HÖLDER REGULARITY OF THE DENSITIES FOR THE NAVIER–STOKES EQUATIONS WITH NOISE

MARCO ROMITO

ABSTRACT. We prove that the densities of the finite dimensional projections of weak solutions of the Navier–Stokes equations driven by Gaussian noise are bounded and Hölder continuous, thus improving the results of Debussche and Romito [DR14].

The proof is based on analytical estimates on a conditioned Fokker–Planck equation solved by the density, that has a "non–local" term that takes into account the influence of the rest of the infinite dimensional dynamics over the finite subspace under observation.

1. INTRODUCTION

In this paper we improve the results of [DR14] for the law of the solutions of the Navier–Stokes equations with Gaussian noise in dimension three. We consider the problem

$$
\begin{cases} \dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \dot{\eta}, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}
$$

on the torus with periodic boundary conditions and driven by a Gaussian noise $\dot{\eta}$. In the equations above u is the velocity, p the pressure and ν the viscosity of an incompressible fluid. It is known that the above problem admits global weak solutions, as well as unique local strong solutions, as in the deterministic case. Nevertheless the presence of noise allows to prove additional properties, such as continuous dependence on initial data [DPD03, DO06, FR06, FR07, FR08], as well as convergence to equilibrium [Oda07, Rom08]. See also the recent surveys [FR08, Deb13] for a general introduction to the problem.

Our interest in the existence of densities stems from a series of mathematical motivations. The first and foremost is the investigation of the regularity properties of solutions of the Navier–Stokes equations. We study here the regularity

Date: March 1, 2016.

2010 *Mathematics Subject Classification.* Primary 76M35; Secondary 60H15, 60G30, 35Q30.

Key words and phrases. Density of laws, Navier-Stokes equations, stochastic partial differential equations, Besov spaces, Fokker-Planck equation.

properties of densities associated to the probabilistic distribution of the solution, as existence and regularity of densities can be seen as a different type of regularity.

Our results concern the existence of densities for suitable finite dimensional projections of the solutions of (1.1), and one reason for this is that in infinite dimension there is no standard reference measure (as is the Lebesgue measure in finite dimension), any choice should be necessary tailored to the problem at hand, and in our case we do not know enough of the problem.

An interesting difficulty in proving regularity of the densities emerges as a by–product of the more general and fundamental problem of proving uniqueness and regularity of solutions of the Navier–Stokes equations. Indeed, a classical tool for existence and regularity of densities is the Malliavin calculus, and it is easy to be convinced that it is not available here. Formally, the equation satisfied by the Malliavin derivative is the linearisation of (1.1) and thus, estimates on the linearized equation are as good for the density as for uniqueness. We remark that in the case of the two dimensional Navier-Stokes equation, existence and smoothness of densities for the finite dimensional projections of the solutions are proved in [MP06] with Malliavin calculus.

This settles the need of methods to prove existence and regularity of the density that do not rely on this calculus, as done in [DR14]. For other works in this direction, see for instance [DM11, BC14, KHT12, HKHY13, HKHY14].

Existence of densities and their regularity in Besov spaces has been proved in [DR14] (see also [Rom13, Rom14, Rom16a]), by extending and generalising a one dimensional idea from [FP10]. Time regularity of the density has been proved in [Rom16b]. The method introduced in [DR14] is simple but effective and has been already used in other problems (see for instance [DF13, Fou15, SSS15a, SSS15b]).

The results of [DR14] ensure that the density of the projection of solution at some fixed time on some finite dimensional sub–space is in the Besov space $B^{1-}_{1,\infty}$. Roughly speaking, this says that densities (almost) have integrable derivative (see Section 2.1 for a short introduction to Besov spaces).

In this paper we show a proof of Hölder regularity of densities of finite dimensional projections that is completely analytic and, unlike [DR14], does not rely on probabilistic ideas. In fact we follow a classical approach to existence and regularity of densities, namely the Fokker–Planck equation. The Fokker–Planck equation describes the evolution of the density of the Itô process solution of a stochastic equation. Here we only look at a partial information on the solution (a finite dimensional projection), thus we derive in Section 3 a Fokker–Planck equation with a "non–local" term that takes into account the effect of the dynamics outside the finite–dimensional space under observation. The non–local term is indeed a conditional expectation and its regularity is known only in terms of the unknown density itself. This makes our Fokker–Planck equation slightly non–standard. We re–derive in this framework the results of [DR14] (see Proposition 4.1), we then prove the core result of the paper, namely boundedness of the densities, in Proposition 4.4, and finally the Hölder regularity.

Our proof of boundedness requires that, at least at the level of the Galerkin approximations we work with, we already know that the densities are bounded, possibly with bounds depending on the approximation (and so useless for the limiting problem). We derive these bounds on the approximations in the appendix by standard results for hypo–elliptic diffusions and to do so we need to assume that the noise is "sufficiently non–degenerate" (see Section 2.3). With periodic boundary conditions the problem has been already thoroughly analyzed in [Rom04] and this is the reason we mainly focus on the problem on the torus. There is in principle no limitation for the problem with Dirichlet boundary conditions, once smoothness of the densities at the level of approximations is settled.

We believe that the assumption of hypo–ellipticity is just a technical requirement (indeed it is not needed in [DR14]) that depends on the approach we have followed. Moreover, there is an inherent limitation in the Fokker–Planck approach, in that it is less flexible than the method developed in [DR14] and thus cannot be used, in general, to evaluate the density of quantities that do not have an associated evolution equation, as for instance in [SSS15a, SSS15b], as well as for nonlinear functionals of the solution of a diffusion process. On the other hand the method is by no means limited to the Navier–Stokes equations and can be, in principle, applied to other stochastic PDEs with similar features. The development of a probabilistic proof of the results of this paper via a generalization of [DR14] is currently the subject of an on–going work.

2. Main result

2.1. **Notations.**

We shall use the following notations. If K is a Hilbert space, and $F \subset K$ a subspace, we denote by $\pi_F : K \to K$ the orthogonal projection of K onto F, and by $\operatorname{span}[x_1, \dots, x_n]$ the subspace of K spanned by its elements $x_1, \dots, x_n$. Given a linear operator $Q : K \to K'$, we denote by $Q^\star$ its adjoint.

We recall the definition of Besov spaces. Given $f : \mathbf{R}^d \to \mathbf{R}$, define

$$(\Delta_h^1 f)(x) := f(x+h) - f(x),$$

$$(\Delta_h^n f)(x) := \Delta_h^1(\Delta_h^{n-1} f)(x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(x+jh),$$

and, for $s > 0$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$,

$$[f]_{B^s_{p,q}} := \Big(\int_{\{|h|\leqslant 1\}} \frac{\|\Delta_h^n f\|_{L^p}^q}{|h|^{sq}} \frac{dh}{|h|^d} \Big)^{\frac{1}{q}},$$

and for $q = \infty$,

$$[f]_{B^s_{p,\infty}} := \sup_{|h|\leqslant 1} \frac{\|\Delta^n_h f\|_{L^p}}{|h|^s},$$

where n is any integer strictly larger than s (the above semi–norms are independent of the choice of n, as long as $n > s$). Given $s > 0$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, define

$$B^s_{p,q}(\mathbf{R}^d) := \{f : \|f\|_{L^p} + [f]_{B^s_{p,q}} < \infty\}.$$

This is a Banach space when endowed with the norm $\|f\|_{B^s_{p,q}} := \|f\|_{L^p} + [f]_{B^s_{p,q}}$.

When in particular $p = q = \infty$ and $s \in (0,1)$, the Besov space $B^s_{\infty,\infty}(\mathbf{R}^d)$ coincides with the Hölder space $C^s_b(\mathbf{R}^d)$, and in that case we will denote by $\|\cdot\|_{C^s_b}$ and $[\cdot]_{C^s_b}$ the corresponding norm and semi–norm. Notice that a more general definition of Besov spaces, that includes also the case $s \leqslant 0$, is based on the Littlewood–Paley decomposition. We refer to [Tri83, Tri92] for more details on the definitions we have given and the connection with the general definition.

