \nabla^\cup \left((h_1 \cup_f h_2)(s_1 \cup_{(f, \tilde{f})} s_2) \right).$$

Let $x \in X_1 \cup_f X_2$; between the cases $x \in i_1(X_1 \setminus Y)$ and $x \in i_2(X_2 \setminus f(Y))$ it suffices to consider one, as they are symmetric. Let us consider $x \in i_1(X_1 \setminus Y)$:

$$\begin{aligned} (\nabla^\cup(hs))(x) &= ((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1) ((\nabla^1(h_1 s_1))(i_1^{-1}(x))) = ((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1) ((dh_1 \otimes s_1 + h_1 \nabla^1 s_1)(i_1^{-1}(x))) = \\ &= ((\tilde{\rho}_1^\Lambda)^{-1}(dh_1) \otimes s)(x) + ((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1) ((h_1 \nabla^1 s_1)(i_1^{-1}(x))) = (dh \otimes s)(x) + (h \nabla^\cup s)(x), \end{aligned}$$

as wanted. It thus remains to consider a point in $i_2(f(Y))$.

Let $x \in i_2(f(Y))$. Consider

$$\begin{aligned} &(\nabla^\cup(hs))(x) = \\ &= \left(\text{Incl}_{\Lambda^1_{f^{-1}(i_2^{-1}(x))}}(X_1) \otimes (j_2 \circ \tilde{f}) \right) ((\nabla^1(h_1 s_1))(f^{-1}(i_2^{-1}(x)))) \oplus \\ &\quad \oplus \left(\text{Incl}_{\Lambda^1_{i_2^{-1}(x)}}(X_2) \otimes j_2 \right) ((\nabla^2(h_2 s_2))(i_2^{-1}(x))) = \\ &= \left(\text{Incl}_{\Lambda^1_{f^{-1}(i_2^{-1}(x))}}(X_1) \otimes (j_2 \circ \tilde{f}) \right) ((dh_1 \otimes s_1 + h_1 \nabla^1 s_1)(f^{-1}(i_2^{-1}(x)))) \oplus \\ &\quad \oplus \left(\text{Incl}_{\Lambda^1_{i_2^{-1}(x)}}(X_2) \otimes j_2 \right) ((dh_2 \otimes s_2 + h_2 \nabla^2 s_2)(i_2^{-1}(x))) = \\ &= \left(\left(\text{Incl}_{\Lambda^1_{f^{-1}(i_2^{-1}(x))}}(X_1) + \text{Incl}_{\Lambda^1_{i_2^{-1}(x)}}(X_2) \right) (dh_1 + dh_2) \right) (x) \otimes s(x) + (h \nabla^\cup s)(x). \end{aligned}$$

It thus remains to check that for any $x \in i_2(f(Y))$ we have

$$dh(x) = \left(\left(\text{Incl}_{\Lambda^1_{f^{-1}(i_2^{-1}(x))}}(X_1) + \text{Incl}_{\Lambda^1_{i_2^{-1}(x)}}(X_2) \right) (dh_1 + dh_2) \right) (x).$$

This is equivalent to

$$\tilde{\rho}_1^\Lambda(dh(x)) = dh_1(f^{-1}(i_2^{-1}(x))) \quad \text{and} \quad \tilde{\rho}_2^\Lambda(dh(x)) = dh_2(i_2^{-1}(x)),$$

and this has already been established above, which completes the proof. \square

From the proof just finished, we can extract the following description of the differential of a function $h \in C^\infty(X_1 \cup_f X_2, \mathbb{R})$ in terms of the differentials of its factors.

Corollary 5.11. *The following is true:*

$$(d(h_1 \cup_f h_2))(x) = \begin{cases} (\tilde{\rho}_1^\Lambda)^{-1}(dh_1(i_1^{-1}(x))) & \text{if } x \in i_1(X_1 \setminus Y), \\ (\text{Incl}_{\Lambda_{f^{-1}(i_2^{-1}(x))}^1}(X_1) \oplus \text{Incl}_{\Lambda_{i_2^{-1}(x)}^1}(X_2))(dh_1(f^{-1}(i_2^{-1}(x))) \oplus dh_2(i_2^{-1}(x))) & \text{if } x \in i_2(f(Y)), \\ (\tilde{\rho}_2^\Lambda)^{-1}(dh_2(i_2^{-1}(x))) & \text{if } x \in i_2(X_2 \setminus f(Y)). \end{cases}$$

Proposition 5.12. *The operator ∇^\cup is smooth as a map*

$$\nabla^\cup : C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2) \rightarrow C^\infty(X_1 \cup_f X_2, \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2))$$

for the usual functional diffeologies on the two spaces.

