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ON WAVE OPERATOR IMAGE OF PERTURBED HAMILTONIANS ON THE LINE

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We consider the following perturbed Hamiltonian $\mathcal{H} = -\partial_x^2 + V(x)$ on the real line. The potential $V(x)$, satisfies a short range assumption of type

$$(1+|x|)^\gamma V(x) \in L^1(\mathbb{R}),\ \gamma > 1.$$

We study the wave operator images of classical homogeneous Sobolev type spaces $\dot{H}_p^s(\mathbb{R})$, $p \in (1, \infty)$. It is shown that the assumption zero is not a resonance guarantees that the corresponding wave operators leave classical homogeneous Sobolev spaces of order $s \in [0, 1/p)$ invariant.

1. Introduction and motivation

A typical perturbation of the free Hamiltonian $\mathcal{H}_0 = -\Delta$ in $\mathbb{R}^n, n \geq 1$ is a Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V(x)$ with a short range real-valued potential $V(x)$. The appearance of eigenvectors of $\mathcal{H}$ is an obstacle to establish the existence and

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completeness of the wave operators. The wave operators in $L^p(\mathbb{R}^n)$ space can be defined correctly provided sufficiently strong decay of the potential is fulfilled.

Starting with standard Hilbert space (typically Lebesgue space L^2) one can prove existence and completeness of wave operators in case of short range perturbations (see [13], [14], [11] and the references therein). The functional calculus for the absolutely continuous part $\mathcal{H}_{ac} = P_{ac}(\mathcal{H})\mathcal{H}$ of the perturbed non-negative operator $\mathcal{H}$ can be introduced with a relation involving $W_\pm$

$$g(\mathcal{H}_{ac}) = W_+ g(\mathcal{H}_0) W_+^* = W_- g(\mathcal{H}_0) W_-^*, \tag{1}$$

for any function $g \in L^\infty_{loc}(0,\infty)$. Moreover, the wave operators map unperturbed Sobolev spaces in the perturbed ones,

$$W_\pm : D(\mathcal{H}_0^{s/2}) \to D(\mathcal{H}_{ac}^{s/2})$$

and we have

$$W_\pm : \dot{H}^s_p(\mathbb{R}) \to \dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R}), \ \forall s \geq 0, \ 1 < p < \infty,$$

where $\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})$ is the perturbed homogeneous Sobolev space generated by the Hamiltonian $\mathcal{H}_{ac}$. More precisely, $\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})$ is the homogeneous Sobolev spaces associated with the absolutely continuous part $\mathcal{H}_{ac}$ of the perturbed Hamiltonian $\mathcal{H} = \mathcal{H}_0 + V$. This is the closure of functions $f \in S(\mathbb{R})$ orthogonal[1] to the eigenvectors of $\mathcal{H}$ with respect to the norm

$$\|f\|_{\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})} = \left\| \mathcal{H}_{ac}^{s/2} f \right\|_{L^p(\mathbb{R})}. \tag{2}$$

In this work we study the invariance of the action of the wave operators

$$W_\pm = s - \lim_{t \to \pm\infty} P_{ac}(\mathcal{H}) e^{it\mathcal{H}} e^{-it\mathcal{H}_0}$$

on classical homogeneous Sobolev spaces.

It is clear that the key point to show this invariance property is to verify the equivalence of the fractional energy norms

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})} \sim \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})}. \tag{3}$$

The equivalence property (3) implies that the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R})$ is invariant under the action of the wave operators $W_\pm$ for $0 \leq s < 1/p$.

[1] The precise definition of eigenvectors is given below in (6).

2. Assumptions and main results

The study of the dispersive properties of the evolution flow in some cases of short range perturbed Hamiltonians $\mathcal{H}$ shows (see [2], [7]) that homogeneous Sobolev norms for perturbed and unperturbed Hamiltonians are equivalent

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^2(\mathbb{R})} \sim \|\mathcal{H}_0^{s/2} f\|_{L^2(\mathbb{R})}, \tag{4}$$

provided $s < 1/2$. Our goal is to extend this equivalence to the case

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})} \sim \|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})}, \tag{5}$$

with $s < 1/p$.

Our key assumption on V is that zero is not a resonance point. The precise definition of the notion of resonance point at the origin is given in Definition 1 by the aid of the relation

$$T(0) = 0.$$

The point spectrum of $\mathcal{H}$ consists of real numbers $\lambda \in (-\infty, 0)$, such that

$$\mathcal{H} f - \lambda f = 0, \; f \in L^2(\mathbb{R}), \tag{6}$$

and absolutely continuous part $[0, \infty)$. We shall denote by $L^2_{pp}(\mathbb{R})$ the linear space generated by the eigenvectors f in (6). This is finite dimensional space and its orthogonal complement in L^2 is the invariant subspace, where the perturbed Hamiltonian $\mathcal{H}$ is absolutely continuous.

The key tool to prove (5) is the following estimate.

Theorem 1. *Suppose*

$$V \in L^1_\gamma(\mathbb{R}), \; \gamma > 1, \; s < \gamma - 1 < 1/p, \; p \in (1, \infty)$$

and the perturbed Hamiltonian $\mathcal{H}$ has no resonance at the origin. Then there exists a positive constant $C = C(s,p) > 0$ so that we have

$$\|(\mathcal{H}_{ac}^{s/2} - \mathcal{H}_0^{s/2}) f\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^q(\mathbb{R})},$$

for $1/p - 1/q = s$ and $f \in S(\mathbb{R})$.

It is natural to use a Paley-Littlewood localization associated with the perturbed Hamiltonian. Here and below $\varphi(\tau) \in C_0^\infty(\mathbb{R} \setminus 0)$ is a non-negative even function, such that

$$\sum_{j\in\mathbb{Z}} \varphi\left(\frac{\tau}{2^j}\right) = 1\ , \quad \forall\, \tau \in \mathbb{R} \setminus 0 \tag{7}$$

and

$$\varphi\left(\frac{\tau}{2^k}\right)\varphi\left(\frac{\tau}{2^\ell}\right) = 0,\ \forall\, k,\ell \in \mathbb{Z},\ |k-\ell| \geq 2. \tag{8}$$

We set

$$\pi_k^{ac} = \varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{2^k}\right),\ \pi_k^0 = \varphi\left(\frac{\sqrt{\mathcal{H}_0}}{2^k}\right). \tag{9}$$

We have the following equivalent norm (see [19])

$$\|f\|_{\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})} \sim \left\| \left(\sum_{k=-\infty}^{\infty} 2^{2ks}\, |\pi_k^{ac} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}. \tag{10}$$

Our approach to prove Theorem 1 is based on establishing estimate of the type.