For a Hilbert space K and a finite dimensional sub–space F of K, we shall denote by $L^p(F)$, $C^s_b(F)$ and $B^s_{p,q}(F)$ the Lebesgue, Hölder and Besov spaces, respectively, on F, when F is identified with a Euclidean space.

2.2. **The Navier–Stokes equations.** Let H be the standard space of periodic square summable divergence free vector fields, defined as the closure of periodic divergence free smooth vector fields with zero spatial mean, with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Define likewise V as the closure of the same space of functions with respect to the H^1 norm. Let Π_L be the Leray projector, and let $A = -\Pi_L\Delta$ be the Stokes operator.

In view of Section 2.3 we specify in details an orthonormal basis of H. Let $\mathbf{Z}^3_\star = \mathbf{Z}^3 \setminus \{0\}$ and consider for every $k \in \mathbf{Z}^3_\star$ an orthonormal basis x^1_k, x^2_k of the subspace $k^\perp$ of $\mathbf{R}^3$ orthogonal to the vector k. Choose moreover x^1_k, x^2_k so that $x^i_{-k} = x^i_k$, $i = 1,2$. An orthonormal basis of H is given by $\mathcal{E} = (e^i_k)_{i=1,2,k\in\mathbf{Z}^3_\star}$, where $e^i_k = x^i_k e^{ik\cdot x}$. Clearly $\mathcal{E}$ is a basis of eigenvectors of the Stokes operator A, and different choices of $(x^i_k)_{i=1,2,k\in\mathbf{Z}^3_\star}$ yield different bases.

Define the bi–linear operator $B : V\times V \to V'$ as $B(u,v) = \Pi_L(u\cdot\nabla v)$, $u,v \in V$, and recall that $\langle u_1, B(u_2,u_3)\rangle = -\langle u_3, B(u_2,u_1)\rangle$. We will use the shorthand $B(u)$ for $B(u,u)$. We refer to Temam [Tem95] for a detailed account of all the above definitions.

Let $\mathcal{S} : H \to H$ be a Hilbert–Schmidt operator, then with the above notations, we can consider the following Navier–Stokes equations

$$du + (\nu Au + B(u))\,dt = \mathcal{S}\,dW, \tag{2.1}$$

with initial condition $u(0) = x \in H$, where W is a cylindrical Wiener process (see [DPZ92] for further details) in H. It is well–known [Fla08] that for every

$x \in H$ there exists a martingale solution of this equation, that is a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}, (\widetilde{\mathscr{F}}_t)_{t\geqslant 0})$, a cylindrical Wiener process $\widetilde{W}$ and a process u with trajectories in $C([0,\infty); D(A)') \cap L^\infty_{\text{loc}}([0,\infty), H) \cap L^2_{\text{loc}}([0,\infty); V)$ adapted to $(\widetilde{\mathscr{F}}_t)_{t\geqslant 0}$ such that the above equation is satisfied with $\widetilde{W}$ replacing W.

2.2.1. *Galerkin approximations.* Given an integer $N \geqslant 1$, denote by H_N the sub–space $H_N = \text{span}[e^i_k : i = 1, 2, |k| \leqslant N]$ and denote by $\pi_N = \pi_{H_N}$ the projection onto H_N. It is standard (see for instance [Fla08]) to verify that the problem

$$du^N + \big(\nu A_N u^N + B_N(u^N)\big)\,dt = \pi_N \mathcal{S}\,dW, \tag{2.2}$$

where $B_N(\cdot) = \pi_N B(\pi_N \cdot)$, admits a unique strong solution u^N for every initial condition $x^N \in H_N$. The proof is based on the simple fact that in finite dimension all norms are equivalent, thus given any finite dimensional sub–space F of H, there is $c_1 > 0$ such that

$$\|\pi_F Ax\|_H \leqslant c_1\|x\|_H, \qquad \|\pi_F B(x_1, x_2)\|_H \leqslant c_1\|x_1\|_H\|x_2\|_H.$$

If $x \in H$, $x^N = \pi_N x$ and $\mathbb{P}^N_x$ is the distribution of the solution of the problem above with initial condition x^N, then any limit point of $(\mathbb{P}^N_x)_{N\geqslant 1}$ is a solution of the martingale problem associated to (2.1) with initial condition x. In the rest of the paper we will consider only solutions of (2.1) of this type, as specified by the following definition.

Definition 2.1. A solution of (2.1) with initial condition $x \in H$ is any process u with $u(0) = x$ and with trajectories in $C([0,\infty); D(A)') \cap L^\infty_{\text{loc}}([0,\infty), H) \cap L^2_{\text{loc}}([0,\infty); V)$, such that there are a sequence of integers $N_k \uparrow \infty$ and a sequence $x_k \in H_{N_k}$ such that $x_k \to u(0)$, and u is a limit point, in distributions, of $(u^{N_k})_{k\geqslant 1}$, where u^N is solution of (2.2).

2.3. **Assumptions.** Consider again the Hilbert–Schmidt operator $\mathcal{S} : H \to H$. We will assume

$$\mathcal{S} \text{ is diagonal in a basis } \mathcal{E}, \tag{2.3}$$

where bases $\mathcal{E}$ have been defined in Section 2.2. Under this assumption the operators A, $\mathcal{S}$ commute, $\mathcal{E}$ is also an orthonormal basis of eigenvectors of the covariance $\mathcal{S}\mathcal{S}^\star$, and the noise $\mathcal{S}\dot{W}$ is homogeneous in space. This assumption is taken for the sake of simplicity (See Remark 2.2 below), in view of applying the results of [Rom04].

We shall also need the following global non–degeneracy condition,

$$\text{let } \mathcal{K} = \{k : \langle \mathcal{S}e^i_k, e^i_k\rangle > 0, i = 1, 2\}, \text{ then } \mathcal{K} \text{ is an algebraic system of generators of the group } (\mathbf{Z}^3, +). \tag{2.4}$$

This assumption has been introduced in [Rom04] to ensure that Galerkin approximations (2.2), with N large enough, are hypo–elliptic diffusions. We will use this fact to deduce, see Theorem A.1, that the solution of (2.2) has a smooth density with respect to the Lebesgue measure on H_N. Notice that the assumption essentially requires that the noise is stirring all directions in H, albeit indirectly. We believe this assumption is technical and depends on the way we have proved our results, namely to ensure that the computations in the next section are rigorous. As already mentioned in the introduction, a probabilistic proof of our main results would definitely get rid of this assumption.

We will consider densities for the projections of solutions of (2.1) over finite dimensional sub–spaces of H. Given a finite dimensional subspace F of H, we consider the following conditions,

$$F \text{ is the span of a finite subset of } \mathcal{E}, \tag{2.5}$$

where the basis $\mathcal{E}$ is the same of assumption (2.3), and that the noise is non–degenerate on F, namely,

$$\pi_F \mathcal{S}\mathcal{S}^\star \pi_F \text{ is a non–singular matrix.} \tag{2.6}$$

This condition has been introduced in [DR14, Rom16b], and amounts to say that the covariance has full range in F or, in different words, that the noise is *directly* stirring all directions of F.

Remark 2.2. In general, there is nothing special with the bases $\mathcal{E}$ provided by the eigenvectors of the Stokes operator and our results would work when applied to Galerkin approximations generated by any (smooth enough) orthonormal basis of H. On the other hand we are using the results of [Rom04] and the setting with the bases $\mathcal{E}$ is the most suitable.

One could rather work with a general basis of eigenvectors and assume that the spectral Galerkin approximations are hypo–elliptic diffusions, thus ensuring the conclusions of Theorems A.1 and A.4. We have preferred to proceed with the explicit version of the assumptions. Similar considerations apply for the problem on a bounded domain with Dirichlet boundary conditions.