Proof. Let $p : U \rightarrow C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ be a plot of $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$; we need to check that $u \mapsto \nabla^\cup(p(u))$ is a plot of $C^\infty(X_1 \cup_f X_2, \Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2))$. Since the latter has functional diffeology, we need to check that for any plot $q : U' \rightarrow X_1 \cup_f X_2$ of the base space $X_1 \cup_f X_2$, the evaluation map

$$\epsilon_{p,q} : (u, u') \mapsto (\nabla^\cup(p(u)))(q(u'))$$

is a plot of $\Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$. As usual, it suffices to assume that both U and U' are connected. This in particular implies that q lifts to either a plot q_1 of X_1 or a plot q_2 of X_2 .

Assume first that q lifts to q_1 and consider $\epsilon_{p,q}(u, u')$ for an arbitrary point $(u, u') \in U \times U'$. Recall as a preliminary consideration that, since by assumption p is a plot of $C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$, the following map (the corresponding version of the evaluation map) is smooth:

$$(u, u') \mapsto (p(u))(\tilde{i}_1^{-1}(q(u'))) = (p(u))(q_1(u')).$$

Since for each $u \in U$ the image $p(u)$ is a smooth section of $V_1 \cup_{\tilde{f}} V_2$, it decomposes as $p(u) = p(u)_1 \cup_{(f, \tilde{f})} p(u)_2$, where $p(u)_1 = \tilde{j}_1^{-1} \circ p(u) \circ \tilde{i}_1 \in C^\infty(X_1, V_1)$ and $p(u)_2 = j_2^{-1} \circ p(u) \circ i_2 \in C^\infty(X_2, V_2)$.

We have by construction

$$\epsilon_{p,q}(u, u') = \begin{cases} ((\tilde{\rho}_1^\Lambda)^{-1} \otimes j_1)((\nabla^1 p(u)_1)(q_1(u'))) & \text{if } q_1(u') \in X_1 \setminus Y, \\ \left(\text{Incl}_{\Lambda_{f^{-1}(i_2^{-1}(x))}^1}(X_1) \otimes (j_2 \circ \tilde{f}) \right) ((\nabla^1 p(u)_1)(q_1(u'))) \oplus \\ \oplus \left(\text{Incl}_{\Lambda_{i_2^{-1}(x)}^1}(X_2) \otimes j_2 \right) ((\nabla^2 p(u)_2)(f(q_1(u')))) & \text{if } q_1(u') \in Y. \end{cases}$$

Recall that by Theorem 2.29 the two assignments $u \mapsto p(u)_1$ and $u \mapsto p(u)_2$ defined shortly above are plots of $C^\infty(X_1, V_1)$ and $C^\infty(X_2, V_2)$ respectively. In particular, by assumption we have that $(u, u') \mapsto (\nabla^1 p(u)_1)(q_1(u'))$ is smooth as a map into $\Lambda^1(X_1) \otimes V_1$ onto the set of all pairs (u, u') such that the expression $(\nabla^1 p(u)_1)(q_1(u'))$ makes sense.

On the other hand, we cannot immediately make a similar claim regarding $(\nabla^2 p(u)_2)(f(q_1(u')))$; indeed, f is smooth for the subset diffeology on Y , to which $q_1|_{q_1^{-1}(Y)}$ might not belong. To draw the desired conclusion nonetheless, consider a plot $h : U'' \rightarrow \text{Domain}(\epsilon_{p,q}) \subset U \times U'$, which is just an ordinary smooth function. We need to show that $\epsilon_{p,q} \circ h$ is a plot of $\Lambda^1(X_1 \cup_f X_2) \otimes (V_1 \cup_{\tilde{f}} V_2)$.