Lemma 1. *If the assumptions of Theorem 1 are fulfilled, then for any* $s \in (0, 1/p)$ *and* $q \in (1,\infty)$ *defined by*

$$\frac{1}{p} - \frac{1}{q} = s$$

we have

$$\left\| \left\| 2^{ks} \left(\pi_k^{ac} - \pi_k^0\right) f \right\|_{\ell^2_k} \right\|_{L^p_x(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})}. \tag{11}$$

Indeed if this estimate is verified, then we can use (10) and see that (11) implies the assertion of Theorem 1.

Therefore, the estimate (11) is the key point in the proof of Theorem 1.

Corollary 1. *If the assumptions of Theorem* 1 *are fulfilled, then the equivalence property* (3) *holds.*

Proof. The results in [4, 15, 1, 3, 19] imply the existence and continuity of the wave operators in L^p, $1 < p < \infty$, so one can deduce Bernstein inequality

$$\|\pi_k^{ac} f\|_{L^q(\mathbb{R})} \leq C(2^k)^{1/p-1/q}\|f\|_{L^p(\mathbb{R})}, \quad 1 \leq p \leq q \leq \infty, \ k \in \mathbb{Z} \tag{12}$$

and via the equivalence property (10) we deduce the Sobolev estimate

$$\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})}, \quad 1 < p < q < \infty, \quad s = \frac{1}{p} - \frac{1}{q}. \tag{13}$$

From the estimate of Theorem 1 now we can write

$$\|(\mathcal{H}_{ac}^{s/2} - \mathcal{H}_0^{s/2}) f\|_{L^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})},$$

so we have

$$\|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})} \leq C\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})}.$$

The opposite estimate can be deduced in the same way from Theorem 1 and the "free" Sobolev estimate

$$\|f\|_{L^q(\mathbb{R})} \leq C\|\mathcal{H}_0^{s/2} f\|_{L^p(\mathbb{R})}, \quad 1 < p < q < \infty, \quad s = \frac{1}{p} - \frac{1}{q}. \tag{14}$$

This completes the proof. □

3. Idea to prove the key Lemma 1

Our main tool to study the kernel

$$\varphi\left(\frac{\sqrt{\mathcal{H}_{ac}}}{M}\right)(x, y)$$

is the following representation of the kernel as filtered Fourier transform

$$\mathcal{F}_{\varphi,M}(a)(\xi) = \int \varphi\left(\frac{\tau}{M}\right) a(\tau) e^{-\mathrm{i}\xi\tau} d\tau \tag{15}$$

of symbols $a(\tau)$ represented as linear combinations with constant coefficients of functions in the set

$$\mathcal{A} = \{\ 1,\ T(\tau),\ R_{\pm}(\tau)\ \}, \tag{16}$$

or more generally of symbols involving functions $a(x,\tau)$ represented as linear combinations with constant coefficients of functions in the set

$$\mathcal{B} = \{\widetilde{m_\pm}(x,\tau),\ T(\tau)\widetilde{m_\pm}(x,\tau),\ R_\pm(\tau)\widetilde{m_\pm}(x,\tau)\ \}, \tag{17}$$

where $\widetilde{m_\pm}(x,\tau) = m_\pm(x,\tau) - 1$, $m_\pm$ are modified Jost functions, while $T, R_\pm$ are the transmission and reflection coefficients.

It is simple to establish that the kernel $\varphi(\sqrt{\mathcal{H}}/M)(x,y)$ can be decomposed as follows (one can see [6]):

Lemma 2. *If φ is an even non-negative function, such that $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$, then for any $M > 0$ we have*

$$\varphi\left(\frac{\sqrt{\mathcal{H}}}{M}\right)(x,y) = K_M^0(x,y) + \widetilde{K}_M(x,y), \tag{18}$$

where $K_M^0(x,y)$ can be represented as sum of the terms

$$\mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(a)(\epsilon_3 x + \epsilon_4 y) \tag{19}$$

and the term $\widetilde{K}_M(x,y)$ is represented as sum of the terms

$$\begin{aligned} &\mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(b_1(x,\cdot))(\epsilon_3 x + \epsilon_4 y) + \\ &\mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(b_2(y,\cdot))(\epsilon_3 x + \epsilon_4 y) + \\ +&\mathbb{1}_{\epsilon_1 x>0}\mathbb{1}_{\epsilon_2 y>0}\mathcal{F}_{\varphi,M}(b_3(x,\cdot)b_4(y,\cdot))(\epsilon_3 x + \epsilon_4 y), \end{aligned} \tag{20}$$

where $\epsilon_i = \pm 1$, for $i = 1,\ldots,4$, $a(\tau)$ represents a linear combination with constant coefficients of functions in the set $\mathcal{A}$ in (16) *and b_i, for $i = 1,\ldots,4$, are linear combinations with constant coefficients of functions in the set $\mathcal{B}$ in* (17).

Remark. We shall call the term $K_M^0(x,y)$ the leading one, with the following exact representation

$$K_M^0(x,y) = c\int_{\mathbb{R}} e^{-\mathrm{i}\tau(x-y)}\varphi\left(\frac{\tau}{M}\right)\alpha(x,y,\tau)\,d\tau \tag{21}$$

with symmetric kernel $\alpha(x,y,\tau) = \alpha(y,x,\tau)$ and

$$\alpha(x,y,\tau) = T(\tau),\ \text{if } x < 0 < y,$$

$$\alpha(x,y,\tau) = (R_+(\tau)+1)e^{2i\tau x} - e^{2i\tau x} + 1,\ \text{if } 0 < x < y,$$

$$\alpha(x,y,\tau) = (R_-(\tau)+1)e^{-2i\tau y} - e^{-2i\tau y} + 1, \text{ if } x < y < 0.$$

The term $\widetilde{K}_M(x,y)$ will be called the remainder one. In Lemma 2 to simplify the notation we neglected the symbolism $a^\pm$, $b_i^\pm$.

A priori estimates for the remainder term are obtained using the estimates of the filtered Fourier transform established in Lemma 11 and Lemma 12.