Remark 2.3. The two non–degeneracy assumptions given above are, in a way, independent. An easy example where assumption (2.4) holds while assumption (2.6) does not hold, is provided in [RX11], where all but the "low modes" are forced by the noise and (2.4) holds. But if F contains low modes components, (2.6) is not true.

On the other hand, fix $k_0 \in \mathbf{Z}^3_\star$ and set

$$\mathcal{S} = \sum_{n=1}^\infty \sigma_n^1 \langle \cdot, e^1_{nk_0}\rangle e^1_{nk_0} + \sigma_n^2 \langle \cdot, e^2_{nk_0}\rangle e^2_{nk_0},$$

so that the set $\mathcal{K}$ introduced in assumption (2.4) is $\mathcal{K} = \{nk_0 : n \in \mathbf{Z}, n \neq 0\}$. It is easy to check that, formally, the Navier–Stokes dynamics (2.1) is closed in the subspace $\operatorname{span}[e^i_k : i = 1, 2, k \in \mathcal{K}]$, and in particular is not hypo–elliptic. If F is spanned by elements of $\mathcal{K}$, then (2.6) holds.

We do believe (and this is the subject of a work in progress) that also in this case the projections on F have Hölder densities. Notice that Proposition 4.1 below only ensures that these densities are in $B^1_{1,\infty}$.

2.4. **Main result.** The main result of the paper is as follows.

Theorem 2.4. *Consider equation* (2.1) *and assume* $\mathcal{S}$ *satisfies assumptions* (2.3) *and* (2.4). *Let* u *be a solution of* (2.1) *as in Definition 2.1 and let* F *be a finite dimensional subspace of* H *satisfying conditions* (2.5) *and* (2.6). *For every* $t > 0$ *denote by* $f_F(t)$ *the density of* $\pi_F u(t)$ *with respect to the Lebesgue measure on* F. *Then for every* $T > 0$ *and every* $\alpha \in (0,1)$,

$$\sup_{(0,T)} t^{\frac{1}{2}(d+\alpha)} \|f_F(t)\|_{C^\alpha_b} < \infty,$$

where $d = \dim F$.

In particular the density f_F is bounded and Hölder continuous for every exponent $\alpha < 1$. The proof of this theorem will be given at the end of Section 4.

3. Formulation of the conditioned Fokker–Planck equation

Under our assumptions (see Theorem A.1) we know that, if u^N is a solution of (2.2), then for every $t > 0$ the law of $u^N(t)$ has a smooth density $f_N(t)$ with respect to the Lebesgue measure in the Schwartz space. Then it is a standard fact that f_N satisfies a Fokker–Planck equation, that in our notations reads

$$\partial_t f_N = \frac{1}{2}\mathscr{A}_N f_N + \nabla_N \cdot \big((\nu A_N x + B_N(x)) f_N\big), \tag{3.1}$$

where

$$\mathscr{A}_N g = \operatorname{Tr}(\mathcal{S}_N \mathcal{S}_N^\star D^2 g),$$

and $\nabla_N \cdot$ is the divergence on H_N.

Fix a subspace F of H_N such that conditions (2.5) and (2.6) hold and consider the projection $\pi_F u^N$ of u^N on F. The marginal density is given by

$$f_{F,N}(t,x') = \int_{F^{\perp_N}} f_N(t, x' + x'')\,dx'',$$

where $F^{\perp_N}$ is the space orthogonal to F in H_N. In the sequel we will understand $x \in H_N$ as $x = (x', x'')$ with $x' \in F$ and $x'' \in F^{\perp_N}$. We wish to derive now an equation satisfied by $f_{F,N}$. By integrating the equation (3.1) over $F^{\perp_N}$, it is not difficult to see that

$$\partial_t f_{F,N} = \frac{1}{2}\mathscr{A}_F f_{F,N} + \nabla_F \cdot (G_{F,N} f_{F,N}), \tag{3.2}$$

where $\nabla_F \cdot$ is the divergence on F, $\mathscr{A}_F = \pi_F \mathscr{A}_N \pi_F$ (the resulting operator is independent of N due to the assumption (2.3) that the covariance is diagonal in the Galerkin basis), and

$$G_{F,N}(t,x') = \nu \pi_F A_N x' + \mathbb{E}[\pi_F B_N(u^N(t)) | \pi_F u^N(t) = x'].\tag{3.3}$$

Indeed, set for brevity $G_N(x) = \nu A_N x + B_N(x)$, then

$$\int_{F^{\perp_N}} \nabla_N \cdot (G_N f_N) = \int_{F^{\perp_N}} \nabla_F \cdot (\pi_F G_N f_N) + \int_{F^{\perp_N}} \nabla_{F^{\perp_N}} \cdot (\pi_{F^{\perp_N}} G_N f_N)$$

where the second integral in the displayed formula above is zero by integration by parts, and the first integral can be reinterpreted as

$$\begin{aligned}\int_{F^{\perp_N}} \nabla_F \cdot (\pi_F G_N f_N)\, dx'' &= \int_{F^{\perp_N}} \nabla_F \cdot \big(\pi_F G_N f_{F^{\perp_N}|F}(t, x|x') f_{F,N}(t, x')\big)\, dx'' \\ &= \nabla_F \cdot \big(\mathbb{E}[\pi_F G_N(u(t))|\pi_F u(t) = x'] f_{F,N}(t, x')\big).\end{aligned}$$

Here $f_{F^{\perp_N}|F}(t, x|x')$ is the conditional density of $u^N(t)$ given $\pi_F u^N(t)$. An additional simplification, due to the fact that A_N is also diagonal in the Galerkin basis, yields (3.3). Similar but simpler computations show that the contribution of $\mathscr{A}_N f_N$, when averaged over $F^{\perp_N}$, is the term $\mathscr{A}_F f_{F,N}$.

Lemma 3.1. *Let* N *be large enough (that* $F \subset H_N$*) and* u^N *be a solution of* (2.2)*. Then* $G_{F,N}(t) \in L^p(F; f_{F,N}\, dx')$ *for every* $p \geqslant 1$ *and* $t > 0$*. Moreover, for every* $T > 0$ *there is* $c_2 = c_2(p, \nu, T, F, \|u^N(0)\|_H) > 0$ *such that*

$$\mathcal{G}_p(T)^p := \sup_{[0,T]} \int_F |G_{F,N}(t, x')|^p f_{F,N}(t, x')\, dx' \leqslant c_2. \tag{3.4}$$

Proof. Fix $p \geqslant 1$ and $t > 0$. Let $f_{F^{\perp_N}|F}(t, x''|x')$ be the conditional density of $u^N(t)$ given $\pi_F u^N(t)$, then by the Hölder inequality,

$$|G_{F,N}(t, x')|^p \leqslant \int_{F^{\perp_N}} |\pi_F G_N(x', x'')|^p f_{F^{\perp_N}|F}(t, x''|x')\, dx'',$$

hence

$$\begin{aligned}\int_F |G_{F,N}(t, x')|^p f_{F,N}(t, x')\, dx' &\leqslant \int_{H_N} |\pi_F G_N(x)|^p f_N(t, x)\, dx \\ &= \mathbb{E}[\|\pi_F G_N(u^N(t))\|_H^p] \\ &\leqslant c_3\big(\nu^p \mathbb{E}[\|u^N(t)\|_H^p] + \mathbb{E}[\|\pi_F B_N(u^N(t))\|_H^p]\big) \\ &\leqslant c_4\big(\nu^p \mathbb{E}[\|u^N(t)\|_H^p] + \mathbb{E}[\|u^N(t)\|_H^{2p}]\big) \\ &\leqslant c_2,\end{aligned}$$

where we have used the fact that on F all norms are equivalent and the (uniform in N) estimate (A.1). □

Remark 3.2 (exponential bound for $G_{F,N}$). A slightly (although useless so far) better estimate can be obtained using (A.2). By the Jensen inequality

$$e^{\lambda |G_{F,N}(t,x')|} \leqslant \int_{F^{\perp_N}} e^{\lambda|\pi_F G_N(x)|} f_{F^{\perp_N}|F}(t, x''|x')\, dx'',$$

hence, as in the proof of the lemma above, and by (A.2),

$$\int_F e^{\lambda|G_{F,N}(t,x')|} f_{F,N}(t,x')\,dx' \leqslant \mathbb{E}[e^{\lambda\|\pi_F G_N(u^N(t))\|_H}] \leqslant \mathbb{E}[e^{\lambda c_5(1+\|u^N(t)\|_H^2)}],$$

that is finite for λ small enough.