To do so, present h as a pair of smooth functions $(h_U, h_{U'})$, where h_U is the composition of h with the projection of its range on U and likewise $h_{U'}$ is its composition with the projection on U' . The composition $\epsilon_{p,q} \circ h$ is then the evaluation of $p \circ h_U$ on $q \circ h_{U'}$. It then remains to notice that $q \circ h_{U'}$ also lifts to a plot $(q \circ h_{U'})_1$ of X_1 , and this lift is a plot for the subset diffeology on Y . Thus,

$$(\epsilon_{p,q} \circ h)(u'') = \left(\text{Incl}_{\Lambda_{f^{-1}(i_2^{-1}(x))}^1}(X_1) \otimes (j_2 \circ \tilde{f}) \right) ((\nabla^1 p(h_U(u''))_1)((q_1 \circ h_{U'}(u'')))) \oplus$$

$$\oplus \left(\text{Incl}_{\Lambda_{i_2^{-1}(x)}^1(X_2)} \otimes j_2 \right) \left((\nabla^2 p(h_U(u''))_2)(f((q_1 \circ h_{U'}) (u'')) \right),$$

and in particular $u'' \mapsto (\nabla^2 p(h_U(u''))_2)(f((q_1 \circ h_{U'}) (u''))$ is now smooth by assumption on ∇^2 . We can therefore conclude that $\epsilon_{p,q} \circ h$ is indeed smooth, which completes the consideration of the case when q lifts to a plot of X_1 .

The treatment of the case when q lifts to a plot q_2 of X_2 is completely analogous, so we omit it. \square

This sequence of statements now trivially yields the following.

Corollary 5.13. *The operator ∇^\cup is a connection on $V_1 \cup_{\tilde{f}} V_2$.*

Proof. This is a consequence of Lemmas 5.8, 5.9, 5.10, and of Proposition 5.12. \square

Theorem 5.14. *Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let (\tilde{f}, f) be a gluing between them such that both \tilde{f} and f are diffeomorphisms of their domains with their images, and let ∇^1 and ∇^2 be compatible connections on V_1 and V_2 respectively. Then $V_1 \cup_{\tilde{f}} V_2$ can be endowed with a connection, that over $i_1(X_1 \setminus Y)$ is naturally equivalent to ∇^1 and over $i_2(X_2 \setminus f(Y))$, to ∇^2 .*

Proof. This is the content of Corollary 5.13; the operator ∇^\cup corresponding to ∇^1 and ∇^2 is a connection and satisfies the claim of the theorem. \square

5.3 Compatibility of the induced connection ∇^\cup with the induced pseudo-metric \tilde{g}

Assume now that the two pseudo-bundles $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ are endowed with pseudo-metrics g_1 and g_2 respectively, and that these pseudo-metrics are compatible with the gluing along (\tilde{f}, f) :

$$g_1(y)(\cdot, \cdot) = g_2(f(y))(\tilde{f}(\cdot), \tilde{f}(\cdot)) \quad \text{for all } y \in Y.$$

Let ∇^1 be a connection on V_1 compatible with g_1 , and let ∇^2 be a connection on V_2 compatible with g_2 . We can then consider the pseudo-metric \tilde{g} on $V_1 \cup_{\tilde{f}} V_2$ obtained by gluing together g_1 and g_2 , and the connection ∇^\cup on it. We wish to show that ∇^\cup is compatible with \tilde{g} .

Recall first that \tilde{g} is defined by

$$\tilde{g}(x)(\cdot, \cdot) = \begin{cases} g_1(i_1^{-1}(x))(j_1^{-1}(\cdot), j_1^{-1}(\cdot)) & \text{if } x \in i_1(X_1 \setminus Y) \\ g_2(i_2^{-1}(x))(j_2^{-1}(\cdot), j_2^{-1}(\cdot)) & \text{if } x \in i_2(X_2). \end{cases}$$

Thus, at least over $i_1(X_1 \setminus Y)$ and $i_2(X_2 \setminus f(Y))$ the compatibility would follow from the assumption on ∇^1 and ∇^2 respectively.

Theorem 5.15. *Let $\pi_1 : V_1 \rightarrow X_1$ and $\pi_2 : V_2 \rightarrow X_2$ be two diffeological vector pseudo-bundles, let (\tilde{f}, f) be a gluing between them such that both \tilde{f} and f are diffeomorphisms of their domains with their images, and let ∇^1 and ∇^2 be compatible connections on V_1 and V_2 respectively. Suppose furthermore that V_1 and V_2 are endowed with pseudo-metrics g_1 and g_2 that are compatible with the gluing along f and \tilde{f} . Assume finally that ∇^1 is compatible with g_1 , and ∇^2 is compatible with g_2 . Then the induced connection ∇^\cup on $V_1 \cup_{\tilde{f}} V_2$ is compatible with the induced pseudo-metric \tilde{g} .*

Proof. Let $s, t \in C^\infty(X_1 \cup_f X_2, V_1 \cup_{\tilde{f}} V_2)$ be two sections. We need to prove the following:

$$d(\tilde{g}(s, t)) = \tilde{g}(\nabla^\cup s, t) + \tilde{g}(s, \nabla^\cup t).$$