Lemma 3. *Suppose the condition*

$$\|\langle x\rangle^\gamma V\|_{L^1(\mathbb{R})} < \infty, \quad \gamma \geq 1, \tag{22}$$

is fulfilled with $\gamma > 1+s$, $s \in (0,1)$, *the operator* $\mathcal{H}$ *has no point spectrum and* 0 *is not a resonance point for* $\mathcal{H}$. *If* φ *is an even non-negative function, such that* $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$, *then for any* $p \in (1, 1/s)$, *any* $M \in (0,\infty)$ *and for any* $b^\pm(x,\tau)$, $b_1^\pm(x,\tau)$, $b_2^\pm(x,\tau)$ *in the set* (17) *we have*

$$\begin{gathered}\left\|\int_{\mathbb{R}} \mathbb{1}_{\pm x>0}\mathcal{F}_{\varphi,M}(b^\pm(x,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})} + \\ + \left\|\int_{\mathbb{R}} \mathbb{1}_{\pm y>0}\mathcal{F}_{\varphi,M}(b^\pm(y,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})} \leq \frac{C}{\langle M\rangle}\|f\|_{L^q(\mathbb{R})},\end{gathered} \tag{23}$$

and

$$\begin{gathered}\left\|\int_{\mathbb{R}} \mathbb{1}_{\pm x>0}\mathbb{1}_{\pm y>0}\mathcal{F}_{\varphi,M}(b_1^\pm(x,\cdot)b_2^\pm(y,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})} \leq \\ \leq \frac{C}{\langle M\rangle}\|f\|_{L^q(\mathbb{R})},\end{gathered} \tag{24}$$

where $\frac{1}{q} = \frac{1}{p} - s$.

According with the notation introduced in (9), we set

$$\pi^{ac}_{\leq k} = \sum_{j\leq k}\pi^{ac}_j, \quad \pi^{ac}_{\geq k} = \sum_{j\geq k}\pi^{ac}_j. \tag{25}$$

$$f_k = \pi^{ac}_k f, \quad f_{\leq k} = \sum_{j\leq k}\pi^{ac}_j f, \quad f_{\geq k} = \sum_{j\geq k}\pi^{ac}_j f, \quad f_{k_1,k_2} = \sum_{k_1\leq j\leq k_2}\pi^{ac}_j f$$

and respectively f^0_k, $f^0_{\leq k}$, $f^0_{\geq k}$, $f^0_{k_1,k_2}$ defined as before replacing π^{ac}_j with π^0_j.

Hence, the decomposition (18) can be rewritten as follows

$$\pi^{ac}_k = I_k - (\pi^{ac}_k - I_k),$$

where the operator I_k represents the operators involved in the leading kernel and $(\pi_k^{ac} - I_k)$ is the remainder term.

To prove Lemma 1 we will establish the following inequalities:

$$\left\| \left\| 2^{ks} \left(\pi_k^{ac} - I_k\right) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})}, \tag{26}$$

$$\left\| \left\| 2^{ks} \left(I_k - \pi_k^0\right) f \right\|_{\ell_k^2} \right\|_{L_x^p(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})}, \tag{27}$$

with $1/p = 1/q + s$ and I_k are the operators

$$I_k(f)(x) = \int_{\mathbb{R}} K_{2^k}^0(x, y) f(y) dy$$

with kernels representing the leading term (19) in the expansion of Lemma 2 of π_k.

4. Sup and Hölder type arpiori estimates

4.1. Estimates for the modified Jost functions

In this section we recall some classical results concerning the spectral decomposition of the perturbed Hamiltonian. Recall that the Jost functions are solutions $f_\pm(x, \tau) = e^{\pm i\tau x} m_\pm(x, \tau)$ of $\mathcal{H}u = \tau^2 u$ with

$$\lim_{x \to +\infty} m_+(x, \tau) = 1 = \lim_{x \to -\infty} m_-(x, \tau).$$

We set $x_+ := \max\{0, x\}$, $x_- := \max\{0, -x\}$.

The estimate and the asymptotic expansions of $m_\pm(x, \tau)$ are based on the following integral equations

$$m_\pm(x, \tau) = 1 + K_\pm^{(\tau)}(m_\pm(\cdot, \tau))(x), \tag{28}$$

where $K_\pm^{(\tau)}$ is the integral operator defined as follows

$$K_\pm^{(\tau)}(f)(x) = \pm \int_x^{\pm\infty} D(\pm(t - x), \tau) V(t) f(t) dt$$

and

$$D(t, \tau) = \frac{e^{2it\tau} - 1}{2i\tau} = \int_0^t e^{2iy\tau} dy; \tag{29}$$

The following lemma is well known.

Lemma 4. (see Lemma 1 p. 130 [4] and Lemma 2.1 in [16]) *Assume* $V \in L^1_\gamma(\mathbb{R})$, $\gamma \in [1,2]$. *Then we have the properties:*

a) for any $x \in \mathbb{R}$ *the function*

$$\tau \in \overline{\mathbb{C}_\pm} \mapsto m_\pm(x,\tau), \quad \mathbb{C}_\pm = \{\tau \in \mathbb{C}; \mathrm{Im}\tau \gtrless 0\} \tag{30}$$

is analytic in $\mathbb{C}_\pm$ *and can be extended as a* $C^1(\overline{\mathbb{C}_\pm})$ *function;*

b) there exist constants C_1 *and* $C_2 > 0$ *such that for any* $x, \tau \in \mathbb{R}$*:*

$$\mathbb{1}_{\pm x>0}|m_\pm(x,\tau) - 1| \leq C_1 \langle \tau \rangle^{-1} \ ;$$
$$\mathbb{1}_{\pm x>0}|\partial_\tau m_\pm(x,\tau)| \leq \frac{C_2}{|\tau|^{2-\gamma}\langle \tau \rangle^{\gamma-1}}.$$

A slight improvement is given in the next Lemma.

Lemma 5. (see [6]) *Suppose* $V \in L^1_\gamma(\mathbb{R})$ *with* $\gamma \geq 1$. *Then we have the following properties:*

a) There exists a constant $C > 0$ *such that for any* $x \in \mathbb{R}$, $\tau \in \overline{\mathbb{C}_\pm}$, *we have*

$$|m_\pm(x,\tau) - 1| \leq C \frac{\langle x_\mp \rangle}{\langle x_\pm \rangle^{\gamma-1}}; \tag{31}$$

b) There exists a constant $C > 0$ *such that for any* $x \in \mathbb{R}$, $\tau \in \overline{\mathbb{C}_\pm} \setminus \{0\}$, *we have*

$$|m_\pm(x,\tau) - 1| \leq C \frac{\langle x_\mp \rangle}{\langle x_\pm \rangle^{\gamma}|\tau|}; \tag{32}$$

c) Let $\sigma \in [0,1)$. *Then there exists a constant* $C > 0$ *such that for any* $x \in \mathbb{R}$ *we have*

$$\|m_\pm(x,\tau) - 1\|_{C^{0,\sigma}(\mathbb{C}_\pm)} \leq C \frac{\langle x_\mp \rangle^{1+\sigma}}{\langle x_\pm \rangle^{\gamma-1-\sigma}}, \ \gamma > 1, \ 0 \leq \sigma \leq \gamma - 1; \tag{33}$$

d) Let $\sigma \in [0,1)$. *Then there exists a constant* $C > 0$ *such that for any* $x \in \mathbb{R}$ *we have*

$$\|\tau(m_\pm(x,\tau) - 1)\|_{C^{0,\sigma}(\mathbb{C}_\pm)} \leq C \frac{\langle x_\mp \rangle^{1+\sigma}}{\langle x_\pm \rangle^{\gamma-\sigma}}, \ \gamma > 1. \tag{34}$$