Remark 3.3. The equation (3.2) can be recast in a way that shows more explicitly how the contribution of modes in $F^{\perp_N}$ enter in the evolution of $f_{F,N}$. More precisely, for $x \in H_N$,

$$\pi_F B_N(x) = \pi_F B_N(x') + \pi_F B_N(x', x'') + \pi_F B_N(x'', x') + \pi_F B_N(x''),$$

and so

$$\begin{aligned} G_{F,N}(t,x') = \nu A_F x' + B_F(x') &+ \mathbb{E}[\pi_F B_N(\pi_F u^N(t), \pi_{F^{\perp_N}} u^N(t)) \,|\, \pi_F u^N(t) = x'] \\ &+ \mathbb{E}[\pi_F B_N(\pi_{F^{\perp_N}} u^N(t), \pi_F u^N(t)) \,|\, \pi_F u^N(t) = x'] \\ &+ \mathbb{E}[\pi_F B_N(\pi_{F^{\perp_N}} u^N(t)) \,|\, \pi_F u^N(t) = x'], \end{aligned}$$

that is $f_{F,N}$ solves an equation analogous to (3.1) with additional terms that take into account the influence of the evolution of modes from $F^{\perp_N}$.

3.1. **Integral formulation of (3.2).** Due to our assumptions (2.5) and (2.6) on F, the operator $\mathscr{A}_F$ is elliptic and with constant coefficients. Let $\wp^F$ be its kernel, then the integral formulation of equation (3.2) is given by,

$$f_{F,N}(t,x') = \wp_t^F(x' - x_0') - \int_0^t (\nabla_F \wp_{t-s}^F \star \big(G_{F,N}(s,\cdot) f_{F,N}(s,\cdot)\big)(x')\,ds, \tag{3.5}$$

where $x_0' = \pi_f u^N(0)$.

Proposition 3.4. *Under assumptions (2.5) and (2.6) on F, for every p with $1 \leqslant p \leqslant \infty$ and every $n \geqslant 1$ there is $c_6 > 0$ such that for every $h \in F$ with $|h| \leqslant 1$, and $t > 0$,*

$$\|\wp_t^F\|_{L^p} \leqslant c_6 t^{-\frac{d}{2q}} \qquad \|\nabla \wp_t^F\|_{L^p} \leqslant c_6 t^{-(\frac{d}{2q}+\frac{1}{2})}, \tag{3.6}$$

$$\|\Delta_h^n \wp_t^F\|_{L^p} \leqslant c_6 t^{-\frac{d}{2q}} \Big(1 \wedge \frac{|h|}{\sqrt{t}}\Big)^n, \qquad \|\Delta_h^n \nabla \wp_t^F\|_{L^p} \leqslant c_6 t^{-(\frac{d}{2q}+\frac{1}{2})} \Big(1 \wedge \frac{|h|}{\sqrt{t}}\Big)^n, \tag{3.7}$$

where q is the Hölder conjugate exponent of p and $d = \dim F$.

Proof. The operator $\mathscr{A}_F$ is a second order elliptic constant coefficients operator, so its kernel $\wp_t^F$ is, up to an invertible linear change of variables, the standard heat kernel. In particular, by parabolic scaling, $\wp_t^F(x) = t^{-d/2}\wp_1^F(x/\sqrt{t})$. With these observation at hand the proof of the proposition follows from similar computations for the heat kernel. The computations are elementary and thus omitted. □

4. Hölder regularity

In this section we prove our main Theorem 2.4. Prior to this, we give a proof of the result of [DR14] using the Fokker–Planck formulation (see Proposition 4.1). Notice that, consistently with [DR14], we do not need assumption (2.4) to do so. Then, as an intermediate step, we prove boundedness of the densities. This is, not surprisingly in fact, the crucial step and with boundedness at hand the Hölder regularity follows easily. In order to obtain L^∞ bounds we need to know though that densities are smooth (albeit with bounds not necessarily uniform in N). We will take care of this in the appendix.

Notice finally that, unlike in [DR14], we have not been able to derive better bounds for stationary solutions. This may have a twofold reason. On the one hand the higher summability needed to derive better bounds for stationary solutions are not as good as those of Lemma 3.1. On the other hand the method of [DR14] exploits non–trivially some correlations in time that are harder to use in this framework.

4.1. **Basic Besov regularity.** In this section we wish to illustrate a proof of the Besov regularity of the density f_F alternative to [DR14], using the formulation with the conditioned Fokker–Planck (3.5). This is to emphasize that in [DR14] Lemma 3.1 is (implicitly) used only with $p = 1$ and thus that there is space for improvement. This will be then the subject of the rest of the section.

Proposition 4.1. *Let F be a subspace of H satisfying (2.5) and (2.6). Then there is $c_7 > 0$ such that for every $N \geqslant 1$, $x_0 \in H_N$, and $t > 0$*

$$\|f_{F,N}(t)\|_{B^1_{1,\infty}} \leqslant \frac{c_7}{\sqrt{1\wedge t}}\big(1 + \mathcal{G}_1(t)\big),$$

where $f_{F,N}$ is the density of the solution of (2.2) with initial condition x_0.

Proof. Clearly for each $t > 0$ $\|f_{F,N}(t)\|_{L^1} = 1$, so we need to estimate only the Besov semi–norm $[f_{F,N}(t)]_{B^1_{1,\infty}}$. Let $h \in F$, with $|h|_F \leqslant 1$, then by Proposition 3.4

$$\|\Delta_h^2\wp_t^F\|_{L^1} \leqslant c_6\Big(1\wedge\frac{|h|}{\sqrt{t}}\Big) \leqslant \frac{c_6}{\sqrt{1\wedge t}}|h|, \qquad \|\Delta_h^2\nabla\wp_t^F\|_{L^1} \leqslant \frac{c_6}{\sqrt{t}}\Big(1\wedge\frac{|h|}{\sqrt{t}}\Big)^2.$$

We have by (3.5) that

$$\Delta_h^2 f_{F,N}(t) = \Delta_h^2\wp_t^F - \int_0^t (\Delta_h^2\nabla\wp_{t-s}^F)\star(G_{F,N}(s)f_{F,N}(s))\,ds,$$

and, by the Hölder inequality and Lemma 3.1, for every $s \leqslant t$,

$$\begin{aligned}\|(\Delta_h^2\nabla\wp_{t-s}^F)\star(G_{F,N}(s)f_{F,N}(s))\|_{L^1} &\leqslant \|\Delta_h^2\nabla\wp_{t-s}^F\|_{L^1}\|G_{F,N}(s)f_{F,N}(s)\|_{L^1}\\ &\leqslant \frac{c_6}{\sqrt{t-s}}\Big(1\wedge\frac{|h|}{\sqrt{t-s}}\Big)^2\mathcal{G}_1(t),\end{aligned}$$

where $\mathcal{G}_1$ is defined in (3.4). Therefore,

$$\|\Delta_h^2 f_{F,N}(t)\|_{L^1} \leqslant \frac{c_6}{\sqrt{1\wedge t}}|h| + \mathcal{G}_1(t)\int_0^t \frac{c_6}{\sqrt{t-s}}\Big(1\wedge\frac{|h|}{\sqrt{t-s}}\Big)^2\,ds$$
$$\leqslant \frac{4c_6}{\sqrt{1\wedge t}}(1+\mathcal{G}_1(t))|h|,$$

and, by definition, $[f_{F,N}]_{B^1_{1,\infty}} \leqslant 4c_6(1\wedge t)^{-1/2}(1+\mathcal{G}_1(t))$. □

In the above proof we have only used that $G_{F,N} \in L^1(F)$. As long as we only know this, the previous result is essentially optimal. Indeed, the term $\nabla_F\cdot(G_{F,N}f_{F,N})$ is, roughly, in $W^{-1,1}$ hence, by convolution with the heat kernel, we can expect that $f_{F,N}$ is at most in $W^{1,1}$ (that is very close to what we have proved).