Consider the usual splittings of s and t as $s = s_1 \cup_{(f, \tilde{f})} s_2$ and $t = t_1 \cup_{(f, \tilde{f})} t_2$, where $s_1, t_1 \in C^\infty(X_1, V_1)$ and $s_2, t_2 \in C^\infty(X_2, V_2)$. For these splittings, we have by assumption

$$d(g_1(s_1, t_1)) = g_1(\nabla^1 s_1, t_1) + g_1(s_1, \nabla^1 t_1) \quad \text{and} \quad d(g_2(s_2, t_2)) = g_2(\nabla^2 s_2, t_2) + g_2(s_2, \nabla^2 t_2).$$

Since the differential is involved, and by Corollary 5.11, we need to consider three cases, those of a point in $i_1(X_1 \setminus Y)$, a point in $i_2(f(Y))$, and one in $i_2(X_2 \setminus f(Y))$, although the definition of \tilde{g} only has two parts. We also express the function

$$h_{\tilde{g},s,t} : X_1 \cup_f X_2 \ni x \mapsto \tilde{g}(x)(s(x), t(x)) \in \mathbb{R}$$

as the result of gluing of the following two functions:

$$h_{g_1,s_1,t_1} : X_1 \ni x_1 \mapsto g_1(x_1)(s_1(x_1), t_1(x_1)) \quad \text{and} \quad h_{g_2,s_2,t_2} : X_2 \ni x_2 \mapsto g_2(x_2)(s_2(x_2), t_2(x_2)).$$

It is then trivial to check that the gluing of these two functions along f is well-defined (that is, that they are compatible with f , which in turn follows from the compatibility of g_1 with g_2), and that

$$h_{\tilde{g},s,t} = h_{g_1,s_1,t_1} \cup_f h_{g_2,s_2,t_2}.$$

Consider now the first case, $x \in i_1(X_1 \setminus Y)$. Then by Corollary 5.11 and the observation just made

$$d(\tilde{g}(s, t))(x) = (\tilde{\rho}_1^\Lambda)^{-1}(d(g_1(s_1, t_1)(i_1^{-1}(x)))) = (\tilde{\rho}_1^\Lambda)^{-1}((g_1(\nabla^1 s_1, t_1) + g_1(s_1, \nabla^1 t_1))(i_1^{-1}(x))).$$

It is thus sufficient to show that at a point $x \in i_1(X_1 \setminus Y)$ we have

$$(\tilde{\rho}_1^\Lambda)^{-1}((g_1(\nabla^1 s_1, t_1)(i_1^{-1}(x))) = \tilde{g}(\nabla^\cup s, t)(x),$$

and this is a direct consequence of the construction of ∇^\cup . The completely analogous reasoning holds also in the case of $x \in i_2(X_2 \setminus f(Y))$.

It thus remains to consider the case of $x \in i_2(f(Y))$. For such an x we have, first of all,

$$d(\tilde{g}(s, t))(x) = (\text{Incl}_{f^{-1}(i_2^{-1}(x))}^{\Lambda^1}(X_1) \oplus \text{Incl}_{i_2^{-1}(x)}^{\Lambda^1}(X_2))((dh_{g_1,s_1,t_1})(f^{-1}(i_2^{-1}(x))) + (dh_{g_2,s_2,t_2})(i_2^{-1}(x))).$$

As follows from the assumptions on ∇^i , and the linearity properties, what we now need to check is that for any $x \in i_2(f(Y))$ we have

$$(\text{Incl}_{f^{-1}(i_2^{-1}(x))}^{\Lambda^1}(X_1) \oplus \text{Incl}_{i_2^{-1}(x)}^{\Lambda^1}(X_2))(g_1(\nabla^1 s_1, t_1)(f^{-1}(i_2^{-1}(x))) + g_2(\nabla^2 s_2, t_2)(i_2^{-1}(x))) = \tilde{g}(\nabla^\cup s, t).$$

This is also explicit from the construction of ∇^\cup , which completes the proof. \square

Remark 5.16. *One might also consider the potential interplay between the two compatibility notions, one for connections and the other for pseudo-metrics, along the lines of whether one would imply the other (likely, the former, the latter). The proof just given indeed strongly suggests this possibility, at least as long as there are local bases. However, since in general diffeological pseudo-bundles do not have to have them, we do not follow through on this issue.*

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