4.2. Estimates for transmision and reflection coefficients

The transmission coefficient $T(\tau)$ and the reflection coefficients $R_\pm(\tau)$ are defined by the formula

$$T(\tau)m_\mp(x,\tau) = R_\pm(\tau)e^{\pm 2\mathrm{i}\tau x}m_\pm(x,\tau) + m_\pm(x,-\tau). \tag{35}$$

From [4] and from [16] we have the following lemma.

Lemma 6. *We have the following properties of the transmissions and reflection coefficients.*

a) $T, R_\pm \in C(\mathbb{R})$*.*

b) There exists $C_1, C_2 > 0$ *such that:*

$$|T(\tau) - 1| + |R_\pm(\tau)| \le C_1\langle\tau\rangle^{-1}$$
$$|T(\tau)|^2 + |R_\pm(\tau)|^2 = 1.$$

c) If $T(0) = 0$, *(i.e. zero is not a resonance point), then for some* $\alpha \in \mathbb{C}\setminus\{0\}$ *and for some* $\alpha_+, \alpha_- \in \mathbb{C}$

$$\begin{aligned} T(\tau) &= \alpha\tau + o(\tau), \quad 1 + R_\pm(\tau) = \alpha_\pm\tau + o(\tau) \quad \text{as } \tau \to 0, \\ T(\tau) &= 1 + O(|\tau|^{-1}), \quad R_\pm(\tau) = O(|\tau|^{-1}) \quad \text{as } \tau \to \infty. \end{aligned}$$

d) there exists a constant $C > 0$ *such that for any* $\tau \in \mathbb{R}$*:*

$$T'(\tau) \le C\langle\tau\rangle^{-1}.$$

The property c) in the last Lemma suggests the following.

Definition 1. *The origin is a resonance point for the hamiltonian* $\mathcal{H}$ *if and only if*

$$T(0) \neq 0.$$

Therefore, taking a bump function $\varphi \in C_0^\infty((0,\infty))$ (with support in $[1/2, 2]$ for example), we have estimates in the the algebra $C([0,4])$ of the terms of type

$$\|\varphi(\cdot)T(M\cdot)\|_{C^0([0,4])} + \|\varphi(\cdot)\,(R_\pm(M\cdot)+1)\|_{C^0([0,4])} \le CM \tag{36}$$

and

$$\left\| \frac{\varphi(\cdot)}{T(M\cdot)} \right\|_{C^0([0,4])} + \left\| \frac{\varphi(\cdot)}{(R_\pm(M\cdot)+1)} \right\|_{C^0([0,4])} \leq CM^{-1} \tag{37}$$

for $M \in (0,1]$.

We can use the assumption $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1$, to get some more precise Hölder type bounds.

Lemma 7. *Suppose $V \in L^1_\gamma(\mathbb{R})$ with $\gamma > 1$ and $T(0) = 0$. Then for any $\sigma \in (0, s]$ and $M \in (0, 1]$ we have:*

a) $T, R_\pm \in C^{0,\sigma}(\mathbb{R})$;

b) for $M \in (0,1)$ we have

$$\|\varphi(\cdot)T(M\cdot)\|_{C^{0,\sigma}((0,+\infty))} + \|\varphi(\cdot)\,(R_\pm(M\cdot)+1)\|_{C^{0,\sigma}((1/2,2))} \leq CM; \tag{38}$$

c) for $M \in [1,\infty)$ we have

$$\begin{aligned} \|\varphi(\cdot)\,(T(M\cdot)-1)\|_{C^{0,\sigma}((0,+\infty))} + \|\varphi(\cdot)R_\pm(M\cdot)\|_{C^{0,\sigma}((1/2,2))} \leq \\ \leq CM^{-1}. \end{aligned} \tag{39}$$

P r o o f. The proof is based on the relations

$$\frac{\tau}{T(\tau)} = \tau - \frac{1}{2i}\int_{\mathbb{R}} V(t)m_+(t,\tau)dt, \quad \tau \in \mathbb{R}\setminus\{0\}, \tag{40}$$

$$R_\pm(\tau) = \frac{T(\tau)}{2i\tau}\int_{\mathbb{R}} e^{\mp 2it\tau}V(t)m_\mp(t,\tau)dt, \ \tau \in \mathbb{R}\setminus\{0\} \tag{41}$$

and the properties of the functions $m_\mp(t,\tau)$ from Lemma 5. Indeed, we can get the estimates

$$\left\| \frac{\varphi(\cdot)}{T(M\cdot)} \right\|_{C^{0,\sigma}([0,4])} + \left\| \frac{\varphi(\cdot)}{(R_\pm(M\cdot)+1)} \right\|_{C^{0,\sigma}([0,4])} \leq CM^{-1} \tag{42}$$

first. Further, we can use the fact[2] that we can control the norm of the inverse of f in the subalgebra $C^{0,\sigma}$ by the norm of f in $C^{0,\sigma}$ and the norm of $1/f$ in $C(T)$

$$\left\| \frac{\varphi(\cdot)}{f(\cdot)} \right\|_{C^{0,\sigma}([0,4])} \leq C\left\| \frac{\widetilde{\varphi}(\cdot)}{f(\cdot)} \right\|_{C^0([0,4])} + \frac{\|\widetilde{\varphi}(\cdot)f\|_{C^{0,\sigma}([0,4])}}{\|f(\cdot)\|^2_{C^0([0,4])}},$$

[2] The problem to have norm-controlled inversion in smooth Banach algebra is well-known and some more general results and references can be found in [9].

where $\widetilde{\varphi} \in C_0^\infty((0,\infty))$ has slightly larger support in $[1/2-\delta, 2+\delta]$ with $\delta > 0$ sufficiently small. Applying this estimate and the estimate (37) and (42) with φ replaced by a cut-off function with slightly larger support, we complete the proof. $\square$

5. Estimates of the filtered Fourier transform of $m_\pm - 1$

Given a bump function $\varphi \in C_0^\infty(\mathbb{R})$, we define the corresponding filtered Fourier transform as in (15). We shall distinguish two different cases. If the bump function $\varphi \in C_0^\infty((0,\infty))$ is such that (7) and (8) are satisfied, then we can assert that $\varphi(\tau/M)$ has a support with $\tau \sim M$.