Since by Lemma 3.1 the quantity $\mathcal{G}_1$ is uniformly bounded in N, in the limit $N\to\infty$ we can derive a result for solutions of the infinite dimensional problem, as in [DR14].

Corollary 4.2. *Given a weak martingale solution u of the Navier–Stokes equations as in Definition 2.1 and a finite dimensional sub–space F of H such that conditions (2.5) and (2.6) hold, for every $t>0$ the random variable $\pi_F u(t)$ admits a density in $B^1_{1,\infty}(F)$ with respect to the Lebesgue measure on F.*

Proof. Let u be a solution of (2.1) according to Definition 2.1. Then u is a limit point, in distribution, of the sequence $(u^N)_{N\geqslant 1}$ of solutions of (2.2). By Proposition 4.1 and the embedding of $B^1_{1,\infty}$ in L^p (for a $p>1$ depending on the dimension of F), the densities $f_{F,N}(t)$ of $\pi_F u^N(t)$ are uniformly integrable. We can thus find sub–sequences, that we will keep denoting by $(u^N)_{N\geqslant1}$ and $(f_{F,N})_{N\geqslant1}$, such that u^N converges in distribution to u and $f_{F,N}(t)$ converges weakly in L^1 to the density $f_F(t)$ of $\pi_F u(t)$. The $B^1_{1,\infty}$ bound on f_F follows now easily because of the weak convergence. □

4.2. **Boundedness of the densities.** As an intermediate step in the proof of our main result we prove that under our assumptions the densities are bounded. This is the crucial step and having boundedness at hand, Hölder regularity then is not difficult to prove.

Given a finite dimensional sub–space F of H satisfying conditions (2.5) and (2.6), an integer $N\geqslant 1$ large enough that $F\subset H_N$, and a solution u^N of (2.2), define for every $T>0$ and $\alpha>0$,

$$\mathcal{F}^{\alpha}_{F,N}(T) := \sup_{t\in(0,T]} t^{\alpha}\|f_{F,N}(t)\|_\infty. \tag{4.1}$$

Lemma 4.3. *Under assumptions (2.3) and (2.4), if $N\geqslant 1$ is large enough (that $F\subset H_N$), there is $\alpha_0\geqslant\frac{d}{2}$ such that if u^N is a solution of (2.2) and $f_{F,N}(\cdot)$ is the density of $\pi_F u^N(\cdot)$, then $\mathcal{F}^{\alpha_0}_{F,N}(T)<\infty$ for every $T>0$.*

Proof. Fix $k > N$. By Theorem A.4 we know that there is $\alpha_k > 0$ such that $M := \sup_{t,x}(1\wedge t)^{\alpha_k}(1+|x|^k)f_N(t,x) < \infty$. Hence

$$(1\wedge t)^{\alpha_k} f_{F,N}(t,x') = (1\wedge t)^{\alpha_k}\int_{F^{\perp_N}} f_N(t,x)\,dx'' \leqslant \\ \leqslant \int_{F^{\perp_N}} \frac{1}{1+|x''|^k}(1\wedge t)^{\alpha_k}(1+|x|^k)f_N(t,x)\,dx'' \leqslant c_9 M.$$

Therefore $\mathcal{F}^{\alpha_k}_{F,N} < \infty$. □

The next result provides a uniform (in N) estimate of $\mathcal{F}^{d/2}_{F,N}(T)$.

Proposition 4.4. *Under assumptions (2.3) and (2.4), given $T > 0$ and $M > 0$, if F is a subspace of H satisfying (2.5) and (2.6), there is $c_{10} = c_{10}(\nu, T, F, M) > 0$ such that if N is large enough (that $F \subset H_N$), and if u^N is a solution of (2.2) with $\|u^N(0)\|_H \leqslant M$, then*

$$\mathcal{F}^{d/2}_{F,N}(T) \leqslant c_{10}.$$

Proof. Fix $T > 0$ and a solution u^N of (2.2) with initial condition x_0, and set $x_0' = \pi_F x_0$. By (3.5), for every $x' \in F$ and $t \in (0,T]$,

$$|f_{F,N}(t,x')| \leqslant \|\wp^F_t\|_{L^\infty} + \int_0^t \|\nabla\wp^F_{t-s}\star(G_{F,N}(s)f_{F,N}(s))\|_{L^\infty}\,ds. \tag{4.2}$$

Fix $\epsilon \in (0,\frac12)$ and let $d = \dim F$. On the one hand, by Proposition 3.4,

$$\begin{aligned}\int_0^{t-t\epsilon}\|\nabla\wp^F_{t-s}\star(G_{F,N}(s)f_{F,N}(s))\|_{L^\infty}\,ds &\leqslant \int_0^{t-t\epsilon}\|\nabla\wp^F_{t-s}\|_{L^\infty}\|(G_{F,N}f_{F,N})(s)\|_{L^1}\,ds\\ &\leqslant \mathcal{G}_1(T)\int_0^{t-t\epsilon}\frac{c_6}{(t-s)^{\frac{d+1}{2}}}\,ds\\ &\leqslant c_{11}(t\epsilon)^{-\frac12(d-1)}\mathcal{G}_1(T).\end{aligned} \tag{4.3}$$

On the other hand, if $\alpha > 0$, $p \in (1, \frac{d}{d-1})$, and q is such that $\frac1p + \frac1q = 1$, by the Hölder inequality and again Proposition 3.4,

$$\begin{aligned}\int_{t-t\epsilon}^t\|\nabla\wp^F_{t-s}\star(G_{F,N}(s)f_{F,N}(s))\|_{L^\infty}\,ds &\leqslant \int_{t-t\epsilon}^t\|\nabla\wp^F_{t-s}\|_{L^p}\|(G_{F,N}f_{F,N})(s)\|_{L^q}\,ds\\ &\leqslant \mathcal{F}^\alpha_{F,N}(T)^{\frac1p}\mathcal{G}_q(T)\int_{t-t\epsilon}^t\frac{c_6}{s^{\frac{\alpha}{p}}(t-s)^{\frac{d}{2q}+\frac12}}\,ds\\ &\leqslant c_{12}\epsilon^{\frac12-\frac{d}{2q}}t^{\frac12-\frac{\alpha}{p}-\frac{d}{2q}}\mathcal{F}^\alpha_{F,N}(T)^{\frac1p}\mathcal{G}_q(T).\end{aligned} \tag{4.4}$$

Take $\beta \geqslant \frac{d}{2} \vee \left(\frac{d}{2q} + \frac{\alpha}{p} - \frac{1}{2}\right)$, then by using together (4.3) and (4.4) into (4.2) we obtain

$$(4.5)\quad \begin{aligned}\mathcal{F}^{\beta}_{F,N}(T) \leqslant c_6 T^{\beta - \frac{d}{2}} + c_{11}\epsilon^{-\frac{1}{2}(d-1)} T^{\beta - \frac{1}{2}(d-1)} \mathcal{G}_1(T)\\ + c_{12}\epsilon^{\frac{1}{2} - \frac{d}{2q}} T^{\beta + \frac{1}{2} - \frac{\alpha}{p} - \frac{d}{2q}} \mathcal{F}^{\alpha}_{F,N}(T)^{\frac{1}{p}} \mathcal{G}_q(T).\end{aligned}$$

We can now prove that $\mathcal{F}^{d/2}_{F,N}(T) < \infty$. Indeed, let α_0 be the exponent given by Lemma 4.3, so that $\mathcal{F}^{\alpha_0}_{F,N}(T) < \infty$. If $\alpha_0 = \frac{d}{2}$ there is nothing to prove. Otherwise we can take $\alpha_1 = \frac{d}{2} \vee \left(\frac{d}{2q} + \frac{\alpha_0}{p} - \frac{1}{2}\right)$ and, by (4.5), $\mathcal{F}^{\alpha_1}_{F,N}(T) < \infty$ as well. By the choice of p (that gives $q > d$), it turns out that $\alpha_1 < \alpha_0$. It is also clear that by iterating the above procedure with exponents $\alpha_2, \alpha_3, \ldots$, in a finite number of steps we will obtain the exponent $\frac{d}{2}$ and thus that $\mathcal{F}^{d/2}_{F,N}(T) < \infty$.