The integral equation (28) with sign + can be rewritten as

$$\widetilde{m_+}(x,\tau) = \int_x^\infty \int_0^{t-x} e^{2\mathrm{i}\tau y} V(t) dy dt + \int_x^\infty \int_0^{t-x} e^{2\mathrm{i}\tau y} V(t) \widetilde{m_+}(t,\tau) dy dt, \tag{43}$$

where

$$\widetilde{m_+}(x,\tau) = m_+(x,\tau) - 1.$$

If we assume that $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1+s$, then the assertion of Lemma 5 guarantees that $\widetilde{m_+}(x,\tau) = m_+(x,\tau) - 1$ is in $L^1_{x>0}(\mathbb{R})$.

Applying the filtered Fourier transform and setting

$$g_M(\xi;x) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi} \widetilde{m_+}(x,\tau) \varphi\left(\frac{\tau}{M}\right) d\tau = \mathcal{F}_{\varphi,M}(\widetilde{m_+}(x,\cdot))(\xi),$$

we get

$$\begin{aligned} g_M(\xi;x) = \; & \underbrace{M \int_x^\infty \int_0^{t-x} V(t) \widehat{\varphi}(M(\xi - 2y)) dy dt}_{a_M(\xi;x)} + \\ + \; & \int_x^\infty \int_0^{t-x} V(t) g_M(\xi - 2y; t) dy dt. \end{aligned} \tag{44}$$

We have the following pointwise estimates.

Lemma 8. *If $\varphi \in C_0^\infty(\mathbb{R})$, satisfies (7), (8) and $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$, $s \in (0,1)$, then for $M \in (0,1)$ the filtered Fourier transform*

$$\mathcal{F}_{\varphi,M}\left(\widetilde{m_\pm}(x,\cdot)\right)(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi} \widetilde{m_\pm}(x,\tau) \varphi\left(\frac{\tau}{M}\right) d\tau$$

satisfies the estimate

$$\mathbb{1}_{\{\pm x>0\}}\langle x\rangle^s\,|\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi)| \le F_M^\pm(\xi), \tag{45}$$

where

$$F_M^\pm(\xi)\in L^1(\mathbb{R}),\quad \|F_M^\pm\|_{L^1(\mathbb{R})}\le C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})}.$$

P r o o f. We choose the sign $+$ in (45) for determinacy. To prove (45) we set

$$G_M(\xi;x)=\mathbb{1}_{\{x>0\}}\sup_{\eta<\xi}|g_M(\eta;x)|\langle x\rangle^s,$$

where $g_M(\xi;x)$ is the Filtered Fourier transform of the remainder $\widetilde{m_+}(x,\tau)=m_+(x,\tau)-1$, satisfying the integral equation (44). The function

$$F_M(\xi)=M\int_0^\infty\langle t\rangle^\gamma|V(t)|\int_0^t|\widehat{\varphi}(M(\xi-2y))|dydt, \tag{46}$$

satisfies

$$F_M(\xi)\in L^1(\mathbb{R}),\quad \|F_M\|_{L^1(\mathbb{R})}\le\|V\|_{L^1_\gamma(\mathbb{R})}\|\widehat{\varphi}\|_{L^1(\mathbb{R})}. \tag{47}$$

Moreover, since we are considering the case $x>0$ we get easily the following estimates

$$|\mathbb{1}_{x>0}\langle x\rangle^s a_M(\xi;x)|\le F_M(\xi),$$

where $a_M(\xi;x)$ is defined in (44). Hence, coming back to $G_M(\xi;x)$ and recalling (44) we have

$$G_M(\xi;x)\le F_M(\xi)+\int_x^\infty\langle t\rangle|V(t)|G_M(\xi;t)dt,\ \forall x>0. \tag{48}$$

Applying the Gronwall lemma we get

$$G_M(\xi;x)\le CF_M(\xi),$$

where C is a positive constant depending on $\|V\|_{L^1_1(\mathbb{R})}$ and $F_M(\xi)$ satisfies (46) and (47). This completes the proof. $\square$

If $M\ge 1$ and φ satisfying (7) and (8), then we can improve the results of Lemma 8. Indeed, the term $a_M(\xi;x)$ in (44) can be rewritten as follows

$$a_M(\xi;x)=M\int_x^\infty dt\int_{\mathbb{R}}d\tau V(t)e^{-i\tau M\xi}\varphi(\tau)\frac{e^{2iM\tau(x-y)}-1}{2iM\tau}.$$

Hence we have that

$$|\mathbb{1}_{x>0}\langle x\rangle^s a_M(\xi;x)| \le F_M^{(1)}(\xi),$$

where

$$F_M^{(1)}(\xi) = \int_x^\infty \langle t\rangle^s |V(t)||\hat{\varphi}(M\xi)|\,dt \tag{49}$$

and

$$\|F_M^{(1)}(\xi)\|_{L^1(\mathbb{R})} \le \frac{1}{M}\|V\|_{L^1_s(\mathbb{R})}\|\hat{\varphi}\|_{L^1(\mathbb{R})}.$$

Proceeding as in the proof of Lemma 8 we get the following result.

Lemma 9. *If φ satisfies* (7) *and* (8) *and $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1+s$, $s \in (0,1)$, then for $M \in (0,\infty)$ the filtered Fourier transform*

$$\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi}\,(\widetilde{m_\pm}(x,\tau))\,\varphi\left(\frac{\tau}{M}\right)d\tau$$

satisfies the pointwise estimate

$$\mathbb{1}_{\{\pm x>0\}}\langle x\rangle^s\,|\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi)| \le F_M^\pm(\xi), \tag{50}$$

where

$$F_M^\pm(\xi) \in L^1(\mathbb{R}),\quad \|F_M^\pm\|_{L^1(\mathbb{R})} \le \frac{1}{\langle M\rangle}C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})}.$$

One can use a Wiener type argument and deduce estimates for $T(\tau)$, $R_\pm(\tau)+1$.

Lemma 10. (see [3], [19]) *If $\varphi \in C_0^\infty(\mathbb{R})$ obeys* (7), (8) *and $V \in L^1_\gamma(\mathbb{R})$, $\gamma = 1+s$, $s \in (0,1)$, then for $M \in (0,\infty)$ the filtered Fourier transforms*

$$\mathcal{F}_{\varphi,M}(T(\cdot))(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi}T(\tau)\varphi\left(\frac{\tau}{M}\right)d\tau$$

and

$$\mathcal{F}_{\varphi,M}(R_\pm(\cdot)+1)(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi}(R_\pm(\tau)+1)\varphi\left(\frac{\tau}{M}\right)d\tau$$

are in $L^1(\mathbb{R})$ *and the following inequality are satisfied*

$$\begin{aligned}(51)\qquad &\|\mathcal{F}_{\varphi,M}(T(\cdot))(\xi)\|_{L^1(\mathbb{R})}+\|\mathcal{F}_{\varphi,M}(R_\pm(\cdot)+1)(\xi)\|_{L^1(\mathbb{R})}\leq\\ &\qquad C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})},\ M\in(0,1),\\ &\|\mathcal{F}_{\varphi,M}(T(\cdot)-1)(\xi)\|_{L^1(\mathbb{R})}+\|\mathcal{F}_{\varphi,M}R_\pm(\cdot)(\xi)\|_{L^1(\mathbb{R})}\leq\\ &\qquad \frac{1}{\langle M\rangle}C(\|V\|_{L^1_{1+s}(\mathbb{R})})\|\widehat{\varphi}\|_{L^1(\mathbb{R})},\ M>1.\end{aligned}$$

Turning to the estimates (45), we see that

$$a(x,\xi)=\mathbb{1}_{\{\pm x>0\}}\mathcal{F}_{\varphi,M}(\widetilde{m_\pm}(x,\cdot))(\xi)$$

satisfies estimate

$$(52)\qquad |a(x,\xi)|\leq a_1(x)a_2(\xi),\ a_1\in L^{1/s,\infty}(\mathbb{R}),\ a_2\in L^1(\mathbb{R}),$$

where $a_1(x)=\langle x\rangle^{-s}$. Lemma 10 guarantees that

$$b(\xi)=\mathcal{F}_{\varphi,M}(T(\cdot))(\xi)\in L^1(\mathbb{R}).$$

Since

$$\mathbb{1}_{\{\pm x>0\}}\mathcal{F}_{\varphi,M}(T(\cdot)(\widetilde{m_\pm}(x,\cdot)))(\xi)=a(x,\cdot)*b(\cdot)(\xi),$$

we see that

$$|a(x,\cdot)*b(\cdot)(\xi)|\leq a_1(x)\underbrace{a_2*|b|}_{\widetilde{a_2}}(\xi),\ a_1\in L^{1/s,\infty}(\mathbb{R}),\ \widetilde{a_2}\in L^1(\mathbb{R}),$$

since

$$L^1*L^1\subset L^1$$

due to the Young inequality.

The above inclusion actually can be modified in a way suitable for our a priori estimates as follows

$$(53)\qquad \left(L^1\cap L^\infty\right)*\left(L^1\cap L^\infty\right)\subset\left(L^1\cap L^\infty\right).$$

This observation leads to the following.

Lemma 11. *If* $\varphi \in C_0^\infty(\mathbb{R})$, $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1+s$, $s \in (0,1)$, *and* $a^\pm(x,\tau)$ *is any function in the set*

$$\{\widetilde{m_\pm}(x,\tau),\ T(\tau)\widetilde{m_\pm}(x,\tau),\ (R_\pm(\tau)+1)\widetilde{m_\pm}(x,\tau)\}, \tag{54}$$

then for $M \in (0,\infty)$ *the filtered Fourier transform*

$$\mathcal{F}_{\varphi,M}(a^\pm(x,\cdot))(\xi) = \int_{\mathbb{R}} e^{-\mathrm{i}\tau\xi} a^\pm(x,\tau)\varphi\left(\frac{\tau}{M}\right) d\tau$$

satisfies the pointwise estimates:

$$\mathbb{1}_{\{\pm x>0\}} \left|\mathcal{F}_{\varphi,M}(a^\pm(x,\cdot))(\xi)\right| \leq f_1(x) f_2^{(M)}(\xi), \tag{55}$$

where

$$f_1(x) \in L^{1/s,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ f_2^{(M)}(\xi) \in L^1(\mathbb{R})$$

and $\|f_2^{(M)}\|_{L^1(\mathbb{R})} \leq C/\langle M\rangle$.

Finally we consider products of type $a^\pm(x,\tau)b^\pm(y,\tau)$, where a, b are in the set (54) and we have the following estimates.

Lemma 12. *If* $\varphi \in C_0^\infty(\mathbb{R})$ *is a bump function satisfying* (7), (8), $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1+s$, $s \in (0,1)$, *then for* $M \in (0,\infty)$ *the filtered Fourier transform of* $a^\pm(x,\tau)b^\pm(y,\tau)$ *satisfies the pointwise estimate:*

$$\mathbb{1}_{\pm x>0}\mathbb{1}_{\pm y>0} \left|\mathcal{F}_{\varphi,M}(a^\pm(x,\cdot)b^\pm(y,\cdot))(\xi)\right| \leq f_1(x) f_2^{(M)}(\xi) f_3(y), \tag{56}$$

where

$$f_1, f_3 \in L^{1/s,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ f_2^{(M)}(\xi) \in L^1(\mathbb{R}),\ \|f_2^{(M)}\|_{L^1(\mathbb{R})} \leq \frac{C}{\langle M\rangle}$$

with some constant $C > 0$ *independent of* M.

Now we can proceed with the proof of Lemma 2.