Finally, we prove the uniform bound on $\mathcal{F}^{d/2}_{F,N}(T) < \infty$. Consider again (4.5) with $\alpha = \beta = \frac{d}{2}$, and use the Young inequality,

$$\begin{aligned}\mathcal{F}^{d/2}_{F,N}(T) &\leqslant c_6 + c_{11}\epsilon^{-\frac{1}{2}(d-1)}\sqrt{T}\mathcal{G}_1(T) + c_{12}\epsilon^{\frac{1}{2} - \frac{d}{2q}}\sqrt{T}\mathcal{G}_q(T)\mathcal{F}^{d/2}_{F,N}(T)^{\frac{1}{p}}\\ &\leqslant c_6 + c_{11}\epsilon^{-\frac{1}{2}(d-1)}\sqrt{T}\mathcal{G}_1(T) + \frac{1}{q}\left(c_{12}\epsilon^{\frac{1}{2} - \frac{d}{2q}}\sqrt{T}\mathcal{G}_q(T)\right)^q + \frac{1}{p}\mathcal{F}^{d/2}_{F,N}(T).\end{aligned}$$

Since $p > 1$ we deduce that $\mathcal{F}^{d/2}_{F,N}(T) \leqslant C(T, \mathcal{G}_1(T), \mathcal{G}_q(T))$ and the conclusion of the theorem finally follows. □

The singularity in time of the L^∞ norm in the previous result clearly originates only from the singularity in the initial condition. It is reasonable then that, when we look for an initial distribution for u^N with a smoother law, the same result should hold with a smaller power for the time. As an example, we give a direct proof of the L^∞ bound of the density of the (unique, see [Rom04]) invariant measure. Denote by $k_{F,N}$ its density, then as in Section 3, the density satisfies the equation

$$\frac{1}{2}\mathscr{A}_F k_{F,N} + \nabla_F \cdot (G_{F,N} k_{F,N}) = 0.$$

Notice that when the averaging in (3.3) is done with respect to the invariant measure, $G_{F,N}$ does not depend on time. Lemma 3.1 holds though and $G_{F,N} \in L^p(F; k_{F,N}\,dx')$ for every $p \geqslant 1$.

Proposition 4.5. *Under assumptions* (2.3) *and* (2.4)*, given a subspace* F *of* H *satisfying* (2.5) *and* (2.6)*, and* $N \geqslant 1$ *large enough, let* $k_{F,N}$ *be the density of the unique invariant measure of* (2.2)*. Then there is* $c_{13} > 0$ *depending only on* ν, F *and some polynomial moment of the invariant measure in* H *such that* $\|k_{F,N}\|_{L^\infty} \leqslant c_{13}$.

Proof. Let $d = \dim F$ and denote by g_F the Green function of $\mathscr{A}_F$. As in Proposition 3.4, g_F can be obtained by an invertible linear transformation from the

Poisson kernel. Hence $\nabla g_F \leqslant c_{14}|x|^{1-d}$. We have

$$\begin{aligned}k_{F,N}(x') &= -(\nabla g_F)\star(G_{F,N}k_{F,N})(x')\\ &= -\big(\nabla g_F \mathbb{1}_{B_\epsilon(0)}\big)\star(G_{F,N}k_{F,N})(x') - \big(\nabla g_F \mathbb{1}_{B^c_\epsilon(0)}\big)\star(G_{F,N}k_{F,N})(x')\\ &= \boxed{\mathrm{i}} + \boxed{\mathrm{o}},\end{aligned}$$

where $\epsilon > 0$ will be chosen later. Fix $q > d$ and let p be its Hölder conjugate exponent. Then

$$\|\boxed{\mathrm{i}}\| \leqslant \|\nabla g_F\|_{L^p(B_\epsilon(0))}\|G_{F,N}k_{F,N}\|_{L^q} \leqslant c_{15}\epsilon^{\frac{d}{p}-(d-1)}\mathcal{G}_q\|k_{F,N}\|_{L^\infty}^{1-\frac{1}{q}},$$

and likewise

$$\|\boxed{\mathrm{o}}\| \leqslant \|\nabla g_F\|_{L^q(B^c_\epsilon(0))}\|G_{F,N}k_{F,N}\|_{L^p} \leqslant \frac{c_{16}}{\epsilon^{(d-1)-\frac{d}{q}}}\mathcal{G}_p\|k_{F,N}\|_{L^\infty}^{1-\frac{1}{p}}.$$

where $\mathcal{G}_p$, $\mathcal{G}_q$ are the quantities in (3.4) but computed on the stationary solution (so there is no need to evaluate the supremum in time).

Collecting the two estimates above and choosing $\epsilon = \|k_{F,N}\|_{L^\infty}^{-1/d}$ yields

$$\begin{aligned}|k_{F,N}(x')| &\leqslant c_{15}\epsilon^{\frac{d}{p}-(d-1)}\mathcal{G}_q\|k_{F,N}\|_{L^\infty}^{1-\frac{1}{q}} + \frac{c_{16}}{\epsilon^{(d-1)-\frac{d}{q}}}\mathcal{G}_p\|k_{F,N}\|_{L^\infty}^{1-\frac{1}{p}}\\ &\leqslant c_{17}(\mathcal{G}_q+\mathcal{G}_p)\|k_{F,N}\|_\infty^{\frac{d-1}{d}},\end{aligned}$$

and hence the statement of the proposition, since $\|k_{F,N}\|_{L^\infty}$ is non–zero and finite by Lemma 4.3. □

Clearly both results above immediately extend to solutions of the infinite dimensional problem that are limit point of Galerkin approximations as in Corollary 4.2.

4.3. **Hölder regularity.** Using the boundedness of the densities proved so far we can give an estimate on $G_{F,N}$ different from (3.4). Indeed, if $p > 1$ and $t > 0$,

$$\begin{aligned}\|G_{F,N}(t)f_{F,N}(t)\|_{L^p} &= \Big(\int_F |G_{F,N}(t,x')f_{F,N}(t,x')|^p\,dx'\Big)^{\frac{1}{p}}\\ &\leqslant \|f_{F,N}(t)\|_{L^\infty}^{1-\frac{1}{p}}\Big(\int_F |G_{F,N}(t,x')|^p f_{F,N}(t,x')\,dx'\Big)^{\frac{1}{p}}\\ &\leqslant \mathcal{G}_p(t)\mathcal{F}_{F,N}^{d/2}(t)^{\frac{1}{q}}t^{-\frac{d}{2q}},\end{aligned}\tag{4.6}$$

where q is the conjugate Hölder exponent of p. In particular it is elementary to verify that $G_{F,N}f_{F,N} \in L^r([0,T];L^p(F))$ for every $T > 0$ and every $p, r \geqslant 1$, with $\frac{2}{r}+\frac{d}{p} > d$. In other words $G_{F,N}f_{F,N}$ has classical summability.