Proof of Lemma 2. To fix the idea and to simplify the notation we consider the case involving $b^+(y,\tau) = b(y,\tau)$. We separate two cases: $M \in (0,1]$ and $M \geq 1$. For $M \in (0,1]$ our first step is to prove

$$\left\|\textstyle\int_{\mathbb{R}} \mathbb{1}_{y>0}\mathcal{F}_{\varphi,M}(b(y,\cdot))(x \pm y) f(y) dy\right\|_{L^p_x(\mathbb{R})} \leq C\|f\|_{L^q(\mathbb{R})}. \tag{57}$$

We use the pointwise estimate (55) so we can write

$$\mathbb{1}_{y>0}\,|\mathcal{F}_{\varphi,M}(b(y,\cdot))(x\pm y)| \le B_1^{(M)}(x\pm y)B_2(y),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \le C, \quad B_2 \in L^{1/s,\infty}(\mathbb{R})$$

and (57) follows from Young inequality

$$\left\|B_1^{(M)} * (B_2 f)\right\|_{L^p_x(\mathbb{R})} \le C\|B_1^{(M)}\|_{L^1(\mathbb{R})}\|B_2 f\|_{L^p(\mathbb{R})}, \tag{58}$$

and the Hölder estimate

$$\|B_2 f\|_{L^p(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})}, \quad B_2 \in L^{1/s,\infty}(\mathbb{R}), \ \frac{1}{q} = \frac{1}{p} - s. \tag{59}$$

Similarly, to prove

$$\left\|\int_{\mathbb{R}} \mathbb{1}_{x>0}\mathcal{F}_{\varphi,M}(b(x,\cdot))(x\pm y)f(y)dy\right\|_{L^p_x(\mathbb{R})} \le C\|f\|_{L^q(\mathbb{R})} \tag{60}$$

we use the pointwise estimate (55) again, so we can write

$$\mathbb{1}_{x>0}\,|\mathcal{F}_{\varphi,M}(b(x,\cdot))(x\pm y)| \le B_1^{(M)}(x\pm y)B_2(x),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \le C, \ B_2 \in L^{1/s,\infty}(\mathbb{R}).$$

This time we have to estimate the term

$$\left\|B_2(B_1^{(M)} * f)\right\|_{L^p_x(\mathbb{R})}$$

so first we apply Hölder estimate (59) and then the Young convolution inequality.

Finally, the estimate (24) follows from (56) since we have

$$\mathbb{1}_{x>0}\mathbb{1}_{y>0}\,|\mathcal{F}_{\varphi,M}(b_1(x,\cdot)b_2(y,\cdot))(x\pm y)| \le B_1^{(M)}(x\pm y)B_2(y)B_3(x),$$

where

$$B_1^{(M)} \in L^1(\mathbb{R}), \quad \|B_1^{(M)}\|_{L^1(\mathbb{R})} \le C, \quad B_2(y), B_3(x) \in L^{1/s,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

This completes the proof for the case $M \in (0,1]$. For $M \geq 1$ we simply use the fact that we have better estimate

$$\|B_1^{(M)}\|_{L^1(\mathbb{R})} \leq CM^{-1}$$

and we prove (23) and (24) assuming $V \in L^1_1(\mathbb{R})$ only. This completes the proof. $\square$

6. Equivalence of homogeneous Sobolev norms

In this section we are going to prove Lemma 1.

Proof of the inequality (26). The relation (20) guarantees that

$$\pi_k^{ac}(f)(x) - I_k(f)(x)$$

can be represented as a sum of remainder terms of the form

$$\sum_{\epsilon_1,\ldots,\epsilon_4=\pm1} \mathbb{1}_{\epsilon_1 x>0} \int_{\mathbb{R}} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(b_1(x,\cdot))(\epsilon_3 x + \epsilon_4 y) f(y)\, dy +$$

$$+ \sum_{\epsilon_1,\ldots,\epsilon_4=\pm1} \mathbb{1}_{\epsilon_1 x>0} \int_{\mathbb{R}} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(b_2(y,\cdot))(\epsilon_3 x + \epsilon_4 y) f(y)\, dy +$$

$$+ \sum_{\epsilon_1,\ldots,\epsilon_4=\pm1} \mathbb{1}_{\epsilon_1 x>0} \int_{\mathbb{R}} \mathbb{1}_{\epsilon_2 y>0} \mathcal{F}_{\varphi,M}(b_3(x,\cdot)b_4(y,\cdot))(\epsilon_3 x + \epsilon_4 y) f(y)\, dy,$$

such that the estimates of Lemma 2 imply

$$\|(\pi_k - I_k)\, f\|_{L^p(\mathbb{R})} \leq \frac{C}{\langle 2^k \rangle} \|f\|_{L^q(\mathbb{R})},$$

with

$$\frac{1}{q} = \frac{1}{p} - s.$$

Using the inequalities

$$\left\| \left\| 2^{ks} (\pi_k - I_k)\, f \right\|_{\ell^2_k} \right\|_{L^p_x(\mathbb{R})} \leq \left\| \left\| 2^{ks} (\pi_k - I_k)\, f \right\|_{\ell^1_k} \right\|_{L^p_x(\mathbb{R})} \leq$$

$$\leq \left\| \left\| 2^{ks} \left(\pi_k - I_k \right) f \right\|_{L^p_x(\mathbb{R})} \right\|_{\ell^1_k} \leq \left\| \frac{2^{ks}}{\langle 2^k \rangle} \right\|_{\ell^1_k} \| f \|_{L^q_x(\mathbb{R})} ,$$

and so we deduce (26).

This completes the proof. □

Proof of Lemma 1. Our main goal is to establish the following estimate

$$\left\| \left\| 2^{ks}(\pi_k - \pi_k^0) f \right\|_{\ell^2_k} \right\|_{L^p(\mathbb{R})} \leq C \| f \|_{L^q(\mathbb{R})}, \tag{61}$$

with $1/q = 1/p - s$.

We start proving that

$$\left\| \left\| 2^{ks}(\pi_k - \pi_k^0) f \right\|_{\ell^2_{k\leq 0}} \right\|_{L^p(\mathbb{R})} \leq C \| f \|_{L^q(\mathbb{R})}. \tag{62}$$

In particular, it will be enough to prove the inequality (27), i.e.

$$\left\| \left\| 2^{ks}(I_k - \pi_k^0) f \right\|_{\ell^2_{k\leq 0}} \right\|_{L^p(\mathbb{R})} \leq C \| f \|_{L^q(\mathbb{R})},$$

since the estimate (26) has been just established above.

Using the decomposition

$$f = \sum_{j\in\mathbb{Z}} f_j^0,$$

we have that

$$\left(I_k - \pi_k^0 \right) f = \left(I_k - \pi_k^0 \right) f^0_{k-2,k+2}. \tag{63}$$

Indeed, if follows from

$$\left(I_k - \pi_k^0 \right) f^0_{\leq k-2}(x) = \int \int e^{i(x+y)\tau} \varphi \left(\frac{\tau}{2^k} \right) f^0_{\leq k-2}(y) \, d\tau \, dy = 0$$

and

$$\left(I_k - \pi_k^0 \right) f^0_{\geq k-2}(x) = \int \int e^{i(x+y)\tau} \varphi \left(\frac{\tau}{2^k} \right) f^0_{\geq k-2}(y) \, d\tau \, dy = 0.$$

Moreover, the expression of the leading term shows that the kernel $\left(I_k - \pi_k^0 \right)(x,y)$ can be represented as sum of the terms

$$\mathbb{1}_{\epsilon_1 x > 0} \mathbb{1}_{\epsilon_2 y > 0} \mathcal{F}_{\varphi,M}(a)(\epsilon_3 x + \epsilon_4 y),$$

with $\epsilon_j = \pm 1, j = 1, \ldots, 4$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1$ and $a \in \mathcal{A}$, defined in (16).