Proposition 4.6. *Under assumptions (2.3) and (2.4), given a subspace F of H satisfying (2.5) and (2.6), $\alpha \in (0,1)$, $T > 0$ and $M > 0$, there is $c_{18} = c_{18}(\nu,\alpha,T,F,M) > 0$*

such that if N *is large enough (that* $F \subset H_N$*), and if* u^N *is a solution of* (2.2) *with* $\|u^N(0)\|_H \leqslant M$*, then*

$$\sup_{t\in[0,T]} t^{\frac{1}{2}(d+\alpha)}\|f_{F,N}(t)\|_{C_b^\alpha} \leqslant c_{18},$$

where $d = \dim F$ *and* $f_{F,N}(\cdot)$ *is the density of* $u^N(\cdot)$*.*

Proof. Fix $\alpha \in (0,1)$, $T > 0$ and a solution u^N of (2.2) with initial condition x_0, and set $x_0' = \pi_F x_0$. By (3.2), for every $x' \in F$, $t \in (0,T]$ and $h \in F$ with $|h| \leqslant 1$,

$$\|\Delta_h f_{F,N}(t)\|_{L^\infty} \leqslant \|\Delta_h \wp_t^F\|_{L^\infty} + \int_0^t \|(\Delta_h \nabla \wp_{t-s}^F) \star (G_{F,N}(s) f_{F,N}(s))\|_{L^\infty}\,ds.$$

Fix $\epsilon \in (0,\frac{1}{2}]$ and set $q = \frac{d}{1-\alpha}$. On the one hand, by Proposition 3.4,

$$\begin{aligned}\int_0^{t-t\epsilon} &\|(\Delta_h \nabla \wp_{t-s}^F) \star (G_{F,N}(s) f_{F,N}(s))\|_{L^\infty}\,ds\\ &\leqslant \int_0^{t-t\epsilon} \|\Delta_h \nabla \wp_{t-s}^F\|_{L^\infty}\|(G_{F,N}(s) f_{F,N}(s)\|_{L^1}\,ds\\ &\leqslant \mathcal{G}_1(T)|h|^\alpha \int_0^{t-t\epsilon} \frac{c_6}{(t-s)^{\frac{d+1+\alpha}{2}}}\,ds\\ &\leqslant c_{19}(t\epsilon)^{-\frac{d+\alpha-1}{2}}\mathcal{G}_1(T)|h|^\alpha.\end{aligned}$$

On the other hand, by the Hölder inequality, the estimate (4.6) and again Proposition 3.4,

$$\begin{aligned}\int_{t-t\epsilon}^t &\|(\Delta_h \nabla \wp_{t-s}^F) \star (G_{F,N}(s) f_{F,N}(s)\|_{L^\infty}\,ds\\ &\leqslant \int_{t-t\epsilon}^t \|\Delta_h \nabla \wp_{t-s}^F\|_{L^p}\|G_{F,N}(s) f_{F,N}(s)\|_{L^q}\,ds\\ &\leqslant \mathcal{F}_{F,N}^{d/2}(T)^{\frac{1}{p}}\mathcal{G}_q(T)\int_{t-t\epsilon}^t \frac{c_6}{s^{\frac{d}{2p}}(t-s)^{\frac{d}{2q}+\frac{1}{2}}}\Big(1 \wedge \frac{|h|}{\sqrt{t-s}}\Big)\,ds\\ &\leqslant c_{20} t^{-\frac{d}{2}(1-\frac{1}{q})}\mathcal{F}_{F,N}^{d/2}(T)^{\frac{1}{p}}\mathcal{G}_q(T)|h|^{1-\frac{d}{q}},\end{aligned}$$

where p is the conjugate Hölder exponent of q. Since by definition $1 - \frac{d}{q} = \alpha$, the two estimates above together yield

$$\begin{aligned}\|\Delta_h f_{F,N}(t)\|_{L^\infty} \leqslant c_6 t^{-\frac{1}{2}(d+\alpha)}|h|^\alpha + c_{19}(t\epsilon)^{-\frac{1}{2}(d+\alpha-1)}\mathcal{G}_1(T)|h|^\alpha&\\ + c_{20} t^{-\frac{1}{2}(d+\alpha-1)}\mathcal{G}_q(T)\mathcal{F}_{F,N}^{d/2}(T)^{\frac{1}{p}}|h|^\alpha&,\end{aligned}$$

and therefore

$$\sup_{[0,T]}(t^{\frac{1}{2}(d+\alpha)}[f_{F,N}(t)]_{B_{\infty,\infty}^\alpha}) \leqslant c_{21}\big(1 + \sqrt{T}\mathcal{G}_1(T) + \sqrt{T}\mathcal{G}_{\frac{d}{1-\alpha}}(T)\mathcal{F}_{F,N}^{d/2}(T)^{\frac{d+\alpha-1}{d}}\big).$$

The conclusion follows since $B_{\infty,\infty}^\alpha = C_b^\alpha$. □

Proof of Theorem 2.4. Let u be a solution of (2.1) according to Definition 2.1. Then u is a limit point, in distribution, of the sequence $(u^N)_{N\geqslant 1}$ of solutions of (2.2), and the density f_F of $\pi_F u$ is a limit point for the weak convergence in L^1 of $(f_{F,N})_{N\geqslant 1}$ by Corollary 4.2.

Proposition 4.6 above states that the sequence $(f_{F,N})_{N\geqslant 1}$ is bounded in $C^\alpha_b(F)$, hence by the Ascoli–Arzelà theorem each sub–sequence admits a further sub–sequence converging uniformly on compact sets of F (and weakly in L^1 since it is also uniformly integrable) to a $C^\alpha_b(F)$ function. This proves the theorem. □

Appendix A. Technical estimates

In this appendix we derive some (non–uniform in N) quantitative bounds on the density f_N of u^N that are necessary as an intermediate step in the proof of our main result. To this end we recall that the solutions of the Galerkin approximations (2.2) satisfy

(A.1)

$$\mathbb{E}\Big[\sup_{[0,T]}\|u^N(t)\|_H^{2p} + \nu\int_0^T \|u^N(t)\|_V^2\|u^N(t)\|_H^{2p-2}\,dt\Big] \leqslant c_{22}(p,\nu,T,\|u^N(0)\|_H,\mathcal{S}),$$

for every $N\geqslant 1$, $u^N(0)\in H$, $T>0$ and $p\geqslant 1$, and the number on the right hand side depends on N only through $\|u^N(0)\|_H$. In fact there is a stronger estimate: there is $\lambda>0$ such that for every $N\geqslant 1$, $u^N(0)\in H$ and $T>0$,

$$\mathbb{E}\Big[\exp\Big(\lambda\sup_{[0,T]}\|u^N(t)\|_H^2 + \nu\lambda\int_0^T \|u^N(t)\|_V^2\,dt\Big)\Big] \leqslant c_{23}(\nu,T,\|u^N(0)\|_H,\mathcal{S}), \tag{A.2}$$

See for instance [Fla08] for details.

The main point in our estimates of Section 4 is that, in order to prove uniform (in N) bounds in L^∞ for the marginal density $f_{F,N}$, we already need to know that $f_{F,N}$ is regular. This in principle does not follow immediately from the regularity of f_N (see Example A.3 below) unless the joint density f_N is in the Schwartz space, as shown in the result below.

Theorem A.1. *Under assumptions (2.3) and (2.4), for every $x_0\in H_N$ the density f_N of the solution u^N of (2.2), with initial condition x_0, is in the Schwartz space.*

Proof. The conclusion follows immediately from [Nua06, Proposition 2.1.5] once we show that $u^N(t)$ is a non–degenerate random variable (see [Nua06, Definition 2.1.1]), namely that $u^N(t)\in\mathbb{D}^\infty$, where $\mathbb{D}^\infty$ is the space of random variable with Malliavin derivatives of arbitrary order, and that $(\det\mathcal{M}_{N,t})^{-1}\in L^p$ for every $p\geqslant 1$, where $\mathcal{M}_{N,t}$ is the Malliavin matrix of $u^N(t)$.

The fact that $u^N(t)\in\mathbb{D}^\infty$ for every $t>0$ can be proved as in [Nua06, Theorem 2.2.2]. The quoted theorem requires that the coefficients should have bounded derivatives of any order larger or equal than one. The coefficients of our equation (2.2) are second order polynomials, and it is not difficult to verify, by going through the proof of the quoted theorem, that the boundedness condition can be replaced by (A.1) to prove existence and uniqueness, and by (A.2) to prove existence of the Malliavin derivatives, since the equations for the Malliavin derivatives are linear.

Finally, the fact that the inverse of the Malliavin matrix has all polynomial moments finite follows as in [Nua06, Theorem 2.3.3] (where again we use (A.2) to replace the boundedness condition on the derivatives), once the Hörmander condition (see [Nua06, Section 2.3.2]) holds. Under assumptions (2.3) and (2.4) the Hörmander condition follows from [Rom04, Lemma 4.2]. □

From the previous theorem we immediately deduce the following consequence, whose proof is based on elementary computations. Notice that, in view of Example A.3, the fact that the density f_N is in the Schwartz space is crucial for the proof of boundedness of $f_{F,N}$.

Corollary A.2. *Under the same assumptions of the previous theorem, the marginal density* $f_{F,N}$ *is in the Schwartz space.*

Example A.3. It is easy to construct a density $\phi \in C^\infty(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$ such that at least one of its marginals is not bounded. Indeed, take $d = 2$ and let $\varphi \in C^\infty(\mathbf{R}^2)$ such that $\varphi \geqslant 0$, $\operatorname{Supp}\varphi \subset (-\frac{1}{2}, \frac{1}{2})^2$, $\|\varphi\|_{L^1} = 1$, and $\|\varphi(0,\cdot)\|_{L^1} + \|\varphi(\cdot, 0)\|_{L^1} > 0$. Set for every $k \in \mathbf{Z}^2$,

$$\phi_k(x) = \begin{cases} \varphi(k_1 k_2 x), & k_1 \neq 0, k_2 \neq 0, \\ 0, & \text{otherwise}, \end{cases}$$

and $\phi = \sum_k \phi_k(x - k)$. Then ϕ is a smooth, bounded density (up to a re–normalization constant), but both marginals are unbounded.

We provide a more quantitative version of the previous theorem by giving the explicit dependence of the supremum norm of the density with respect to time. This is necessary for the proof of Proposition 4.4. Similar bounds can be also given for the derivatives of the densities, although we do not need these estimates in the article.

Theorem A.4. *Under assumptions* (2.3) *and* (2.4), *let* $x_0 \in H_N$ *and let* f_N *be the density of the solution* u^N *of* (2.2) *with initial condition* x_0. *Then for every* $k \geqslant 1$ *there is* $\alpha_k > 0$ *such that for every* $T > 0$,

$$\sup_{t\in(0,T), x\in H_N} \big((1\wedge t)^{\alpha_k}(1 + |x|^k) f_N(t, x)\big) < \infty.$$

Proof. By the previous theorem we know that f_N is in the Schwartz space, so we only have to understand the singularity at $t = 0$. By [Nua06, Proposition 2.1.5] we have the following formulas for the density,

$$f_N(t,x) = \mathbb{E}\Big[\prod_{j=1}^{D_N} \mathbb{1}_{\{u_{N,j}(t)>x_j\}} H_{N,t}\Big] = \mathbb{E}\Big[\prod_{j\neq i} \mathbb{1}_{\{u_{N,j}(t)>x_j\}} \mathbb{1}_{\{u_{N,i}(t)<x_i\}} H_{N,t}\Big],$$

for $x \in H_N$ and $t > 0$, where $D_N = \dim H_N$, $H_{N,t} \in \mathbb{D}^\infty$ is a suitable random variable arising from integration by parts, and $(u_{N,j})_{j=1,\dots,D_N}$ are the components of $u^N(t)$ with respect to a basis of H_N. By the Cauchy–Schwartz inequality, the representation formulas above and (A.1),

$$\begin{aligned}(1+|x|^k) f_N(x) &\leqslant c_{27}\mathbb{E}[(1+\|u^N(t)\|_H^k)|H_t|] \\ &\leqslant c_{28}\mathbb{E}[(1+\|u^N(t)\|_H^k)^2]^{\frac12}\mathbb{E}[|H_t|^2]^{\frac12} \leqslant c_{29}\mathbb{E}[|H_t|^2]^{\frac12},\end{aligned}$$

for every $k \geqslant 1$. By [Nua06, formula (2.32)], given $p \geqslant 1$ there are $\beta, \gamma > 1$ and integers $n, m \geqslant 1$ such that

$$\mathbb{E}[|H_t|^2]^{\frac1p} \leqslant c_{30}\|\det \mathcal{M}_{N,t}^{-1}\|_{L^\beta(\Omega)}^m \|\mathcal{D}u^N(t)\|_{D_N,\gamma},$$

where $\mathcal{D}u^N(t)$ and $\mathcal{M}_{N,t}$ are respectively the Malliavin derivative and the Malliavin matrix of $u^N(t)$, and $\|\cdot\|_{k,p}$, for $k \geqslant 1$ and $p \geqslant 1$ is the norm of the Malliavin–Sobolev space $\mathbb{D}^{k,p}$, see [Nua06, Section 1.2]. By [Nua06, Theorem 2.2.2], together with our exponential bound (A.2), as in the proof of the previous theorem, we see that

$$\sup_{[0,T]} \|\mathcal{D}u^N(t)\|_{D_N,\gamma} < \infty,$$

and that the singularity at $t = 0$ originates, not unexpectedly, from the inverse Malliavin matrix, or equivalently, from the smallest eigenvalue $\lambda_{N,t}$ of $\mathcal{M}_{N,t}$. Indeed, $|\det \mathcal{M}_{N,t}^{-1}| \leqslant \lambda_{N,t}^{-D_N}$, hence

$$\|\det \mathcal{M}_{N,t}^{-1}\|_{L^\beta(\Omega)}^m \leqslant \mathbb{E}[\lambda_{N,t}^{-\beta D_N}]^{\frac{m}{\beta}}.$$

To conclude the proof, we briefly go through the proof of [Nua06, Theorem 2.3.3] and point out only, for brevity, the arguments where the dependence in time arises. To show the finiteness of the moments of $\lambda_{N,t}^{-1}$, we look for the tail probabilities $\mathbb{P}[\lambda_{N,t} \leqslant \epsilon]$ at 0. By [Nua06, Lemma 2.3.1]

$$\mathbb{P}[\lambda_{N,t} \leqslant \epsilon] \leqslant c_{31}\epsilon^{-2D_N} \sup_{|v|=1} \mathbb{P}[v^T \mathcal{M}_{N,t} v \leqslant \epsilon] + \mathbb{P}[|\mathcal{M}_{N,t}| \geqslant \epsilon^{-1}].$$

The second term on the right hand side is not the source of singularity in time and can be easily bounded since $\mathcal{M}_{N,t}$ has all the polynomial moments finite. The first term is bounded using the Norris lemma ([Nua06, Lemma 2.3.2]). Indeed for every $q \geqslant 2$ there is $c_{32} > 0$ such that

$$\sup_{|v|=1} \mathbb{P}[v^T \mathcal{M}_{N,t} v \leqslant \epsilon] \leqslant c_{32}\epsilon^q,$$

if $\epsilon \leqslant \epsilon_0$, for a suitable $\epsilon_0 > 0$. The value of ϵ_0 is identified through the aforementioned Norris lemma, and here one has to look for the singularity in time. It is elementary to see, just by going through the proof of the lemma, that $\epsilon_0 \approx (1 \wedge t)^\alpha$, for some α that depends on q. By this, it follows that

$$\mathbb{E}[\lambda_{N,t}^{-\beta D_N}] \leqslant c_{33}(1 \wedge t)^{-\alpha\beta D_N},$$

as needed. □

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, LARGO BRUNO PONTECORVO 5, I–56127 PISA, ITALIA

E-mail address: romito@dm.unipi.it

URL: http://www.dm.unipi.it/pages/romito