For simplicity we consider the case $a = 1$, $\epsilon_j = 1$, $\forall j = 1, \ldots, 4$, and we shall estimate the term

$$\int \mathbb{1}_{x>0}\mathbb{1}_{y>0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{M}\right) d\tau.$$

Then, we can proceed similarly for the other terms.

Integrating by parts and using Lemma 5, we get

$$\left\| 2^{ks} \int\int \mathbb{1}_{x>0}\mathbb{1}_{y>0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{2^k}\right) f_k^0(y)\, d\tau\, dy \right\|_{\ell^2_{k\leq 0}} \leq$$

$$\leq C \int \left\| \frac{2^{k(s+1)} \mathbb{1}_{x>0}\mathbb{1}_{y>0}}{\langle 2^k(x+y)\rangle^{1+s}} f_k^0(y)\, dy \right\|_{\ell^2_{k\leq 0}} dy$$

$$\leq C \int \left\| \frac{2^{k(s+1)} \mathbb{1}_{x>0}\mathbb{1}_{y>0}}{\langle 2^k(x+y)\rangle^{1+s}} \right\|_{\ell^\infty_{k\leq 0}} \|f_k^0\|_{\ell^2_{k\leq 0}}\, dy.$$

From the trivial inequality

$$\left\| \frac{2^{k(s+1)}}{\langle 2^k x\rangle^{1+s}} \right\|_{\ell^\infty_{k\leq 0}} \leq \frac{C}{|x|^{1+s}}$$

combined with the Young inequality in Lorentz spaces we have

$$\left\| \left\| 2^{ks} \int\int \mathbb{1}_{x>0}\mathbb{1}_{y>0} e^{i\tau(x+y)} \varphi\left(\frac{\tau}{2^k}\right) f_k^0(y)\, d\tau\, dy \right\|_{\ell^2_{k\leq 0}} \right\|_{L^p(\mathbb{R})}$$
$$\leq C \left\| \|f_k^0(y)\|_{\ell^2_{k\leq 0}} \right\|_{L^q(\mathbb{R})},$$

with $1/q = 1/p - s$ and $0 < s < 1/p$.

The case $k \geq 0$ follows similarly using the estimate

$$\left|(\pi_k - \pi_k^0) f(x)\right| \leq C \int \frac{f(y)}{\langle 2^k(x \pm y)\rangle^s} \left(\frac{1}{\langle x\rangle} + \frac{1}{\langle y\rangle}\right) dy.$$

This complete the proof. $\square$

7. Appendix II: Equivalence of homogeneous Triebel–Lizorkin spaces and existence of wave operators

In this section we recall some of the properties of Triebel–Lizorkin spaces associated with diadic system of functions $\varphi_j(x) \in C_0^\infty(\mathbb{R})$ satisfying

$$\begin{gathered} \operatorname{supp} \varphi_j \subseteq \{x; |x| \in [2^{j-1}, 2^{j+1}]\}; \\ |\varphi_j^{(k)}(x)| \leq C_k 2^{-kj}, \quad \forall k \in \mathbb{Z}; \\ \textstyle\sum_{j\in\mathbb{Z}} \varphi_j(x) \sim c > 0. \end{gathered} \tag{64}$$

The functional calculus for the absolutely continuous part of $\mathcal{H}$ can be defined by the spectral calculus

$$\phi(\mathcal{H}_{ac}) = \int_0^\infty \phi(\lambda) dE_\lambda^{ac},$$

where dE_λ^{ac} is the absolutely continuous part of the spectral measure for $\mathcal{H}$.

Theorem 1.3 in [20] asserts that the conditions

$$|\varphi_j(\mathcal{H}_{ac})(x,y)| \leq \frac{C 2^{j/2}}{(1 + 2^{j/2}|x-y|)^N}; \tag{65}$$

$$\left|\varphi_j'(\mathcal{H}_{ac})(x,y)\right| \leq \frac{C 2^{j}}{(1 + 2^{j/2}|x-y|)^N} \tag{66}$$

imply

$$\|f\|_{L^p(\mathbb{R})} \sim \left\| \|\varphi_j(\mathcal{H}_{ac}) f\|_{\ell^2_j} \right\|_{L^p(\mathbb{R})} \tag{67}$$

for any $p \in (1, \infty)$ and any $f \in L^p \cap H_{ac}$, where H_{ac} in the absolutely continuous part of L^2. Since the properties of type (67) in the case of free Hamiltonians can be established by using Michlin Fourier multiplier theorem, one can follow the same idea and using Theorem 1.4 in [19] one can conclude that (67) holds for any $H = -\partial_x^2 + V$ with $V \in L^1_1(\mathbb{R})$, such that $\mathcal{H}$ has no resonance at the origin. The results in [19] imply also that Triebel–Lizorkin norms

$$\|f\|_{\dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})} \sim \left\| \left(\sum_{j=-\infty}^{\infty} 2^{2js} \left|\varphi_j(\mathcal{H}_{ac}) f\right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})}, \tag{68}$$

are independent of the choice of the Paley–Littlewood system (64) so two different Paley–Littlewood systems (64) give equivalent Triebel–Lizorkin norms.

The property (67) implies also

$$\|\mathcal{H}_{ac}^{s/2} f\|_{L^p(\mathbb{R})} \sim \left\| 2^{js} \, \|\varphi_j(\mathcal{H}_{ac}) f\|_{\ell^2_j} \right\|_{L^p(\mathbb{R})} . \tag{69}$$

The splitting property

$$\mathcal{H}_{ac}^{s/2} W_+ = W_+ \mathcal{H}_0^{s/2}$$

and the fact that W_+ is L^p bounded imply

$$\|\mathcal{H}_{ac}^{s/2} W_+(g)\|_{L^p} \leq C \|\mathcal{H}_0^{s/2} g\|_{L^p}$$

and we see that $W_+(H^s(\mathbb{R})) \subseteq \dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R})$, i.e.

$$\dot{H}^s_p(\mathbb{R}) \to \dot{H}^s_{p,\mathcal{H}_{ac}}(\mathbb{R}), \ \forall s \geq 0, \ 1 < p < \infty,$$

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